ON CONTINUITY AND OPENNESS OF LOCALE MAPPINGS

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ABSTRACT. This note reviews some adjoint situations, in the algebraic (pointfree) setting of frames and locales, that describe fundamental properties of mappings such as residuation, continuity and openness.

Given posets X, Y and maps $h: X \to Y$ and $f: Y \to X$ such that

$$h(x) \le y \iff x \le f(y),\tag{1}$$

the pair (h, f) is said to be a *Galois adjunction* (and one writes $h \dashv f$), with h the *left adjoint*, and f the *right adjoint*. Galois adjunctions are a basic tool in order theory that provide descriptions of many phenomena in order theory, algebra and topology ([2, 3, 4]). For instance, the continuity of a map f between topological spaces ([10]) or, more generally, closure spaces (X, \mathcal{C}_X) and (Y, \mathcal{C}_Y) , is characterized by the fact that the pair $(f_{\mathcal{C}}^{\rightarrow}, f_{\mathcal{C}}^{\leftarrow})$ is an adjunction ([1, 2, 3]), for

$$f_{\mathcal{C}}^{\rightarrow} \colon \mathcal{C}_X \to \mathcal{C}_Y, \ A \mapsto f^{\rightarrow}(A)^-$$

$$f_{\mathcal{C}}^{\leftarrow} \colon \mathcal{C}_Y \to \mathcal{C}_X, \ B \mapsto f^{\leftarrow}(B)^-,$$

$$(2)$$

where $\bar{}$ denotes closure and the symbols $f \rightarrow$ and $f \leftarrow$ are used for the image and preimage map, respectively.

Similar adjunctions, characterizing continuity and openness, that interchange images and preimages with closure and interior operators, occur in the algebraic setting of frames and locales (and, more generally, in the setting of implicative semilattices; see the forerunner [5] for a complete study). It is our aim with this note to review them.

Let L be a *locale* (=*frame*), that is, a complete lattice in which binary meets distribute over arbitrary joins (equivalently, a complete Heyting algebra with Heyting operator \rightarrow). For general background, notation and terminology concerning the category of locales and continuous (=localic) maps and its dual category of frames and frame homomorphisms, we refer to [11]. The subsets $S \subseteq L$ such that

- (S1) for every $M \subseteq S$, $\bigwedge M \in S$, and
- (S2) for every $s \in S$ and $x \in L, x \to s \in S$,

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form the system S(L) of all sublocales of L. This is a coframe (that is, the dual of frame) with a fairly transparent structure:

$$\bigwedge S_i = \bigcap S_i \quad \text{and} \quad \bigvee S_i = \{\bigwedge M \mid M \subseteq \bigcup S_i\}.$$
(3)

The least sublocale $\bigvee \emptyset = \{1\}$ is denoted by O (the *void sublocale*); the largest sublocale is, of course, L. For each $a \in L$ there are the open and closed sublocales

$$\mathbf{o}a = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\} \quad \text{and} \quad \mathbf{c}a = \uparrow a = \{x \in L \mid x \ge a\}.$$
(4)

They are complements of each other in S(L) and satisfy the properties

$$\mathfrak{o}0 = \mathsf{O}, \ \mathfrak{o}1 = L, \ \mathfrak{o}a \cap \mathfrak{o}b = \mathfrak{o}(a \wedge b) \text{ and } \bigvee \mathfrak{o}a_i = \mathfrak{o}(\bigvee a_i),$$
(5)
$$\mathfrak{c}1 = \mathsf{O}, \ \mathfrak{c}0 = L, \ \mathfrak{c}a \vee \mathfrak{c}b = \mathfrak{c}(a \wedge b) \text{ and } \bigcap \mathfrak{c}a_i = \mathfrak{c}(\bigvee a_i).$$

Denoting by $\mathfrak{o}L$ and $\mathfrak{c}L$ respectively the sets of all open and closed sublocales of L, the formulas above show that $\mathfrak{o}: L \to \mathfrak{o}L$ and $\mathfrak{c}: L \to \mathfrak{c}L$ are frame isomorphisms between L and $(\mathfrak{o}L, \subseteq)$ resp. $(\mathfrak{c}L, \supseteq)$.

In the following, a right-adjoint (=meet-preserving) map $f: L \to M$ between locales will be *continuous* if

(C1) $f(a) = 1 \implies a = 1$, and

(C2) $f(h(b) \to a) = b \to f(a)$ for all $a \in L, b \in M$,

(where h is the left adjoint of f). Continuous maps $L \to M$ are precisely the right-adjoints whose left adjoints are frame homomorphisms ([11]).

To begin with, for each plain map $f: L \to M$ between locales consider the maps $f_*: M \to L$ and $f_!: L \to M$ defined by

$$f_*(b) = \bigvee \{ a \in L \mid f^{\to}(\mathfrak{o}a) \subseteq \mathfrak{o}b \} \text{ and } f_!(a) = \bigvee \{ b \in M \mid \mathfrak{o}b \subseteq f^{\to}(\mathfrak{o}a) \}.$$
(6)

One may transfer these maps from the level of the ordered structures L and M into the $\mathfrak{o}L$ (and $\mathfrak{c}L$) and $\mathfrak{o}M$ (and $\mathfrak{c}M$) by defining *closure* and *interior* of a subset S of a locale as follows. For any $S \subseteq L$, let S^- denote the closed sublocale

$$\bigcap \{ \mathfrak{c}a \in \mathfrak{c}L \mid S \subseteq \mathfrak{c}a \} = \mathfrak{c}(\bigvee \{ a \in L \mid S \subseteq \mathfrak{c}a \}) = \mathfrak{c}(\bigwedge S).$$
(7)

This defines a map from the power set $\mathcal{P}(L)$ to $\mathfrak{c}L$ with the properties of a closure operator:

- extensivity:
$$S \subseteq S^-$$
.

- isotonicity: $S \subseteq T \Rightarrow S^- \subseteq T^-$. idempotency: $(S^-)^- = S^-$.

Similarly, we may consider the open sublocale S° defined as

$$\bigvee \{ \mathfrak{o}a \in \mathfrak{o}L \mid \mathfrak{o}a \subseteq S \} = \mathfrak{o}(\bigvee \{ a \in L \mid \mathfrak{o}a \subseteq S \}).$$
(8)

This defines now a map from $\mathcal{P}(L)$ to $\mathfrak{o}L$ with two of the properties of interior operators:

- isotonicity: $S \subseteq T \Rightarrow S^{\circ} \subseteq T^{\circ}$.
- *idempotency*: $(S^{\circ})^{\circ} = S^{\circ}$.

However, intensivity does not hold generally; indeed, S° is not necessarily contained in S because the join in (8) is taken in S(L). Nevertheless, for S in the complete lattice M(L) of all *meet-subsets* of L (i.e. subsets closed under meets) it is clear from the formula for joins in (3) that $S^{\circ} \subseteq S$ and, in particular, that $\mathfrak{o}a \subseteq S$ iff $\mathfrak{o}a \subseteq S^{\circ}$.

The following diagram emerges

where \neg denotes complementation in S(L) and S(M), and the upper and lower horizontal maps are defined by

$$f_{\mathfrak{o}}^{\Rightarrow}(\mathfrak{o}a) = f^{\rightarrow}(\mathfrak{o}a)^{\circ}, \qquad f_{\mathfrak{o}}^{\leftarrow}(\mathfrak{o}b) = f^{\leftarrow}(\mathfrak{o}b)^{\circ}, \qquad f_{\mathfrak{o}}^{\rightarrow}(\mathfrak{o}a) = \neg(f^{\rightarrow}(\mathfrak{c}a)^{-}), \qquad (9)$$
$$f_{\mathfrak{c}}^{\Rightarrow}(\mathfrak{c}a) = \neg(f^{\rightarrow}(\mathfrak{o}a)^{\circ}), \qquad f_{\mathfrak{c}}^{\leftarrow}(\mathfrak{c}b) = f^{\leftarrow}(\mathfrak{c}b)^{-}, \qquad f_{\mathfrak{c}}^{\rightarrow}(\mathfrak{c}a) = f^{\rightarrow}(\mathfrak{c}a)^{-}.$$

Note that

- squares (1), (2), (4) do commute always,
- (3) and (6) commute iff f is order-preserving, and
- (5) commutes iff

$$f^{\leftarrow}(\mathfrak{c}b)^{-} = \neg(f^{\leftarrow}(\mathfrak{o}b)^{\circ}). \tag{10}$$

(Indeed, $\mathfrak{c}(f_{*}(b)) = \neg \mathfrak{o}(\bigvee\{a \in L \mid \mathfrak{o}a \subseteq f^{\leftarrow}(\mathfrak{o}b)\}) = \neg(f^{\leftarrow}(\mathfrak{o}b)^{\circ}).)$

Of course, whenever the squares commute, any adjunction in the middle level of the ground locales yields a corresponding adjunction in the upper and lower levels (and vice-versa). It is also clear that the exterior rectangles (1)-(4) and (3)-(6) commute always but again (2)-(5) commutes iff condition (10) holds so only under this condition one may transfer, by complementation, any adjunction in the top/bottom levels to an adjunction in the opposite level.

Notes. (a) If f is continuous then f_* is precisely the frame homomorphism h left adjoint to f. To see that directly, recall from [11] that for sublocales $S \subseteq L$ and $T \subseteq M$, the standard set theoretic image $f^{\rightarrow}(S)$ is a sublocale of M but the standard preimage $f^{\leftarrow}(T)$ is generally not a sublocale. However, $f^{\leftarrow}(T)$ is closed under meets and hence, by (3), there is the largest sublocale contained in it, namely the join $\bigvee \{S \in \mathsf{S}(L) \mid S \subseteq f^{\leftarrow}(T)\}$, that we denote here by $f_{\leftarrow}(T)$

and call it the continuous preimage of T (as opposed to the set theoretic preimage $f^{\leftarrow}(T)$). There is the obvious adjunction $f^{\rightarrow}(S) \subseteq T$ iff $S \subseteq f_{\leftarrow}(T)$. In particular, $f^{\rightarrow}(-)$ preserves joins and $f_{\leftarrow}(-)$ preserves meets of sublocales. Furthermore, $f_{\leftarrow}(-)$ is a coframe homomorphism that preserves complements and $f^{\rightarrow}(-)$ is a cocontinuous map (see [11] for details). Moreover, $f_{\leftarrow}(0) = 0$ (because already $f^{\leftarrow}(0) = 0$, by (C1)). Continuous preimages of closed (resp. open) sublocales are closed (resp. open), more precisely,

$$f_{\leftarrow}(\mathfrak{c}b) = \mathfrak{c}(h(b)) \quad \text{and} \quad f_{\leftarrow}(\mathfrak{o}b) = \mathfrak{o}(h(b)).$$
 (11)

Hence, in the definition of f_* , we have $f^{\rightarrow}(\mathfrak{o}a) \subseteq \mathfrak{o}b$ iff $\mathfrak{o}a \subseteq f_{\leftarrow}(\mathfrak{o}b) = \mathfrak{o}(h(b))$ iff $a \leq h(b)$, thus $f_* = h$.

(b) Since f_* preserves finite meets, the continuity of f is completely described by the adjunction $f_* \dashv f$ (since f^* will be then a frame homomorphism) and thus, by the commutativity of (2) and (3),

an order-preserving
$$f$$
 is continuous iff $(f_{\mathfrak{o}}^{\leftarrow}, f_{\mathfrak{o}}^{\rightarrow})$ is an adjoint pair. (12)

The following facts about the mappings in the diagram above are also well known. Firstly, by general facts for adjoint maps between complete lattices in [2, 3], one has:

P1. A plain f is meet-preserving (with left adjoint h) iff it is residual (i.e. $f^{\leftarrow}(\mathfrak{c}b) = \mathfrak{c}(h(b))$ for every $b \in M$) iff $(f_{\mathfrak{c}}^{\rightarrow}, f_{\mathfrak{c}}^{\leftarrow})$ is an adjoint pair.

Moreover, as an instance of [5, Thm. 4.6], one has the following characterization of continuous maps:

P2. A plain f is continuous iff $f^{\leftarrow}(\mathfrak{c}b)$ is closed for every $b \in M$, $f^{\leftarrow}_{\mathfrak{c}}(\mathsf{O}) = \mathsf{O}$, and $\neg f^{\leftarrow}(\mathfrak{c}b) \subseteq f^{\leftarrow}(\mathfrak{o}b)$ for every $b \in M$.

Regarding openness, open continuous maps are naturally modelled in pointfree topology as continuous maps f such that the image $f^{\rightarrow}(\mathfrak{o}a)$ of every open sublocale is open. They are characterized by the celebrated Joyal-Tierney Theorem [9] (it follows also from the more general Prop. 5.1 and Thm. 5.3 in [5]):

P3. A continuous f is open iff the adjoint frame homomorphism $h: M \to L$ is a complete Heyting homomorphism (i.e. if it preserves also arbitrary meets and the Heyting operation) iff h admits a left adjoint g that satisfies the identity $f(a \to h(b)) = g(a) \to b$ for all $a \in L$ and $b \in M$.

There is also the following result contained in Thm. 6.3 and Prop. 6.5 of [5]:

- **P4.** Consider the following conditions about a continuous $f: L \to M$:
 - (a) f is open. (b) $f_{\leftarrow}(T)^{\circ} = f_{\leftarrow}(T^{\circ})$ for every $T \in \mathsf{S}(M)$. (c) $f_{\leftarrow}(T)^{-} = f_{\leftarrow}(T^{-})$ for every $T \in \mathsf{S}(M)$. Then $(a) \Leftrightarrow (b) \Rightarrow (c)$.

We begin with a result that rephrases P2 in terms of a single condition. For that we need a basic lemma:

Lemma 1. Let $f: L \to M$ be a meet-preserving map between locales. Then: (a) $f^{\leftarrow}(\mathsf{O}) = \mathsf{O}$ iff $f^{\leftarrow}(\mathfrak{c}b) \subseteq \neg(f^{\leftarrow}(\mathfrak{o}b)^{\circ})$ for every $b \in M$. (b) $f^{\leftarrow}(T)^{\circ} = f_{\leftarrow}(T)^{\circ}$ for every $T \in \mathsf{M}(M)$.

Proof. (a) Let $f^{\leftarrow}(\mathsf{O}) = \mathsf{O}$. Since $f^{\leftarrow}(\mathfrak{o}b)$ is a meet-subset we have $f^{\leftarrow}(\mathfrak{c}b) \cap f^{\leftarrow}(\mathfrak{o}b) \cap f^{\leftarrow}(\mathfrak{o}b) = f^{\leftarrow}(\mathfrak{c}b \cap \mathfrak{o}b) = f^{\leftarrow}(\mathsf{O}) = \mathsf{O}$. We may then conclude that $f^{\leftarrow}(\mathfrak{c}b) \subseteq \neg(f^{\leftarrow}(\mathfrak{o}b)^{\circ})$. For the converse, take just the case b = 1: $f^{\leftarrow}(\mathsf{O}) = f^{\leftarrow}(\mathfrak{o}) = f^{\leftarrow}(\mathfrak{o}) = \mathfrak{o}$.

(b) It suffices to check the inclusion " \subseteq ". Since f is meet-preserving, $f^{\leftarrow}(T)$ is a meet-subset and $f^{\leftarrow}(T)^{\circ} \subseteq f^{\leftarrow}(T)$. On the other hand, since $f^{\leftarrow}(T)^{\circ}$ is a sublocale, the former condition implies that $f^{\leftarrow}(T)^{\circ} \subseteq f_{\leftarrow}(T)$ and thus $f^{\leftarrow}(T)^{\circ} \subseteq f_{\leftarrow}(T)^{\circ}$.

Proposition 2. A plain $f: L \to M$ is continuous iff

 $\neg f_{\mathfrak{o}}^{\leftarrow}(\mathfrak{o}b) = f^{\leftarrow}(\mathfrak{c}b) \quad for \ every \ b \in M.$

Proof. By Lemma 1(b), the condition holds for any continuous f. The reverse implication follows using P2: each $f^{\leftarrow}(\mathfrak{c}b)$ is closed hence f is meet-preserving; this guarantees that $f^{\leftarrow}[\mathsf{O}] = \mathsf{O}$ (by Lemma 1(a)) and that $\neg f^{\leftarrow}(\mathfrak{c}b) \subseteq f^{\leftarrow}(\mathfrak{o}b)$ (note that the intensivity property $f^{\leftarrow}(\mathfrak{o}b)^{\circ} \subseteq f^{\leftarrow}(\mathfrak{o}b)$ holds here since f is meet-preserving).

Remark 3. The condition in Proposition 2 is very close to the condition (10) that characterizes the commutativity of square (5) in the diagram. We see now, from Proposition 2 and P1, that an f is continuous iff it preserves meets and satisfies (10).

The equivalence (12) in Note (b) follows also immediately from Proposition 2:

Corollary 4. An order-preserving $f: L \to M$ is continuous iff $(f_{\mathfrak{o}}^{\leftarrow}, f_{\mathfrak{o}}^{\rightarrow})$ is an adjoint pair. Hence a plain f is continuous iff it is order-preserving and $(f_{\mathfrak{o}}^{\leftarrow}, f_{\mathfrak{o}}^{\rightarrow})$ is an adjoint pair.

Proof. For each $a \in L$ and $b \in M$, $a \in \neg(f^{\leftarrow}(\mathfrak{o}b)^{\circ}) \Leftrightarrow \mathfrak{c}a \subseteq \neg(f^{\leftarrow}(\mathfrak{o}b)^{\circ}) \Leftrightarrow f_{\mathfrak{o}}^{\leftarrow}(\mathfrak{o}b) \subseteq \mathfrak{o}a$ while $a \in f^{\leftarrow}(\mathfrak{c}b) \Leftrightarrow f^{\rightarrow}(\mathfrak{c}a) \subseteq \mathfrak{c}b \Leftrightarrow f^{\rightarrow}(\mathfrak{c}a)^{-} \subseteq \mathfrak{c}b \Leftrightarrow \mathfrak{o}b \subseteq f_{\mathfrak{o}}^{\rightarrow}(\mathfrak{o}a).$

Regarding *openness* in a broad sense, as (plain) maps between locales such that the image of every open sublocale is still open, the proof of the following wellknown fact is an easy exercise:

P5. A meet-preserving $f: L \to M$ is open iff $(f_{\mathfrak{o}}^{\Rightarrow}, f_{\mathfrak{o}}^{\leftarrow})$ is an adjoint pair.

It follows, in particular, that an order-preserving $f: L \to M$ is an open continuous map iff

$$f_{\mathfrak{o}}^{\Rightarrow} \dashv f_{\mathfrak{o}}^{\leftarrow} \dashv f_{\mathfrak{o}}^{\Rightarrow},\tag{13}$$

which by the commutativity of the top squares of the diagram is equivalent to

$$f_! \dashv f_* \dashv f \tag{14}$$

on the level of the involved ground locales.

Proposition 5. (a) If f is open then $f^{\leftarrow}(T)^{\circ} \subseteq f^{\leftarrow}(T^{\circ})^{\circ}$ for every $T \in \mathsf{M}(M)$. (b) If f is meet-preserving then it is open iff

 $f_{\leftarrow}(T)^{\circ} \subseteq f_{\leftarrow}(T^{\circ}) \quad \text{for every } T \in \mathsf{M}(M).$

Proof. (a) It suffices to show that $\mathfrak{o}a \subseteq f^{\leftarrow}(T)$ implies $\mathfrak{o}a \subseteq f^{\leftarrow}(T^{\circ})$. So let $\mathfrak{o}a \subseteq f^{\leftarrow}(T)$. Then $f^{\leftarrow}(\mathfrak{o}a) \subseteq T$ hence $f^{\rightarrow}(\mathfrak{o}a) \subseteq T^{\circ}$ (since $f^{\rightarrow}(\mathfrak{o}a)$ is open) and thus $\mathfrak{o}a \subseteq f^{\leftarrow}(T^{\circ})$.

(b) Let $T = f^{\rightarrow}(\mathfrak{o}a)$. Since T is a meet-subset, $T^{\circ} \subseteq T$. On the other hand, $f_{\leftarrow}(T^{\circ}) \supseteq f_{\leftarrow}(T)^{\circ} = f_{\leftarrow}f^{\rightarrow}(\mathfrak{o}a)^{\circ} \supseteq \mathfrak{o}a$ and thus $T = f^{\rightarrow}(\mathfrak{o}a) \subseteq T^{\circ}$. \Box

There is a natural alternative to f_* in the main diagram, namely the mapping $\tilde{f}_* \colon M \to L$ defined by

$$\tilde{f}_*(b) = \bigvee \{ a \in L \mid f^{\leftarrow}(\mathfrak{c}b) \subseteq \mathfrak{c}a \}.$$
(15)

With f_* in the place of f_* then square (5) is commutative but, on the other hand, it makes (2) to commute iff f satisfies (10). Also, $f_* = \tilde{f}_*$ iff (10) holds. Using \tilde{f}_* , one gets:

Proposition 6. A meet-preserving $f: L \to M$ is open continuous iff $(f_{\mathfrak{c}}^{\leftarrow}, f_{\mathfrak{c}}^{\Rightarrow})$ is an adjoint pair.

Proof. By the diagram, $f_{\mathfrak{c}}^{\leftarrow} \dashv f_{\mathfrak{c}}^{\Rightarrow}$ iff $f_! \dashv \tilde{f}_*$, and since f is meet-preserving, $\tilde{f}_* = h$ (the left adjoint of f). In this case h is a complete frame homomorphism, hence f is continuous and $h = f_*$; by P5, f is open. Conversely, if f is open continuous then, again, $f_! \dashv f_*$ and $f_* = h$.

Combining this with P1 one gets the parallel result to P5 that a plain $f\colon L\to M$ is an open continuous map iff

$$f_{\mathfrak{c}}^{\rightarrow} \dashv f_{\mathfrak{c}}^{\leftarrow} \dashv f_{\mathfrak{c}}^{\Rightarrow}.$$
 (16)

It is also possible to describe open continuity by a single condition on preimages:

Proposition 7. A plain map $f: L \to M$ is an open continuous map iff

$$\neg(f^{\leftarrow}(T)^{\circ}) = f^{\leftarrow}(\neg(T^{\circ})) \quad \text{for all } T \in \mathsf{S}(M).$$

Proof. Let $f: L \to M$ be an open continuous map. By Lemma 1(b) and P4, we conclude that $f_{\leftarrow}(T^{\circ}) = f_{\leftarrow}(T)^{\circ} = f^{\leftarrow}(T)^{\circ}$. Since f is continuous and $\neg(T^{\circ})$ is a closed sublocale it follows that $f^{\leftarrow}(\neg(T^{\circ})) = f_{\leftarrow}(\neg(T^{\circ})) = \neg f_{\leftarrow}(T^{\circ}) = \neg(f^{\leftarrow}(T)^{\circ})$.

Conversely, applying the hypothesis to $T = \mathfrak{o}b$ we get $f^{\leftarrow}(\mathfrak{c}b) = \neg(f^{\leftarrow}(\mathfrak{o}b)^\circ)$. Then f is continuous, by Prop. 2. Moreover, for any sublocale T of M, we have, by the previous lemma, $f_{\leftarrow}(T)^\circ = \neg(\neg(f^{\leftarrow}(T)^\circ)) = \neg f^{\leftarrow}(\neg(T^\circ)) = \neg(f_{\leftarrow}(\neg(T^\circ))) = f_{\leftarrow}(T^\circ)$. Hence f is also an open map (again by P4). **Proposition 8.** Let $f: L \to M$ be an open continuous map and $T \in S(M)$. The least sublocale of L that contains $(f_*)^{\to}(T)$ is contained in $f^{\leftarrow}(T)$ hence in $f_{\leftarrow}(T)$. In particular, $(f_*)^{\to}(T) \subseteq f_{\leftarrow}(T)$.

Proof. Let S be the meet-closure of the set $\{a \to f_*(t)) \mid a \in L, t \in T\}$. This is a sublocale of L. Indeed, for each $b \in L$ and $\bigwedge_{i \in I} (a_i \to f_*(t_i)) \in S$, using basic properties of the Heyting operator we get

$$b \to \left(\bigwedge_{i \in I} (a_i \to f_*(t_i))\right) = \bigwedge_{i \in I} (b \to (a_i \to f_*(t_i))) = \bigwedge_{i \in I} ((b \land a_i) \to f_*(t_i)) \in S.$$

Hence it is clearly the least sublocale of L that contains $(f_*)^{\rightarrow}(T)$. Moreover, it is contained in $f^{\leftarrow}(T)$. In fact, by P3 we know that the left adjoint $f_!$ of f_* satisfies the identity $f(a \rightarrow f_*(b)) = f_!(a) \rightarrow b$ for every $a \in L$ and $b \in M$, hence

$$f(\bigwedge_{i\in I}(a_i\to f_*(t_i)))=\bigwedge_{i\in I}f(a_i\to f_*(t_i))=\bigwedge_{i\in I}(f_!(a_i)\to t_i)\in T.$$

Continuous maps satisfying the containment $f_{\leftarrow}(T^-) \subseteq f_{\leftarrow}(T)^-$ for all $T \in S(M)$ (hence, the equality, since the converse containment is always true) are the *hereditary skeletal maps* (Johnstone [8]). Johnstone characterized them as the $f: L \to M$ such that $f_*(b \to c) \to f_*(c) = (f_*(b) \to f_*(c)) \to f_*(c)$ for every $b, c \in M$ ([8, Lemma 4.1]), and proved that an hereditarily skeletal map $f: L \to M$ is open whenever $f^{\to}(L)$ is a complemented sublocale of M.

Prop. 8 extends from open maps to hereditary skeletal maps, as follows:

Proposition 9. The following are equivalent for a continuous $f: L \to M$:

- (i) f is hereditarily skeletal.
- (ii) $\bigwedge f_{\leftarrow}(T) \leq f_*(\bigwedge T)$ for every $T \in \mathsf{S}(M)$.
- (iii) $f_*(\bigwedge T) \in f_{\leftarrow}(T)$ for every $T \in \mathsf{S}(M)$.

Proof. (i) \Leftrightarrow (ii): Since $f_{\leftarrow}(T^-) = f_{\leftarrow}(\bigcap \{ \mathfrak{c}b \mid \mathfrak{c}b \supseteq T \}) = \bigcap \{ \mathfrak{c}(f_*(b)) \mid \mathfrak{c}b \supseteq T \}$ and $f_{\leftarrow}(T)^- = \bigcap \{ \mathfrak{c}a \mid \mathfrak{c}a \supseteq f_{\leftarrow}(T) \}$, the hypothesis amounts to the implication $f_{\leftarrow}(T) \subseteq \mathfrak{c}a \Rightarrow \bigcap \{ \mathfrak{c}(f_*(b)) \mid \mathfrak{c}b \supseteq T \} \subseteq \mathfrak{c}a$ that is, $a \leq \bigwedge f_{\leftarrow}(T) \Rightarrow a \leq \bigvee \{ f_*(b) \mid b \leq \bigwedge T \}$; in other words, $\bigwedge f_{\leftarrow}(T) \leq f_*(\bigwedge T)$.

(ii) \Rightarrow (iii): The inequality $f_*(\bigwedge T) \leq \bigwedge f_{\leftarrow}(T)$ is always true (indeed, for $x = \bigwedge f_{\leftarrow}(T) \in f_{\leftarrow}(T), f(x) \in T$ so $f_*(\bigwedge T) \leq f_*f(x) \leq x$). Hence $f_*(\bigwedge T) = x \in f_{\leftarrow}(T)$.

 $(iii) \Rightarrow (ii)$ is obvious.

Remark 10. Let Loc denote the category of locales and continuous maps. Consider the functor $T: \text{Loc} \rightarrow \text{Loc}$ defined by

$$\mathsf{T}(L) = \mathsf{S}(L)^{\mathrm{op}}$$
 and $\mathsf{T}(f) = f^{\rightarrow}(-),$

and the dissolving maps ([6]) $\gamma_L \colon \mathsf{T}(L) \to L$ defined by $\gamma_L(S) = \bigwedge S$ (see [12] for more information on the dissolution of a locale). The dissolving maps are continuous (indeed, they are the right adjoints of the natural frame embeddings

 $\mathbf{c}_L = (a \mapsto \mathbf{c}a) \colon L \to \mathsf{T}(L)$ and establish a natural transformation $\gamma \colon \mathsf{T} \to \mathrm{Id}$ ([6, 1.4]). Therefore, for each $f \colon L \to M$ in **Loc**, the following diagram commutes



as well as the square of their left adjoints (dotted arrows in the diagram below)

In the latter diagram, one has always the inequality $\gamma_L \circ f_{\leftarrow}(-) \geq f_* \circ \gamma_M$ (since $\gamma_L \circ \mathfrak{c}_L \geq 1$ and $\mathfrak{c}_M \circ \gamma_M \leq 1$). Prop. 9 shows that f is hereditarily skeletal precisely when one has the other inequality, hence the equality

$$\gamma_L \circ f_{\leftarrow}(-) = f_* \circ \gamma_M. \tag{17}$$

When f is open, we have moreover the adjunction $f_! \dashv f_*$ and then (17) implies $f_! \circ \gamma_L \circ f_{\leftarrow}(-) \leq \gamma_M$, that is, $f_!(\bigwedge f_{-1}(T)) \leq \bigwedge T$ for every $T \in \mathsf{S}(M)$.

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