FDM/FEM for nonlinear convection-diffusion-reaction equations with Neumann boundary conditions - convergence analysis for smooth and nonsmooth solutions

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Abstract

This paper aims to present in a systematic form the stability and convergence analysis of a numerical method defined in nonuniform grids for nonlinear elliptic and parabolic convection-diffusion-reaction equations with Neumann boundary conditions. The method proposed can be seen simultaneously as a finite difference scheme and as a fully discrete piecewise linear finite element method. We establish second convergence order with respect to a discrete H^1-norm which shows that the method is simultaneously supraconvergent and superconvergent. Numerical results to illustrate the theoretical results are included.

1 Introduction

In this work we consider the one-dimensional initial boundary value problem (IBVP)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(A(u)\frac{\partial u}{\partial x} + f(u)\right) - qu + g \text{ in } \Omega \times (0, T], \tag{1}$$

where $\Omega = (0, 1)$ and $A, f : \mathbb{R} \longrightarrow \mathbb{R}$, $q, g : \overline{\Omega} \times [0, T] \longrightarrow \mathbb{R}$ are smooth enough, complemented with the homogeneous Neumann boundary conditions

$$A(u)\frac{\partial u}{\partial x} + f(u) = 0, \text{ on } \partial\Omega \times (0, T], \tag{2}$$

and with the initial condition

$$u(x, 0) = u_0(x), \text{ for all } x \in \overline{\Omega}. \tag{3}$$

We also assume that there exist positive constants A_0, q_0 such that

$$q(x, t) \geq q_0 > 0, \text{ for all } x \in \overline{\Omega},\ t \in [0, T] \quad A(x) \geq A_0 > 0, \text{ for all } x \in \mathbb{R}.$$

The differential equation (1) is a convection-diffusion-reaction equation, where the diffusion and convection terms are possibly nonlinear. The analysis of a version of coupled equations of a similar type of (1)-(3) with Dirichlet boundary conditions was recently carried in [6], although its scope was focused on stability and error estimates for a semidiscretization scheme of the system.

The study of numerical methods for Neumann boundary value problems (BVP) or Neumann IBVP has not received much attention recently. However, some studies have been conducted in the past to show proper error estimates for problems similar to (1)-(3). Without being exhaustive we mention [8] where, for a second-order linear differential equation, involving the Laplace operator, defined in a square, a finite difference method based on nonuniform grids was proposed, and its convergence analyzed. We also mention [4] for a an integro-differential equation coupled with a parabolic equation defined in an interval and [10] for quasilinear parabolic equations defined in a square. In both cases, the convergence analysis assumes smooth solutions.

The goal of this paper is to propose and analyse the convergence properties of a numerical scheme for problem (1)-(3). The paper aims to introduce, for nonlinear elliptic and parabolic convection-diffusion-reaction equations with Neumann boundary conditions, supraconvergent finite difference schemes that can be seen as superconvergent piecewise linear finite element method and, simultaneously, present two different approaches for the convergence analysis depending on the solution's smoothness. As it can be seen later, for smooth solutions we use Taylor's formula and this will require the use of a discrete version of the so called trace inequality ([1]). However, the approach used for nonsmooth solutions, relying on the Bramble-Hilbert Lemma ([5]) for the derivation of the error estimates, greatly simplifies the analysis and avoids the use of this discrete trace inequality. We will show that the methods are second order convergent (w.r.t. space) for a discrete H^1-norm. This fact is surprising because the problems are nonlinear and, as finite difference methods, they present first order truncation error with respect to the norm $\|\cdot\|_\infty$, defined in nonuniform meshes, while, as finite element methods, the piecewise linear finite element method is only of first order with respect to the usual H^1 norm.

We recall that the term supraconvergence was introduced in the literature in the 1980s by the finite difference method's community to identify finite difference schemes that present convergence order greater that the order of the truncation error. Without being exhaustive we mention [11, 12, 7, 14, 15, 17, 9]. The term superconvergence was introduced in the literature by the finite element method's community to identify methods that present unexpected convergence order. This phenomena was initially introduced associated with the identification of certain points of the spatial domain where the convergence order is higher than in the rest of the spatial domain. In what concerns this subject we recommend [16] and the references contained there.

In what concerns stability, as we dealing with nonlinear problems, it is a local property. We observe that to have stability for a certain discrete solution we need to impose smoothness assumptions on the local solution. We conclude that we should consider open balls with stepsize dependent radius and the required condition is consequence of the convergence results.

The paper is divided in two parts. In the first part we will start by analysing the convergence properties under the assumption of a smooth solution. We will first consider the fully stationary problem, leading to a nonlinear elliptic boundary value problem and subsequently, analyse a semidiscretization and full discretization of problem (1)-(3). The conducted analysis will be based on the study of the truncation error. The second part of the paper aims at proving similar error estimates under less restrictive regularity assumptions on the solution of problem (1)-(3). Here we will follow the ideas in [6] and carefully use the Bramble-Hilbert Lemma [5] to show convergence of the numerical schemes for stationary and nonstationary problems. The techniques considered in the first part will provide crucial guidelines on the analysis performed on the second part.

2 Some notations and results

Let Λ be a sequence of vectors $h = (h_1, \ldots, h_N)$ with positive entries such that $\sum_{i=1}^{N} h_i = 1$. For each partition h, we define

$$h_{max} = \max_{i=1,\ldots,N} h_i, \quad h_{min} = \min_{i=1,\ldots,N} h_i.$$

We introduce the grids

$$\overline{\Omega}_h = \{x_i : \ i = 0, \ldots, N, \ x_0 = 0, \ x_N = 1, \ x_i = x_{i-1} + h_i, \ i = 1, \ldots, N\},$$

$$\overline{\Omega}_h^* = \overline{\Omega}_h \cup \{x_{-1} = -h_1, x_{N+1} = 1 + h_N\},$$

$$\overline{\Omega}_h^{up} = \overline{\Omega}_h \backslash \{x_0\},$$

$$\partial\Omega_h = \partial\Omega \cap \overline{\Omega}_h$$

and $h_{i+1/2} = \frac{h_i + h_{i+1}}{2}$, $i = 1, \ldots, N-1$, $h_{1/2} = \frac{h_1}{2}$, $h_{N+1/2} = \frac{h_N}{2}$ and $h_0 = h_1$. We say that a family of grids $\left\{\overline{\Omega}_h, \ h \in \Lambda\right\}$ is *quasiuniform* if there exists a positive constant K such that

$$\frac{\max_i h_i}{\min_i h_i} \leq K,$$

for all $h \in \Lambda$.

Let $\mathbb{W}_h$ denote the space of grid functions defined in $\overline{\Omega}_h$, $\mathbb{W}_h^*$ the space of grid functions defined in $\overline{\Omega}_h^*$ and $\mathbb{W}_h^{up}$ the space of grid functions defined in $\overline{\Omega}_h^{up}$. We introduce in $\mathbb{W}_h$ the inner product

$$(v_h, w_h)_h = \sum_{i=0}^{N} h_{i+1/2} v_h(x_i) w_h(x_i)$$

for all $v_h, w_h \in \mathbb{W}_h$ and its corresponding induced norm, $\|\cdot\|_h$.

In $\mathbb{W}_h^{up}$ we introduce the inner product

$$(v_h, w_h)_+ = \sum_{i=1}^{N} h_i v_h(x_i) w_h(x_i), \ v_h, w_h \in \mathbb{W}_h^{up}$$

and its corresponding norm $\|\cdot\|_+ = \sqrt{(\cdot,\cdot)_+}$. We also introduce in $\mathbb{W}_h$ the norm $\|\cdot\|_{h,\infty}$ as a discrete version of the L^∞-norm

$$\|v_h\|_{h,\infty} = \max_{x \in \overline{\Omega}_h} |v_h(x)|.$$

2.1 Difference operators

To describe the spatial discretization on a more compact fashion, we shall introduce some difference operators to ease the notation. Let $u_h \in \mathbb{W}_h^*$ and $i \in \{0, \ldots, N\}$. We define

$$D_x^* u_h(x_i) = \frac{u_h(x_{i+1}) - u_h(x_i)}{h_{i+1/2}}, \tag{4}$$

$$D_{-x} u_h(x_i) = \frac{u_h(x_i) - u_h(x_{i-1})}{h_i}, \tag{5}$$

$$D_c u_h(x_i) = \frac{u_h(x_{i+1}) - u_h(x_{i-1})}{h_i + h_{i+1}}, \tag{6}$$

$$M_h u_h(x_i) = \frac{u_h(x_i) + u_h(x_{i-1})}{2}. \tag{7}$$

Remark 1. Although the difference operators are defined for grid functions in $\mathbb{W}_h^*$, they can be also applied to functions of $\mathbb{W}_h$ (or $\mathbb{W}_h^{up}$) as long as the definitions of the difference operators make sense.

The previous operators allow to introduce a new norm in $\mathbb{W}_h$, denoted by $\|\cdot\|_{1,h}$, defined by

$$\|v_h\|_{1,h} = \sqrt{\|v_h\|_h^2 + \|D_{-x}v_h\|_+^2},\ v_h \in \mathbb{W}_h,$$

which can be seen as a discrete version of the usual H^1-norm, where $D_{-x} : \mathbb{W}_h \longrightarrow \mathbb{W}_h^{up}$ is defined through (5).

A central point in the convergence analysis carried out in the upcoming sections is the relationship between the inner products $(\cdot,\cdot)_h$, $(\cdot,\cdot)_+$ and the difference operators. To this end, we introduce the operators $D_x^* : \mathbb{W}_h^* \longrightarrow \mathbb{W}_h$, $D_c : \mathbb{W}_h^* \longrightarrow \mathbb{W}_h$ and $M_h : \mathbb{W}_h^* \longrightarrow \mathbb{W}_h$ defined through (4), (6) and (7), respectively. The following result establishes a discrete version of the known formulas of integration by parts and can be shown using summation by parts.

Proposition 1. *If $u_h \in \mathbb{W}_h^*$ and $v_h \in \mathbb{W}_h$ then the following formulas hold*

$$\begin{aligned}
-(D_x^* u_h, v_h)_h &= (u_h, D_{-x}v_h)_+ + (M_h u_h(x_1))v_h(x_0) - (M_h u_h(x_{N+1}))v_h(x_N) \\
-(D_c u_h, v_h)_h &= (M_h u_h, D_{-x}v_h)_+ + \frac{1}{4}(u_h(x_1) + 2u_h(x_0) + u_h(x_{-1}))v_h(x_0) \\
&\quad - \frac{1}{4}(u_h(x_{N+1}) + 2u_h(x_N) + u_h(x_{N-1}))v_h(x_N).
\end{aligned}$$

The next result is a discrete trace inequality for functions in $\mathbb{W}_h$. This result will be fundamental in establishing convergence in section 3.

Proposition 2. *For $v_h \in \mathbb{W}_h$ we have*

$$\max_{x\in\partial\Omega_h} |v_h(x)| \le 2\,\|v_h\|_{1,h}\,.$$

Proof. Let $i \in \{1,\ldots,N\}$. From the representation

$$v_h(x_0) = -\sum_{j=1}^{i} h_j D_{-x}v_h(x_j) + v_h(x_i),$$

we obtain

$$v_h(x_0)^2 \le 2\left(\|D_{-x}v_h\|_+^2 + v_h(x_i)^2\right).$$

Consequently,

$$\sum_{i=1}^{N-1} h_{i+1/2}v_h(x_0)^2 \le 2\left(\|D_{-x}v_h\|_+^2 + \sum_{i=1}^{N-1} h_{i+1/2}v_h(x_i)^2\right),$$

that concludes the proof for $x = x_0$. The proof for $x = x_N$ follows the same steps.

□

2.2 Functional spaces

In the following sections we will require certain functionspaces which we now introduce. Let m be a nonnegative integer and $1 \le p \le \infty$. By $L^p(\Omega)$, $H^m(\Omega)$ and $W^{m,\infty}(\Omega)$, we denote the usual Lebesgue and Sobolev spaces with their respective norms, $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{H^m(\Omega)}$

and $\|\cdot\|_{W^{m,\infty}(\Omega)}$. We also introduce $C_B^m(\mathbb{R})$, the space of real differentiable functions with derivative up to order m bounded in $\mathbb{R}$ and its corresponding norm

$$\|u\|_{C_B^m(\mathbb{R})} = \max_{i=0,\dots,m} \|u^{(i)}\|_{L^\infty(\mathbb{R})}.$$

We denote by $C^m([0,T],V)$, $m \in \mathbb{N}_0$, where V is a normed vector space, the space of functions $v : [0,T] \longrightarrow V$ such that $v^{(j)} : [0,T] \longrightarrow V$, $j = 0,\dots,m$, are continuous functions, imbued with the norm

$$\|u\|_{C^m([0,T],V)} = \max_{t\in[0,T]} \|u(t)\|_V$$

where $\|\cdot\|_V$ is a norm in V.

3 Convergence analysis with smooth solutions

We now turn to the analysis of problem (1)-(3). With the aim of analysing a fully discrete scheme to approximate the solution of (1)-(3), we will start by analysing a spatial discretization of the stationary version of (1)-(3). We will then proceed to the analysis of the corresponding semidiscretization of the problem and finally study the fully proposed discrete scheme.

3.1 An elliptic nonlinear boundary value problem

This section starts by considering the elliptic boundary value problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(A(u)\frac{\mathrm{d}u}{\mathrm{d}x} + f(u)\right) + qu = g \text{ in } \Omega, \tag{8}$$

with the boundary conditions

$$A(u)\frac{\mathrm{d}u}{\mathrm{d}x} + f(u) = 0 \text{ on } \partial\Omega, \tag{9}$$

where $A, f \in C^1(\mathbb{R})$, $q, g \in C^0(\overline{\Omega})$. For this boundary value problem, we propose a finite difference method, that can be seen as fully discrete piecewise linear finite element method. We shall prove that it leads to a second-order approximation with respect to a discrete H^1-norm.

We now introduce the operator that will be used to discretize the boundary conditions (9). For $u_h \in \mathbb{W}_h^*$, let D_η denote the operator defined by

$$\begin{aligned}
D_\eta u_h(x_0) = &-\frac{1}{2}\left(A(M_h u_h(x_1))D_{-x}u_h(x_1) + A(M_h u_h(x_0))D_{-x}u_h(x_0)\right)\\
&-\frac{1}{4}\left(f(u_h(x_1)) + 2f(u_h(x_0)) + f(u_h(x_{-1}))\right)\\
D_\eta u_h(x_N) = &\ \frac{1}{2}\left(A(M_h u_h(x_{N+1}))D_{-x}u_h(x_{N+1}) + A(M_h u_h(x_N))D_{-x}v_h(x_N)\right)\\
&+\frac{1}{4}\left(f(u_h(x_{N+1})) + 2f(u_h(x_N)) + f(u_h(x_{N-1}))\right).
\end{aligned}$$

The finite difference scheme to approximate the solution of system (8)-(9) is then established by the set of equations

$$-\left(D_x^*(A(M_h u_h)D_{-x}u_h) + D_c f(u_h)\right) + (R_h q)u_h = R_h g, \quad \text{in } \overline{\Omega}_h \tag{10}$$

and

$$D_\eta u_h(x_0) = D_\eta u_h(x_N) = 0, \tag{11}$$

holding for $u_h \in \mathbb{W}_h^*$, where $R_h : C^0(\overline{\Omega}) \longrightarrow \mathbb{W}_h$ denotes the standard restriction operator.

An important piece in the convergence analysis is the link between the discrete operator $D_x^*(A(M_h u_h)D_{-x}u_h + D_c f(u_h)$ and the boundary operator D_η, which is a direct consequence of Proposition 1.

Proposition 3. *For $u_h \in \mathbb{W}_h^*$ and $v_h \in \mathbb{W}_h$, it holds*

$$\begin{aligned}-(D_x^*(A(M_h u_h)D_{-x}u_h) + D_c f(u_h), v_h)_h = {} & (A(M_h u_h)D_{-x}u_h, D_{-x}v_h)_+ \\ & + (M_h f(u_h), D_{-x}v_h)_+ \\ & - D_\eta u_h(x_0)v_h(x_0) - D_\eta u_h(x_N)v_h(x_N).\end{aligned}$$

This allows to show that our proposed discretization is adequate to approximate the solution of the original elliptic problem. Indeed, let $u \in H^1(\Omega)$ denote the weak solution for problem(8)-(9), *i.e.*, u satisfies

$$(A(u)u' + f(u), v') + (qu, v) = (g, v), \;\forall v \in H^1(\Omega).$$

If P_h denotes de piecewise linear interpolation operator and $u_h \in \mathbb{W}_h$, then

$$(A(P_h u_h)P_h u_h' + f(P_h u_h), P_h v_h') + (qP_h u_h, P_h v_h) = (g, P_h v_h), \;\forall v_h \in \mathbb{W}_h.$$

This leads us to the discrete version of looking for $u_h \in \mathbb{W}_h$ such that

$$(A(M_h u_h)D_{-x}u_h + M_h f(u_h), D_{-x}v_h)_+ + ((R_h q)u_h, v_h)_h = (R_h g, v_h)_h, \;\forall v_h \in \mathbb{W}_h. \tag{12}$$

The following result is a direct consequence of Proposition 3 and establishes the connection between the FEM formulation (12) and the FDM formulation (10)-(11).

Proposition 4. *If $u_h \in \mathbb{W}_h^*$ is solution of problem* (10)-(11)*, then u_h satisfies equation* (12)*.*

The convergence analysis presented in what follows assumes that the solution of the BVP (8)-(9) belongs to $C^4(\overline{\Omega}^*)$ where $\Omega^* = \bigcup_{h\in\Lambda}(-x_1, x_{N+1})$. Let us denote by $R_h^* : C^0(\overline{\Omega}^*) \longrightarrow \mathbb{W}_h^*$ the restriction operator and $T_h \in \mathbb{W}_h$ the truncation error induced by discretization (10) in $\overline{\Omega}_h$, *i.e.*,

$$T_h = -D_x^*(A(M_h R_h^* u)D_{-x}R_h^* u) - D_c f(R_h^* u) + (R_h q)R_h^* u - R_h g$$

and $T_{h,\partial\Omega}$ denote the truncation error associated with the discretization of the boundary conditions (11), *i.e.*,

$$T_{h,\partial\Omega}(x_i) = D_\eta R_h^* u(x_i), \; i = 0, N.$$

Proposition 5. *Let $A \in C_B^4(\mathbb{R})$, $f \in C_B^3(\mathbb{R})$, $g, q \in C^0(\overline{\Omega})$. If $u \in C^4(\overline{\Omega}^*)$ denotes the solution of equation* (8) *then the truncation error $T_h(x_i)$, $i = 1, \ldots, N-1$ can be decomposed as*

$$T_h(x_i) = T_h^{(1)}(x_i) + T_h^{(2)}(x_i)$$

where

1. $T_h^{(1)}(x_i) = (h_{i+1} - h_i)R(x_i)$, *for $R \in C^1(\overline{\Omega})$ defined as*

$$R(x) = -\frac{1}{3}\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(A\left(\frac{u(x) + R_h u(x_i)}{2}\right)(u(x) - R_h u(x_i))\right) - \frac{1}{2}\frac{\mathrm{d}^2 f(u)}{\mathrm{d}x^2}(x)$$

2. $T_h^{(2)}(x_i)$ *is of the order of* h_{max}^2*, and there exists a positive constant* C*, h-independent, such that*

$$|T_h^{(2)}(x_i)| \leq Ch_{max}^2 \|u\|_{C^4(\overline{\Omega})}, \quad i = 1, \dots, N-1.$$

Moreover, the following estimates hold

$$|T_h(x_0)| \leq Ch_1^2 \|u\|_{C^4(\overline{\Omega}^*)},$$
$$|T_h(x_N)| \leq Ch_N^2 \|u\|_{C^4(\overline{\Omega}^*)}$$

for some positive constant, h-independent, C*.*

Proposition 6. *Let* $A \in C_B^3(\mathbb{R})$*,* $f \in C_B^2(\mathbb{R})$ *and* $u \in C^3(\overline{\Omega}^*)$ *denote the solution of equation* (8)*. Then there exists a positive constant* C*, h-independent, such that*

$$|T_{h,\partial\Omega}(x_0)| \leq Ch_1^2 \|u\|_{C^3(\overline{\Omega}^*)} \quad \text{and} \quad |T_{h,\partial\Omega}(x_N)| \leq Ch_N^2 \|u\|_{C^3(\overline{\Omega}^*)}.$$

We are now able to establish a convergence result for the discretization (10)-(11) and provide a bound for the discretization error $E_u = u_h - R_h u \in \mathbb{W}_h$.

Theorem 1. *Let* $u \in C^4(\overline{\Omega}^*)$ *denote the solution of problem* (8)-(9) *and* $u_h \in \mathbb{W}_h^*$ *denote the solution of system* (10)-(11)*. If* $A \in C_B^4(\mathbb{R})$*,* $f \in C_B^3(\mathbb{R})$*,* $g, q \in C^0(\overline{\Omega})$ *then there exists a positive constant* C*, h-independent, such that*

$$\|E_u\|_{1,h} \leq Ch_{max}^2 \|u\|_{C^4(\overline{\Omega}^*)}, \tag{13}$$

for $h \in \Lambda$*, provided that*

$$\|u'\|_{L^\infty(\Omega)} \|A'\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})} < \min\left\{A_0, \frac{q_0}{2}\right\}. \tag{14}$$

Proof. It can be easily shown that the following equation holds for E_u

$$\begin{aligned}(A(M_h u_h)D_{-x}E_u, D_{-x}E_u)_+ &+ ((R_h q)E_u, E_u)_h \\ &= ((A(M_h R_h u) - A(M_h u_h))D_{-x}R_h u, D_{-x}E_u)_+ \\ - (M_h(f(u_h) - f(R_h u)), D_{-x}E_u)_+ &- (T_h, E_u)_h - \sum_{i=0,N} T_{h,\partial\Omega}(x_i)E_u(x_i). \end{aligned} \tag{15}$$

We now need to estimate all terms in the right hand side of equation (15). Let

$$\begin{aligned}\tau_1 &= ((A(M_h R_h u) - A(M_h u_h))D_{-x}R_h u, D_{-x}E_u)_+ \\ \tau_2 &= -(T_h, E_u)_h - \sum_{i=0,N} T_{h,\partial\Omega}(x_i)E_u(x_i) \\ \tau_3 &= -(M_h(f(u_h) - f(R_h u)), D_{-x}E_u)_+.\end{aligned}$$

- For τ_1, it easily follows that

$$|\tau_1| \leq \|u'\|_{L^\infty(\Omega)} \|A'\|_{L^\infty(\mathbb{R})} \|E_u\|_h \|D_{-x}E_u\|_+.$$

- Regarding τ_2, using Proposition 5, the following representation holds

$$\begin{aligned}\tau_2 = &\sum_{i=0,N} \left(T_{h,\partial\Omega}(x_i) - \frac{h_i}{2}T_h(x_i) + s_i\frac{h_i^2}{2}R(x_i)\right)E_u(x_i) \\ &+ \frac{1}{2}\sum_{i=1}^{N} h_i^3(R(x_i)D_{-x}E_u(x_i) + D_{-x}R(x_i)E_u(x_{i-1}) \\ &- \sum_{i=1}^{N-1} h_{i+1/2}T_h^{(2)}(x_i)E_u(x_i),\end{aligned}$$

where $s_0=-1$, $s_N=1$ and, to simplify the presentation, we consider that $h_0=h_1$.

Therefore, using the bounds from Propositions 2, 5 and 6, there exists a positive constant $\tilde{C}$, h-independent, such that, for all $\epsilon\neq 0$,

$$\begin{aligned}|\tau_2| \leq& \frac{\tilde{C}}{4\epsilon^2}h_{max}^4\|u\|_{C^4(\overline{\Omega}^*)}+2\epsilon^2\|E_u\|_{1,h}^2\\&+\frac{h_{max}^2}{2}\|R\|_{L^\infty(\Omega)}\|D_{-x}E_u\|_+ +\frac{\sqrt{2}}{2}h_{max}^2\left\|R'\right\|_{L^\infty(\Omega)}\|E_u\|_h\\&+\tilde{C}h_{max}^2\|u\|_{C^4(\overline{\Omega}^*)}\|E_u\|_h\end{aligned}$$

- Finally, for τ_3 we can establish that

$$|\tau_3|\leq\left\|f'\right\|_{L^\infty(\mathbb{R})}\|E_u\|_h\|D_{-x}E_u\|_+.$$

From the lower bounds for A and q and combining equation (15) with the estimates for τ_i, $i=1,2,3$, we finally obtain

$$\begin{aligned}\left(A_0-4\epsilon^2\right)\|D_{-x}E_u\|_+^2+\left(q_0-4\epsilon^2-\frac{1}{4\epsilon^2}\left(\left\|u'\right\|_{L^\infty(\Omega)}\left\|A'\right\|_{L^\infty(\mathbb{R})}+\left\|f'\right\|_{L^\infty(\mathbb{R})}\right)^2\right)\|E_u\|_h^2\\ \leq Ch_{max}^4\|u\|_{C^4(\overline{\Omega}^*)}^2.\end{aligned}$$

for some positive constant C, h-independent. Choosing

$$\epsilon^2=\frac{\|u'\|_{L^\infty(\Omega)}\|A'\|_{L^\infty(\mathbb{R})}+\|f'\|_{L^\infty(\mathbb{R})}}{4}$$

and using condition (14), we finally conclude estimate (13).

□

3.2 Parabolic IBVP

3.2.1 A semidiscrete approximation

This section aims to extend the results of the previous section to the IBVP (1)-(3). Let $u_h(t)\in\mathbb{W}_h^*$, $t\in[0,T]$, denote the semidiscrete approximation for the previous IBVP, defined by the spatial discretization studied in the last section. This means that $u_h(t)\in\mathbb{W}_h^*$ is defined by

$$\begin{cases}\dfrac{\mathrm{d}u_h}{\mathrm{d}t}(t)=D_x^*(A(M_hu_h(t))D_{-x}u_h(t))+D_cf(u_h(t))-R_hq(t)u_h(t)+R_hg(t),\\ D_\eta u_h(t)=0,\end{cases}\tag{16}$$

for all $t\in(0,T]$, with initial condition $u_h(0)$

Proposition 7. *Let $u_h,\tilde{u}_h\in C^0([0,T],\mathbb{W}_h^*)\cap C^1([0,T],\mathbb{W}_h)$ be solutions of problem (16) with initial conditions $u_h(0),\tilde{u}_h(0)\in\mathbb{W}_h$, respectively. If $A\in C_B^1(\mathbb{R})$, $f\in C_B^1(\mathbb{R})$ and $\omega_h(t)=u_h(t)-\tilde{u}_h(t)$ then for all $t\in[0,T]$,*

$$\|\omega_h(t)\|_h^2+A_0\int_0^t e^{\theta_h(s)-\theta_h(t)}\|D_{-x}\omega_h(s)\|_+^2\,\mathrm{d}s\leq e^{\theta_h(t)}\|\omega_h(0)\|_h,\tag{17}$$

where

$$\theta_h(t)=2q_0t-\frac{1}{A_0}\int_0^t\left(\left\|A'\right\|_{L^\infty(\mathbb{R})}\|D_{-x}u_h(s)\|_{h,\infty}+\left\|f'\right\|_{L^\infty(\mathbb{R})}\right)^2\mathrm{d}s.$$

Proof. We start by remarking that $\omega_h(t)$ is solution of the following differential problem

$$\begin{cases} \dfrac{\mathrm{d}\omega_h}{\mathrm{d}t}(t) = D_x^* \left(A(M_h u_h(t))D_{-x}u_h(t) - A(M_h\tilde{u}_h(t))D_{-x}\tilde{u}_h(t)\right) & \\ \qquad\qquad + D_c\left(f(u_h(t)) - f(\tilde{u}_h(t))\right) - R_h q(t)\omega_h(t), & \text{in } \overline{\Omega}_h \times (0,T], \\ D_\eta u_h(t) = D_\eta \tilde{u}_h(t) = 0, & \text{on } \partial\Omega_h \times (0,T], \\ \omega_h(0) = R_h u_0 - \tilde{u}_h(0), & \text{in } \overline{\Omega}_h. \end{cases} \tag{18}$$

From system (18), taking into account Proposition 3, the following equality is valid for $t \in (0,T]$,

$$\begin{aligned} \left(\frac{\mathrm{d}\omega_h}{\mathrm{d}t}(t), \omega_h(t)\right)_h &= -(A(M_h u_h(t))D_{-x}u_h(t) - A(M_h\tilde{u}_h(t))D_{-x}\tilde{u}_h(t), D_{-x}\omega_h(t))_+ \\ &\quad - (M_h(f(u_h(t)) - f(\tilde{u}_h(t))), D_{-x}\omega_h(t)) - (R_h q(t)\omega_h(t), \omega_h(t))_h. \end{aligned} \tag{19}$$

Moreover, the following upper bounds hold

$$\begin{aligned} &-(A(M_h u_h(t))D_{-x}u_h(t) - A(M_h\tilde{u}_h(t))D_{-x}\tilde{u}_h(t), D_{-x}\omega_h(t))_+ \\ &\qquad \leq -A_0 \|D_{-x}\omega_h(t)\|_+^2 + \|A'\|_{L^\infty(\mathbb{R})} \|D_{-x}u_h(t)\|_{h,\infty} \|\omega_h(t)\|_+ \|D_{-x}\omega_h(t)\|_+, \end{aligned} \tag{20}$$

and

$$-(M_h(f(u_h(t)) - f(\tilde{u}_h(t))), D_{-x}\omega_h(t)) \leq \|f'\|_{L^\infty(\mathbb{R})} \|\omega_h(t)\|_+ \|D_{-x}u_h(t)\|_+, \tag{21}$$

Combining equation (19) and inequalities (20)-(21), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_h(t)\|_+^2 + (A_0 - \epsilon^2)\|D_{-x}\omega_h(t)\|_+^2 \\ &\qquad \leq \left(-q_0 + \frac{1}{4\epsilon^2}\left(\|A'\|_{L^\infty(\mathbb{R})}\|D_{-x}u_h(t)\|_{h,\infty} + \|f'\|_{L^\infty(\mathbb{R})}\right)^2\right)\|\omega_h(t)\|_+^2 \ \text{ in } (0,T]. \end{aligned}$$

for all $\epsilon \neq 0$. Choosing ϵ such that $\epsilon^2 = \frac{A_0}{2}$ and integrating the previous inequality, we conclude the upper bound (17). □

Remark 2. Inequality (17) guarantees that the IVP (16) has at most one solution in $C^0([0,T], \mathbb{W}_h^*) \cap C^1([0,T], \mathbb{W}_h)$. In fact, if $u_h, \tilde{u}_h$ are solutions in $C^0([0,T], \mathbb{W}_h^*) \cap C^1([0,T], \mathbb{W}_h)$, then, from (17), we conclude that $u_h(t) = \tilde{u}_h(t)$ in $\overline{\Omega}_h$.

Since the upper bound (17) depends on h, to conclude stability of the semidiscrete solution, it is sufficient to prove a uniform upper bound for $\int_0^t \|D_{-x}u_h(\mu)\|_{h,\infty}\,\mathrm{d}\mu, t \in [0,T], h \in \Lambda$ while imposing that the perturbations are around $R_h u_0$.

Remark 3. We highlight that, if $u_h(t)$, solution of system (16) satisfies $\int_0^t \|D_{-x}u_h(s)\|_+^2\,\mathrm{d}s \leq C_B,\ t \in [0,T], h \in \Lambda$ for some positive constant C_B, h and t independent, then it follows that

$$\int_0^t \|D_{-x}u_h(s)\|_{h,\infty}^2\,\mathrm{d}s \leq \frac{C_B}{h_{min}^2}, t \in [0,T], h \in \Lambda.$$

Therefore, in order to prove the stability of the scheme, it is sufficient to show that $\int_0^t \|D_{-x}u_h(s)\|_+^2\,ds, t \in [0,T], h \in \Lambda$, can be suitably bounded (uniformly).

Let $t \in [0,T]$ and $E_u(t) = u_h(t) - R_h u(t) \in \mathbb{W}_h$ denote the spatial discretization error where u is solution of the IBVP (1)-(3) and $u_h(t) \in \mathbb{W}_h$ is the semidiscrete approximation defined by system (16). Through a straightforward calculation, $E_u(t)$ is solution of the IVP

$$\begin{cases} \dfrac{\mathrm{d}E_u}{\mathrm{d}t}(t) = D_x^*(A(M_h u_h(t))D_{-x}u_h(t) - A(M_h R_h^* u(t))D_{-x}R_h^* u(t)) & \\ \qquad + D_c(f(u_h(t)) - f(R_h^* u(t))) - (R_h q(t))E_u(t) - T_h(t), & \text{in } \overline{\Omega}_h \times (0,T], \\ T_{h,\partial\Omega}(t) = 0, & \text{on } \partial\Omega \times (0,T], \\ E_u(0) \text{ given}, & \text{in } \overline{\Omega}_h, \end{cases} \tag{22}$$

where $T_h(t) \in \mathbb{W}_h$ with

$$T_h(t) = \frac{\mathrm{d}}{\mathrm{d}t}R_h^* u(t) - D_x^*(A(M_h R_h^* u(t))D_{-x}R_h^* u(t)) - D_c f(R_h^* u(t)) + (R_h q)R_h^* u(t) - R_h g(t)$$

and $T_{h,\partial\Omega}(t)(x_i) = -D_\eta R_h^* u(x_i,t)$, $i = 0, N$.

Remark 4. Under the assumptions $A \in C_B^4(\mathbb{R})$, $f \in C_B^3(\mathbb{R})$ and $u \in C^0([0,T], C^4(\overline{\Omega}^*)) \cap C^1([0,T], C^0(\overline{\Omega}))$, it can be shown that $T_h(t)$ and $T_{h,\partial\Omega}(t)$, satisfy similar bounds to those on Propositions 5 and 6.

Theorem 2. *Let $u \in C^0([0,T], C^4(\overline{\Omega}^*)) \cap C^1([0,T], C^0(\overline{\Omega}))$ denote the solution of the IBVP (1)-(3) and $u_h(t) \in \mathbb{W}_h^*$ the semidiscrete approximation defined by (16). If $A \in C_B^4(\mathbb{R})$, $f \in C_B^3(\mathbb{R})$, $g, q \in C^0([0,T], C^0(\overline{\Omega}))$ then there exists a positive constant C, h and t independent, such that*

$$\|E_u(t)\|_h^2 + A_0 \int_0^t e^{\theta(u(s)) - \theta(u(t))} \|D_{-x}E_u(s)\|_+^2 \, \mathrm{d}s \le e^{-\theta(t)} \|E_h(0)\|_h^2 + Ch_{max}^4 \int_0^t e^{\theta(u(s)) - \theta(u(t))} \|u(s)\|_{C^4(\overline{\Omega}^*)}^2 \, \mathrm{d}s, \tag{23}$$

where

$$\theta(u(t)) = 2(q_0 - A_0)t - \frac{4}{A_0} \int_0^t \left(\|u(s)\|_{C^1(\overline{\Omega})} \|A'\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})} \right)^2 \mathrm{d}s. \tag{24}$$

for all $t \in [0,T]$.

Proof. From system (22), the following inequality is easily obtained

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|E_u(t)\|_h^2 \le & -(A(M_h u_h(t))D_{-x}u_h(t) + M_h f(u_h(t)), D_{-x}E_u(t))_+ \\ & + (A(M_h R_h u(t))D_{-x}R_h u(t) + M_h f(R_h u(t)), D_{-x}E_u(t))_+ \\ & - q_0 \|E_u(t)\|_h^2 - (T_h(t), E_u)_h - \sum_{i=0,N} T_{h,\partial\Omega}(x_i,t)E_u(x_i,t) \end{aligned}$$

for $t \in (0,T]$. Following the proof of Theorem 1, it can be shown that there exists a positive constant C, h and t independent, such that

$$2\left(q_0 - 2A_0 - \frac{2}{A_0}\left(\|u(t)\|_{C^1(\overline{\Omega})} \|A'\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})} \right)^2\right) \|E_u(t)\|_h^2 + \frac{\mathrm{d}}{\mathrm{d}t} \|E_u(t)\|_h^2 + A_0 \|D_{-x}E_u(t)\|_+^2 + \le Ch_{max}^4 \|u(t)\|_{C^4(\overline{\Omega}^*)}^2, \text{ in } (0,T], \tag{25}$$

where θ is defined by equation (24). Inequality (25) leads to estimate (23).

□

As mentioned before, the stability of the IVP (16) in $(u_h(t))_{h\in\Lambda}$ follows from estimate (17) if there exists positive constant C_s, h and t independent, such that

$$\int_0^t \|D_{-x}u_h(s)\|_{h,\infty}^2 \, \mathrm{d}s \le C_s \; t \in [0,T],$$

for $h \in \Lambda$ with h_{max} small enough, which is established in the next corollary.

Corollary 1. *If the sequence of grids* $\{\overline{\Omega}_h,\, h\in\Lambda\}$ *is quasiuniform, under the assumptions of Theorem 2, the solution* $u_h(t) \in W_h^*$ *of* (16) *is a stable solution in* $[0,T]$ *provided that* $u_h(0) \in B_{h_{max}}(R_h u(0))$.

Proof. Applying Theorem 2, the following estimate holds for $E_u(t)$

$$\|E_u(t)\|_h^2 + \int_0^t \|D_{-x}E_u(s)\|_+^2 \, \mathrm{d}s \le C_u\big(h_{max}^2 + h_{max}^4\big),$$

for $h \in \Lambda$ and h_{max} small enough. Therefore, from

$$\begin{aligned}\int_0^t \|D_{-x}u_h(s)\|_{h,\infty}^2 \, \mathrm{d}s &\le 2\int_0^t \|D_{-x}E_u(s)\|_{h,\infty}^2 \, \mathrm{d}s + 2\int_0^t \left\|\frac{\partial u}{\partial x}(s)\right\|_{L^\infty(\Omega)}^2 \mathrm{d}s \\ &\le \frac{2}{h_{min}^2}\int_0^t \|D_{-x}E_u(s)\|_+^2 \, \mathrm{d}s + 2\int_0^t \left\|\frac{\partial u}{\partial x}(s)\right\|_{L^\infty(\Omega)}^2 \mathrm{d}s\end{aligned}$$

we conclude that

$$\int_0^t \|D_{-x}u_h(s)\|_{h,\infty}^2 \, \mathrm{d}s \le 2C\frac{h_{max}^2 + h_{max}^4}{h_{min}^2} + 2\int_0^t \left\|\frac{\partial u}{\partial x}(s)\right\|_{L^\infty(\Omega)}^2 \mathrm{d}s$$

and stability follows from the quasiuniformity of the grid.

□

3.2.2 A fully-discrete scheme

Let $M \in \mathbb{N}$ and $\Delta t = \frac{T}{M}$. We introduce a uniform grid in $[0,T]$ of timestep Δt defined by $t_n = n\Delta t$, $n = 0,\ldots,M$. We now propose a fully discrete scheme, derived from system (16) by applying an implicit-explicit approach to the nonlinear terms and a standard backward Euler discretization of the time derivative. For $n = 0,\ldots,M-1$, let $u_h^n \in \mathbb{W}_h^*$ be defined by

$$\begin{cases} D_{-t}u_h^{n+1} = D_x^*(A(M_h u_h^n)D_{-x}u_h^{n+1}) + D_c(f(u_h^{n+1})) \\ \qquad\qquad - R_h q(t_{n+1})u_h^{n+1} + R_h g(t_{n+1}), \\ D_\eta^n u_h^{n+1}(x_i) = 0,\; i = 0, N \\ \qquad\quad u_h^0 = R_h u_0, \end{cases} \tag{26}$$

where D_{-t} denotes the backward finite difference operator to approximate the first partial derivative with respect to t and

$$\begin{aligned} D_\eta^k u_h^j(x_0) &= -\frac{1}{2}\Big(A(M_h u_h^k(x_1))D_{-x}u_h^j(x_1) + A(M_h u_h^k(x_0))D_{-x}u_h^j(x_0)\Big) \\ &\quad - \frac{1}{4}\Big(f(u_h^k(x_1)) + 2f(u_h^k(x_0)) + f(u_h^k(x_{-1}))\Big), \\ D_\eta^k u_h^j(x_N) &= \frac{1}{2}\Big(A(M_h u_h^k(x_{N+1}))D_{-x}u_h^j(x_{N+1}) + A(M_h u_h^k(x_N))D_{-x}u_h^j(x_N)\Big) \\ &\quad + \frac{1}{4}\Big(f(u_h^k(x_{N+1})) + 2f(u_h^k(x_N)) + f(u_h^k(x_{N-1}))\Big). \end{aligned}$$

for $k,j=1,\dots,M$.

In what follows, $C^{n,m}(\overline{\Omega}^* \times [0,T])$ represents the space of functions defined in $\overline{\Omega}^* \times [0,T]$, with continuous partial derivatives with respect to x and t until order n and m, respectively.

For $t \in [\Delta t, T]$, let $\tilde{T}_h(t) \in \mathbb{W}_h^*$ denote the truncation error associated with the discretization of the differential equation, that is,

$$\begin{aligned}\tilde{T}_h(t) = &\frac{R_h^* u(t) - R_h^* u(t-\Delta t)}{\Delta t} - D_x^*(A(M_h R_h^* u(t-\Delta t)) D_{-x} R_h^* u(t)) \\ &- D_c(f(R_h^* u(t-\Delta t))) + (R_h q(t)) R_h^* u(t) - R_h g(t)\end{aligned}$$

and $\tilde{T}_{h,\partial\Omega}(t)$ denote the truncation error associated with the discretization of the boundary conditions.

In order to establish convergence, we will require the next result.

Lemma 1. *Let $\alpha, \beta \in \mathbb{R}$ and $\gamma, \Delta t \in \mathbb{R}^+$ such that $1 + \Delta t \alpha > 0$ and $\alpha \leq \beta$. If $(x_n)_{n \in \mathbb{N}_0}, (y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ are sequences of nonnegative numbers satisfying*

$$(1+\Delta t\alpha)x_n + \Delta t \gamma y_n \leq (1+\Delta t \beta) x_{n-1} + \Delta t z_n,\ n \geq 1$$

then

$$x_n + \Delta t \sum_{i=1}^n y_i \leq \left((1+\Delta t\beta)x_0 + \Delta t \sum_{i=1}^n z_i\right) \exp\left(\frac{n\Delta t(\beta-\alpha)}{m}\right),\ n\geq 1$$

where $m = \min\{1+\Delta t\alpha, \gamma\}$.

Proof. From the hypothesis, we can show using induction that for $n \geq 1$

$$(1+\Delta t \alpha)x_n + \Delta t\gamma \sum_{i=1}^n y_i \leq (1+\Delta t\beta)x_0 + \Delta t(\beta-\alpha)\sum_{i=1}^{n-1} x_i + \Delta t \sum_{i=1}^n z_i,$$

which leads to

$$x_n + \Delta t \sum_{i=1}^n y_i \leq \frac{1+\Delta t\beta}{m} x_0 + \Delta t \frac{(\beta-\alpha)}{m} \sum_{i=1}^{n-1} x_i + \frac{\Delta t}{m} \sum_{i=1}^n z_i,$$

A direct application of a discrete Gronwall lemma concludes the proof. □

We are now able to prove a convergence result for the proposed method. Let $E_u^n = u_h^n - R_h u(t_n) \in \mathbb{W}_h$ denote the global error for each $n = 0, \dots, M$.

Theorem 3. *Let $u \in C^{4,0}(\overline{\Omega}^* \times [0,T]) \cap C^2(\overline{\Omega} \times [0,T]) \cap C^1(\overline{\Omega}^* \times [0,T])$ be solution of the IBVP (1)-(3) and let $u_h^n \in W_h^*, n = 0,\dots,M$, be defined by (26). If $A \in C_B^4(\mathbb{R})$, $f \in C_B^3(\mathbb{R})$, $g, q \in C^0([0,T], C^0(\overline{\Omega}))$,*

$$\gamma = 3A_0 + \frac{2}{A_0}\left(\|u\|^2_{C^1(\overline{\Omega}\times[0,T])}\left\|A'\right\|^2_{L^\infty(\mathbb{R})} + \left\|f'\right\|^2_{L^\infty(\mathbb{R})}\right) - 2q_0 \geq 0 \ \text{ and } \ 2q_0 - 3A_0 \neq 0, \quad (27)$$

then there exists $\Delta t_0 > 0$ and a positive constant C (h independent) such that for all $\Delta t \leq \Delta t_0$ it holds

$$\|E_u^n\|_h^2 + \Delta t \sum_{j=1}^n \left\|D_{-x}E_u^j\right\|_+^2 \leq C(h_{max}^4 + \Delta t^2),\ n = 1,\dots,M, \quad (28)$$

Proof. Let $\Delta t_0 \leq \frac{1}{|2q_0-3A_0|}$ and take $\Delta t \leq \Delta t_0$. From (26) it can be shown, for $n = 0,\ldots,M-1$, that the following inequality holds

$$
\begin{aligned}
\left\|E_u^{n+1}\right\|_h^2 \leq (E_u^n, E_u^{n+1})_h - \Delta t A_0 \left\|D_{-x}E_u^{n+1}\right\|_+^2 + \\
&+ \Delta t((A(M_hR_hu(t_n)) - A(M_hu_h^n))D_{-x}R_hu(t_{n+1}), D_{-x}E_u^{n+1})_+ \\
&- \Delta t(M_h(f(u_h^n) - f(R_hu(t_n))), D_{-x}E_u^n)_+ \\
&- \Delta t q_0 \left\|E_u^{n+1}\right\|_h^2 - \Delta t(\tilde{T}_h(t_{n+1}), E_u^{n+1})_h \\
&+ \Delta t \sum_{i=0,N} \tilde{T}_{h,\partial\Omega_h}(x_i, t_{n+1})E_u^{n+1}(x_i),
\end{aligned}
\tag{29}
$$

with $E_u^0 = 0$ in $\overline{\Omega}_h$. Observe that for all $\epsilon \neq 0$,

$$
\begin{aligned}
&((A(M_hR_hu(t_n)) - A(M_hu_h^n))D_{-x}R_hu(t_{n+1}), D_{-x}E_u^{n+1})_+ \\
&\qquad - (M_h(f(u_h^n) - f(R_hu(t_n))), D_{-x}E_u^{n+1})_+ \\
&\leq \frac{1}{4\epsilon^2}\left(\|u\|^2_{C^1(\overline{\Omega}\times[0,T])}\left\|A'\right\|^2_{L^\infty(\mathbb{R})} + \left\|f'\right\|^2_{L^\infty(\mathbb{R})}\right)\|E_u^n\|_h^2 + \epsilon^2\left\|D_{-x}E_u^{n+1}\right\|_+^2.
\end{aligned}
$$

On the other hand, proceeding with a similar proof of Theorem 1, it follows that there exists a positive constant $\tilde{C}$, independent of $h, \Delta t$, such that

$$
-(\tilde{T}_h(t_{n+1}), E_u^{n+1})_h \leq \frac{\tilde{C}}{\epsilon^2}(h_{max}^4 + \Delta t^2) + \epsilon^2\left\|D_{-x}E_u^{n+1}\right\|_+^2 + 2\epsilon^2\left\|E_u^{n+1}\right\|_h^2.
$$

and

$$
\sum_{i=0,N} \tilde{T}_{h,\partial\Omega_h}(x_i, t_{n+1})E_u^{n+1}(x_i) \leq 2\tilde{C}(\Delta t^2 + h_{max}^4) + 4\Delta t\epsilon^2\left\|E_u^{n+1}\right\|_h^2,
$$

where C is a positive constant, only dependent on A, f and u.

Taking in (29) the last estimates, we conclude

$$
\begin{aligned}
&\left[1 + \Delta t\left(2q_0 - 3\epsilon^2\right)\right]\left\|E_u^{n+1}\right\|_h^2 + 2\Delta t(A_0 - 2\epsilon^2)\left\|D_{-x}E_u^{n+1}\right\|_+^2 \\
&\leq \left(1 + \frac{\Delta t}{2\epsilon^2}\left(\|u\|^2_{C^1(\overline{\Omega}\times[0,T])}\left\|A'\right\|^2_{L^\infty(\mathbb{R})} + \left\|f'\right\|^2_{L^\infty(\mathbb{R})}\right)\right)\|E_u^n\|_h^2 + \Delta t C(h_{max}^4 + \Delta t^2).
\end{aligned}
\tag{30}
$$

Taking $\epsilon^2 = \frac{A_0}{4}$ and under the assumption (27), we can now apply Lemma 1 and from (30), we get

$$
\|E_u^n\|_h^2 + \Delta t\sum_{j=1}^{n}\left\|D_{-x}E_u^j\right\|_+^2 \leq C(h_{max}^4 + \Delta t^2)\exp\left(\frac{\gamma T}{m}\right),\ j = 1,\ldots,M,
$$

where $m = \min\left\{1 + \Delta t\left(2q_0 - 3A_0\right), A_0\right\}$. Finally, from the hypothesis on Δt and Δt_0, it follows that

$$
\exp\left(\frac{\gamma T}{m}\right) \leq \exp\left(\gamma T \max\left\{\frac{1}{A_0}, \frac{1}{1 - |2q_0 - 3A_0|\Delta t_0}\right\}\right)
$$

which leads to inequality (28). □

Theorem 3 establishes the following estimate for the error

$$
\|E_u^n\|_h^2 + \Delta t\,\|D_{-x}E_u^n\|_+^2 \leq C(h_{max}^4 + \Delta t^2),\ n = 1,\ldots,M.
$$

where C is a positive constant h and Δt independent, ensuring that the scheme (26) is of second order in space and first order in time for Δt small enough.

4 Convergence analysis for less smooth solutions

The previous study of convergence properties of the schemes for the stationary and time-dependent problems, using smooth enough solutions, provides useful insights on how to study these problems. In the following sections, we want to reduce the regularity assumptions on the exact solution of the problem, while still proving second order convergence with respect to space. This will require a careful application of the Bramble-Hilbert Lemma, a technique suited for this scenario. In this setting, for $g \in L^1(\Omega)$, we introduce the averaging operator $(g)_h : \overline{\Omega}_h \longrightarrow \mathbb{R}$ defined as

$$(g)_h(x_i) = \frac{1}{h_{i+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} g(x)\,\mathrm{d}x,\ 0 = 1, \dots, N,$$

where $x_{i+1/2} = x_i + \frac{h_{i+1}}{2}$, $i = 0, \dots, N-1$, and we take $x_{-1/2} = x_0$ and $x_{N+1/2} = x_N$.

4.1 Revisiting the elliptic problem

We recall the nonlinear elliptic differential problem (8)-(9)

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(A(u)\dfrac{\mathrm{d}u}{\mathrm{d}x} + f(u)\right) + qu = g \text{ in } \Omega, \\ A(u)\dfrac{\mathrm{d}u}{\mathrm{d}x} + f(u) = 0 \text{ on } \partial\Omega, \end{cases}$$

and introduce a new discrete scheme, mostly the same as (10)-(11), but with a slight modification on the discretization of the right hand side

$$-\left(D_x^*(A(M_h u_h)D_{-x}u_h) + D_c f(u_h)\right) + (R_h q)u_h = (g)_h, \quad \text{in } \overline{\Omega}_h \tag{31}$$

and

$$D_\eta u_h(x_0) = D_\eta u_h(x_N) = 0. \tag{32}$$

Before proving a convergence result, we establish the following proposition, which will be useful going forward.

Proposition 8. *If $w \in H^2(\Omega)$ and $v_h \in \mathbb{W}_h$ then there exists a positive constant C, h-independent, such that*

$$|((w)_h - R_h w, v_h)_h| \le C h_{max}^2 \left(\|w\|_{H^2(\Omega)} \|v_h\|_h + \|w\|_{H^1(\Omega)} \|D_{-x}v_h\|_+ \right). \tag{33}$$

Moreover, the previous inequality also holds for $w(t)$, $t \in [0,T]$, if $w \in C^0([0,T], H^2(\Omega))$ with C a positive constant independent of h and t.

Proof. We only prove the result for $w \in H^2(\Omega)$ since the proof is similar in the other case. To determine an upper bound for $|((w)_h - R_h w, v_h)_h|$, we follow the proof of Theorem 2 from [3]. We start by noticing that $((w)_h - R_h w, v_h)_h = -\frac{T_1 + T_2}{2}$ where

$$T_1 = \sum_{i=0}^{N-1} \left[\frac{h_{i+1}}{2}(w(x_{i+1}) + w(x_i)) - \int_{x_i}^{x_{i+1}} w(x)\,\mathrm{d}x \right] (E_u(x_{i+1}) + E_u(x_i)) \tag{34}$$

and

$$T_2 = \sum_{i=0}^{N-1} \left[\frac{h_{i+1}}{2}(w(x_{i+1}) - w(x_i)) + \int_{x_i}^{x_{i+1/2}} w(x)\,\mathrm{d}x - \int_{x_{i+1/2}}^{x_{i+1}} w(x)\,\mathrm{d}x \right] (E_u(x_{i+1}) - E_u(x_i)). \tag{35}$$

We now apply Lemma 1.4 from [2] to each term $\frac{h_{i+1}}{2}(w(x_{i+1}) + w(x_i)) - \int_{x_i}^{x_{i+1}} w(x)\,\mathrm{d}x$ of (34) to establish that there exists a positive constant $C_{1,1}$, h-independent, such that

$$\begin{aligned}
|T_1| &\leq C_{1,1} \sum_{i=1}^{N} h_i^{5/2} \|w\|_{H^2(x_{i-1},x_i)} |E_u(x_{i-1}) + E_u(x_i)| \\
&\leq C_{1,1} \left(\sum_{i=1}^{N} h_i^4 \|w\|^2_{H^2(x_{i-1},x_i)} \right)^{1/2} \left(\sum_{i=1}^{N} h_i (E_u(x_{i-1}) + E_u(x_i))^2 \right)^{1/2} \\
&\leq 2C_{1,1} h^2_{max} \|w\|_{H^2(\Omega)} \|E_u\|_h
\end{aligned}$$

For (35) it can be shown, through the Bramble-Hilbert Lemma that there exists a positive constant $C_{1,2}$, h-independent, such that

$$\begin{aligned}
|T_2| &\leq C_{1,2} \sum_{i=1}^{N} h_i^{3/2} \|w\|_{H^1(x_{i-1},x_i)} |E_u(x_{i-1}) - E_u(x_i)| \\
&\leq C_{1,2} \sum_{i=1}^{N} h_i^{5/2} \|u\|_{H^1(x_{i-1},x_i)} |D_{-x} E_u(x_i)| \\
&\leq C_{1,2} \left(\sum_{i=1}^{N} h_i^4 \|w\|^2_{H^1(x_{i-1},x_i)} \right)^{1/2} \left(\sum_{i=1}^{N} h_i (D_{-x} E_u(x_i))^2 \right)^{1/2} \\
&\leq C_{1,2} h^2_{max} \|w\|^2_{H^1(\Omega)} \|D_{-x} E_u\|_+ .
\end{aligned}$$

Taking $C = \max\{2C_{1,1}, C_{1,2}\}$, this establishes estimate (33). □

For the discrete scheme (31)-(32), the following convergence result holds for the error $E_u = u_h - R_h u \in \mathbb{W}_h$.

Theorem 4. *Let $u \in H^3(\Omega^*)$ denote the solution of problem (8)-(9) and $u_h \in \mathbb{W}_h^*$ denote the solution of system (31)-(32). If $A, f \in C_B^1(\mathbb{R})$ with $f(u) \in H^2(\Omega)$, $q \in W^{2,\infty}(\Omega)$ and $g \in L^1(\Omega)$ then there exists a positive constant C, h-independent, such that*

$$\|E_u\|_{1,h} \leq C h^2_{max} \|u\|_{H^3(\Omega)} ,$$

for all $h \in \Lambda$, provided that

$$\|u'\|_{L^\infty(\Omega)} \|A'\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})} < \min\{q_0, A_0\} .$$

Proof. Calculating the discrete inner product $(\cdot,\cdot)_h$ of both sides of (31) with E_u, applying Proposition 3 and using the boundary conditions (32), we easily establish, as before, that

$$\begin{aligned}
(A(M_h u_h) D_{-x} E_u, D_{-x} E_u)_+ + (R_h q E_u, E_u)_h = {} & ((g)_h, E_u)_h \\
& - (M_h(f(u_h) - f(R_h u)), D_{-x} E_u)_+ \\
& - (A(M_h u_h) D_{-x} R_h u, D_{-x} E_u)_+ \\
& - (M_h(f(R_h u)), D_{-x} E_u)_+ \\
& - (R_h(qu), E_u)_h
\end{aligned}$$

A straightforward development of $((g)_h, E_u)_h$ and using the lower bounds for A and q, leads to

$$\begin{aligned}
A_0 \|D_{-x} E_u\|_+ + q_0 \|E_u\|_h \leq {} & ((qu)_h - R_h(qu), E_u)_h \\
& - (M_h(f(u_h) - f(R_h u)), D_{-x} E_u)_+ \\
& - (A(M_h u_h) D_{-x} R_h u - \hat{R}_h \left(A(u) u'\right), D_{-x} E_u)_+ \\
& - (M_h(f(R_h u)) - \hat{R}_h f(u), D_{-x} E_u)_+
\end{aligned}$$

where $\hat{R}_h : C^0(\overline{\Omega}) \longrightarrow \mathbb{W}_h^{up}$ is defined as $\hat{R}_h u(x_i) = u(x_{i-1/2})$, $i = 1, \dots, N$, for $u \in C^0(\overline{\Omega})$. Let τ_i, $i = 1, 2, 3, 4$ be defined as

$$\begin{aligned}
\tau_1 &= ((qu)_h - R_h(qu), E_u)_h \\
\tau_2 &= -(M_h(f(u_h) - f(R_h u)), D_{-x} E_u)_+ \\
\tau_3 &= -(A(M_h u_h) D_{-x} R_h u - \hat{R}_h \left(A(u)u'\right), D_{-x} E_u)_+ \\
\tau_4 &= -(M_h(f(R_h u)) - \hat{R}_h f(u), D_{-x} E_u)_+
\end{aligned}$$

We now prove upper bounds for $|\tau_i|$, $i = 1, 2, 3, 4$.

- Bound for τ_1: Given the regularity assumptions on q and u, using Proposition 8, it follows that there exists a positive constant C_1, h-independent, such that

$$|\tau_1| \leq C_1 h_{max}^2 \left(\|q\|_{W^{2,\infty}(\Omega)} \|u\|_{H^2(\Omega)} \|E_u\|_h + \|q\|_{W^{1,\infty}(\Omega)} \|u\|_{H^1(\Omega)} \|D_{-x} E_u\|_+ \right).$$

- Bound for τ_2: As we have seen before in the proof of Theorem 1, this term can be bounded as

$$|\tau_2| \leq \left\|f'\right\|_{L^\infty(\mathbb{R})} \|E_u\|_h \|D_{-x} E_u\|_+$$

- Bound for τ_3: Splitting carefully the term $A(M_h u_h) D_{-x} R_h u - \hat{R}_h \left(A(u)u'\right)$ as

$$\begin{aligned}
&\left(A(M_h u_h) - A(M_h R_h u)\right) D_{-x} R_h u \\
&\qquad + A(M_h R_h u) \left(D_{-x} R_h u - \hat{R}_h u'\right) \\
&\qquad\qquad + \left(A(M_h R_h u) - \hat{R}_h A(u)\right) \hat{R}_h u'
\end{aligned}$$

it follows that there exist a positive constant C_3, h-independent, such that

$$\begin{aligned}
|\tau_3| &\leq \left\|A'\right\|_{L^\infty(\mathbb{R})} \left\|u'\right\|_{L^\infty(\Omega)} \|E_u\|_h \|D_{-x} E_u\|_+ \\
&\quad + C_3 \left[\left(\sum_{i=1}^N h_i^4 \|u\|_{H^3(x_{i-1}, x_i)}^2 \right)^{1/2} + \left(\sum_{i=1}^N h_i^4 \|u\|_{H^2(x_{i-1}, x_i)}^2 \right)^{1/2} \right] \|D_{-x} E_u\|_+
\end{aligned}$$

- Bound for τ_4: Again, following [3], if $f(u) \in H^2(\Omega)$ then there exists a positive constant C_4, h-independent, such that

$$|\tau_4| \leq C_4 \left(\sum_{i=1}^N h_i^4 \|f(u)\|_{H^2(x_{i-1}, x_i)}^2 \right)^{1/2} \|D_{-x} E_u\|_+$$

Combining all the bounds we can now establish

$$\begin{aligned}
\sum_{i=1}^4 |\tau_i| &\leq C_1 \left(h_{max}^2 \|q\|_{W^{2,\infty}(\Omega)} \|u\|_{H^2(\Omega)} \|E_u\|_h + h_{max}^2 \|q\|_{W^{1,\infty}(\Omega)} \|u\|_{H^1(\Omega)} \|D_{-x} E_u\|_+ \right) \\
&\quad + \left(\left\|f'\right\|_{L^\infty(\mathbb{R})} + \left\|A'\right\|_{L^\infty(\mathbb{R})} \left\|u'\right\|_{L^\infty(\Omega)} \right) \|E_u\|_h \|D_{-x} E_u\|_+ \\
&\quad + C_3 h_{max}^2 \left(\|u\|_{H^3(\Omega)} + \|u\|_{H^2(\Omega)} \right) \|D_{-x} E_u\|_+ + C_4 h_{max}^2 \|f(u)\|_{H^2(\Omega)} \|D_{-x} E_u\|_+ \\
&\leq C h_{max}^2 \|u\|_{H^3(\Omega)} \left(\|E_u\|_h + \|D_{-x} E_u\|_+ \right) \\
&\quad + \left(\left\|f'\right\|_{L^\infty(\mathbb{R})} + \left\|A'\right\|_{L^\infty(\mathbb{R})} \left\|u'\right\|_{L^\infty(\Omega)} \right) \|E_u\|_h \|D_{-x} E_u\|_+
\end{aligned}$$

where

$$\tilde{C} = \max\left\{C_1 \|q\|_{W^{2,\infty}(\Omega)}, C_1 \|q\|_{W^{1,\infty}(\Omega)}, C_3\left(\|u\|_{H^3(\Omega)} + \|u\|_{H^2(\Omega)}\right), C_4 \|f(u)\|_{H^2(\Omega)}\right\}$$

Then, for all $\epsilon \neq 0$, it follows

$$(A_0 - 2\epsilon^2)\|D_{-x}E_u\|_+ + \left(q_0 - \epsilon^2 - \frac{\alpha^2}{4\epsilon^2}\right)\|E_u\|_h \leq \frac{\tilde{C}^2}{4\epsilon^2} h_{max}^4 \|u\|_{H^3(\Omega)}^2$$

where $\alpha = \|f'\|_{L^\infty(\mathbb{R})} + \|A'\|_{L^\infty(\mathbb{R})}\|u'\|_{L^\infty(\Omega)}$. Choosing $\epsilon^2 = \frac{\alpha}{2}$, we conclude the proof. □

Remark 5. Following the proof of Theorem 4, the assumption that $A \in C_B^1(\mathbb{R})$ can be weakened. In fact, a similar proof holds, with the due adaptations, assuming A globally Lipschitz in $\mathbb{R}$ and bounded.

Remark 6. The proof of Theorem 4 together with Proposition 8 allow to show that under the regularity assumptions of Theorem 4, as long as $g \in H^2(\Omega)$, then using the restriction operator R_h on the right hand side of (10) guarantees that method (10)-(11) is still second order. This follows from noting that the averaging process is a second order approximation for the pointwise value of function, as observed in [3].

4.2 Revisiting the parabolic problem

We now turn our attention to problem (1)-(3) which we recall to be

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left(A(u)\dfrac{\partial u}{\partial x} + f(u)\right) + qu = g, & \text{in } \Omega \times (0,T], \\ A(u)\dfrac{\partial u}{\partial x} + f(u) = 0, & \text{on } \partial\Omega \times (0,T], \\ u(\cdot, 0) = u_0, & \text{in } \overline{\Omega} \end{cases}$$

where $q, g \in C^0([0,T], C^0(\overline{\Omega}))$.

4.2.1 Semidiscrete scheme

As before, we start by considering a semidiscretization of the full parabolic problem (1)-(3). Let us denote now $u_h \in \mathbb{W}_h^*$ as the solution of a slight variation of problem (16):

$$\begin{cases} \dfrac{\mathrm{d}u_h}{\mathrm{d}t}(t) = D_x^*(A(M_h u_h(t))D_{-x}u_h(t)) + D_c f(u_h(t)) - R_h q(t) u_h(t) + (g(t))_h, \\ D_\eta u_h(t) = 0, \end{cases} \tag{36}$$

for all $t \in (0,T]$, with initial condition $u_h(0) = R_h u_0$ and where $q, g \in C^0([0,T], C^0(\overline{\Omega}))$.

We can now establish the following convergence result for the error $E_u(t) = u_h(t) - R_h u(t)$, for all $t \in [0,T]$.

Theorem 5. *Let $u \in C^0([0,T], H^3(\Omega^*))$ with $\frac{\partial u}{\partial t} \in C^0([0,T], H^2(\Omega))$ denote the solution of the IBVP (1)-(3) and $u_h(t) \in \mathbb{W}_h^*$ the semidiscrete approximation defined by (36). If $A, f \in C_B^1(\mathbb{R})$ with $f(u) \in C^0([0,T], H^2(\Omega))$, $g \in C^0([0,T], L^1(\Omega))$, $q \in C^0([0,T], W^{2,\infty}(\Omega))$ then there exists a positive constant C, h and t independent, such that*

$$\|E_u(t)\|_h^2 + A_0 \int_0^t e^{\theta(u(s)) - \theta(u(t))} \|D_{-x}E_u(s)\|_+^2 \, \mathrm{d}s$$
$$\leq C h_{max}^4 \int_0^t e^{\theta(u(s)) - \theta(u(t))} \left(\|u(s)\|_{H^3(\Omega)}^2 + \left\|\frac{\partial u}{\partial t}(s)\right\|_{H^2(\Omega)}^2\right) \mathrm{d}s,$$

where

$$\theta(u(t)) = \left(2q_0 - \frac{3A_0}{2}\right)t - \frac{2}{3A_0}\int_0^t \left(\|A'\|_{L^\infty(\mathbb{R})}\left\|\frac{\partial u}{\partial x}(s)\right\|_{L^\infty(\Omega)} + \|f'\|_{L^\infty(\mathbb{R})}\right)^2 \mathrm{d}s.$$

for all $t \in [0, T]$.

Proof. The proof follows closely a combination of arguments used on the proofs of Theorems 2 and 4. Let $E_u(t) = u_h(t) - R_h u(t)$ for all $t \in [0, T]$. Following previous arguments, it is easily established that for $u_h(t) \in \mathbb{W}_h^*$ it holds

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_u(t)\|_h^2 \leq &-(M_h f(u_h(t) - f(R_h u(t))), D_{-x}E_u(t))_+ \\ &- (A(M_h u_h(t))D_{-x}R_h u(t) + M_h f(R_h u(t)), D_{-x}E_u(t))_+ \\ &- q_0\|E_u(t)\|_h^2 - A_0\|D_{-x}E_u(t)\|_+^2 \\ &- \left(R_h\frac{\partial u}{\partial t}(t), E_u(t)\right)_h + ((g(t))_h, E_u(t))_h\end{aligned}$$

for all $t \in (0, T]$.

Expanding the term $((g(t))_h, E_u(t))_h$, the previous inequality leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_u(t)\|_h^2 + q_0\|E_u(t)\|_h^2 \leq -A_0\|D_{-x}E_u(t)\|_+^2 + \sum_{i=1}^{5}\tau_i(t)$$

where

$$\begin{aligned}\tau_1(t) &= ((q(t)u(t))_h - R_h(q(t)u(t)), E_u(t))_h \\ \tau_2(t) &= (M_h(f(R_h u(t)) - f(u_h(t))), D_{-x}E_u(t))_+ \\ \tau_3(t) &= \left(\hat{R}_h\left(A(u(t))\frac{\partial u}{\partial x}(t)\right) - A(M_h u_h(t))D_{-x}R_h u(t), D_{-x}E_u(t)\right)_+ \\ \tau_4(t) &= \left(\hat{R}_h\left(f(u(t))\right) - M_h f(R_h u(t)), D_{-x}E_u(t)\right)_+ \\ \tau_5(t) &= \left(\left(\frac{\partial u}{\partial t}(t)\right)_h - R_h\left(\frac{\partial u}{\partial t}(t)\right), E_u(t)\right)_h\end{aligned}$$

Using Proposition 8 it follows that there exist positive constants C_1, C_2, $h{-}t$ independent, such that

$$\begin{aligned}|\tau_1(t)| \leq C_1 h_{max}^2 \Big(&\|q(t)\|_{W^{2,\infty}(\Omega)}\|u(t)\|_{H^2(\Omega)}\|E_u(t)\|_h \\ &+ \|q(t)\|_{W^{1,\infty}(\Omega)}\|u(t)\|_{H^1(\Omega)}\|D_{-x}E_u(t)\|_+\Big).\end{aligned}$$

and

$$|\tau_5(t)| \leq C_2 h_{max}^2\left(\left\|\frac{\partial u}{\partial t}(t)\right\|_{H^2(\Omega)}\|E_u(t)\|_h + \left\|\frac{\partial u}{\partial t}(t)\right\|_{H^1(\Omega)}\|D_{-x}E_u(t)\|_+\right).$$

Regarding τ_2, τ_3 and τ_4, it is straightforward to show that there exist positive constants C_3, C_4, $h - t$ independent, such that

$$|\tau_2(t)| \leq \|f'\|_{L^\infty(\mathbb{R})}\|E_u(t)\|_h\|D_{-x}E_u(t)\|_+,$$

$$|\tau_3(t)| \le \left(\|A'\|_{L^\infty(\mathbb{R})} \left\| \frac{\partial u}{\partial x}(t) \right\|_{L^\infty(\Omega)} \|E_u(t)\|_h + C_3 h_{max}^2 \|u(t)\|_{H^3(\Omega)} \right) \|D_{-x} E_u(t)\|_+$$

and

$$|\tau_4(t)| \le C_4 h_{max}^2 \|f(u(t))\|_{H^2(\Omega)} \|D_{-x} E_u(t)\|_+$$

Combining the previous bounds for $\tau_i(t)$, $i = 1\ldots, 5$ and through suitable applications of Young's inequality, for all $\epsilon \neq 0$, it holds

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \|E_u(t)\|_h^2 + 2\left(q_0 - \epsilon^2 - \frac{\alpha(t)^2}{4\epsilon^2} \right) \|E_u(t)\|_h^2 \\ &\qquad + 2(A_0 - 2\epsilon^2) \|D_{-x} E_u(t)\|_+^2 \le \frac{C h_{max}^4}{2\epsilon^2} \left(\|u(t)\|_{H^3(\Omega)}^2 + \left\| \frac{\partial u}{\partial t}(t) \right\|_{H^2(\Omega)} \right) \end{aligned} \tag{37}$$

where C is a constant positive independent of h and t and

$$\alpha(t) = \|A'\|_{L^\infty(\mathbb{R})} \left\| \frac{\partial u}{\partial x}(t) \right\|_{L^\infty(\Omega)} + \|f'\|_{L^\infty(\mathbb{R})}.$$

Choosing $\epsilon^2 = \frac{3A_0}{4}$ and integrating (37), we conclude the proof.

□

4.2.2 Fully discrete scheme

Our final goal is to establish a convergence result for a fully discrete scheme similar to (26), while reducing the regularity assumptions considered in Theorem 3. Let us introduce the new fully discrete scheme: for $n = 0, \ldots, M-1$, let $u_h^n \in \mathbb{W}_h^*$ be defined by

$$\begin{cases} D_{-t} u_h^{n+1} = D_x^*(A(M_h u_h^n) D_{-x} u_h^{n+1}) + D_c(f(u_h^n)) \\ \qquad\qquad - R_h q(t_{n+1}) u_h^{n+1} + (g(t_{n+1}))_h, \\ D_\eta^n u_h^{n+1} = 0, \end{cases} \tag{38}$$

where $u_h^0 = R_h u_0$.

Let $E_u^n = u_h^n - R_h u(t_n) \in \mathbb{W}_h$ denote the global error for each $n = 0, \ldots, M$.

Theorem 6. *Let $u \in C^0([0,T], H^3(\Omega^*))$ satisfying $\frac{\partial u}{\partial t} \in C^0([0,T], H^2(\Omega))$ and $\frac{\partial^2 u}{\partial t^2} \in u \in C^0([0,T], H^1(\Omega))$ denote solution of the IBVP (1)-(3) and let $u_h^n \in W_h^*, n = 0, \ldots, M$, be defined by (38). If $A, f \in C_B^1(\mathbb{R})$, $f(u) \in C^0([0,T], H^2(\Omega)) \cap C^1([0,T], H^1(\Omega))$, $q \in C^0([0,T], W^{2,\infty}(\Omega))$, $g \in C^0([0,T], C^0(\overline{\Omega}))$,*

$$\frac{A_0}{2} + \frac{4}{A_0}\left(\|A'\|_{L^\infty(\mathbb{R})}^2 \|u\|_{C^0([0,T],H^2(\Omega))}^2 + \|f'\|_{L^\infty(\mathbb{R})}^2 \right) - 2q_0 \ge 0, \quad 2q_0 \neq \frac{A_0}{2} \tag{39}$$

then there exists $\Delta t_0 > 0$ and a positive constant C (h independent) such that for all $\Delta t \le \Delta t_0$ it holds

$$\|E_u^n\|_h^2 + \Delta t \sum_{j=1}^n \|D_{-x} E_u^j\|_+^2 \le C \Gamma(u)(h_{max}^4 + \Delta t^2),\ n = 1, \ldots, M, \tag{40}$$

where

$$\begin{aligned} \Gamma(u) = \|u\|_{C^0([0,T],H^3(\Omega))}^2 + \|f(u)\|_{C^0([0,T],H^2(\Omega))}^2 + \left\| \frac{\partial f(u)}{\partial t} \right\|_{C^0([0,T],H^1(\Omega))}^2 \\ + \left\| \frac{\partial u}{\partial t} \right\|_{C^0([0,T],H^2(\Omega))}^2 + \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{C^0([0,T],H^1(\Omega))}^2. \end{aligned}$$

Proof. The proof follows a similar strategy than the one adopted in the proof of Theorem 4. Let

$$\Delta t_0 \leq \frac{2}{|4q_0 - A_0|}$$

and take $\Delta t \leq \Delta t_0$. It can be shown easily that for $n = 0, \ldots, M-1$, the following representation holds

$$\begin{aligned}
\left\|E_u^{n+1}\right\|_h^2 &= (E_u^n, E_u^{n+1})_h + \Delta t (R_h q(t_{n+1}) E_u^{n+1}, E_u^{n+1})_h \\
&- \Delta t \left(A(M_h u_h^n) D_{-x} E_u^{n+1}, D_{-x} E_u^{n+1}\right)_+ \\
&+ \Delta t \left((q(t_{n+1}) u(t_{n+1}))_h - R_h(q(t_{n+1}) u(t_{n+1})), E_u^{n+1}\right)_h \\
&+ \Delta t \left(M_h(R_h f(u(t_n)) - f(u_h^n)), D_{-x} E_u^{n+1}\right)_+ \\
&+ \Delta t \left(M_h(R_h f(u(t_{n+1})) - R_h f(u(t_n))), D_{-x} E_u^{n+1}\right)_+ \\
&+ \Delta t \left(\hat{R}_h \left(A(u(t_{n+1})) \frac{\partial u}{\partial x}(t_{n+1})\right) - A(M_h u_h^n) D_{-x} R_h u(t_{n+1})), D_{-x} E_u^{n+1}\right)_+ \\
&+ \Delta t \left(\hat{R}_h(f(u(t_{n+1}))) - M_h(R_h f(u(t_{n+1}))), D_{-x} E_u^{n+1}\right)_+ \\
&+ \Delta t \left(\left(\frac{\partial u}{\partial t}(t_{n+1})\right)_h - R_h \left(\frac{u(t_{n+1}) - u(t_n)}{\Delta t}\right), E_u^{n+1}\right)_h
\end{aligned}$$

which leads to

$$(1 + 2\Delta t q_0) \left\|E_u^{n+1}\right\|_h^2 + 2\Delta t A_0 \left\|D_{-x} E_u^{n+1}\right\|_+^2 \leq \|E_u^n\|_h^2 + 2\Delta t \sum_{k=1}^{6} \tau_k^{n+1} \tag{41}$$

where

$$\begin{aligned}
\tau_1^{n+1} &= \left((q(t_{n+1}) u(t_{n+1}))_h - R_h(q(t_{n+1}) u(t_{n+1})), E_u^{n+1}\right)_h \\
\tau_2^{n+1} &= \left(M_h(R_h f(u(t_n)) - f(u_h^n)), D_{-x} E_u^{n+1}\right)_+ \\
\tau_3^{n+1} &= \left(\hat{R}_h \left(A(u(t_{n+1})) \frac{\partial u}{\partial x}(t_{n+1})\right) - A(M_h u_h^n) D_{-x} R_h u(t_{n+1})), D_{-x} E_u^{n+1}\right)_+ \\
\tau_4^{n+1} &= \left(\hat{R}_h(f(u(t_{n+1}))) - M_h R_h f(u(t_{n+1}))), D_{-x} E_u^{n+1}\right)_+ \\
\tau_5^{n+1} &= \left(\left(\frac{\partial u}{\partial t}(t_{n+1})\right)_h - R_h \left(\frac{u(t_{n+1}) - u(t_n)}{\Delta t}\right), E_u^{n+1}\right)_h \\
\tau_6^{n+1} &= \left(M_h(R_h f(u(t_{n+1})) - R_h f(u(t_n))), D_{-x} E_u^{n+1}\right)_+
\end{aligned}$$

Bounds for τ_1^{n+1}, τ_2^{n+1} and τ_4^{n+1} are derived using the same tools as for the semidiscrete case. Hence there exist positive constants C_1 and C_4, independent of h, n and Δt such that

$$|\tau_1^{n+1}| \leq C_1 h_{max}^2 \|u\|_{C^0([0,T], H^2(\Omega))} \left(\|E_u(t_{n+1})\|_h + \|D_{-x} E_u(t_{n+1})\|_+\right),$$

$$|\tau_2^{n+1}| \leq \left\|f'\right\|_{L^\infty(\mathbb{R})} \|E_u^n\|_h \left\|D_{-x} E_u^{n+1}\right\|_+,$$

and

$$|\tau_4^{n+1}| \leq C_4 h_{max}^2 \|f(u)\|_{C^0([0,T], H^2(\Omega))} \left\|D_{-x} E_u^{n+1}\right\|_+.$$

We now establish bounds for τ_3^{n+1} and τ_5^{n+1}.

- Bound for τ_3^{n+1}: We start by noticing that τ_3^{n+1} can rewritten as

$$\tau_3^{n+1} = \sum_{k=1}^{4} \delta_k^{n+1}$$

where

$$\begin{aligned}
\delta_1^{n+1} &= \big((A(M_h R_h u(t_n)) - A(M_h u_h^n))\, D_{-x} R_h u(t_{n+1}), D_{-x} E_u^{n+1}\big)_+ \\
\delta_2^{n+1} &= \big((A(M_h R_h u(t_{n+1})) - A(M_h R_h u(t_n)))\, D_{-x} R_h u(t_{n+1}), D_{-x} E_u^{n+1}\big)_+ \\
\delta_3^{n+1} &= \left(\left(\hat{R}_h(A(u(t_{n+1}))) - A(M_h R_h u(t_{n+1}))\right) \hat{R}_h\left(\frac{\partial u}{\partial t}(t_{n+1})\right), D_{-x} E_u^{n+1}\right)_+ \\
\delta_4^{n+1} &= \left(A(M_h R_h u(t_{n+1}))\left(\hat{R}_h\left(\frac{\partial u}{\partial t}(t_{n+1})\right) - D_{-x} R_h u(t_{n+1})\right), D_{-x} E_u^{n+1}\right)_+
\end{aligned}$$

For $\delta_1^{n+1}, \delta_3^{n+1}$ and δ_4^{n+1} it is easily established, following previously presented arguments, that there exist positive constants $C_{3,3}$ and $C_{3,4}$, independent of h, n and Δt, such that

$$|\delta_1^{n+1}| \leq \left\|A'\right\|_{L^\infty(\mathbb{R})} \|u\|_{C^0([0,T],H^2(\Omega))} \|E_u^n\|_h \left\|D_{-x} E_u^{n+1}\right\|_+$$

$$|\delta_3^{n+1}| \leq C_{3,3} h_{max}^2 \left\|A'\right\|_{L^\infty(\mathbb{R})} \left\|\frac{\partial u}{\partial t}\right\|_{C^0([0,T],H^1(\Omega))} \|u\|_{C^0([0,T],H^2(\Omega))} \left\|D_{-x} E_u^{n+1}\right\|_+$$

$$|\delta_4^{n+1}| \leq C_{3,4} h_{max}^2 \left\|A'\right\|_{L^\infty(\mathbb{R})} \|u\|_{C^0([0,T],H^3(\Omega))} \left\|D_{-x} E_u^{n+1}\right\|_+$$

Regarding δ_2^{n+1}, it follows that

$$\begin{aligned}
|\delta_2^{n+1}| &\leq \sum_{i=1}^{N} h_i |A(M_h R_h u(t_{n+1})(x_i)) - A(M_h R_h u(t_n)(x_i))| \cdot |D_{-x} R_h u(t_{n+1})(x_i)| \\
&\qquad \cdot |D_{-x} E_u^{n+1}(x_i)| \\
&\leq \left\|A'\right\|_{L^\infty(\mathbb{R})} \|u\|_{C^0([0,T],H^2(\Omega))} \sum_{i=1}^{N} h_i |M_h\left(R_h u(t_{n+1})(x_i) - R_h u(t_n)(x_i)\right)| \cdot \\
&\qquad \cdot |D_{-x} E_u^{n+1}(x_i)|
\end{aligned}$$

Applying the Bramble-Hilbert Lemma to estimate $|u(x_i, t_{n+1}) - u(x_i, t_n)|$, for $i = 0, \ldots, N-1$, it follows that there exists a positive constant $C_{3,2}$, independent of h, n and Δt, such that

$$|R_h(u(t_{n+1}) - u(t_n))(x_i)| \leq C_{3,2}\sqrt{\Delta t}\left(\int_{t_n}^{t_{n+1}} \left\|\frac{\partial u}{\partial t}(t)\right\|_{H^1(\Omega)}^2 \mathrm{d}t\right)^{1/2}, \; i = 1, \ldots, N$$

and therefore

$$|\delta_2^{n+1}| \leq C_{3,2}\sqrt{\Delta t}\left\|A'\right\|_{L^\infty(\mathbb{R})} \|u\|_{C^0([0,T],H^2(\Omega))} \left(\int_{t_n}^{t_{n+1}} \left\|\frac{\partial u}{\partial t}(t)\right\|_{H^1(\Omega)}^2 \mathrm{d}t\right)^{1/2} \left\|D_{-x} E_u^{n+1}\right\|_+ .$$

Finally, the following bound holds for τ_3^{n+1}

$$|\tau_3^{n+1}| \leq \left\|A'\right\|_{L^\infty(\mathbb{R})} \|u\|_{C^0([0,T],H^2(\Omega))} \|E_u^n\|_h \left\|D_{-x}E_u^{n+1}\right\|_+ + C_3 \left\|D_{-x}E_u^{n+1}\right\|_+ \left[\sqrt{\Delta t}\, \|u\|_{C^0([0,T],H^2(\Omega))} \left(\int_{t_n}^{t_{n+1}} \left\|\frac{\partial u}{\partial t}(t)\right\|_{H^1(\Omega)}^2 \mathrm{d}t\right)^{1/2} + h_{max}^2 \left(\|u\|_{C^0([0,T],H^3(\Omega))} + \|u\|_{C^0([0,T],H^2(\Omega))} \left\|\frac{\partial u}{\partial t}\right\|_{C^0([0,T],H^1(\Omega))}\right)\right]$$

where $C_3 = \left\|A'\right\|_{L^\infty(\mathbb{R})} \cdot \max_{i=2,3,4} C_{3,i}$.

- Bound for τ_5^{n+1}: We remark that τ_5^{n+1} can be written as

$$\begin{aligned}\tau_5^{n+1} &= \left(\left(\frac{\partial u}{\partial t}(t_{n+1})\right)_h - R_h\left(\frac{u(t_{n+1}) - u(t_n)}{\Delta t}\right), E_u^{n+1}\right)_h \\ &= \left(\left(\frac{\partial u}{\partial t}(t_{n+1})\right)_h - R_h\frac{\partial u}{\partial t}(t_{n+1}), E_u^{n+1}\right)_h \\ &\quad + \left(R_h\left(\frac{\partial u}{\partial t}(t_{n+1}) - \left(\frac{u(t_{n+1}) - u(t_n)}{\Delta t}\right)\right), E_u^{n+1}\right)_h.\end{aligned}$$

The first term can be bounded as in the semidiscrete case through Proposition 8. Therefore, there exists positive constants C_5, h, n and Δt independent, such that

$$\left|\left(\left(\frac{\partial u}{\partial t}(t_{n+1})\right)_h - R_h\frac{\partial u}{\partial t}(t_{n+1}), E_u^{n+1}\right)_h\right| \leq C_5 h_{max}^2 \left\|\frac{\partial u}{\partial t}\right\|_{C^0([0,T],H^2(\Omega))} \left(\left\|E_u^{n+1}\right\|_h + \left\|D_{-x}E_u^{n+1}\right\|_+\right)$$

and

$$\left|\left(R_h\left(\frac{\partial u}{\partial t}(t_{n+1}) - \left(\frac{u(t_{n+1}) - u(t_n)}{\Delta t}\right)\right), E_u^{n+1}\right)_h\right| \leq C_5\sqrt{\Delta t}\left(\int_{t_n}^{t_{n+1}} \left\|\frac{\partial^2 u}{\partial t^2}(t)\right\|_{H^1(\Omega)}^2 \mathrm{d}t\right)^{1/2} \left\|E_u^{n+1}\right\|_h$$

leading to

$$|\tau_5^{n+1}| \leq C_5\left[h_{max}^2 \left\|\frac{\partial u}{\partial t}\right\|_{C^0([0,T],H^2(\Omega))} \left(\left\|E_u^{n+1}\right\|_h + \left\|D_{-x}E_u^{n+1}\right\|_+\right) + \sqrt{\Delta t}\left(\int_{t_n}^{t_{n+1}} \left\|\frac{\partial^2 u}{\partial t^2}(t)\right\|_{H^1(\Omega)}^2 \mathrm{d}t\right)^{1/2} \left\|E_u^{n+1}\right\|_h\right]$$

- Bound for τ_6^{n+1}: Proceeding as in the previous estimates, there exists a positive constant C_6, independent of h and Δt, such that

$$|\tau_6^{n+1}| \leq C_6\sqrt{\Delta t}\left(\int_{t_n}^{t_{n+1}} \left\|\frac{\partial f(u)}{\partial t}(t)\right\|_{H^1(\Omega)}^2 \mathrm{d}t\right)^{1/2} \left\|D_{-x}E_u^{n+1}\right\|_+$$

Defining

$$R_1^n = C_1 h_{max}^2 \|u\|_{C^0([0,T],H^2(\Omega))} + C_5 h_{max}^2 \left\|\frac{\partial u}{\partial t}\right\|_{C^0([0,T],H^2(\Omega))} + C_5\sqrt{\Delta t}\left(\int_{t_n}^{t_{n+1}} \left\|\frac{\partial^2 u}{\partial t^2}(t)\right\|_{H^1(\Omega)}^2 \mathrm{d}t\right)^{1/2}$$

and

$$\begin{aligned} R_2^n = {} & C_1 h_{max}^2 \|u\|_{C^0([0,T],H^2(\Omega))} + C_4 h_{max}^2 \|f(u)\|_{C^0([0,T],H^2(\Omega))} \\ & + C_3 h_{max}^2 \left(\|u\|_{C^0([0,T],H^3(\Omega))} + \|u\|_{C^0([0,T],H^2(\Omega))}\left\|\frac{\partial u}{\partial t}\right\|_{C^0([0,T],H^1(\Omega))}\right) \\ & + C_3\sqrt{\Delta t}\,\|u\|_{C^0([0,T],H^2(\Omega))}\left(\int_{t_n}^{t_{n+1}}\left\|\frac{\partial u}{\partial t}(t)\right\|_{H^1(\Omega)}^2 \mathrm{d}t\right)^{1/2} \\ & + C_5 h_{max}^2\left\|\frac{\partial u}{\partial t}\right\|_{C^0([0,T],H^2(\Omega))} + C_6\sqrt{\Delta t}\left(\int_{t_n}^{t_{n+1}}\left\|\frac{\partial f(u)}{\partial t}(t)\right\|_{H^1(\Omega)}^2 \mathrm{d}t\right)^{1/2} \end{aligned}$$

inequality (41) leads to

$$\begin{aligned} (1+2\Delta t q_0)\left\|E_u^{n+1}\right\|_h^2 + 2\Delta t A_0\left\|D_{-x}E_u^{n+1}\right\|_+^2 & \\ \leq \|E_u^n\|_h^2 + 2\Delta t\left(R_1^n\left\|E_u^{n+1}\right\|_h + R_2^n\left\|D_{-x}E_u^{n+1}\right\|_+\right) & \\ + 2\Delta t\left(\left\|A'\right\|_{L^\infty(\mathbb{R})}\|u\|_{C^0([0,T],H^2(\Omega))} + \left\|f'\right\|_{L^\infty(\mathbb{R})}\right)\|E_u^n\|_h\left\|D_{-x}E_u^{n+1}\right\|_+ & \quad (42) \end{aligned}$$

Applying Young's inequality to each term on the right hand side of (42), for all $\epsilon \neq 0$, we conclude

$$\begin{aligned} & \left(1+2\Delta t\left(q_0-\epsilon^2\right)\right)\left\|E_u^{n+1}\right\|_h^2 + 2\Delta t\left(A_0-2\epsilon^2\right)\left\|D_{-x}E_u^{n+1}\right\|_+^2 \\ \leq & \left(1+\frac{\Delta t}{\epsilon^2}\left(\left\|A'\right\|_{L^\infty(\mathbb{R})}^2\|u\|_{C^0([0,T],H^2(\Omega))}^2 + \left\|f'\right\|_{L^\infty(\mathbb{R})}^2\right)\right)\|E_u^n\|_h^2 + \frac{\Delta t}{2\epsilon^2}\left((R_1^n)^2 + (R_2^n)^2\right) \end{aligned}$$

Taking $\epsilon^2 = \frac{A_0}{4}$ and applying Lemma 1 we obtain inequality (40).

□

Remark 7. Remark 5 holds true for the regularity assumptions on A of Theorems 5 and 6.

5 Numerical tests

We now present a number of numerical results to illustrate the convergence behaviour of the two different approaches followed in this paper. Although the convergence results were obtained under specific regularity assumptions, we shall provide tests where such regularity is reduced. The goal with this approach is to assess the sharpness of the theoretical results.

5.1 Elliptic problem

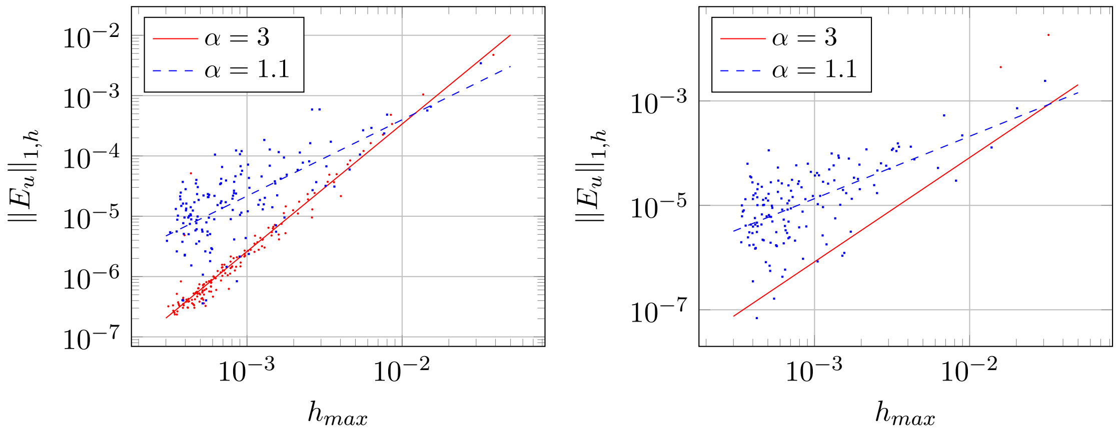
We start by turning our attention to the numerical schemes (10)-(11) and (31)-(32) to approximate the solution of (8)-(9). Conceptually, both schemes differ only by the way the right hand side is discretized. However, we recall that on one hand, we showed second order convergence for scheme (10)-(11) under the assumption that the exact solution of the continuous problem was $C^4(\overline{\Omega}^*)$. On the other hand, we showed that scheme (31)-(32) is also second order convergent for solutions in $H^3(\Omega)$. The results we now present aim at comparing both approaches whilst satisfying the remaining regularity assumptions on A, f, q and g. Let $\alpha \in \mathbb{R}^+$ denote a parameter. We define

$$u_\alpha(x) = |2x-1|^{1+\alpha} - 2(1+\alpha)x(x-1),\ q(x) = x+6,$$

$$A(u) = \frac{1}{1+u^2} + 3,\ f(u) = \frac{\sin(\pi u)}{5},$$

for $x \in [0,1]$ and $u \in \mathbb{R}$.

Imposing A, f and q as coefficients in (8)-(9) and assuming $\alpha \geq 1$, it follows that $u_\alpha \in C^2(\overline{\Omega}^*)$ satisfies the homogeneous Neumann boundary conditions and we can calculate g imposing u_α as the exact solution of (8)-(9). All results were obtained with Matlab [13].



(a) Results for scheme (10)-(11).

(b) Results for scheme (31)-(32).

Figure 1: Log-log plots of $\|E_u\|_{1,h}$ *versus* h_{max} for the elliptic equation. The lines represent least-squares fittings to the data.

We plot in Figure 1 the errors associated with both methods and different exact solutions. For the sake of comparison, we tested both methods assuming $\alpha = 3$. This implies that $u_\alpha \in H^3(\Omega^*)$ and the corresponding right hand side of (8) is $H^2(\Omega)$. Following Remark 6 and since all assumptions on Theorems 1 and 4, both methods are second order. This is clearly illustrated in both plots in Figure 1: the solid lines correspond to the least square fittings to the error data and in both cases, the estimated error of roughly of second order. To provide some clarity on the sharpness of the estimates from both convergence theorems, we also plot in Figure 1 the error for the exact solution corresponding to $\alpha = 1.1$. In this case, $u_\alpha \in H^2(\Omega^*)$. The estimated convergence rate drops to 1.26 and 1.19, respectively, for the numerical schemes (10)-(11) and (31)-(32).

We finally remark that choosing $\alpha = 1.6$ implies that $u_\alpha \in H^3(\Omega^*)$. In this case, method (31)-(32) continues to exhibit second order convergence, while the estimated convergence rate drops for method (31)-(32). This appears to indicate that Remark 5 from [3] might also hold for problems under homogeneous Neumann boundary conditions.

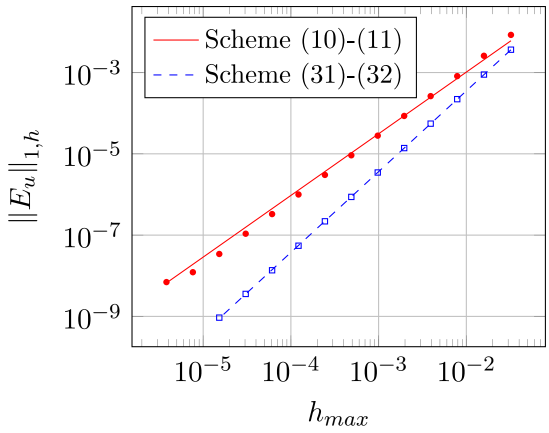


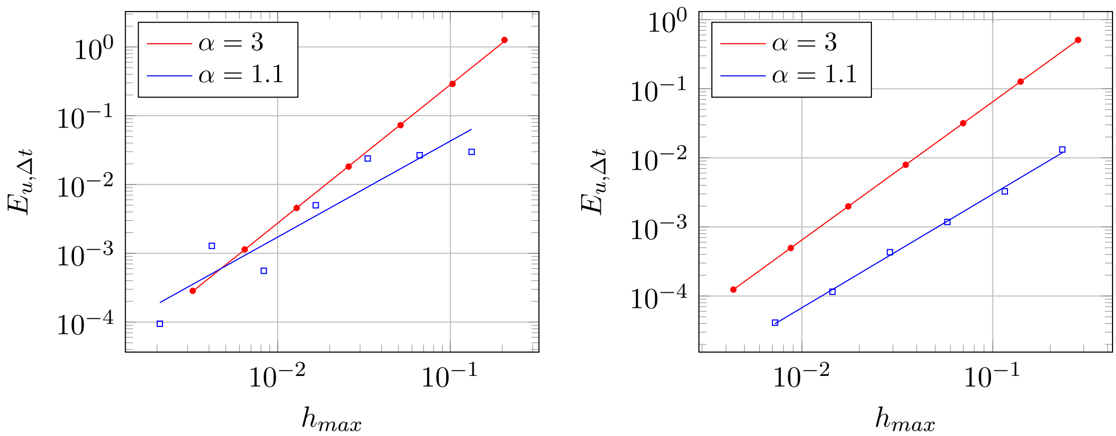
Figure 2: Log-log plots of $\|E_u\|_{1,h}$ *versus* h_{max} for the elliptic equation using $\alpha = 1.6$. The lines represent least-squares fittings to the data.

5.2 Parabolic problem

We now turn our attention to approximating the solution of the differential problem (1)-(3). In this context we define, for $\alpha \in \mathbb{R}^+$,

$$u_\alpha(x,t) = e^t \left(|2x-1|^{1+\alpha} - 2(1+\alpha)x(x-1) - 1 \right),\; q(x,t) = e^{-t}(x+6),\; x \in [0,1],\; t \geq 0.$$

As in the setting of the previous section, for $\alpha \geq 1$, $u_\alpha \in C^2(\overline{\Omega}^* \times [0,T])$ and the homogeneous Neumann boundary conditions (3) hold for u_α. Function g is determined as the right hand side of (1) imposing u_α as the exact solution of the problem.



(a) Results for scheme (26).

(b) Results for scheme (38).

Figure 3: Log-log plots of the error $E_{u,\Delta t}$ *versus* h_{max} for the parabolic problem. The lines represent least-squares fittings to the data.

To measure the error associated with the discretizations, we shall use, as per the results in Theorems 3 and 6, the quantity

$$E_{u,\Delta t} = \max_n \left(\|E_u^n\|_h^2 + \Delta t \sum_{j=1}^{n} \|D_{-x} E_u^j\|_+^2 \right).$$

Under the assumptions of Theorems 3 and 6, $E_{u,\Delta t}$ is of order $h_{max}^2 + \Delta t$. In the numerical tests performed, we decided to generate grids following the iterative refinement strategy detailed in [6]. The parameter Δt is chosen to be, for each grid considered, $\Delta t = h_{max}^2$, in order for the time associated discretization error to not pollute the spatial error.

Examining the error plots in Figure 3, much like in the elliptic case, we validate numerically the error estimates for smooth enough solutions (case $\alpha = 3$). The test with $\alpha = 1.1$ shows that when the solution $u_\alpha \in C^0([0,T], H^2(\Omega^*))$ the estimated convergence rate (with respect to space) goes down to 1.64.

6 Conclusions

The main goal of this paper is to present a robust numerical method discretizing the set of equations (1)-(3), comprising a convection-diffusion-reaction partial differential equation with two nonlinear components (the diffusion coefficient and the convective flux) subject to homogeneous Neumann boundary conditions. We proposed two similar finite difference spatial discretization approaches whose differences lie only on the discretization of the right hand side: one uses a standard restriction operator, (10)-(11), and the other uses an average operator, (31)-(32). Under suitable regularity conditions, we showed second order convergence for these numerical methods associated with the steady problem. Method (10)-(11) is shown to be second order if $u \in C^4(\overline{\Omega}^*)$ and method (31)-(32) exhibits second order convergence if $u \in H^3(\Omega^*)$.

Fully discrete schemes for system (1)-(3) are also proposed and are based on the spatial discretization approach followed for the steady case as well as an IMEX approach for the nonlinear terms (to avoid solving nonlinear problems in each timestep). Under similar spatial regularity assumptions as used in the steady problem, both methods, (26) and (38), are shown to be convergent and the associated error being of order $h_{max}^2 + \Delta t$ in both cases. The major difference in both results is essentially the required regularity on the exact solution of the problem.

Finally, we illustrated the convergence properties of all methods, for both the steady and unsteady cases. A drop on the estimated convergence order is observed if the exact solutions have less regularity than the required in Theorems 1, 3, 4 and 6, thus showing the sharpness of our estimates.

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References

[1] R. Adams and J. Fournier. *Sobolev Spaces*, volume 140. Elsevier, 2 edition, 2003.

[2] S. Barbeiro. *Normas Duais Discretas em Problemas Elípticos.* PhD thesis, Universidade de Coimbra, 2005.

[3] S. Barbeiro, J.A. Ferreira, and R. Grigorieff. Supraconvergence of a finite difference scheme for solutions in $H^s(0, L)$. *IMA Journal of Numerical Analysis*, 25:797–811, 2005.

[4] J.S. Borges, J.A. Ferreira, G. Romanazzi, and E. Abreu. Drug release from viscoelastic polymeric matrices - a stable and supraconvergent FDM. *Computers & Mathematics with Applications*, 99:257–269, 2021.

[5] J. Bramble and S. Hilbert. Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation. *SIAM Journal on Numerical Analysis*, 7:112–124, 1970.

[6] S. Carvalho, J.A. Ferreira, and G. Pena. Nonlinear systems of parabolic IBVP: A stable super-supraconvergent fully discrete piecewise linear FEM. *Applied Mathematics and Computation*, 419:126857, 2022.

[7] Frank De Hoog and David Jackett. On the rate of convergence of finite difference schemes on nonuniform grids. *The ANZIAM Journal*, 26(3):247–256, 1985.

[8] J.A. Ferreira. *Métodos adaptativos para problemas parabólicos: Estudo da convergência*. PhD thesis, 1995.

[9] J.A. Ferreira and R.D. Grigorieff. On the supraconvergence of elliptic finite difference schemes. *Applied Numerical Mathematics*, 28(2):275–292, 1998.

[10] J.A. Ferreira, H.P. Gómez, and L. Pinto. A Mathematical Model for NIR Light Protocol Optimization in Controlled Transdermal Drug Delivery. *Applied Mathematical Modelling*, 112:1–17, 2022.

[11] P.A. Forsyth and P.H. Sammon. Quadratic convergence for cell-centered grids. *Applied Numerical Mathematics*, 4(5):377–394, 1988.

[12] Rolf Dieter Grigorieff. Some Stability Inequalities for Compact Finite Difference Schemes. *Mathematische Nachrichten*, 135(1):93–101, 1988.

[13] The MathWorks Inc. Matlab version: 9.9.0.1467703 (r2022b), 2022.

[14] H.-O. Kreiss, T. A. Manteuffel, B. Swartz, B. Wendroff, and A. B. White. Supra-Convergent Schemes on Irregular Grids. *Mathematics of Computation*, 47(176):537–554, 1986.

[15] Thomas A. Manteuffel and Andrew B. White. The Numerical Solution of Second-Order Boundary Value Problems on Nonuniform Meshes. *Mathematics of Computation*, 47(176):511–535, 1986.

[16] L. B. Wahlbin. *Superconvergence in Galerkin Finite Element Methods*, volume 1605. Lecture Notes in Mathematics, 1995.

[17] Alan Weiser and Mary Fanett Wheeler. On Convergence of Block-Centered Finite Differences for Elliptic Problems. *SIAM Journal on Numerical Analysis*, 25(2):351–375, 1988.