A NOTE ON IDEMPOTENT SEMIRINGS

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Dedicated to Themba Dube on the occasion of his 65th birthday

ABSTRACT. For a commutative semiring S, by an S-algebra we mean a commutative semiring A equipped with a homomorphism $S \to A$. We show that the subvariety of S-algebras determined by the identities $1+2x=1$ and $x^2=x$ is closed under non-empty colimits. The (known) closedness of the category of Boolean rings and of the category of distributive lattices under non-empty colimits in the category of commutative semirings both follow from this general statement.

1. INTRODUCTION

Let us recall:

1.1. A *commutative semiring* is an algebraic structure of the form $S = (S, 0, +, 1, \cdot)$ in which $(S, 0, +)$ and $(S, 1, \cdot)$ are commutative monoids with

$$x0 = 0, \;\; x(y+z) = xy + xz$$

for all $x, y, z \in S$. Here and below we use standard notational agreements: e.g. $xy + xz$ means $(x \cdot y) + (x \cdot z)$. The category of commutative semirings will be denoted by CSR; as expected, its morphisms are semiring homomorphisms, that is, maps $f : S \to S'$ of (commutative) semirings with

$$f(0) = 0, \;\; f(s+t) = f(s) + f(t), \;\; f(1) = 1, \;\; f(st) = f(s)f(t)$$

for all $s, t \in S$.

1.2. For a (commutative) semiring S, an S-*module* (or S-*semimodule*) is an algebraic structure of the form $A = (A, 0, +, h)$ in which $(A, 0, +)$ is a commutative monoid and $h : S \times A \to A$ is a map written as $(s, a) \mapsto sa$ that has

$$1a = a, \;\; s(ta) = (st)a, \;\; s0 = 0, \;\; s(a+b) = sa + sb, \;\; 0a = 0, \;\; (s+t)a = sa + ta$$

for all $s, t \in S$ and $a, b \in A$. The category of S-modules will be denoted by S-mod.

1.3. For S above, a *commutative* S-*algebra* is a pair (A, f) in which A is a commutative semiring, and $f : S \to A$ is a semiring homomorphism. Accordingly, the category of commutative S-algebras is just the comma category $(S \downarrow \mathsf{CSR})$. Equivalently, a commutative S-algebra can be defined as an algebraic structure of the form $A = (A, 0, +, h, 1, \cdot)$ in which $A = (A, 0, +, h)$ is an S-module and $A = (A, 0, +, 1, \cdot)$ is a commutative semiring with $s(ab) = (sa)b$ for all $s \in S$ and $a, b \in A$; we will then have $sa = f(s)a$.

1.4. If S is a commutative ring, then $(S \downarrow \mathsf{CSR})$ is the ordinary category of commutative (unital) S-algebras. In particular: if $S = \mathbb{Z}$ is the ring of integers, then $(S \downarrow \mathsf{CSR})$ is the category CRings of commutative rings; if $S = \{0, 1\}$ with

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$1+1=0$, then $(S\downarrow \mathsf{CSR})$ is the category CRings_2 of commutative rings of characteristic 2 (=the category of commutative algebras over the two-element field). The category CRings_2 contains the category BRings of Boolean rings (=commutative rings satisfying the identity $x^2=x$).

1.5. If $S=\{0,1\}$ with $1+1=1$, then $(S\downarrow \mathsf{CSR})$ is the category AICSR of additively idempotent commutative semirings (=commutative semirings satisfying the identity $2=1$, or, equivalently, the identity $2x=x$). This category contains the category DLat of distributive lattices.

In this paper we present a general result (Corollary 2.3 in the next section) on S-semialgebras, which implies (known) closedness of the categories of Boolean rings and of distributive lattices under non-empty colimits (or, equivalently, just under binary coproducts) in the category of commutative semirings.

2. The general case

Let us first mention an obvious general fact:

Lemma 2.1. *Let $\mathcal{V}$ be a variety of universal algebras, $\mathcal{W}$ a subvariety of $\mathcal{V}$, and I the initial object of $\mathcal{W}$ (=the free algebra in $\mathcal{W}$ on the empty set). If $\mathcal{W}\approx(I\downarrow\mathcal{W})$ is coreflective in $(I\downarrow\mathcal{V})$, then $\mathcal{W}$ is closed under non-empty colimits in $\mathcal{V}$.* □

Then we take:

- $\mathcal{V}=(S\downarrow \mathsf{CSR})$;
- $\mathcal{W}=(S\downarrow \mathsf{CSR})^*$ to be the subvariety of $(S\downarrow \mathsf{CSR})$ consisting of all S-algebras satisfying the identities $1+2x=1$ and $x^2=x$. This makes $I\approx S/E$, where E is the smallest congruence on S containing $(1+2s,1)$ and (s^2,s) for each $s\in S$.

Theorem 2.2. *Let $I\approx S/E$ be as above. The variety*

$$(S\downarrow \mathsf{CSR})^*\approx(I\downarrow(S\downarrow \mathsf{CSR})^*)$$

of commutative S-algebras satisfying the identities $1+2x=1$ and $x^2=x$ is a coreflective subcategory of $(I\downarrow \mathsf{CSR})\approx(I\downarrow(S\downarrow \mathsf{CSR}))$.

Proof. It suffices to show that, for each $A\in(I\downarrow\mathcal{V})$, the set

$$A'=\{a\in A\mid 1+2a=1\ \&\ a^2=a\}$$

is a subalgebra of A, that is, to show the following:

(i) $a,b\in A'\Rightarrow a+b\in A'$;
(ii) for each $s\in S$, $a\in A'\Rightarrow sa\in A'$;
(iii) $1\in A'$;
(iv) $a,b\in A'\Rightarrow ab\in A'$;

Indeed, (i): Suppose a and b are in A'. Then $1+2(a+b)=1+2a+2b=1+2b=1$ and $(a+b)^2=a^2+2ab+b^2=a+2ab+b=a(1+2b)+b=a+b$.

(iii): $1+2\cdot 1=1$ since this equality holds in I.

(iv): Suppose a and b are in A'. Then

$$1+2ab=1+2a+2ab=1+a+a(1+2b)=1+a+a=1+2a=1$$

and $(ab)^2=a^2b^2=ab$.

(ii) follows from (iv) since $sa=(s1)a$ and $(s1)$ is in A' (since $s1$ is the image of class of s under the homomorphism $I\to A$). □

From Lemma 3.1 and Theorem 3.2 we obtain:

Corollary 2.3. *The variety $(S\downarrow \mathsf{CSR})^*$ of commutative S-algebras satisfying the identities $1+2x=1$ and $x^2=x$ is closed under non-empty colimits in the variety $(S\downarrow \mathsf{CSR})$ of all commutative S-algebras.* □

Taking S to be the ring of natural numbers, we obtain the following special cases of Theorem 2.2 and Corollary 2.3:

Corollary 2.4. *The variety* CSR^* *of commutative semirings satisfying the identities* $1 + 2x = 1$ *and* $x^2 = x$ *is coreflective in the variety* $(\{0,1,2\} \downarrow \mathsf{CSR})$*, where* $1 + 2 = 1$ *in* $\{0,1,2\}$. □

Corollary 2.5. *The variety* CSR^* *above is closed under non-empty colimits in* CSR. □

3. Boolean rings and distributive lattices

If an object A of $(\{0,1,2\} \downarrow \mathsf{CSR})$ belongs to $(\{0,1\} \downarrow \mathsf{CSR})$ with $1 + 1 = 0$ in $\{0,1\}$ making $(\{0,1\} \downarrow \mathsf{CSR}) = \mathsf{CRings}_2$, then

$$\{a \in A \mid 1 + 2a = 1 \ \&\ a^2 = a\} = \{a \in A \mid 2a = 0 \ \&\ a^2 = a\}.$$

But if it is the case with $1 + 1 = 1$ making $(\{0,1\} \downarrow \mathsf{CSR}) = \mathsf{AICSR}$, then

$$\{a \in A \mid 1 + 2a = 1 \ \&\ a^2 = a\} = \{a \in A \mid 1 + a = 1 \ \&\ a^2 = a\}.$$

Therefore we obtain the commutative diagram

$$\begin{array}{ccccc} \mathsf{CRings}_2 & \longrightarrow & (\{0,1,2\} \downarrow \mathsf{CSR}) & \longleftarrow & \mathsf{AICSR} \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{BRings} & \longrightarrow & \mathsf{CSR}^* & \longleftarrow & \mathsf{DLat} \end{array}$$

where the horizontal arrows are inclusion functors while the left-hand and right-hand vertical arrows are the coreflections induced by the coreflection of Corollary 2.4 represented by the middle vertical arrow. Since CRings_2 and AICSR both being of the form $(\{0,1\} \downarrow \mathsf{CSR})$ (with different $1 + 1$ in $\{0,1\}$) are closed in CSR under non-empty colimits, we conclude that both BRings and DLat are also closed in CSR under non-empty colimits. That is, as promised in our Introduction, these two known results follow from what we have done in general (in Section 2).

4. Two additional remarks

4.1. The Reader might ask, what is special about $(S \downarrow \mathsf{CSR})$? The answer consists of the following observations:

- $(S \downarrow \mathsf{CSR})$ is the category of commutative monoids in the monoidal category S-mod having therefore 'good' colimits; indeed, its binary coproducts are given by tensor products.
- The monoidal category structure of S-mod is determined by the fact that it is a commutative variety of universal algebras.
- A commutative variety of universal algebras is semi-additive if and only if it is of the form S-mod for some commutative semiring S. This immediately follows from the equivalence 1.$\Leftrightarrow$ 5. in Theorem 2.1 of [2], which refers to [1] for the proof.

4.2. The coreflectivity of DLat in AICSR is a 'finitary copy' of the coreflectivity of the category of frames in the category of quantales, see Section C1.1 of [3]: in fact A_f on Page 479 there is the same as our $\{a \in A \mid 1 + a = 1 \ \&\ a^2 = a\}$.

References

[1] B. Csákány, Primitive classes of algebras which are equivalent to classes of semimodules and modules (Russian), Acta Sci. Math. (Szeged) 24 (1963), 157-164

[2] J. S. Johnson and E. G. Manes, On modules over a semiring, J. Algebra 15 (1970), 57-67

[3] P. T. Johnstone, Sketches of an elephant: a topos theory compendium, Vol. 2. Oxford Logic Guides 44, The Clarendon Press, Oxford University Press, Oxford, 2002.

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