Notes on sublocales and dissolution

Jorge Picado and Aleš Pultr

Abstract

The dissolution (introduced by Isbell in [3], discussed by Johnstone in [5] and later exploited by Plewe in [12, 13]) is here viewed as the relation of the geometry of L with that of the more dispersed $\mathsf{T}(L) = \mathsf{S}(L)^{\mathsf{op}}$ mediated by the natural embedding $\mathfrak{c}_L = (a \mapsto \uparrow a)$ and its adjoint localic map $\gamma_L \colon \mathsf{T}(L) \to L$. The associated image-preimage adjunction $(\gamma_L)_{-1}[-] \dashv \gamma_L[-]$ between the frames $\mathsf{T}(L)$ and $\mathsf{TT}(L)$ is shown to coincide with the adjunction $\mathfrak{c}_{\mathsf{T}(L)} \dashv \gamma_{\mathsf{T}(L)}$ of the second step of the assembly (tower) of L. This helps to explain the role of $\mathsf{T}(L) = \mathsf{S}(L)^{\mathsf{op}}$ as an "almost discrete lift" (sometimes used as a sort of model of the classical discrete lift $DL \to L$) as a dispersion going halfway to Booleanness. Consequent use of the concrete sublocales technique simplifies the reasoning. We illustrate it on the celebrated Plewe's Theorem on ultranormality (and ultraparacompactness) of $\mathsf{S}(L)$ which becomes (we hope) substantially more transparent.

Introduction

Dissolution of a frame, introduced by Isbell in [3] (see also [4]), discussed by Johnstone in [5] (see also [6, 7]) and later exploited by Plewe ([12, 13]), is, roughly speaking, a representation of subobjects (sublocales) of a frame L by closed sublocales of the systems of subobjects. In [5], where the term "dissolution" was first used, Johnstone remarks that it can be to some extent viewed as a pointfree variant of the lift of a topological space to the discrete space with the same underlying set. This sounds like a metaphor, but in

2020 *Mathematics Subject Classification.* 06D22, 18F70.

Key words and phrases. Frame, locale, sublocale, sublocale lattice, dissolution, supplement, dense sublocale, Booleanization, ultranormality, ultraparacompactness.

fact it is a very deep observation. The facts can be viewed as follows. One of the ways of modelling the discrete modification $DX \to X$ in various applications is the localic map $\mathsf{S}(L)^{\mathsf{op}} \to L$ associated with the natural embedding of a frame L into the frame $\mathsf{S}(L)^{\mathsf{op}}$ of its sublocales. This frame is indeed "much more discrete" (dispersed) then the L itself [1]. We propose to view the dissolution, rather than as just the frame L_d from literature, as the relation of the geometry of L with that of the more dispersed $\mathsf{S}(L)^{\mathsf{op}}$ as seen in the behaviour of the systems of subobjects of these two generalized spaces.

What we wish to emphasize is particularly lucid when we view the category of locales **Loc**, the dual of the category of frames, as a concrete one, with morphisms the right Galois adjoints to frame homomorphisms. The subobjects are then naturally represented as well-defined subsets with which it is very easy to work. Since we have often encountered the misunderstanding that this approach is just a shorthand for working with other representations of subobjects we present in some detail the computational aspects, namely that one really works with concrete subsets using concrete transparent operations and formulas (meets coinciding with the set intersections, joins having a very simple formula, images of subobjects coinciding with standard set-theoretic images, preimages being only slightly modified, localic maps being continuous in the sense of preserving openness and closedness by preimage, etc.). To persuade the reader that the reckoning is really easy and not based on harder facts quoted from elsewhere we present the facts (which we would have to recall anyway) with short and easy proofs (expanding the text by less than a page).

Then we study the dissolution by comparing two adjoint situations. We show that the image/preimage adjunction associated with the natural embedding of L to $\mathsf{S}(L)^{\mathsf{op}}$ coincides with the localic map/frame homomorphism adjunction of the second step of the assembly

$$L \to \mathsf{S}(L)^{\mathsf{op}} \to \mathsf{S}(\mathsf{S}(L)^{\mathsf{op}})^{\mathsf{op}} \to \cdots \ .$$

This allows (a.o.) to show how far the dissolution relaxes the generalized topology. The continuity of localic maps $f\colon M \to L$ is characterized, similarly as in the classical case, by preserving closedness and openness by the preimage function, and the relative dispersedness is expressed by the question how often a preimage of S is closed resp. open even if S is not. Here

[1] Not "quite discrete" which in the pointfree context should be Boolean; Boolean lifts do exist and make sense (see e.g. [10]), but here we are particularly interested in the not quite Boolean nature of the $\mathsf{S}(L)$.

every S is a preimage of a closed one, but the "discreteness" goes only half way. Unlike in the classical case, preserving closedness and openness do not imply each other.

The fact that one does not have a full discreteness is in fact in some respects beneficial. A particularly interesting application is the proof of the fact that the frame of sublocales (the inclusion inverted) is ultranormal[2] (Plewe [13]). In the last section we present a simplified proof; we follow the Plewe's procedure but using the explicit sublocale technique makes it simpler and, we hope, quite transparent.

1. Preliminaries

1.1. Our posets will be typically complete lattices and we will use the standard notation for meets (infima) and joins (suprema) in posets (partially ordered sets): $a \wedge b$, $\bigwedge A$ or $\bigwedge_{a\in A} a$, $a \vee b$, $\bigvee A$ or $\bigvee_{a\in A} a$. The least resp. largest element (if it exists) will be denoted by 0 resp. 1. We write

$$\uparrow\! a \text{ for } \{x \,|\, x \geq a\} \quad\text{and}\quad \uparrow\! A = \{x \,|\, \exists\; a \in A,\; x \geq a\}.$$

1.2. A *pseudocomplement* of a, denoted by a^*, is the largest b such that $a \wedge b = 0$, if it exists. Dually, a *supplement* of a, denoted by $a^{\#}$, is the smallest b such that $a \vee b = 1$, if it exists. Recall the standard facts that

$$a \leq a^{**} \quad\text{and}\quad (a \wedge b)^{**} = a^{**} \wedge b^{**}.$$

A *complement* of a is a b such that $a \wedge b = 0$ and $a \vee b = 1$. In a distributive lattice, each complement is a pseudocomplement; therefore we may write a^* also for complement, but if there is a danger of misunderstanding we prefer a^{c}. An element which has a complement is said to be *complemented.*

1.3. Monotone maps $f\colon X \to Y$ and $g\colon Y \to X$ are (*Galois*) *adjoint, f to the left and g to the right* if

$$f(x) \leq y \iff x \leq g(y),$$

It is standard that

(1) left adjoints preserve all existing suprema and right adjoints preserve all existing infima,

(2) and if X, Y are complete lattices then each $f\colon X \to Y$ preserving all suprema is a left adjoint, and each $g\colon Y \to X$ preserving all infima is a right adjoint.

[2] That is, that any two disjoint subobjects can be separated by a complemented one.

1.4. A *frame* (resp. *coframe*) is a complete lattice L satisfying the distributivity law

$$\big(\bigvee A\big) \wedge b = \bigvee\{a \wedge b \,|\, a \in A\} \tag{frm}$$

$$\text{(resp. } \big(\bigwedge A\big) \vee b = \bigwedge\{a \vee b \,|\, a \in A\}) \tag{cofrm}$$

for all $A \subseteq L$ and $b \in L$. A *frame homomorphism* preserves all joins and all finite meets.

The rule (frm) makes the $(-) \wedge b$ left adjoints; consequently a frame has a *Heyting structure* with the Heyting operation $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c.$$

A frame has pseudocomplements $a^* = a \to 0$.

1.4.1. Note. It is easy to check that

if a is complemented then $a \to b = a^{\mathsf{c}} \vee b$

(recall the $A \Rightarrow B \equiv \text{non}A \vee B$ from classical logic). But one cannot express $a \to b$ as $a^* \vee b$ with a general pseudocomplement.

1.4.2. Dually, a coframe is naturally endowed with a co-Heyting operation, the difference $a \smallsetminus b$ satisfying

$$a \smallsetminus b \leq c \quad \text{iff} \quad a \leq b \vee c.$$

A coframe has *supplements* $a^\# = 1 \smallsetminus a$ (the smallest b such that $a \vee b = 1$).

1.5. A typical frame is the lattice $\Omega(X)$ of open sets of a topological space X, and for continuous maps $f\colon X \to Y$ there are frame homomorphisms $\Omega(f) = (U \mapsto f^{-1}[U])\colon \Omega(Y) \to \Omega(X)$. Thus we have a contravariant functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Frm},$$

where **Top** is the category of topological spaces, and **Frm** that of frames. To make it covariant one considers the *category of locales* $\mathbf{Loc} = \mathbf{Frm}^{\mathsf{op}}$.

It is of advantage to view **Loc** as a concrete category with the opposites of frame homomorphisms $h\colon L \to M$ represented by their right adjoints $f\colon M \to L$, called *localic maps*. They are characterized by

1.5.1. Proposition. *A meet-preserving $f\colon M \to L$ is a* localic map*, that is, a right adjoint to a frame homomorphism $h\colon L \to M$, iff $f(a) = 1$ (if and) only if $a = 1$ and*

$$f(h(a) \to b) = a \to f(b) \quad \textit{for all} \quad a, b.\text{[3]}$$

[3] This is often referred to as the Frobenius identity.

1.6. Sublocales. In **Frm**, extremal epimorphisms are precisely the onto frame homomorphisms and hence the extremal monomorphisms in **Loc** are the one-to-one localic maps. This leads to the following approach to subobjects of frames (locales).

A *sublocale* of a frame L is a subset $S \subseteq L$ such that the embedding map $j\colon S \subseteq L$ is a localic one. Such S are characterized by the requirements that

(S1) for every $M \subseteq S$ the meet $\bigwedge M$ lies in S, and

(S2) for every $s \in S$ and every $x \in L$, $x \to s$ lies in S

(see e.g. [9]). The system $\mathsf{S}(L)$ of all sublocales of L is a coframe with a fairly transparent structure:

$$\bigwedge S_i = \bigcap S_i \quad \text{and} \quad \bigvee S_i = \{\bigwedge M \mid M \subseteq \bigcup S_i\}. \tag{1.6.1}$$

The least sublocale $\bigvee \emptyset = \{1\}$ is designated by O and referred to as the *void sublocale.*

1.6.1. Note. A sublocale S of a frame L is itself a frame (the adjoint to the embedding is a frame homomorphism). By (S2) we easily see that its Heyting operation is the restriction of the $\to$ from L.

1.6.2. Sublocales are upsets only if they are closed. But generally we have

Proposition. *If S is a sublocale then for every x^* holds the implication*

$$x^* \geq s_0 = \bigwedge S \quad \Rightarrow \quad x^* \in S.$$

Proof. $x^* = x \to 0 = x \to (x \wedge s_0) = (x \to x) \wedge (x \to s_0) = (x \to s_0) \in S.$ □

1.7. Images and preimages. Let $f\colon L \to M$ be a localic map and $S \subseteq L$ and $T \subseteq M$ sublocales. Then the (standard set theoretic) image $f[S]$ is easily seen to be a sublocale of M. The standard $f^{-1}[T]$ is generally not a sublocale, but it is closed under meets and hence, by (1.6.1) there is the largest sublocale contained in this set, which we denote by $f_{-1}[T]$ and call the (localic) preimage of T. There is the obvious (Galois) adjunction

$$f[S] \subseteq T \quad \text{iff} \quad S \subseteq f_{-1}[T].$$

Consequently, in particular, $f_{-1}[-]$ preserves intersections. Moreover, $f_{-1}[\mathsf{O}] = \mathsf{O}$ (because already $f^{-1}[\mathsf{O}] = \mathsf{O}$, recall 1.5.1).

2. A short course in sublocale computing

In this section we collect some facts about sublocales that we will need in the following. The aim in presenting them with proofs (that makes this section hardly more then half a page longer) is to persuade the reader that the sublocale reckoning is not a shorthand for (more involved) reckoning with other representations of subobject (nuclei, congruences, onto homomorphisms). One really calculates with the formulas given for the sublocales in the subsections above – and usually obtains very simple and transparent proofs.

2.1. We will use simple Heyting rules like

$$a \to \bigwedge_{i\in J} b_i = \bigwedge_{i\in J}(a \to b_i), \tag{$distr$}$$

$$(\bigvee_{i\in J} a_i) \to b = \bigwedge_{i\in J}(a_i \to b), \tag{$distr^{\mathsf{op}}$}$$

$$a \le b \to c \quad \text{iff} \quad b \le a \to c, \tag{$exch$}$$

$$1 \to a = a, \qquad a \le b \quad \text{iff} \quad a \to b = 1, \qquad a \le b \to a, \tag{2.1.1}$$

$$a \wedge (a \to b) = a \wedge b, \quad (a \wedge b) \to c = a \to (b \to c) = b \to (a \to c), \tag{2.1.2}$$

$$a \le (a \to b) \to b \tag{2.1.3}$$

all of them trivial, and only a slightly less simple

$$a = (a \vee b) \wedge (b \to a) \tag{2.1.4}$$

($a \le (a \vee b) \wedge (b \to a)$ and on the other hand $(a \vee b) \wedge (b \to a) = (a \wedge (b \to a)) \vee (b \wedge (b \to a)) \le a \vee (b \wedge a) = a$).

2.2. The nucleus and a handy formula. Recalling (S1) we see that we have, for a sublocale S

$$\nu_S(a) = \bigwedge\{s \in S \mid a \le s\} \in S.$$

The resulting map $\nu_S\colon L \to L$ is called the *nucleus* of S. Obviously $a \in S$ iff $\nu_S(a) = a$.

2.2.1. Proposition. *If $s \in S$ then for all a, $\nu_S(a) \to s = a \to s$.*

Proof. $x \le a \to s$ iff $a \le x \to s$ ($\in S$) iff $\nu(a) \le x \to s$ iff $x \le \nu(a) \to s$. □

2.3. Open and closed sublocales. With $a \in L$ we associate the *open* and *closed* sublocales

$$\mathfrak{o}(a) = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\} \quad \text{and} \quad \mathfrak{c}(a) = \uparrow a$$

(for the equality in the first see (2.1.2)).

2.3.1. Proposition. *$\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements of each other.*

Proof. I. If $x \in \mathfrak{c}(a) \cap \mathfrak{o}(a)$ then $a \leq x = a \to x$ and hence $1 = a \to x = x$. Thus, $\mathfrak{c}(a) \cap \mathfrak{o}(a) = \{1\} = \mathsf{O}$.

II. For each $x \in L$ we have, by (2.1.4), $x = (a \vee x) \wedge (a \to x)$ with $a \vee x \in \mathfrak{c}(a)$ and $a \to x \in \mathfrak{o}(a)$. □

2.3.2. A frame is *zero-dimensional* (or *totally disconnected*) if it is join-generated by complemented elements. We have

Proposition. *For every sublocale $S \in L$,*

$$S = \bigcap_{x \in L} \mathfrak{c}(\nu_S(x)) \vee \mathfrak{o}(x).$$

Thus in particular, $\mathsf{S}(L)^{\mathsf{op}}$ is a zero-dimensional frame.

Proof. $\subseteq$: Let $a \in S$ and $x \in L$ arbitrary. Then $x \to a = \nu(x) \to a$ and hence

$$a = (\nu(x) \vee a) \wedge (\nu(x) \to a) = (\nu(x) \vee a) \wedge (x \to a) \in \mathfrak{c}(\nu(x)) \vee \mathfrak{o}(x).$$

$\supseteq$: Let $a \in \bigcap_{x \in L} \mathfrak{c}(\nu_S(x)) \vee \mathfrak{o}(x)$. In particular $a \in \mathfrak{c}(\nu_S(a)) \vee \mathfrak{o}(a)$ and hence $a = x \wedge y$ for some $x \geq \nu(a)$ and $y = a \to y$. Then $a \leq a \to y$, hence $a = a \wedge a \leq y$ and $y = a \to y = 1$. Thus, $a = x \geq \nu(a)$ and $a \in S$. □

2.3.3. Proposition. *We have*

$$\mathfrak{o}(0) = \mathsf{O}, \mathfrak{o}(1) = L, \quad \mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b) \quad \textit{and} \quad \bigvee \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee a_i),$$
$$\mathfrak{c}(1) = \mathsf{O}, \mathfrak{c}(0) = L, \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b) \quad \textit{and} \quad \bigcap \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee a_i).$$

Proof. The statements on the closed sublocales are straightforward, those on the open ones then follow immediately from 2.3.1. □

2.3.4. Open and closed sublocales in sublocales. A sublocale S is itself a frame (recall 1.6.1). Denoting the open and closed sublocales in S by $\mathfrak{o}_S$ and $\mathfrak{c}_S$ we obtain for $a \in S$

$$\mathfrak{o}_S(a) = \mathfrak{o}(a) \cap S \quad \text{and} \quad \mathfrak{c}_S(a) = \mathfrak{c}(a) \cap S.$$

(the latter is trivial and the former follows from 1.6.1).

2.3.5. A *cover* of a frame L (more generally, of a sublocale $S \subseteq L$) is a subset $\mathcal{C} \subseteq \mathsf{S}(L)$ such that $\bigvee\{C \mid C \in \mathcal{C}\} = L$ (resp. $\bigvee\{C \mid C \in \mathcal{C}\} \supseteq S$).

Note that in literature one often speaks of covers of L as of subsets $A \subseteq L$ such that $\bigvee A = 1$. This can be viewed as a special case of the above thinking of A as a simplified encoding of an *open cover* $\{\mathfrak{o}(a) \mid a \in A\}$.

2.3.6. Preimages of open and closed sublocales. *For a localic map $f\colon L \to M$ and its adjoint homomorphism h we have*

$$f_{-1}[\mathfrak{c}(a)] = \mathfrak{c}(h(a)) \quad \textit{and} \quad f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(h(a)).$$

Proof. We have $x \in f^{-1}[\mathfrak{c}(a)]$ iff $f(x) \geq a$ iff $x \geq h(a)$ iff $x \in \mathfrak{c}(h(a))$ which is already a sublocale.

Next, by 1.5.1, $x \in \mathfrak{o}(h(a))$ makes $f(x) = f(h(a) \to x) = a \to f(x)$, hence $\mathfrak{o}(h(a)) \subseteq f^{-1}[\mathfrak{o}(a)]$. On the other hand, since $f_{-1}[-]$ preserves meets and O we have

$$f_{-1}[\mathfrak{o}(a)] \cap \mathfrak{c}(h(a)) = f_{-1}[\mathfrak{o}(a)] \cap f_{-1}[\mathfrak{c}(a)] = f_{-1}[\mathfrak{o}(a) \cap \mathfrak{c}(a)] = f_{-1}[\mathsf{O}] = \mathsf{O}$$

and hence $f_{-1}[\mathfrak{o}(a)] \subseteq \mathfrak{o}(h(a))$. □

2.4. Closure, interior and density. We have the *closure*, the smallest closed sublocale containing S,

$$\overline{S} = \bigcap\{\mathfrak{c}(a) = \uparrow a \mid S \subseteq \uparrow a\} = \uparrow(\bigwedge S). \tag{2.4.1}$$

Obviously the meet (the minimum) of $\mathfrak{o}(a)$ is $a \to 0 = a^*$ so that

$$\overline{\mathfrak{o}(a)} = \uparrow(a^*). \tag{2.4.2}$$

Using the fact that $S^{\#\#} \subseteq S$ we immediately see that $\mathfrak{o}(a) \subseteq S$ iff $S^{\#} \subseteq \mathfrak{c}(a)$ and obtain the formula for the *interior* of S,

$$\operatorname{int} S = \mathfrak{o}(\bigwedge S^{\#}). \tag{2.4.3}$$

2.4.1. Observations. 1. *A sublocale S is dense, that is, $\overline{S} = L$, iff $0 \in S$.*

2. *S is dense iff the intersection with each non-void open sublocale is non-void.*

3. $\bigwedge S^{\#} \leq (\bigwedge S)^*$, *and the equality holds iff* $\operatorname{int} S = \operatorname{int} \overline{S}$.

(1 is trivial, and 2 follows from 2.3.1. For 3: realize that because $S^{\#} \vee S = L$, $\bigwedge S^{\#} \wedge \bigwedge S = 0$; moreover $\bigwedge S^{\#} = (\bigwedge S)^*$ iff $\mathfrak{o}(\bigwedge S^{\#}) = \mathfrak{o}((\bigwedge S)^*)$ iff $\operatorname{int} S = \operatorname{int}(\mathfrak{c}(\bigwedge S))$.)

Obviously, the *Booleanization* of L

$$\mathfrak{B}L = \{x \in L \,|\, x = x^{**}\} = \{x \to 0 = x^* \,|\, x \in L\}$$

is a sublocale of L and it is the smallest sublocale containing 0. Thus we have another

2.4.2. Observation. (Isbell's Density Theorem) *A sublocale $S \subseteq L$ is dense iff it contains $\mathfrak{B}L$.*

2.4.3. Lemma. *For any complemented S, $(\bigwedge S)^* \le (\bigwedge S^{\mathsf{c}})^{**}$.*

Proof. By 1.6.2, $(\bigwedge S \vee \bigwedge S^{\mathsf{c}})^{**} \in S \cap S^{\mathsf{c}} = \mathsf{O}$ hence $0 = (\bigwedge S \vee \bigwedge S^{\mathsf{c}})^* = (\bigwedge S)^* \wedge (\bigwedge S^{\mathsf{c}})^*$. □

2.4.4. Lemma. *For any complemented S, $\operatorname{int} S \vee \operatorname{int} S^{\mathsf{c}}$ is dense.*

Proof. From $0 = \bigwedge(S \vee S^{\mathsf{c}}) = \bigwedge S \wedge \bigwedge S^{\mathsf{c}}$ and 2.4.3 we get

$$0 = (\bigwedge S)^{**} \wedge (\bigwedge S^{\mathsf{c}})^{**} \ge (\bigwedge S^{\mathsf{c}})^* \wedge (\bigwedge S)^* = \bigwedge(\operatorname{int} S \vee \operatorname{int} S^{\mathsf{c}}). \quad \square$$

2.4.5. Proposition. *For any complemented S, $\mathfrak{B}L \cap \operatorname{int} S = \mathfrak{B}L \cap S$.*

Proof. $S \cap \mathfrak{B}L = S \cap (\operatorname{int} S \vee \operatorname{int} S^{\mathsf{c}}) \cap \mathfrak{B}L = (\operatorname{int} S \vee \mathsf{O}) \cap \mathfrak{B}L = \operatorname{int} S \cap \mathfrak{B}L$. □

2.4.6. Note. On the other hand, by 1.6.2 we have generally that $\mathfrak{B}L \cap \overline{S} = \mathfrak{B}L \cap S$.

3. Dissolution and the frame of sublocales

3.1. The system of sublocales of L is a coframe. We will work also with its dual and to simplify the notation we will write

$$\mathsf{T}(L) = \mathsf{S}(L)^{\mathsf{op}}$$

and speak of the *frame of sublocales.* We have the natural frame embedding (recall 2.3.3)

$$\mathfrak{c}_L = (a \mapsto \mathfrak{c}(a)) \colon L \to \mathsf{T}(L)$$

(the index L in $\mathfrak{c}_L$ will be omitted if obvious)[4]. The localic map associated with the $\mathfrak{c}_L$ in the adjunction

$$L \underset{\gamma_L}{\overset{\mathfrak{c}_L}{\rightleftarrows}} \mathsf{T}(L) \qquad (\perp) \qquad\qquad (\mathfrak{c}\text{-}\gamma)$$

[4]The T, used to simplify the notation is an allusion to the first step in the Tower (assembly). Note that the order of $\mathsf{T}(L)$ follows the order of the associated frames of congruences, or of the nuclei, see e.g. [9].

is obviously given by the formula

$$\gamma_L(S) = \bigwedge S.$$

This $\mathfrak{c}_L$ resp. the associated localic γ_L are often (mostly satisfactory) used as a surrogate for the lifting of the topological space X to DX in which one replaces the topology by the discrete one on the same set (for instance, the "not necessarily continuous" localic maps from L to M are modelled by the localic maps $\mathsf{T}(L) \to M$ mimicking the not necessarily continuous map $X \to Y$ represented as $f\colon DX \to Y$).

In this section we will point out how far the $\gamma_L\colon \mathsf{T}(L) \to L$ emulates the discrete lifting.

3.2. Two coinciding adjunctions. Recall the image-preimage adjunction

$$\mathsf{T}(L) \underset{\gamma_L[-]}{\overset{(\gamma_L)_{-1}[-]}{\rightleftarrows}} \mathsf{T}(\mathsf{T}(L))\ . \qquad \text{(dis)}$$

The ($\mathfrak{c}$-γ) provides another one between $\mathsf{T}(L)$ and $\mathsf{T}(\mathsf{T}(L))$, namely

$$\mathsf{T}(L) \underset{\gamma_{\mathsf{T}(L)}}{\overset{\mathfrak{c}_{\mathsf{T}(L)}}{\rightleftarrows}} \mathsf{T}(\mathsf{T}(L))\ .$$

We will show that these two adjunctions coincide.

3.2.1. Proposition. *Let $\mathcal{S}$ be a subcolocale of $\mathsf{S}(L)$ and let $S_0 = \bigvee\{S \mid S \in \mathcal{S}\}$. Then for each $t \in S_0$ there is a $T \in \mathcal{S}$ such that $t = \bigwedge T$.*

Proof. For $S \in \mathcal{S}$ and $x \in S$ set $S_x = S \smallsetminus \mathfrak{o}(x) = S \cap \mathfrak{c}(x) = \{s \in S \mid s \geq x\}$. Then by (S2) in 1.6 and 1.4.1,

$$S_x \in \mathcal{S} \quad \text{and} \quad \bigwedge S_x = x.$$

For $s \in S_0 = \bigvee\{S \mid S \in \mathcal{S}\}$ we have $s = \bigwedge_{S\in\mathcal{S}} x_S$ for some $x_S \in S$. Set $T = \bigvee\{S_{x_S} \mid S \in \mathcal{S}\}$. Then $\bigwedge T = S$. □

3.2.2. Theorem. $\gamma_L[-] = \gamma_{\mathsf{T}(L)}$ *and hence also* $(\gamma_L)_{-1}[-] = \mathfrak{c}_{\mathsf{T}(L)}$.

Proof. We have

$$\gamma_{\mathsf{T}(L)}(\mathcal{S}) = \bigvee\{S \mid S \in \mathcal{S}\} = \bigvee\{\bigwedge S \mid S \in \mathcal{S}\} = \gamma_L[\mathcal{S}]$$

by 3.2 (the inclusion $\gamma_{\mathsf{T}(L)}(\mathcal{S}) \subseteq \bigvee\{S \mid S \in \mathcal{S}\}$ is trivial), and the adjoints are mutually uniquely determined. □

3.2.3. Notes. 1. The fact that $(\gamma_L)_{-1}[-] = \mathfrak{c}_{\mathsf{T}(L)}$ can be also proved directly, although not quite as easily.

To this end let us recall the fact that $f_{-1}[-]$, for any localic map f, is a coframe homomorphism (see, e.g., [9] – the proof is slightly more involved than the sublocale facts presented in Section 2: preserving meets and O is straightforward, see 1.7, but proving that the join $S \vee T$ is preserved needs half a page of computing).

Further, considering the maps $\mathfrak{c}_L$ as above, and

$$\mathfrak{o}_L = (a \mapsto \mathfrak{o}(a)) \colon L \to \mathsf{S}(L) \quad \text{or} \quad L \to \mathsf{T}(L)$$

we get in the first case lattice homomorphisms and in the second case anti-homomorphisms, hence in any case maps preserving complements and hence in particular

$$\mathfrak{c}(\mathfrak{c}(a)) = \mathfrak{o}(\mathfrak{o}(a)), \ \mathfrak{c}(\mathfrak{o}(a)) = \mathfrak{o}(\mathfrak{c}(a)) \text{ and } \mathfrak{o}(\mathfrak{c}(a)) = \mathfrak{c}(\mathfrak{o}(a)).$$

Now if we represent $S \in \mathsf{S}(L)$ as $\bigcap_i(\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))$ (recall 2.3.2) we can compute (recall 2.3.6)

$$\begin{aligned}\gamma_{-1}[S] &= \bigcap(\gamma_{-1}[\mathfrak{o}(a_i)] \vee \gamma_{-1}[\mathfrak{c}(b_i)]) = \bigcap(\mathfrak{o}(\mathfrak{c}(a_i)) \vee \mathfrak{c}(\mathfrak{c}(b_i))) = \\ &= \bigcap(\mathfrak{c}(\mathfrak{o}(a_i)) \vee \mathfrak{c}(\mathfrak{c}(b_i))) = \mathfrak{c}(\bigcap(\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i)) = \mathfrak{c}(S).\end{aligned}$$

Note how much more complicated this approach is: we needed the non-trivial fact that $f_{-1}[-]$ was a homomorphism, the representation of S from 2.3.2 and the not quite trivial $f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(h(a))$ from 2.3.6, while 3.2.2 needed just a straightforward use of the fact that $\mathcal{S}$ was a sub(co)locale.

2. For an alternate categorical proof of this fact, consider the assignment $L \mapsto \mathsf{T}(L)$ as a functor

$$\mathsf{T} \colon \mathbf{Loc} \to \mathbf{Loc}$$

defined on morphisms by $\mathsf{T}(f) = f[-]$ and recall that the dissolving maps γ_L are monic and are the components of a natural transformation $\mathsf{T} \to \mathrm{Id}$ ([3, 1.4]). Hence

$$\gamma_L \cdot \gamma_{\mathsf{T}(L)} = \gamma_L \cdot \mathsf{T}(\gamma_L)$$

for every L, and thus $\gamma_{\mathsf{T}(L)} = \mathsf{T}(\gamma_L) = \gamma_L[-]$.

3.3. Corollary. *Thus in particular, for every $S \in \mathsf{S}(L)$ the preimage $(\gamma_L)_{-1}[S]$ is closed in $\mathsf{T}(L)$.*

The system of closed sublocales of $\mathsf{T}(L)$ is sometimes called the *dissolution* of L and denoted by

$$L_d.$$

Thus, L_d is an isomorphic copy of $\mathsf{T}(L)$. The point is in the specific concrete representation of this frame which can be used to advantage.

3.3.1. Corollary. *A sublocale S of L is complemented iff $\mathfrak{c}_{\mathsf{T}(L)}(S) = (\gamma_L)_{-1}[S]$ is clopen in $\mathsf{T}(L)$.*

3.4. The "weak discreteness" of the $\mathsf{T}(L)$ can be now elucidated by the following comparison. The classical discrete lifting $\delta = (x \mapsto x)\colon DX \to X$ is characterized by the facts that

- δ is one-one and onto, and
- $\delta^{-1}[A]$ is closed for each subset $A \subseteq X$,

while for the $\gamma = \gamma_L\colon \mathsf{T}(L) \to L$ we have that

- γ is monic and epic, and
- $\gamma_{-1}[S]$ is closed for each sublocale $S \subseteq L$.

The former is a very good analogy: of course a localic map cannot be one-one and onto unless it is an isomorphism. The real difference is in the latter: while in spaces one obtains automatically that also $\delta^{-1}[A]$ is open for each subset $A \subseteq X$, for localic maps one has that preserving closed and open maps for preimages are two independent facts[5] (and $\gamma_{-1}[S]$ are not generally open).

3.5. To the characterization of complemented sublocales and other facts about the dissolution from the literature (e.g. [12, 13]) let us add the characterization of the differences (dual Heyting arrows) $B \smallsetminus A$ and the supplements $A^\#$ in $\mathsf{S}(L)$.

3.5.1. Proposition. $(\gamma_L)_{-1}[B \smallsetminus A] = \bigcap\{$*clopen* $\mathcal{C} \mid \mathfrak{c}_{\mathsf{T}(L)}(B) \cap \mathfrak{o}_{\mathsf{T}(L)}(A) \subseteq \mathcal{C}\}$.

Proof. Since

$$B \smallsetminus A = \bigcap\{S \in \mathsf{S}(L) \mid S \text{ is complemented }, B \subseteq S \vee A\} \qquad \text{(cf. [2])}$$

[5]Somewhat surprisingly, being a localic map is characterized in among plain maps by preserving both open and closed sublocales, see [1, 11].

and $(\gamma_L)_{-1}[-]$ is a homomorphism, $(\gamma_L)_{-1}[B \smallsetminus A]$ is given by

$$\begin{aligned}
&\bigcap\{(\gamma_L)_{-1}[S] \mid S \text{ is complemented}, B \subseteq S \vee A\} \\
&\qquad = \bigcap\{\mathfrak{c}_{\mathsf{T}(L)}(S) \mid S \text{ is complemented}, B \subseteq S \vee A\} \\
&\qquad = \bigcap\{\mathfrak{c}_{\mathsf{T}(L)}(S) \mid S \text{ is complemented}, \mathfrak{c}_{\mathsf{T}(L)}(B) \subseteq \mathfrak{c}_{\mathsf{T}(L)}(S) \vee \mathfrak{c}_{\mathsf{T}(L)}(A)\} \\
&\qquad = \bigcap\{\text{clopen } \mathcal{C} \mid \mathfrak{c}_{\mathsf{T}(L)}(B) \subseteq \mathcal{C} \vee \mathfrak{c}_{\mathsf{T}(L)}(A)\} \\
&\qquad = \bigcap\{\text{clopen } \mathcal{C} \mid \mathfrak{c}_{\mathsf{T}(L)}(B) \cap \mathfrak{o}_{\mathsf{T}(L)}(A) \subseteq \mathcal{C}\}. \qquad \square
\end{aligned}$$

3.5.2. In particular, for $B = L$, the formula yields

$$(\gamma_L)_{-1}[L \smallsetminus A] = \bigcap\{\text{clopen } \mathcal{C} \mid \mathfrak{o}_{\mathsf{T}(L)}(A) \subseteq \mathcal{C}\}$$

which by the zero-dimensionality of $\mathsf{T}(L)$ is equal to $\overline{\mathfrak{o}_{\mathsf{T}(L)}(A)}$. This follows also from 3.2.2 and (2.4.2):

$$(\gamma_L)_{-1}[L \smallsetminus A] = \mathfrak{c}_{\mathsf{T}(L)}(A^*) = \overline{\mathfrak{o}_{\mathsf{T}(L)}(A)}.$$

4. An application: Plewe's Theorem revisited

4.1. The formula (frm) does not generally hold in coframes and the formula (cofrm) does not generally hold in frames for general b. In the exceptional cases when it does one speaks of a *distributive* system A ([13]).

4.1.1. Proposition. *Let L be a frame. Then $\{a_i \,|\, i \in J\} \subseteq L$ is distributive iff $\bigvee \mathfrak{c}(a_i) = \mathfrak{c}(\bigwedge a_i)$ in the coframe $\mathsf{S}(L)$.*

Proof. $\Rightarrow$: Let $x \geq \bigwedge a_i$. Then $x = x \vee \bigwedge a_i = \bigwedge(x \vee a_i)$ and, of course, $\{x \vee a_i\}_i \subseteq \bigcup {\uparrow} a_i$, hence $x \in \bigvee \mathfrak{c}(a_i)$. The other inclusion is obvious.
$\Leftarrow$: Let $b \vee \bigwedge a_i \in \mathfrak{c}(\bigwedge a_i) = \bigvee \mathfrak{c}(a_i)$. Then there is some $B = \{b_j\}_j \subseteq \bigcup_i {\uparrow} a_i$ such that $b \vee \bigwedge a_i = \bigwedge B = \bigwedge_j b_j$ (in particular, $b \leq b_j$ for every j and for each j there is some i_j such that $b_j \geq a_{i_j}$). Finally, we have

$$b \vee \bigwedge a_i = \bigwedge_j b_j = \bigwedge_j (b_j \vee a_{i_j}) \geq \bigwedge_j (b \vee a_{i_j}) \geq \bigwedge_i (b \vee a_i). \quad \square$$

4.2. Open sublocales dense in closed ones. We say that $\mathfrak{o}(b)$ is *dense in* $\mathfrak{c}(a)$ if $\mathfrak{o}(b) \cap \mathfrak{c}(a)$ is dense in $\mathfrak{c}(a)$. We have

$$\mathfrak{o}(b) \cap \mathfrak{c}(a) = \mathfrak{o}(a \vee b) \cap \mathfrak{c}(a)$$

(if $x \geq a$ then $(a \vee b) \to x = x$ makes $x = (a \to x) \wedge (b \to x) = b \to x$) and hence we may restrict ourselves to the $\mathfrak{o}(b)$ with $b \geq a$ when speaking of dense open sublocales of $\mathfrak{c}(a)$. Then we have

$$b > a \quad \text{unless} \quad a = 1. \tag{4.2.1}$$

4.3. Lemma. *Let* $\mathcal{S} \subseteq \mathsf{S}(L)$ *be such that* $\bigvee\{S \cap \mathfrak{B}L \,|\, S \in \mathcal{S}\} = \mathfrak{B}L$. *Then there are pairwise disjoint* $\mathfrak{o}(b_S)$ *such that* $\bigvee\{S \cap \mathfrak{o}(b_S) \cap \mathfrak{B}L \,|\, S \in \mathcal{S}\} = \mathfrak{B}L$.

Proof. Consider the standard pairwise disjoint cover $\{B_S \,|\, S \in \mathcal{S}\}$ of (the Boolean) $\mathfrak{B}L$ such that $B_S \subseteq S$ for all S [6]. For each S then choose an $\mathfrak{o}(b_S)$ such that $\mathfrak{o}(b_S) \cap \mathfrak{B}L = B_S$. Then we have

$$\mathfrak{o}(b_S) \cap S \cap \mathfrak{B}L = \mathfrak{o}(b_S) \cap \mathfrak{B}L \cap S \cap \mathfrak{B}L = B_S \cap S \cap \mathfrak{B}L = B_S$$

so that $\bigvee\{S \cap \mathfrak{o}(b_S) \cap \mathfrak{B}L \,|\, S \in \mathcal{S}\} = \bigvee_S B_S = \mathfrak{B}L$. □

4.4. Proposition. *For every distributive cover* $\mathcal{C}$ *of* L *by complemented sublocales there exists an open dense* $\mathfrak{o}(a) \subseteq L$ *and a pairwise disjoint cover* $\{\mathfrak{o}(a_C) \,|\, C \in \mathcal{C}\}$ *of* $\mathfrak{o}(a)$ *such that* $\mathfrak{o}(a_C) \subseteq C$ *for all* $C \in \mathcal{C}$.

Proof. By 2.4.5, $C \cap \mathfrak{B}L = \operatorname{int} C \cap \mathfrak{B}L$ for each $C \in \mathcal{C}$, and by the distributivity

$$\bigvee_{C\in\mathcal{C}} (\operatorname{int} C \cap \mathfrak{B}L) = \bigvee_{C\in\mathcal{C}} (C \cap \mathfrak{B}L) = \mathfrak{B}L \cap \bigvee_{C\in\mathcal{C}} C = \mathfrak{B}L.$$

Then use Lemma 4.3 for the system $\mathcal{S} = \{\operatorname{int} C \,|\, C \in \mathcal{C}\}$ and set $\mathfrak{o}(a_C) = \operatorname{int} C \cap \mathfrak{o}(b_C)$ and $a = \bigvee_{C\in\mathcal{C}} a_C$. □

The next results are from [13]. We follow the reasoning from [13] in a way simplified by the sublocale calculus as above. In particular we do not need special definitions and the constructions can be made explicit and hence more transparent.

4.5. Theorem. *Each open cover* $\mathcal{U} = \{U_i \,|\, i \in J\}$ *of* $\mathsf{T}(L)$, *for an arbitrary* L, *has a disjoint refinement* $\{V_i \,|\, i \in J\}$ *such that* $V_i \subseteq U_i$.

Proof. Let $\mathcal{U} = \{U_i \,|\, i \in J\}$ be an open cover of $\mathsf{T}(L)$. Because of zero-dimensionality of $\mathsf{T}(L)$ (recall 2.3.2) we may assume the U_i clopen, and hence $\mathcal{U} = \{\mathfrak{c}_{\mathsf{T}(L)}(C_i) \,|\, i \in J\}$ with complemented $C_i \in \mathsf{S}(L)$. Since

$$\mathfrak{c}_{\mathsf{T}(L)}(\bigvee_i C_i) \geq \bigvee\{\mathfrak{c}_{\mathsf{T}(L)}(C_i) \,|\, i \in J\}$$

we have $\mathfrak{c}_{\mathsf{T}(L)}(\bigvee_i C_i) = \mathfrak{c}_{\mathsf{T}(L)}(L)$, that is, $\bigvee_i C_i = L$ (the bottom of $\mathsf{T}(L)$) and $\{C_i \,|\, i \in J\}$ is by 4.1.1 a distributive cover of L.

Set $a_0 = 0$ (in L) and $S_0 = \mathsf{O}$. If a_α is already defined then consider

$$\mathfrak{c}(a_\alpha) \quad \text{and its cover} \quad \{C_i \cap \mathfrak{c}(a_\alpha) \,|\, i \in J\}$$

[6] If B is Boolean, one can order its elements by an ordinal κ as $(a_\alpha)_{\alpha<\kappa}$ and put $b_0 = a_0$ and $b_\alpha = a_\alpha \wedge \bigvee_{\beta<\alpha} b_\beta$. It is easy to check that $(b_\alpha)_{\alpha<\kappa}$ has the required property.

and choose by 4.4 and 4.3 an $a_{\alpha+1} \geq a_\alpha$ with $S_{\alpha+1} = \mathfrak{o}(a_{\alpha+1}) \cup \mathfrak{c}(a_\alpha)$ dense in $\mathfrak{c}(a_\alpha)$ pairwise disjointly covered by $S_{\alpha+1,i}$, $i \in J$, with $S_{\alpha+1,i} \subseteq C_i$. For a limit ordinal set $S_\lambda = \mathsf{O}$.

We have for all α

$$\bigvee_{\beta \leq \alpha} S_\beta = \mathfrak{o}(a_\alpha).$$

Indeed, $S_0 = \mathsf{O} = \mathfrak{o}(a_0)$,

$$\bigvee_{\beta \leq \alpha+1} S_\beta = \bigvee_{\beta \leq \alpha} S_\beta \vee S_{\alpha+1} = \mathfrak{o}(a_\alpha) \cap (\mathfrak{o}(a_{\alpha+1}) \cap \mathfrak{c}(a_\alpha)) =$$
$$= \mathfrak{o}(a_{\alpha+1}) \cap (\mathfrak{o}(a_\alpha) \vee \mathfrak{c}(a_\alpha)) = \mathfrak{o}(a_{\alpha+1}),$$

and for a limit λ,

$$\bigvee_{\beta \leq \lambda} S_\beta = \bigvee_{\alpha < \lambda} \bigvee_{\beta \leq \alpha} S_\beta \vee S_\lambda = \bigvee_{\alpha < \lambda} \mathfrak{o}(a_\alpha) \vee \mathsf{O} = \mathfrak{o}(\bigvee_{\alpha < \lambda} a_\alpha) = \mathfrak{o}(a_\lambda).$$

Since by (4.2.1), a_α strictly grows until it reaches the value 1, we have for a sufficiently large α

$$\bigvee_{\beta \leq \alpha} S_\beta = L$$

and since obviously the S_β's are pairwise disjoint and $S_{\beta,i}$ complemented (intersections of open and closed), we obtain the desired clopen refinement of $\mathcal{U}$ as $\{\mathfrak{c}_{\mathsf{T}(L)}(S_{\beta,i}) \,|\, \beta \leq \alpha, i \in J\}$. □

4.5.1. The property above is called *ultraparacompactness* ([8]). Hence, we can rewrite 4.5 to

Theorem. $\mathsf{T}(L)$ *of an arbitrary* L *is ultraparacompact.*

4.6. Lemma. *Let there exist a pairwise disjoint cover* $C \subseteq L$ *refining* $\{a, b\}$. *Then there exists a complemented* u *such that* $a \vee u = 1 = b \vee u^{\mathsf{c}}$. *In other words, there exists a clopen* $\mathfrak{o}(u)$ *such that*

$$\mathfrak{c}(a) \subseteq \mathfrak{o}(u) \text{ and } \mathfrak{c}(b) \subseteq \mathfrak{o}(u)^{\mathsf{c}}.$$

Proof. Set $u = \bigvee\{c \in C \,|\, c \nleq a\}$ and $v = \bigvee\{c \in C \,|\, c \leq a\}$. Then $u \vee v = \bigvee C = 1$ and $u \wedge v = 0$ and hence $v = u^{\mathsf{c}}$. By refinement we have $c \nleq a \;\Rightarrow\; c \leq b$ and hence $u \leq b$. Thus, $a \vee u \geq v \vee u = 1$ and $b \vee v \geq u \vee v = 1$. □

4.7. A frame L is *ultranormal* (*strongly zero-dimensional* in [13]) if for any a, b in L such that $a \vee b = 1$ there exists a complemented $c \in L$ such that $c \leq a$ and $c^* \leq b$.

Applying Lemma 4.6 to the ultraparacompact $L_d \cong \mathsf{T}(L)$ we obtain (dissolving back closed sublocales to general ones and clopen sublocales to complemented ones):

Theorem. *Each $\mathsf{T}(L)$ is ultranormal. That is, for disjoint sublocales A, B of an arbitrary frame L there is a complemented sublocale S such that $A \subseteq S$ and $B \subseteq S^{\mathsf{c}}$.*

Acknowledgements. The authors acknowledge financial support from the Centre for Mathematics of the University of Coimbra (UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES) and from the Department of Applied Mathematics (KAM) of Charles University.

References

[1] M. Erné, J. Picado and A. Pultr, Adjoint maps between implicative semilattices and continuity of localic maps, *Algebra Univ.* 83 (2022) article n. 13.

[2] M.J. Ferreira, J. Picado and S. Pinto, *Remainders in pointfree topology*, Topology Appl. 245 (2018) 21–45.

[3] J. Isbell, *Atomless parts of spaces*, Math. Scand. 31 (1972) 5–32.

[4] J. Isbell, *On dissolute spaces*, Topology Appl. 40 (1991) 63–70.

[5] P.T. Johnstone, *Fibrewise separation axioms for locales*, Math. Proc. Cambridge Philos. Soc. 108 (1990) 247–256.

[6] P.T. Johnstone, *On solubility of sublocales*, Topology Proc. 29 (2005) 527–537.

[7] P.T. Johnstone, *Complemented sublocales and open maps*, Ann. Pure Appl. Logic 137 (2006) 240–255.

[8] J. Paseka, *Boolean algebras and ultraparacompactness*, Cahiers Topol. Géom. Differ. Catég. 33 (1992) 15–20.

[9] J. Picado and A. Pultr, *Frames and locales. Topology without points,* Frontiers in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2012.

[10] J. Picado and A. Pultr, *A Boolean extension of a frame and a representation of discontinuity*, Quaest. Math. 40 (2017) 1111–1125.

[11] J. Picado and A. Pultr, *Notes on Point-Free Topology*, in: New Perspectives in Algebra, Topology and Categories (ed. by M.M. Clementino, A. Facchini, M. Gran), Coimbra Mathematical Texts, vol. 1, Springer, pp. 173–223, 2022.

[12] T. Plewe, *Higher order dissolutions and Boolean coreflections of locales*, J. Pure Appl. Algebra 154 (2000) 273–293.

[13] T. Plewe, *Sublocale lattices*, J. Pure Appl. Algebra 168 (2002) 309–326.

CMUC, Department of Mathematics, University of Coimbra, 3000-143 Coimbra, Portugal

E-mail address: picado@mat.uc.pt

Department of Applied Mathematics and ITI, MFF, Charles University, Malostranské nám. 24, 11800 Praha 1, Czech Republic

E-mail address: pultr@kam.mff.cuni.cz