GENERALIZED MULTICATEGORIES: CHANGE-OF-BASE, EMBEDDING, AND DESCENT

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Abstract. Via the adjunction $- \bullet 1 \dashv \mathcal{V}(1,-)\colon \mathsf{Span}(\mathcal{V}) \to \mathcal{V}\text{-}\mathsf{Mat}$ and a cartesian monad T on an extensive category $\mathcal{V}$ with finite limits, we construct an adjunction $- \bullet 1 \dashv \mathcal{V}(1,-)\colon \mathsf{Cat}(T,\mathcal{V}) \to (\overline{T},\mathcal{V})\text{-}\mathsf{Cat}$ between categories of generalized enriched multicategories and generalized internal multicategories, provided the monad T satisfies a suitable condition, which is satisfied by several examples.

We verify, moreover, the left adjoint is fully faithful, and preserves pullbacks, provided that the copower functor $- \bullet 1\colon \mathsf{Set} \to \mathcal{V}$ is fully faithful. We also apply this result to study descent theory of generalized enriched multicategorical structures.

These results are built upon the study of base-change for generalized multicategories, which, in turn, was carried out in the context of categories of horizontal lax algebras arising out of a monad in a suitable 2-category of pseudodouble categories.

Contents

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INTRODUCTION

The systematic study of the dichotomy between enriched categories and internal categories can be traced as far back as [44, Section 2.2]. It was shown in [35, Theorem 9.10] that for a suitable base category $\mathcal{V}$, the category $\mathcal{V}$-Cat of enriched $\mathcal{V}$-categories can be fully embedded into the category $\mathsf{Cat}(\mathcal{V})$ of categories internal to $\mathcal{V}$, enabling us to view enriched $\mathcal{V}$-categories as *discrete* categories internal to $\mathcal{V}$[1]. This observation is, for example, employed in the study of descent theory of enriched categories (see [35, Theorem 9.11] and [37]). The aim of this work is to construct such an embedding in the setting of generalized multicategories, which we recall below.

Multicategories, defined in [31, p. 103], are structures that generalize categories, by allowing the domains of morphisms to consist of a finite list of objects. The most quintessential example is the multicategory Vect, whose objects are vector spaces, and whose morphisms are multilinear maps. Their "multicomposition" and the description of the analogous notions of associativity and identity can succinctly be described via the free monoid monad on Set. More precisely, multicategories can be formalized by considering the equipment $\mathsf{Span}(\mathsf{Set})$ of spans in Set (see [4, p. 22]), and extending the free monoid monad to a suitable monad $(-)^*$ on $\mathsf{Span}(\mathsf{Set})$ (see [22, Corollary A.4]).

Generalized multicategories have since been developed in various contexts, abstracting the notion of ordinary multicategories by replacing the monad $(-)^*$ on $\mathsf{Span}(\mathsf{Set})$ by a *suitable notion of monad on a pseudodouble category.*

Enriched T-categories were first introduced in [12] with the terminology *$(T, \mathcal{V})$-categories.* In this setting, the category of $(T, \mathcal{V})$-categories is obtained out of the so-called *lax extension* of a monad on Set to a suitable monad on $\mathcal{V}$-Mat, the ubiquitous equipment of $\mathcal{V}$-matrices (see [12, Section 2]). For instance, when $\mathcal{V}$ is a suitable quantale, the ultrafilter monad $\mathfrak{U}$ on Set admits a lax extension $\overline{\mathfrak{U}}$ to $\mathcal{V}$-Mat [12, Section 8]. In particular, when $\mathcal{V} = 2$, we have an equivalence $\mathsf{Top} \simeq (\overline{\mathfrak{U}}, 2)$-$\mathsf{Cat}$ (first observed in [3]) and $(\overline{\mathfrak{U}}, [0, \infty])$-$\mathsf{Cat}$ is equivalent to the category of approach spaces.

Internal T-categories were introduced in [8] and [22]. For a category $\mathcal{B}$ with pullbacks, the former defines T-categories for any monad T on $\mathcal{B}$, while the latter considers T to be a *cartesian* monad on $\mathcal{B}$. A cartesian monad T on $\mathcal{B}$ induces a strong monad on the equipment $\mathsf{Span}(\mathcal{B})$ of spans in $\mathcal{B}$. In this setting, we can obtain the category $\mathsf{Cat}(T, \mathcal{B})$ of T-categories internal to $\mathcal{B}$. As examples, we recover the category of ordinary multicategories by considering $\mathsf{Cat}((-)^*.\mathsf{Set})$, and letting $\mathfrak{F}$ be the free category monad on Grph, we obtain the category $\mathsf{VDbCat} = \mathsf{Cat}(\mathfrak{F}, \mathsf{Grph})$ of virtual double categories.

The main goal of this paper is to construct an embedding $(\overline{T}, \mathcal{V})$-$\mathsf{Cat} \to \mathsf{Cat}(T, \mathcal{V})$ from a category $\mathcal{V}$, and a monad T on $\mathcal{V}$, satisfying suitable properties. To this end, it is desirable to work in a general setting where these various notions of generalized multicategories can be uniformly studied and compared with one another. This was, in part, accomplished by the work of [15], where the notion of *T-monoids* was introduced, unifying the several approaches to the theory of generalized multicategories. To be precise, these T-monoids are the *horizontal lax algebras* induced by a monad $T = (\mathbb{E}, T, e, m)$ in the 2-category VDbCat of virtual double categories, lax functors and vertical transformations. These objects have a natural structure of a virtual double category, which we denote here by $\mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$.

This general setting ought to provide us an "internalization" functor $(\overline{T}, \mathcal{V})$-$\mathsf{Cat} \to \mathsf{Cat}(T, \mathcal{V})$ obtained from the comparison $\mathcal{V}$-$\mathsf{Mat} \to \mathsf{Span}(\mathcal{V})$, and the induced monad T on $\mathsf{Span}(\mathcal{V})$. However, [15] does not provide a notion of *change-of-base* induced by an appropriate notion of morphism $S \to T$ of monads, where $S = (\mathbb{D}, S, e, m)$ is another monad in VDbCat. We remark this was left as future work in [15, 4.4].

It should be noted that the study of change-of-base functors has been studied in each specific setting of generalized multicategories. In [33, Section 6.7], the author provides such constructions for the internal case, and [12, Sections 5, 6] treats two particular families of monad morphisms for the enriched case. To establish a relationship between the enriched and internal structures, we expand on the work of these authors, with the goal of providing a convenient environment to produce and study such a functor $(\overline{T}, \mathcal{V})$-$\mathsf{Cat} \to \mathsf{Cat}(T, \mathcal{V})$ from simpler tools.

We must mention that our approach diverges from the techniques and tools developed in [15]. Firstly, we must restrict our scope from virtual double categories to pseudodouble categories, as we need to work with *(op)lax horizontal transformations*, which require horizontal composition to be defined. Secondly, instead of using (op)cartesian 2-cells and their universal properties, we opted to use a "mate theory" of conjoints and companions to prove our results, mostly to obtain explicit formulas. Lastly, this is far

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[1] This embedding result was studied in more detail in [13].

from the full scope of the project mentioned in [15, 4.4], as we merely study the underlying categories and functors of the 2-dimensional structures formed by these horizontal lax algebras. Instead, we fall back on an *ad-hoc* approach for the natural transformations between the functors induced by monad (op)lax morphisms, leaving a treatment of the complete story for future work.

Outline of the paper: We begin by reviewing the notion of *pseudodouble category* in Section 1, first introduced in [20], and the two dimensional structures formed by these, in Section 1. For pseudodouble categories $\mathbb{D}, \mathbb{E}$, the structures consisting of

- *lax functors* $\mathbb{D} \to \mathbb{E}$ as 0-cells,
- *vertical transformations* as vertical[2] 1-cells,
- *(op)lax horizontal transformations* as horizontal 1-cells,
- *generalized modifications* as 2-cells.

are, by Proposition 1.8, pseudodouble categories $\mathsf{Lax}_{\mathrm{lax}}(\mathbb{D}, \mathbb{E})$ ($\mathsf{Lax}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E})$). We also have a third double category $\mathsf{PsDbCat}$ (Proposition 1.4) consisting of

- *pseudodouble categories* as 0-cells,
- *(op)lax* functors as (vertical) horizontal 1-cells,
- *generalized vertical transformations* as 2-cells.

The pseudodouble categories that concern our study are the following:

- the pseudodobule category $\mathcal{V}$-Mat of $\mathcal{V}$-matrices for suitable monoidal categories $\mathcal{V}$,
- the pseudodouble category $\mathsf{Span}(\mathcal{B})$ of spans in $\mathcal{B}$, for categories $\mathcal{B}$ with pullbacks,
- the double category of lax T-algebras, for T a pseudomonad on a 2-category $\mathbb{B}$.

We will furthermore review the double categorical structure of the last item.

Let $\mathcal{V}$ be a distributive category with finite limits. Section 2 is devoted to studying the pseudodouble categories $\mathcal{V}$-Mat and $\mathsf{Span}(\mathcal{V})$, and the (op)lax functors induced by the adjunction $- \cdot 1 \dashv \mathcal{V}(1,-)\colon \mathcal{V} \to \mathsf{Set}$. We confirm these functors induce Cat-graph morphisms

$$- \cdot 1\colon \mathcal{V}\text{-}\mathsf{Mat} \to \mathsf{Span}(\mathcal{V}) \quad \text{and} \quad \mathcal{V}(1,-)\colon \mathsf{Span}(\mathcal{V}) \to \mathcal{V}\text{-}\mathsf{Mat},$$

which give us an adjunction $- \cdot 1 \dashv \mathcal{V}(1,-)$ in the 2-category $\mathsf{Grph}(\mathsf{Cat})$ (Lemma 2.1). We also prove that $- \cdot 1\colon \mathcal{V}$-$\mathsf{Mat} \to \mathsf{Span}(\mathcal{V})$ defines an oplax functor of pseudodouble categories (Proposition 2.2). Using techniques from the following couple of sections, we obtain the following

$$\begin{tikzcd} \mathcal{V}\text{-}\mathsf{Mat} \arrow[r, bend left, "-\cdot 1"] \arrow[r, phantom, "\perp"] & \mathsf{Span}(\mathcal{V}) \arrow[l, bend left, "\mathcal{V}(1,-)"] \end{tikzcd}$$

which is a generalized notion of adjunction – a *conjunction* – in the double category $\mathsf{PsDbCat}$.

Section 3 aims to recall the notions of "adjoint" in pseudodouble categories: *conjoints* and *companions*. These were first introduced in [19], under different terminology. We provide an explicit description of "mate theory" for these objects (also studied in [19, 41]), analogous to the mate theory for adjunctions. We also take the opportunity to work out some known results for three reasons: first, to fix technical notation for subsequent sections; second, to serve as examples on their use; and finally, to keep this work self-contained. This Section culminates in our first contribution, crucial to construct functor between categories of lax horizontal algebras, Theorem 3.6. It states that, if $\mathbb{E}$ is *conjoint closed*, then so is $\mathsf{Lax}_{\mathrm{lax}}(\mathbb{D}, \mathbb{E})$.

In Section 4, we explicitly establish an equivalence (Proposition 4.1) between the double category $\mathsf{PsDbCat}$ and the double category of pseudo-algebras for the free internal Cat-category 2-monad on the 2-category Grph, with the goal of making the tools of two-dimensional algebra [7, 28, 34] available to the theory of pseudodouble categories. In particular, via doctrinal adjunction [36, Theorem 1.4.11], we conclude that $\mathcal{V}(1,-)\colon \mathsf{Span}(\mathcal{V}) \to \mathcal{V}$-$\mathsf{Mat}$ is a lax functor, and is the conjoint of $- \cdot 1\colon \mathcal{V}$-$\mathsf{Mat} \to \mathsf{Span}(\mathcal{V})$ in $\mathsf{PsDbCat}$.

After recalling the notion of horizontal lax algebra from [15], in Section 5 we prove Theorem 5.2; it states that any monad lax morphism $(G, \psi)\colon T \to S$ in $\mathsf{PsDbCat}_{\mathrm{lax}}$ induces a *change-of-base* functor $G_!\colon \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg} \to \mathbb{H}\,\mathsf{Lax}\text{-}S\text{-}\mathsf{Alg}$, and any monad oplax morphism $(F, \phi)\colon S \to T$ satisfying a suitable

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[2] In accordance with [15], we take the vertical arrows to be non-strict ones instead.

conditon also induces a change-of-base functor $F_!\colon \mathbb{H}\,\mathsf{Lax}\text{-}S\text{-}\mathsf{Alg} \to \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$. We close this section by comparing our constructions with the change-of-base functors for generalized multicategories considered in [33] and [12].

In Section 6, we consider a conjunction

$$\begin{tikzcd} S \arrow[r, bend left, "{(F,\phi)}"] \arrow[r, phantom, "\perp"] & T \arrow[l, bend left, "{(G,\psi)}"] \end{tikzcd}$$

in the double category $\mathsf{Mnd}(\mathsf{PsDbCat}_{\mathrm{lax}})$, and we proceed to the existence of an adjunction

$$\begin{tikzcd} \mathbb{H}\,\mathsf{Lax}\text{-}S\text{-}\mathsf{Alg} \arrow[r, bend left, "F_!"] \arrow[r, phantom, "\perp"] & \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg} \arrow[l, bend left, "G_!"] \end{tikzcd}$$

between the induced change-of-base functors; this is Theorem 6.1. We also study the conditions for invertibility of unit and counit of such an adjunction, stated in Lemma 6.2 and Corollary 6.4. Finally, after instanciating these results to the settings considered both in [33] and [12, 24], we take the opportunity to point out some of obstacles to the double pseudofunctoriality of $\mathbb{H}\,\mathsf{Lax}\text{-}(-)\text{-}\mathsf{Alg}$.

We devote Section 7 to the study of extensive categories. When $\mathcal{C}$ is an lextensive category, we provide a description of $\mathsf{Fam}(\mathcal{C})$ via Artin glueing (Lemma 7.1), from which we deduce that the coproduct functor $\sum\colon \mathsf{Fam}(\mathcal{C}) \to \mathcal{C}$ preserves finite limits. Studying limits of fibered categories, we obtain Theorem 7.2: it confirms that, in a lextensive category, the coproduct of a "pullback-indexed" family of pullback diagrams is itself a pullback diagram. This result is extensively employed, as illustrated in the remaining results of this Section as well as subsequent ones.

The final groundwork is laid down in Section 8. Via a "structure transfer"-type of result (Proposition 8.1), we are able to construct a monad $\overline{T}$ on $\mathcal{V}\text{-}\mathsf{Mat}$ from a monad T on $\mathsf{Span}(\mathcal{V})$, which is, in turn, induced by a cartesian monad T on a lextensive category $\mathcal{V}$ [22]. In fact, we obtain a conjunction

$$\begin{tikzcd} (\overline{T}, \mathcal{V}\text{-}\mathsf{Mat}) \arrow[r, bend left, "{(-\cdot 1, \hat{\varepsilon}_{T(-\cdot 1)})}"] \arrow[r, phantom, "\perp"] & (T, \mathsf{Span}(\mathcal{V})) \arrow[l, bend left, "{(\mathcal{V}(1,-), \mathcal{V}(1,T\hat{\varepsilon}))}"] \end{tikzcd} \tag{0.1}$$

in the double category $\mathsf{Mnd}(\mathsf{PsDbCat}_{\mathrm{lax}})$. However, only under a suitable condition does this induce an adjunction

$$\begin{tikzcd} (\overline{T},\, \mathcal{V})\text{-}\mathsf{Cat} \arrow[r, bend left, "-\cdot 1"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}(T,\, \mathcal{V}). \arrow[l, bend left, "{\mathcal{V}(1,-)}"] \end{tikzcd} \tag{0.2}$$

The goal of this Section is to study this extra condition. In the case $- \cdot 1\colon \mathsf{Set} \to \mathcal{V}$ is fully faithful, we obtain Theorem 8.6, characterizing this condition in terms of a notion of *fibrewise discreteness* of a monad. Finally, we check that most of the commonly studied cartesian monads on lextensive categories $\mathcal{V}$ are fibrewise discrete, provided $- \cdot 1\colon \mathsf{Set} \to \mathcal{V}$ is fully faithful.

Section 9 contains our main results. Let $\mathcal{V}$ be a lextensive category such that $- \cdot 1\colon \mathsf{Set} \to \mathcal{V}$ is fully faithful, and let T be a fibrewise discrete, cartesian monad on $\mathcal{V}$. We also denote the induced monad on $\mathsf{Span}(\mathcal{V})$ by T. Via Theorem 6.1, we obtain the (ordinary) adjunction (0.2) from the conjunction (0.1) in $\mathsf{PsDbCat}_{\mathrm{lax}}$ (Theorem 9.2).

We then apply Theorem 9.2 to study effective descent morphisms for enriched categorical structures in Section 10. Under an additional technical condition (satisfied by most of the examples we provided), we confirm that $(\overline{T},\, \mathcal{V})\text{-}\mathsf{Cat}$ is precisely the full subcategory of $\mathsf{Cat}(T,\, \mathcal{V})$ with a *discrete* object-of-objects (Theorem 10.3), generalizing [35, 9.10 Theorem] and [13, Corollary 4.5]. Via this description, we confirm that $-\cdot 1\colon (\overline{T},\, \mathcal{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T,\, \mathcal{V})$ reflects effective descent morphisms (Lemma 10.4), and, with the results of [38] pertaining to effective descent morphisms in internal categorical structures, we provide criteria for an enriched $(\overline{T}, \mathcal{V})$-functor to be effective for descent (Theorem 10.5). We finalize the paper by studying the above examples.

1. Structure of double categories

Double categories were first defined in [18], and the more general pseudodouble categories were introduced in [20], allowing the vertical structure to be non-strict. Here, we will recall this notion of pseudodouble category, following the opposite convention of taking the horizontal structure to be non-strict, instead of the vertical, as in [15].

Furthermore, to fix notation, we also recall the notions of lax functor, and generalized vertical transformation [19, 2.2], and we provide definitions for (op)lax horizontal transformations, and the corresponding notion of modifications. For later reference, we also work out the (pseudo)double categorical structures formed by these objects.

1.1. **Pseudodouble categories:** A pseudodouble category $\mathbb{D}$ consists of:

- A category $\mathbb{D}_0$, denoting its objects as *0-cells*, its morphisms as *vertical 1-cells*, its composition and identities as *vertical.*
- A category $\mathbb{D}_1$, denoting its objects as *horizontal 1-cells*, its morphisms as *2-cells*, its composition and identities as *vertical.*
- *Vertical* domain and codomain functors $\mathrm{dom}, \mathrm{cod}\colon \mathbb{D}_1 \to \mathbb{D}_0$,
- A *horizontal unit* functor $1\colon \mathbb{D}_0 \to \mathbb{D}_1$,
- A *horizontal composition* functor $\cdot\colon \mathbb{D}_2 \to \mathbb{D}_1$ for each triple x, y, z of 0-cells, denoted by *horizontal composition*, where $\mathbb{D}_2$, given by pullback of dom and cod, is the category of *composable pairs* of horizontal 1-cells and 2-cells.

This data must satisfy $\mathrm{dom} \circ 1 = \mathrm{cod} \circ 1 = \mathsf{id}$, and $\mathrm{dom}(s{\cdot}r) = \mathrm{dom}(r)$, $\mathrm{cod}(s{\cdot}r) = \mathrm{cod}(s)$. Furthermore, we say a natural transformation $\phi\colon F \to G$ of functors $\mathcal{C} \to \mathbb{D}_1$ is *globular* if $\mathrm{dom} \cdot \phi$ and $\mathrm{cod} \cdot \phi$ are identities. We also have data

- Globular natural isomorphisms $\lambda\colon 1_{\mathrm{cod}(-)} \cdot - \to -$, $\rho\colon\ - \cdot 1_{\mathrm{dom}(-)} \to -$ of functors $\mathbb{D}_1 \to \mathbb{D}_1$, the *left* and *right unitors*, respectively.
- A globular natural isomorphism $\alpha\colon (-_1 \cdot -_2) \cdot -_3 \to -_1 \cdot (-_2 \cdot -_3)$ of functors $\mathbb{D}_3 \to \mathbb{D}_1$, the *associator*, where $\mathbb{D}_3$ is the category of *composable triples.*

These must also satisfy the following *coherence conditions*:

(a) We have $\gamma_{1_x} = \mathsf{id}_{1_x \cdot 1_x}$, where we define $\gamma = \rho^{-1} \circ \lambda$.

(b) The following diagram commutes

$$\begin{tikzcd}
(1_z \cdot s) \cdot r \arrow[rr, "\alpha_{r,s,1_z}"] \arrow[dr, "\lambda_s \cdot \mathsf{id}_r"'] & & 1 \cdot (s \cdot r) \arrow[dl, "\lambda_{s \cdot r}"] \\
& s \cdot r &
\end{tikzcd}$$

for each pair of horizontal 1-cells $r\colon x \to y$, $s\colon y \to z$.

(c) The following diagram commutes

$$\begin{tikzcd}
& s \cdot r & \\
(s \cdot r) \cdot 1_x \arrow[rr, "\alpha_{1_x,r,s}"'] \arrow[ur, "\rho_{s \cdot r}"] & & s \cdot (r \cdot 1_x) \arrow[ul, "\mathsf{id}_s \cdot \rho_r"']
\end{tikzcd}$$

for each pair of horizontal 1-cells $r\colon x \to y$, $s\colon y \to z$.

(d) The following diagram commutes

$$\begin{tikzcd}
(s \cdot 1_y) \cdot r \arrow[rr, "\alpha_{r,1_y,s}"] \arrow[dr, "\rho_s \cdot \mathsf{id}_r"'] & & s \cdot (1_y \cdot r) \arrow[dl, "\mathsf{id}_s \cdot \lambda_r"] \\
& s \cdot r &
\end{tikzcd}$$

for each pair of horizontal 1-cells $r\colon x \to y$, $s\colon y \to z$.

(e) The following diagram commutes:

$$\begin{tikzcd}
((t \cdot s) \cdot r) \cdot q \arrow[r, "\alpha_{q,r,t \cdot s}"] \arrow[d, "\alpha_{r,s,t} \cdot \mathsf{id}_q"'] & (t \cdot s) \cdot (r \cdot q) \arrow[r, "\alpha_{r \cdot q,s,t}"] & t \cdot (s \cdot (r \cdot q)) \\
(t \cdot (s \cdot r)) \cdot q \arrow[rr, "\alpha_{q,s \cdot r,t}"'] & & t \cdot ((s \cdot r) \cdot q) \arrow[u, "\mathsf{id}_t \cdot \alpha_{q,r,s}"']
\end{tikzcd}$$

for each quadruple of composable horizontal 1-cells q, r, s, t.

We will usually supress the subscripts, unless the need to disambiguate occurs. If λ, ρ and α are the identity transformations, we say $\mathbb{D}$ is a *double category.*

Proposition 1.1. *The coherence conditions (a), (b) and (c) are redundant.*

Proof. First, observe that (b) is the horizontal dual of (c), so it is sufficient to verify (a) and (b).

We may obtain (a) from the remaining conditions: we have an equality of 2-cells $(1_x \cdot 1_x) \cdot 1_x \to 1_x$

$$\lambda \circ (\lambda \cdot 1) = \lambda \circ \lambda \circ \alpha = \lambda \circ (1 \cdot \lambda) \circ \alpha = \lambda \circ (\rho \cdot 1)$$

by (b), naturality of λ, and (d). We deduce that $\lambda \cdot 1 = \rho \cdot 1$, and since ρ is a natural isomorphism, we conclude that $\lambda = \rho$.

To prove (b) given only (d) and (e), we consider the following diagram:

$$\begin{tikzcd}
& & 1_z \cdot (1_z \cdot (s \cdot r)) \arrow[d, "\mathsf{id}\cdot\lambda"] & & \\
& & 1_z \cdot (s \cdot r) & & \\
1_z \cdot ((1 \cdot s) \cdot r) \arrow[uurr, bend left, "\mathsf{id}\cdot\alpha"] \arrow[urr, "\mathsf{id}\cdot(\lambda\cdot\mathsf{id})"] \arrow[d, "\alpha^{-1}"'] & & (1_z \cdot s) \cdot r \arrow[u, "\alpha"] & & (1_z \cdot 1_z) \cdot (s \cdot r) \arrow[uull, bend right, "\alpha"'] \arrow[ull, "\rho\cdot\mathsf{id}"'] \\
(1_z \cdot (1_z \cdot s)) \cdot r \arrow[urr, "(\mathsf{id}\cdot\lambda)\cdot\mathsf{id}"] \arrow[rrrr, "\alpha^{-1}\cdot\mathsf{id}"'] & & & & ((1_z \cdot 1_z) \cdot s) \cdot r \arrow[ull, "(\rho\cdot\mathsf{id})\cdot\mathsf{id}"'] \arrow[u, "\alpha"']
\end{tikzcd}$$

Except for the top left triangle, every inner polygon commutes either by (d) or by naturality of α. The outer pentagon is an instance of (e), so we conclude that the top left triangle commutes. Since λ is a natural isomorphism, the result follows. ☐

1.2. **Lax functors:** Let $\mathbb{D}$, $\mathbb{E}$ be double categories. A lax functor $F \colon \mathbb{D} \to \mathbb{E}$ consists of:

- A functor $F_0 \colon \mathbb{D}_0 \to \mathbb{E}_0$.
- A functor $F_1 \colon \mathbb{D}_1 \to \mathbb{E}_1$.
- A globular natural transformation $\mathsf{e}^F \colon 1 \cdot F_0 \to F_1 \cdot 1$.
- A globular natural transformation $\mathsf{m}^F \colon F_1(-_1) \cdot F_1(-_2) \to F_1(-_1 \cdot -_2)$.

This data must satisfy the following properties:

- $\operatorname{dom} \circ F_1 = F_0 \circ \operatorname{dom}$
- $\operatorname{cod} \circ F_1 = F_0 \circ \operatorname{cod}$,
- Comparison coherences for the unit: the following diagrams commute

$$\begin{tikzcd}
1_{Fy} \cdot Fr \arrow[r, "\mathsf{e}^F \cdot Fr"] \arrow[d, "\lambda"'] & F1_y \cdot Fr \arrow[d, "\mathsf{m}^F"] & Fr \cdot 1_{Fx} \arrow[r, "\mathsf{id}\cdot\mathsf{e}^F"] \arrow[d, "\rho"'] & Fr \cdot F1_x \arrow[d, "\mathsf{m}^F"] \\
Fr \arrow[r, "F\lambda^{-1}"'] & F(1_y \cdot r) & Fr \arrow[r, "F\rho^{-1}"'] & F(r \cdot 1_x)
\end{tikzcd}$$

- Comparison coherence for composition: the following diagram commutes

$$\begin{tikzcd}
(Ft \cdot Fs) \cdot Fr \arrow[r, "\mathsf{m}^F \cdot \mathsf{id}"] \arrow[d, "\alpha"'] & F(t \cdot s) \cdot Fr \arrow[r, "\mathsf{m}^F"] & F((t \cdot s) \cdot r) \arrow[d, "F\alpha"] \\
Ft \cdot (Fs \cdot Fr) \arrow[r, "\mathsf{id}\cdot\mathsf{m}^F"'] & Ft \cdot F(s \cdot r) \arrow[r, "\mathsf{m}^F"'] & F(t \cdot (s \cdot r))
\end{tikzcd}$$

Dually, an *oplax functor* $F \colon \mathbb{B} \to \mathbb{C}$ is the horizontally dual notion (reverse the 2-cells). If the unit comparison transformation is an isomorphism, we say F is *normal*, and if both comparisons are isomorphisms, then we say F is a *strong functor* (which can be seen both as a lax and oplax functor).

Proposition 1.2. *Composition of (op)lax functors is well-defined, associative, and has identities. That is to say,* $\mathsf{PsDbCat}_{lax}$ *(*$\mathsf{PsDbCat}_{opl}$*) with double categories as objects and (op)lax functors as morphisms forms a category.*

Proof. For a double category $\mathbb{D}$, the identity functor is given by the identity function on objects and identity functor hom-categories, with coherence morphisms given by identities. The coherence conditions trivialize, thus we get a strong functor.

For lax functors $F\colon \mathbb{C} \to \mathbb{D}$ and $G\colon \mathbb{D} \to \mathbb{E}$, define the GF to be given by

- $(GF)_0 = G_0F_0$,
- $(GF)_1 = G_1F_1$,
- $\mathsf{e}^{GF} = G\,\mathsf{e}^F \circ \mathsf{e}^G$,
- $\mathsf{m}^{GF} = G\,\mathsf{m}^F \circ \mathsf{m}^G$.

To verify GF is a lax functor, first note that e^{GF} and m^{GF} are natural transformations, and globular, since G is a lax functor. Next, observe that the following diagrams

$$\begin{tikzcd}
\cdot \arrow[r, "\mathsf{e}^G \cdot GFf"] \arrow[dd, "\lambda"'] & \cdot \arrow[r, "G\,\mathsf{e}^F \cdot GFf"] \arrow[d, "\mathsf{m}^G"'] & \cdot \arrow[d, "\mathsf{m}^G"] \\
 & \cdot \arrow[r, "G(\mathsf{e}^F \cdot Fr)"'] & \cdot \arrow[d, "G(\mathsf{m}^F)"] \\
\cdot \arrow[ur, "G\lambda^{-1}"] \arrow[rr, "GF\lambda^{-1}"'] & & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[d, "\mathsf{m}^G"'] & \cdot \arrow[l, "GFf\cdot G\,\mathsf{e}^F"'] \arrow[d, "\mathsf{m}^G"] & \cdot \arrow[l, "GFf\cdot\mathsf{e}^G"'] \arrow[dd, "\rho"] \\
\cdot \arrow[d, "G(\mathsf{m}^F)"'] & \cdot \arrow[l, "G(Fr\cdot\mathsf{e}^F)"'] & \\
\cdot & & \cdot \arrow[ll, "GF\rho^{-1}"] \arrow[ul, "G\rho^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \cdot \arrow[dr, "G(\mathsf{m}^F \cdot Fr)"] & & \\
 & \cdot \arrow[ur, "\mathsf{m}^G"] \arrow[dr, "G\,\mathsf{m}^F \cdot GFr"'] & & \cdot \arrow[dr, "G\,\mathsf{m}^F"] & \\
\cdot \arrow[ur, "\mathsf{m}^G \cdot GFr"] \arrow[d, "\alpha"'] & & \cdot \arrow[ur, "\mathsf{m}^G"'] \arrow[d, "G\alpha"] & & \cdot \arrow[d, "GF\alpha"] \\
\cdot \arrow[dr, "GFt\cdot\mathsf{m}^G"'] & & \cdot \arrow[dr, "\mathsf{m}^G"] & & \cdot \\
 & \cdot \arrow[ur, "GFt\cdot G\,\mathsf{m}^F"] \arrow[dr, "\mathsf{m}^G"'] & & \cdot \arrow[ur, "G\,\mathsf{m}^F"'] & \\
 & & \cdot \arrow[ur, "G(Ft\cdot\mathsf{m}^F)"'] & &
\end{tikzcd}$$

commute, since every inner polygon commutes: either by coherence (of both F and G) or by naturality (only in m^G). Hence, the morphisms on the boundaries are equal, which give the coherences for GF. If e^F, m^F, e^G, m^G are isomorphisms, then so are e^{GF} and m^{GF}.

Finally, note that the identity functors are the units for lax functor composition, and this operation is also associative. This is because all required compositions occur on categories: function composition on a category of sets, functor composition on Cat, and 2-cell composition on the hom-categories (plus the composition preservation by the functors between them). □

1.3. **Vertical transformations.** We fix lax functors $H\colon \mathbb{A} \to \mathbb{B}$, $K\colon \mathbb{C} \to \mathbb{D}$ and oplax functors $F\colon \mathbb{A} \to \mathbb{C}$ and $G\colon \mathbb{B} \to \mathbb{D}$. A *generalized vertical transformation* ϕ, depicted as

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\phi"] & \mathbb{B} \arrow[d, "G"] \\
\mathbb{C} \arrow[r, "K"'] & \mathbb{D}
\end{tikzcd}$$

so that the vertical domain, codomain are F, G respectively, and horizontal domain, codomain given by H, K respectively.

- A natural transformation $\phi_0\colon G_0H_0 \to K_0F_0$,
- A natural transformation $\phi_1\colon G_1H_1 \to K_1F_1$,

satisfying $\mathrm{dom}\cdot\phi_1 = \phi_0 \cdot \mathrm{dom}$ and $\mathrm{cod}\cdot\phi_1 = \phi_0 \cdot \mathrm{cod}$, subject to the following conditions:

$$\begin{tikzcd}
& 1_{GHx} \arrow[r, "1_{\phi_x}"] & 1_{KFx} \arrow[dr, "\mathsf{e}^K_{Fx}"] & \\
G1_{Hx} \arrow[ur, "\mathsf{e}^G_{Hx}"] \arrow[dr, "G\,\mathsf{e}^H_x"'] & & & K1_{Fx} \\
& GH1_x \arrow[r, "\phi_{1_x}"'] & KF1_x \arrow[ur, "K\,\mathsf{e}^F_x"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& GHs\cdot GHr \arrow[r, "\phi_s\cdot\phi_r"] & KFs\cdot KFr \arrow[dr, "\mathsf{m}^K_{Fs,Fr}"] & \\
G(Hs\cdot Hr) \arrow[ur, "\mathsf{m}^G_{Hs,Hr}"] \arrow[dr, "G\,\mathsf{m}^H_{s,r}"'] & & & K(Fs\cdot Fr) \\
& GH(s\cdot r) \arrow[r, "\phi_{s\cdot r}"'] & KF(s\cdot r) \arrow[ur, "K\,\mathsf{m}^F_{s,r}"'] &
\end{tikzcd}$$

We say ϕ is a vertical transformation between lax functors if $F = \mathsf{id}$ and $G = \mathsf{id}$, in which case we denote it simply as $\phi\colon H \to K$, and vertical transformations between oplax functors can be analogously defined.

Proposition 1.3. *Lax functors and (generalized) vertical transformations form a category, and the vertical domain, codomain operations define functors to* $\mathsf{PsDbCat}_{opl}$.

Proof. The identity vertical transformation on a lax functor F is given by id_{F_0} and id_{F_1}, which trivially satisifies the conditions, and has identities functors as vertical domain and codomain.

To define composition, we consider the generalized vertical transformations ϕ and ψ as given below:

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "P"] \arrow[d, "F"'] \arrow[dr, phantom, "\phi"] & \mathbb{B} \arrow[d, "G"] \\
\mathbb{C} \arrow[r, "Q"'] & \mathbb{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \arrow[r, "Q"] \arrow[d, "H"'] \arrow[dr, phantom, "\psi"] & \mathbb{D} \arrow[d, "K"] \\
\mathbb{E} \arrow[r, "R"'] & \mathbb{F}
\end{tikzcd}$$

We define $(\psi \circ \phi)_i = \psi_i F_i \circ K_i \phi_i$ for $i = 0, 1$. Since we have

$$\begin{tikzcd}
& & \cdot \arrow[rr, "K\phi_s\cdot K\phi_r"] & & \cdot \arrow[rr, "\psi_{Fs}\cdot\psi_{Fr}"] & & \cdot \arrow[dr, "\mathsf{m}^R"] & \\
& \cdot \arrow[ur, "\mathsf{m}^K"] \arrow[rr, "K(\phi_s\cdot\phi_r)"'] & & \cdot \arrow[ur, "\mathsf{m}^K"'] \arrow[dr, "K\,\mathsf{m}^Q"] & & & & \cdot \\
\cdot \arrow[ur, "K\,\mathsf{m}^G"] \arrow[dr, "KG\,\mathsf{m}^P"'] & & & & \cdot \arrow[rr, "\psi_{Fs\cdot Fr}"] & & \cdot \arrow[ur, "R\,\mathsf{m}^H"'] & \\
& \cdot \arrow[rr, "K\phi_{s\cdot r}"'] & & \cdot \arrow[ur, "KQ\,\mathsf{m}^F"] \arrow[rr, "\psi_{F(s\cdot r)}"'] & & \cdot \arrow[ur, "RH\,\mathsf{m}^F"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& & \cdot \arrow[rr, "1_{K\phi}"] & & \cdot \arrow[rr, "1_{\psi_F}"] & & \cdot \arrow[dr, "\mathsf{e}^R"] & \\
& \cdot \arrow[ur, "\mathsf{e}^K"] \arrow[rr, "K(1_\phi)"'] & & \cdot \arrow[ur, "\mathsf{e}^K"'] \arrow[dr, "K\,\mathsf{e}^Q"] & & & & \cdot \\
\cdot \arrow[ur, "K\,\mathsf{e}^G"] \arrow[dr, "KG\,\mathsf{e}^P"'] & & & & \cdot \arrow[rr, "\psi_{1_F}"] & & \cdot \arrow[ur, "R\,\mathsf{e}^H"'] & \\
& \cdot \arrow[rr, "K\phi_1"'] & & \cdot \arrow[ur, "KQ\,\mathsf{e}^F"] \arrow[rr, "\psi_{F1}"'] & & \cdot \arrow[ur, "RH\,\mathsf{e}^F"'] & &
\end{tikzcd}$$

we conclude that $\psi\circ\phi$ is a generalized vertical transformation, with vertical domain HF and codomain KG. Hence, if ϕ and ψ are globular, then so is $\psi \circ \phi$, so we obtain a subcategory of lax functors and vertical transformations.

Associativity and identity are obtained via componentwise calculation on the underlying natural transformations. □

Proposition 1.4 ([19, 2.2])**.** *We have a double category* $\mathsf{PsDbCat}$ *with double categories, lax and oplax functors and generalized vertical transformations as 0-cells, horizontal and vertical 1-cells and 2-cells, respectively.*

Proof. The underlying Cat-graph for this double category is described in Proposition 1.3.

For an oplax functor $F\colon \mathbb{A} \to \mathbb{D}$, the identity natural transformation on F defines a generalized vertical transformation $1_F\colon 1_{\mathsf{id}} \to 1_{\mathsf{id}}$, whose vertical domain and codomain is F.

Given generalized vertical transformations ϕ and ψ as given by the following diagrams

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "P"] \arrow[d, "F"'] \arrow[dr, phantom, "\phi"] & \mathbb{B} \arrow[d, "G"] & \mathbb{B} \arrow[r, "R"] \arrow[d, "G"'] \arrow[dr, phantom, "\psi"] & \mathbb{C} \arrow[d, "H"] \\
\mathbb{D} \arrow[r, "Q"'] & \mathbb{E} & \mathbb{E} \arrow[r, "S"'] & \mathbb{F}
\end{tikzcd}$$

we denote their horizontal composite by $\psi \cdot \phi$, and is defined by $(\psi \cdot \phi)_i = S_i\phi_i \circ \psi_i P_i$. Since

$$\begin{tikzcd}
 & \cdot \arrow[rr, "\psi_{Ps} \cdot \psi_{Pr}"] & & \cdot \arrow[rr, "S\phi_s \cdot S\phi_r"] \arrow[dr, "\mathsf{m}^S"] & & \cdot \arrow[dr, "\mathsf{m}^S"] & & \\
\cdot \arrow[ur, "\mathsf{m}^H"] \arrow[dr, "H\,\mathsf{m}^R"'] & & & & \cdot \arrow[rr, "S(\phi_s \cdot \phi_r)"] & & \cdot \arrow[dr, "S\,\mathsf{m}^Q"] & \\
 & \cdot \arrow[rr, "\psi_{Ps \cdot Pr}"] \arrow[dr, "HR\,\mathsf{m}^P"'] & & \cdot \arrow[ur, "S\,\mathsf{m}^G"] \arrow[dr, "SG\,\mathsf{m}^P"] & & & & \cdot \\
 & & \cdot \arrow[rr, "\psi_{P(s \cdot r)}"'] & & \cdot \arrow[rr, "S\phi_{s \cdot r}"'] & & \cdot \arrow[ur, "SQ\,\mathsf{m}^F"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \cdot \arrow[rr, "1_{\psi P}"] & & \cdot \arrow[rr, "1_{S\phi}"] \arrow[dr, "\mathsf{e}^S"] & & \cdot \arrow[dr, "\mathsf{e}^S"] & & \\
\cdot \arrow[ur, "\mathsf{e}^H"] \arrow[dr, "H\,\mathsf{e}^R"'] & & & & \cdot \arrow[rr, "S1_\phi"] & & \cdot \arrow[dr, "S\,\mathsf{e}^Q"] & \\
 & \cdot \arrow[rr, "\psi_{1_P}"] \arrow[dr, "HR\,\mathsf{e}^P"'] & & \cdot \arrow[ur, "S\,\mathsf{e}^G"] \arrow[dr, "SG\,\mathsf{e}^P"] & & & & \cdot \\
 & & \cdot \arrow[rr, "\psi_{P1}"'] & & \cdot \arrow[rr, "S\phi_1"'] & & \cdot \arrow[ur, "SQ\,\mathsf{e}^F"'] &
\end{tikzcd}$$

are commutative diagrams, we conclude $\psi \cdot \phi$ is a generalized vertical transformation, with vertical domain F and codomain H. Associativity and identity conditions hold, via componentwise calculation on the underlying natural transformations, so we may take the associator and unitor isomorphisms to be identites. ☐

1.4. **Horizontal transformations:** Let F, $G\colon \mathbb{D} \to \mathbb{E}$ be lax functors. A *lax horizontal transformation* $\phi\colon F \to G$ is given by data

- a functor $\phi\colon \mathbb{D}_0 \to \mathbb{E}_1$
- a globular natural transformation $\mathsf{n}^\phi\colon G \cdot \phi_{\mathrm{dom}} \to \phi_{\mathrm{cod}} \cdot F$ of functors $\mathbb{D}_1 \to \mathbb{E}_1$.

satisfying the following coherence conditions:

- Comparison coherence for the unit: the following diagram commutes

$$\begin{tikzcd}
1_{Gx} \cdot \phi_x \arrow[r, "\gamma"] \arrow[d, "\mathsf{e}^G \cdot \mathsf{id}"'] & \phi_x \cdot 1_{Fx} \arrow[d, "\mathsf{id} \cdot \mathsf{e}^F"] \\
G1_x \cdot \phi_x \arrow[r, "\phi_{1_x}"'] & \phi_x \cdot F1_x
\end{tikzcd}$$

for all 0-cells x.

– Comparison coherence for the composition: the following diagram commutes

$$\begin{tikzcd}
& G(s\cdot r)\cdot\phi_x \arrow[r, "\mathsf{n}^{\phi}_{s\cdot r}"] & \phi_z\cdot F(s\cdot r) & \\
(Gs\cdot Gr)\cdot\phi_x \arrow[ur, "\mathsf{m}^G\cdot\mathsf{id}"] \arrow[d, "\alpha"'] & & & \phi_z\cdot(Fs\cdot Fr) \arrow[ul, "\mathsf{id}\cdot\mathsf{m}^F"'] \\
Gs\cdot(Gr\cdot\phi_x) \arrow[dr, "\mathsf{id}\cdot\mathsf{n}^{\phi}_r"'] & & & (\phi_z\cdot Fs)\cdot Fr \arrow[u, "\alpha"'] \\
& Gs\cdot(\phi_y\cdot Fr) \arrow[r, "\alpha^{-1}"'] & (Gs\cdot\phi_y)\cdot Fr \arrow[ur, "\mathsf{n}^{\phi}_s\cdot\mathsf{id}"'] &
\end{tikzcd}$$

for each pair $r\colon x\to y$,$s\colon y\to z$ of horizontal 1-cells.

Oplax horizontal transformations between (op)lax functors are obtained by dualizing 2-cells, and *strong horizontal transformations* are those whose comparison 2-cells are invertible, and hence are simultaneously lax and oplax.

Proposition 1.5. *Let $F\colon\mathbb{D}\to\mathbb{E}$ be a lax functor. The data*

(i) $(1_F)_x = 1_{Fx}$ *for each 0-cell x,*
(ii) $(1_F)_f = 1_{Ff}$ *for each vertical 1-cell f,*
(iii) $\mathsf{n}^{1_F}_r = \gamma_{Fr}$ *for each horizontal 1-cell r,*

defines a strong horizontal transformation $1_F\colon F\to F$.

Proof. The data (i) and (ii) tell us that the underlying functor of 1_F is the composite $1\circ F_0\colon\mathbb{D}_0\to\mathbb{E}_1$, and the datum (iii) tells us that the underlying globular natural transformation n^{1_F} is given by the whiskering $\gamma\cdot F_1\colon F\cdot 1_{\mathrm{dom}}\to 1_{\mathrm{cod}}\cdot F$, so the data is well-defined.

We're left with checking coherence. First, we observe that

$$\begin{tikzcd}
1_{Fx}\cdot 1_{Fx} \arrow[r, "\gamma"] \arrow[d, "\mathsf{id}\cdot\mathsf{e}^F"] & 1_{Fx}\cdot 1_{Fx} \arrow[d, "\mathsf{e}^F\cdot\mathsf{id}"] \\
1_{Fx}\cdot F1_x \arrow[r, "\gamma"] & F1_x\cdot 1_{Fx}
\end{tikzcd}$$

commutes by naturality, giving the comparison coherence diagram for the unit. Now, note that we have

$$\alpha^{-1}\circ(\mathsf{id}\cdot\gamma)\circ\alpha\circ(\gamma\cdot\mathsf{id})\circ\alpha^{-1}=\gamma, \tag{1.1}$$

by (b), (d) and (c), so that the following diagram commutes

$$\begin{tikzcd}
& 1_{Fz}\cdot F(s\cdot r) \arrow[r, "\gamma"] & F(s\cdot r)\cdot 1_{Fx} & \\
1_{Fz}\cdot(Fs\cdot Fr) \arrow[ur, "\mathsf{id}\cdot\mathsf{m}^F"] \arrow[rrr, "\gamma"] \arrow[d, "\alpha^{-1}"'] & & & (Fs\cdot Fr)\cdot 1_{Fx} \arrow[ul, "\mathsf{m}^F\cdot\mathsf{id}"'] \\
(1_{Fz}\cdot Fs)\cdot Fr \arrow[dr, "\gamma\cdot\mathsf{id}"'] & & & Fs\cdot(Fr\cdot 1_{Fx}) \arrow[u, "\alpha"'] \\
& (Fs\cdot 1_{Fy})\cdot Fr \arrow[r, "\alpha"'] & Fs\cdot(1_{Fy}\cdot Fr) \arrow[ur, "\mathsf{id}\cdot\gamma"'] &
\end{tikzcd}$$

by naturality of γ. Since $\gamma\cdot F_1$ is invertible, we conclude that $1_F\colon F\to F$ is a strong horizontal transformation. ☐

Proposition 1.6. *Let $\phi\colon F\to G$, $\psi\colon G\to H$ be lax horizontal transformations, where $F,G,H\colon\mathbb{D}\to\mathbb{E}$ are lax functors. The data*

(i) $(\psi\cdot\phi)_x=\psi_x\cdot\phi_x\colon Fx\to Hx$ *for each 0-cell x,*
(ii) $(\psi\cdot\phi)_f=\psi_f\cdot\phi_f$ *for each vertical 1-cell f,*
(iii) $\mathsf{n}^{\psi\cdot\phi}_r=\alpha^{-1}\circ(\mathsf{id}\cdot\mathsf{n}^{\phi}_r)\circ\alpha\circ(\mathsf{n}^{\psi}_r\cdot\mathsf{id})\circ\alpha^{-1}$ *for each horizontal 1-cell r*

defines a lax horizontal transformation $\psi\cdot\phi\colon F\to H$.

Proof. Due to functoriality of horizontal composition and of the underlying functors of ϕ and ψ, it is enough to point out that $\phi_x)$ and ψ_x are a composable pair of horizontal 1-cells to make sure the data (i) and (ii) define a functor $\mathbb{D}_0\to\mathbb{E}_1$.

Furthermore, note that the datum (iii) is a composite of globular natural transformations, so it is enough to verify the coherence conditions are satisfied.

We note the following diagram, in which we have supressed the horizontal 1-cells,

$$\begin{tikzcd}
\cdot \arrow[r, "\alpha^{-1}"] \arrow[d, "\mathsf{e}^H\cdot\mathsf{id}"'] & \cdot \arrow[r, "\gamma\cdot\mathsf{id}"] \arrow[d, dashed, "(\mathsf{e}^H\cdot\mathsf{id})\cdot\mathsf{id}"] & \cdot \arrow[r, "\alpha"] \arrow[d, dashed, "(\mathsf{id}\cdot\mathsf{e}^G)\cdot\mathsf{id}"] & \cdot \arrow[r, "\mathsf{id}\cdot\gamma"] \arrow[d, dashed, "\mathsf{id}\cdot(\mathsf{e}^G\cdot\mathsf{id})"] & \cdot \arrow[r, "\alpha^{-1}"] \arrow[d, dashed, "\mathsf{id}\cdot(\mathsf{id}\cdot\mathsf{e}^F)"] & \cdot \arrow[d, "\mathsf{id}\cdot\mathsf{e}^F"] \\
\cdot \arrow[r, "\alpha^{-1}"'] & \cdot \arrow[r, "\mathsf{n}_1^\psi\cdot\mathsf{id}"'] & \cdot \arrow[r, "\alpha"'] & \cdot \arrow[r, "\mathsf{id}\cdot\mathsf{n}_1^\phi"'] & \cdot \arrow[r, "\alpha^{-1}"'] & \cdot
\end{tikzcd}$$

commutes, by naturality of α and unit comparison coherence for ψ and ϕ. By 1.1, the top composite is γ, so this confirms unit comparsion coherence for $\psi\cdot\phi$.

The next diagram verifies composition coherence for $\psi\cdot\phi$: it is a pasting of composition coherences for ψ and ϕ, a naturality square from the functoriality of $\cdot$, and the remaining diagrams are coherence and naturality of α.

$$\begin{tikzcd}
 & & & \cdot \arrow[r, "\mathsf{m}^H\cdot\mathsf{id}"] \arrow[dll, "\alpha"'] \arrow[dd, "\alpha^{-1}"] & \cdot \arrow[rrrr, dashed, "\mathsf{n}^{\psi\cdot\phi}_{s\cdot r}"] \arrow[dd, "\alpha^{-1}"] & & & & \cdot & \cdot \arrow[l, "\mathsf{id}\cdot\mathsf{m}^F"'] & & & \\
 & \cdot \arrow[ddl, "\mathsf{id}\cdot\alpha^{-1}"'] & & & & & & & & & & \cdot \arrow[ull, "\alpha"'] & \\
 & & & \cdot \arrow[r, "(\mathsf{m}^H\cdot\mathsf{id})\cdot\mathsf{id}"] \arrow[dl, "\alpha\cdot\mathsf{id}"] & \cdot \arrow[dr, "\mathsf{n}^\psi_{s\cdot r}\cdot\mathsf{id}"] & & & & \cdot \arrow[uu, "\alpha^{-1}"] & \cdot \arrow[l, "\mathsf{id}\cdot(\mathsf{id}\cdot\mathsf{m}^F)"'] \arrow[uu, "\alpha^{-1}"] & & & \\
\cdot \arrow[d, "\mathsf{id}\cdot(\mathsf{n}_r^\psi\cdot\mathsf{id})"'] & & \cdot \arrow[ll, "\alpha"] \arrow[d, "(\mathsf{id}\cdot\mathsf{n}_r^\psi)\cdot\mathsf{id}"] & & & \cdot \arrow[rr, "\alpha"] & & \cdot \arrow[ur, "\mathsf{id}\cdot\mathsf{n}^\phi_{s\cdot r}"] & & & \cdot \arrow[ul, "\mathsf{id}\cdot\alpha"] & & \cdot \arrow[ll, "\alpha"] \arrow[uul, "\alpha^{-1}\cdot\mathsf{id}"'] \\
\cdot \arrow[rr, "\alpha^{-1}"] \arrow[ddr, "\mathsf{id}\cdot\alpha"'] & & \cdot \arrow[dr, "\alpha^{-1}\cdot\mathsf{id}"] & & & \cdot \arrow[u, "(\mathsf{id}\cdot\mathsf{m}^G)\cdot\mathsf{id}"] \arrow[rr, "\alpha"'] & & \cdot \arrow[u, "\mathsf{id}\cdot(\mathsf{m}^G\cdot\mathsf{id})"'] \arrow[dr, "\mathsf{id}\cdot\alpha"] & & & \cdot \arrow[u, "\mathsf{id}\cdot(\mathsf{n}_s^\phi\cdot\mathsf{id})"] & & \cdot \arrow[u, "(\mathsf{id}\cdot\mathsf{n}_s^\phi)\cdot\mathsf{id}"'] \arrow[ll, "\alpha"] \\
 & & & \cdot \arrow[r, "(\mathsf{n}_s^\psi\cdot\mathsf{id})\cdot\mathsf{id}"] \arrow[ddr, "\alpha"] & \cdot \arrow[ur, "\alpha\cdot\mathsf{id}"] \arrow[drr, "\alpha"] & & & & \cdot \arrow[r, "\mathsf{id}\cdot(\mathsf{id}\cdot\mathsf{n}_r^\phi)"] & \cdot \arrow[ur, "\mathsf{id}\cdot\alpha^{-1}"] & & & \\
 & \cdot \arrow[drrr, "\alpha^{-1}"] \arrow[dddrr, "\mathsf{id}\cdot(\mathsf{id}\cdot\mathsf{n}_r^\phi)"'] & & & & & \cdot \arrow[urr, "\alpha"] \arrow[drr, "\mathsf{id}\cdot\mathsf{n}_r^\phi"] & & & & & \cdot \arrow[uur, "\alpha\cdot\mathsf{id}"'] & \\
 & & & & \cdot \arrow[urr, "\mathsf{n}_s^\psi\cdot\mathsf{id}"] \arrow[drr, "\mathsf{id}\cdot\mathsf{n}_r^\phi"'] & & & & \cdot \arrow[uur, "\alpha"] \arrow[urrr, "\alpha^{-1}"] & & & & \\
 & & & & & & \cdot \arrow[urr, "\mathsf{n}_s^\psi\cdot\mathsf{id}"] \arrow[drrr, "\alpha^{-1}"] & & & & & & \\
 & & & \cdot \arrow[urrr, "\alpha^{-1}"] \arrow[dr, "\mathsf{id}\cdot\alpha^{-1}"'] & & & & & & \cdot \arrow[uuurr, "(\mathsf{n}_s^\psi\cdot\mathsf{id})\cdot\mathsf{id}"'] & & & \\
 & & & & \cdot \arrow[rrrr, "\alpha^{-1}"'] & & & & \cdot \arrow[ur, "\alpha^{-1}\cdot\mathsf{id}"'] & & & &
\end{tikzcd}$$

Hence, we have confirmed composition coherence for $\psi\cdot\phi$, concluding the proof. ☐

1.5. **Modifications.** Let F, G, H, $K\colon\mathbb{C}\to\mathbb{D}$ be lax functors and let $\zeta\colon F\to H$, $\xi\colon G\to K$ be oplax horizontal transformations, and let $\phi\colon F\to G$, $\psi\colon H\to K$ be vertical transformations. A *modification*

$\Gamma\colon \zeta \to \xi$, depicted as

$$\begin{tikzcd}
F \arrow[r, "\zeta"] \arrow[d, "\phi"'] & H \arrow[d, "\psi"] \\
G \arrow[r, "\xi"'] \arrow[ur, phantom, "\Gamma"] & K
\end{tikzcd}\tag{1.2}$$

is a natural transformation $\Gamma\colon \zeta \to \xi$ on the underlying functors $\zeta, \xi\colon \mathbb{D}_0 \to \mathbb{E}_1$ such that

$$\begin{tikzcd}
\zeta_y \cdot Fr \arrow[r, "\Gamma_y \cdot \phi_r"] \arrow[d, "\mathsf{n}^\zeta_r"'] & \xi_y \cdot Gr \arrow[d, "\mathsf{n}^\xi_r"] \\
Hr \cdot \zeta_x \arrow[r, "\psi_r \cdot \Gamma_x"'] & Kr \cdot \xi_x
\end{tikzcd}\tag{1.3}$$

commutes for all horizontal 1-cells $r\colon x \to y$. We say ϕ and ψ are respectively the vertical domain and codomain of Γ.

Proposition 1.7. *We have a category* $\mathsf{Lax}_{opl}(\mathbb{D}, \mathbb{E})$ *with oplax horizontal transformations of lax functors* $\mathbb{D} \to \mathbb{E}$ *as objects, and modifications as morphisms. Moreover, the vertical domain and codomain operations define functors to the category of lax functors and vertical transformations.*

Proof. Let $\zeta\colon F \to G$ be an oplax transformation of lax functors $\mathbb{D} \to \mathbb{E}$. We take the identity modification id_ζ on ζ to be given by identity natural transformation on the underlying functor of ζ, whose vertical domain and codomain are taken to be the identity vertical transformations id_F and id_G, respectively. The instance of the diagram (1.3) for id_ζ is trivially commutative.

Let Γ, Ξ be modifications given by

$$\begin{tikzcd}
F \arrow[r, "\zeta"] \arrow[d, "\phi"'] & G \arrow[d, "\psi"] \\
H \arrow[r, "\xi"] \arrow[d, "\theta"'] \arrow[ur, phantom, "\Gamma"] & K \arrow[d, "\omega"] \\
L \arrow[r, "\chi"'] \arrow[ur, phantom, "\Xi"] & M
\end{tikzcd}$$

We define the composite $\Gamma \circ \Xi$ to be the vertical composition of the underlying natural transformations. Since

$$\begin{tikzcd}
\zeta_y \cdot Fr \arrow[r, "\Gamma_y \cdot \phi_r"] \arrow[d, "\mathsf{n}^\zeta_r"'] & \xi_y \cdot Hr \arrow[r, "\Xi_y \cdot \theta_r"] \arrow[d, dashed, "\mathsf{n}^\xi_r"] & \chi_y \cdot Lr \arrow[d, "\mathsf{n}^\chi_r"] \\
Gr \cdot \zeta_x \arrow[r, "\psi_r \cdot \Gamma_x"'] & Kr \cdot \xi_x \arrow[r, "\omega_r \cdot \Xi_x"'] & Mr \cdot \chi_x
\end{tikzcd}$$

commutes for all horizontal 1-cells $r\colon x \to y$, we confirm $\Xi \circ \Gamma\colon \phi \to \chi$ is a modification with vertical domain $\theta \circ \phi$ and codomain $\omega \circ \psi$.

Associativity and identity properties are inherited from natural transformations, and functoriality of vertical domain and codomain is an immediate consequence. □

Proposition 1.8. *Let* $\mathbb{D}$, $\mathbb{E}$ *be double categories.* $\mathsf{Lax}_{opl}(\mathbb{D}, \mathbb{E})$ *has the structure of a double category, with lax functors as 0-cells, vertical transformations as vertical 1-cells, oplax horizontal transformations as horizontal 1-cells, and modifications as 2-cells.*

Proof. The underlying categories of cells are provided in Propositions 1.2 and 1.7. moreover, the latter has provided the vertical domain and codomain functors.

We have defined the horizontal unit functor on objects in Proposition 1.5. For a vertical transformation $\phi\colon F \to G$, we define 1_ϕ to be the modification with underlying natural transformation $1 \cdot \phi_0$, with vertical domain 1_F and codomain 1_G; note that

$$\begin{tikzcd}
1_F \cdot Fr \arrow[r, "1_\phi \cdot \phi_r"] \arrow[d, "\gamma"'] & 1_G \cdot Gr \arrow[d, "\gamma"] \\
Fr \cdot 1_F \arrow[r, "\phi_r \cdot 1_\phi"] & Gr \cdot 1_G
\end{tikzcd}$$

commutes by naturality of γ. Since this is just whiskering with $1\colon \mathbb{E}_0 \to E_1$, this describes a functor.

We have defined the horizontal composition functor on objects in Proposition 1.6. For modifications Γ and Ξ as depicted below

$$\begin{tikzcd}
F \arrow[r, "\zeta"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\Gamma"] & H \arrow[r, "\xi"] \arrow[d, "\psi"] \arrow[dr, phantom, "\Xi"] & L \arrow[d, "\chi"] \\
G \arrow[r, "\theta"'] & K \arrow[r, "\omega"'] & M
\end{tikzcd}$$

we define $\Xi \cdot \Gamma$ to be the horizontal composition of the underlying natural transformations. This is a modification, since the following diagram commutes

(1.4)

$$\begin{tikzcd}
(\xi_y \cdot \zeta_y) \cdot Fr \arrow[r, "\alpha"] \arrow[d, "(\Xi_y \cdot \Gamma_y) \cdot \phi_r", "|" marking] & \xi_y \cdot (\zeta_y \cdot Fr) \arrow[r, "\mathsf{id} \cdot n_r^\zeta"] \arrow[d, "\Xi_y \cdot (\Gamma_y \cdot \phi_r)", "|" marking] & \xi_y \cdot (Hr \cdot \zeta_x) \arrow[r, "\alpha^{-1}"] \arrow[d, "\Xi_y \cdot (\psi_r \cdot \Gamma_x)", "|" marking] & \ldots \\
(\omega_y \cdot \theta_y) \cdot Gr \arrow[r, "\alpha"] & \omega_y \cdot (\theta_y \cdot Gr) \arrow[r, "\mathsf{id} \cdot n_r^\theta"] & \omega_y \cdot (Kr \cdot \theta_x) \arrow[r, "\alpha^{-1}"] & \ldots
\end{tikzcd}$$

$$\begin{tikzcd}
\ldots \arrow[r, "\alpha^{-1}"] & (\xi_y \cdot Hr) \cdot \zeta x \arrow[r, "n_r^\xi \cdot \mathsf{id}"] \arrow[d, "(\Xi_y \cdot \psi_r) \cdot \Gamma_x", "|" marking] & (Lr \cdot \xi_x) \cdot \zeta_x \arrow[r, "\alpha"] \arrow[d, "(\chi_r \cdot \Xi_x) \cdot \Gamma_x", "|" marking] & Lr \cdot (\xi_x \cdot \zeta_x) \arrow[d, "\chi_r \cdot (\Xi_x \cdot \Gamma_x)", "|" marking] \\
\ldots \arrow[r, "\alpha^{-1}"] & (\omega_y \cdot Kr) \cdot \theta_x \arrow[r, "n_r^\omega \cdot \mathsf{id}"] & (Mr \cdot \omega_x) \cdot \theta_x \arrow[r, "\alpha"] & Mr \cdot (\omega_x \cdot \theta_x)
\end{tikzcd}$$

and has vertical domain ϕ and codomain χ.

Since horizontal composition in $\mathbb{E}$ is functorial, we obtain functoriality of horizontal composition of modifications. Moreover, both the horizontal unit and horizontal composition have the required behaviour with respect to vertical domains and codomains.

We're left with providing the unitors and associator, and the respective proofs that these satisfy the required coherence conditions. Moreover, we define $\lambda_\zeta \colon 1_H \cdot \zeta \to \zeta$ to be given by $\lambda_{\zeta_x} \colon 1_{Hx} \cdot \zeta_x \to \zeta_x$, and ρ_ζ is similarly defined. These are globular modifications, as the following diagrams commute

$$\begin{tikzcd}
(1_{Hy} \cdot \zeta_y) \cdot Fr \arrow[r, "\lambda_{\zeta_y} \cdot \mathsf{id}"] \arrow[d, "\alpha"'] & \zeta_y \cdot Fr \arrow[ddddd, "\mathsf{n}_r^\zeta"] \\
1_{Hy} \cdot (\zeta_y \cdot Fr) \arrow[d, "\mathsf{id} \cdot \mathsf{n}_r^\zeta"'] & \\
1_{Hy} \cdot (Hr \cdot \zeta_x) \arrow[d, "\alpha^{-1}"'] & \\
(1_{Hy} \cdot Hr) \cdot \zeta_x \arrow[d, "\gamma \cdot \mathsf{id}"'] & \\
(Hr \cdot 1_{Hx}) \cdot \zeta_x \arrow[d, "\alpha"'] & \\
Hr \cdot (1_{Hx} \cdot \zeta_x) \arrow[r, "\mathsf{id} \cdot \lambda_{\zeta_x}"'] & Hr \cdot \zeta_x
\end{tikzcd}
\qquad
\begin{tikzcd}
(\zeta_y \cdot 1_{Fy}) \cdot Fr \arrow[r, "\rho_{\zeta_y} \cdot \mathsf{id}"] \arrow[d, "\alpha"'] & \zeta_y \cdot Fr \arrow[ddddd, "\mathsf{n}_r^\zeta"] \\
\zeta_y \cdot (1_{Fy} \cdot Fr) \arrow[d, "\mathsf{id} \cdot \gamma"'] & \\
\zeta_y \cdot (Fr \cdot 1_{Fx}) \arrow[d, "\alpha^{-1}"'] & \\
(\zeta_y \cdot Fr) \cdot 1_{Fx} \arrow[d, "\mathsf{n}_r^\zeta \cdot \mathsf{id}"'] & \\
(Hr \cdot \zeta_x) \cdot 1_{Fx} \arrow[d, "\alpha"'] & \\
Hr \cdot (\zeta_x \cdot 1_{Fx}) \arrow[r, "\mathsf{id} \cdot \rho_{\zeta_x}"'] & Hr \cdot \zeta_x
\end{tikzcd}$$

by naturality of λ, ρ, and coherence.

Finally, we let $\pi \colon L \to P$ be another oplax horizontal transformation. We define $\alpha \colon (\pi \cdot \xi) \cdot \zeta \to \pi \cdot (\xi \cdot \zeta)$ to be given at x by $\alpha \colon (\pi_x \cdot \xi_x) \cdot \zeta_x \to \pi_x \cdot (\xi_x \cdot \zeta_x)$. Also a natural isomorphism, and is a globular modification since the following diagram commutes:

$$\begin{tikzcd}
((\pi_y \cdot \xi_y) \cdot \zeta_y) \cdot Fr \arrow[r, "\alpha\cdot\mathsf{id}"] \arrow[d, "\alpha"'] & (\pi_y \cdot (\xi_y \cdot \zeta_y)) \cdot Fr \arrow[d, "\alpha"] \\
(\pi_y \cdot \xi_y) \cdot (\zeta \cdot Fr) \arrow[d, "\mathsf{id}\cdot\mathsf{n}_r^{\zeta}"'] \arrow[dr, "\alpha"] & \pi_y \cdot ((\xi_y \cdot \zeta_y) \cdot Hr) \arrow[d, "\mathsf{id}\cdot\alpha"] \\
(\pi_y \cdot \xi_y) \cdot (Hr \cdot \zeta_x) \arrow[d, "\alpha^{-1}"'] \arrow[dr, "\alpha"] & \pi_y \cdot (\xi_y \cdot (\zeta_y \cdot Hr)) \arrow[d, "\mathsf{id}\cdot(\mathsf{id}\cdot\mathsf{n}_r^{\zeta})"] \\
((\pi_y \cdot \xi_y) \cdot Hr) \cdot \zeta_x) \arrow[d, "\alpha\cdot\mathsf{id}"'] & \pi_y \cdot (\xi \cdot (Hr \cdot \zeta_x)) \arrow[d, "\mathsf{id}\cdot\alpha^{-1}"] \\
(\pi_y \cdot (\xi_y \cdot Hr)) \cdot \zeta_x \arrow[r, "\alpha"] \arrow[d, "(\mathsf{id}\cdot\mathsf{n}_r^{\xi})\cdot\mathsf{id}"] & \pi_y \cdot ((\xi_y \cdot Hr) \cdot \zeta_x) \arrow[d, "\mathsf{id}\cdot(\mathsf{n}_r^{\xi}\cdot\mathsf{id})"] \\
(\pi_y \cdot (Lr \cdot \xi_x)) \cdot \zeta_x \arrow[r, "\alpha"] \arrow[d, "\alpha^{-1}\cdot\mathsf{id}"'] & \pi_y \cdot ((Lr \cdot \xi_x) \cdot \zeta_x) \arrow[d, "\mathsf{id}\cdot\alpha"] \\
((\pi_y \cdot Lr) \cdot \xi_x) \cdot \zeta_x \arrow[d, "(\mathsf{n}_r^{\pi}\cdot\mathsf{id})\cdot\mathsf{id}"'] \arrow[dr, "\alpha"] & \pi_y \cdot (Lr \cdot (\xi_x \cdot \zeta_x)) \arrow[d, "\alpha^{-1}"] \\
((Pr \cdot \pi_x) \cdot \xi_x) \cdot \zeta_x \arrow[d, "\alpha\cdot\mathsf{id}"'] \arrow[dr, "\alpha"] & (\pi_y \cdot Lr) \cdot (\xi_x \cdot \zeta_x) \arrow[d, "\mathsf{n}_r^{\pi}\cdot\mathsf{id}"] \\
(Pr \cdot (\pi_x \cdot \xi_x)) \cdot \zeta_x \arrow[d, "\alpha"'] & (Pr \cdot \pi_x) \cdot (\xi_x \cdot \zeta_x) \arrow[d, "\alpha"] \\
Pr \cdot ((\pi_x \cdot \xi_x) \cdot \zeta_x) \arrow[r, "\mathsf{id}\cdot\alpha"'] & Pr \cdot (\pi_x \cdot (\xi_x \cdot \zeta_x))
\end{tikzcd}$$

which is obtained by pasting coherence pentagons and naturality squares of α.

By checking componentwise, we find that λ, ρ and α satisfy the desired coherence conditions. ☐

1.6. **Examples:** The pseudodouble categories studied in this body of work are:

- The pseudodouble category $\mathcal{V}$-Mat of $\mathcal{V}$-matrices, for distributive monoidal categories $\mathcal{V}$ (that is, for $\mathcal{V}$ with coproducts, which preserved by the tensor product; see [6, 12, 15]).
- The pseudodouble category $\mathsf{Span}(\mathcal{B})$ of spans of morphsims in $\mathcal{B}$, for $\mathcal{B}$ a category with pullbacks (see [4, 22, 15]).
- The pseudodouble categories $\mathsf{Lax}_{\mathrm{lax}}(\mathbb{D}, \mathbb{E})$ and $\mathsf{Lax}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E})$ for double categories $\mathbb{D}$, $\mathbb{E}$.
- The double categories $\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$, $\mathsf{Ps}\text{-}T\text{-}\mathsf{Alg}$ of lax and pseudo T-algebras, for T a pseudomonad on a 2-category $\mathbb{B}$.
- The double category $\mathsf{Mnd}(\mathbb{B}) = \mathsf{Lax}\text{-}\mathsf{id}\text{-}\mathsf{Alg}$ of monads in a 2-category $\mathbb{B}$.

We shall specify the double categorical structure of $\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$. First, recall that we have 2-categories $\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}_{\mathrm{lax}}$ and $\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}_{\mathrm{opl}}$ whose 0-cells are lax T-algebras, with their (op)lax morphisms and their respective 2-cells [36]; however, there is a notion of generalized 2-cell which subsumes both structures.

We will be taking the vertical 1-cells to be the oplax morphisms and the horizontal 1-cells to be the lax morphisms. Let $(h, \phi)\colon (w, a, \eta, \mu) \to (x, b, \eta, \mu)$, $(k, \psi)\colon (y, c, \eta, \mu) \to (z, d, \eta, \mu)$ be lax T-algebra lax morphisms and $(f, \zeta)\colon (w, a, \eta, \mu) \to (y, c, \eta, \mu)$, $(g, \xi)\colon (x, b, \eta, \mu) \to (z, d, \eta, \mu)$ be lax T-algebra oplax morphisms. A *generalized lax T-algebra 2-cell*

$$\begin{tikzcd}
(w, a, \eta, \mu) \arrow[r, "{(h,\phi)}"] \arrow[d, "{(f,\zeta)}"'] \arrow[dr, phantom, "\omega"] & (x, b, \eta, \mu) \arrow[d, "{(g,\psi)}"] \\
(y, c, \eta, \mu) \arrow[r, "{(k,\xi)}"'] & (z, d, \eta, \mu)
\end{tikzcd}$$

consists of a 2-cell $\omega\colon g\cdot h\to k\cdot f$ satisfying the following coherence condition

$$\begin{tikzcd}
 & g\cdot b\cdot Th \arrow[dl, "g\cdot\phi"'] \arrow[dr, "\xi\cdot Th"] & \\
g\cdot h\cdot a \arrow[d, "\omega\cdot a"'] & & d\cdot Tg\cdot Th \arrow[d, "d\cdot\omega^T"] \\
k\cdot f\cdot a \arrow[dr, "k\cdot\zeta"'] & & d\cdot Tk\cdot Tf \arrow[dl, "\psi\cdot Tf"] \\
 & k\cdot c\cdot Tf &
\end{tikzcd}$$

where we write $\omega^T=(\mathsf{m}^T)^{-1}\circ T\omega\circ\mathsf{m}^T$. Horizontal and vertical composition is defined as expected: to be explicit, we consider generalized lax T-algebra 2-cells θ,σ given by

$$\begin{tikzcd}
(y,c,\eta,\mu) \arrow[r, "{(k,\xi)}"] \arrow[d, "{(f',\phi')}"'] \arrow[dr, phantom, "\theta"] & (z,d,\eta,\mu) \arrow[d, "{(g',\psi')}"] & (x,b,\eta,\mu) \arrow[r, "{(h',\phi')}"] \arrow[d, "{(g,\psi)}"'] \arrow[dr, phantom, "\sigma"] & (w',a',\eta,\mu) \arrow[d, "{(l,\chi)}"] \\
(x',b',\eta,\mu) \arrow[r, "{(m,\pi)}"'] & (z',d',\eta,\mu) & (z,d,\eta,\mu) \arrow[r, "{(k',\xi')}"'] & (y',c',\eta,\mu)
\end{tikzcd}$$

and we define $\theta\circ\omega=(\theta\cdot f)\circ(g'\cdot\omega)$ and $\sigma\cdot\omega=(k'\cdot\omega)\circ(\sigma\cdot h)$. These provide a double categorical structure to Lax-T-Alg, provided the coherence conditions are satisfied for $\theta\circ\omega$ and $\sigma\cdot\omega$, which are given by the commutativity of the following diagrams:

$$\begin{tikzcd}
 & g'\cdot g\cdot b\cdot Th \arrow[dl, "g'\cdot g\cdot\phi"'] \arrow[dr, "g'\cdot\xi\cdot Th"] & & \\
g'\cdot g\cdot h\cdot a \arrow[d, "g'\omega\cdot a"'] & & g'\cdot d\cdot Tg\cdot Th \arrow[d, "g'\cdot d\cdot\omega^T"'] \arrow[dr, "\psi'\cdot Tg\cdot Th"] & \\
g'\cdot k\cdot f\cdot a \arrow[d, "\theta\cdot f\cdot a"'] \arrow[dr, "g'\cdot k\cdot\zeta"] & & g'\cdot d\cdot Tk\cdot Tf \arrow[dl, "g'\cdot\psi\cdot Tf"] \arrow[dr, "\psi'\cdot Tk\cdot Tf"'] & d\cdot Tg'\cdot Tg\cdot Th \arrow[d, "d\cdot Tg'\cdot\omega^T"] \\
m\cdot f'\cdot f\cdot a \arrow[dr, "m\cdot f'\cdot\zeta"'] & g'\cdot k\cdot c\cdot Tf \arrow[d, "\theta\cdot c\cdot Tf"] & & d\cdot Tg'\cdot Tk\cdot Tf \arrow[d, "d\cdot\theta^T\cdot Tf"] \\
 & m\cdot f'\cdot c\cdot Tf \arrow[dr, "m\cdot\phi'\cdot Tf"'] & & d\cdot Tm\cdot Tf'\cdot Tf \arrow[dl, "\pi\cdot Tf'\cdot Tf"] \\
 & & m\cdot b'\cdot Tf'\cdot Tf &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & l\cdot a'\cdot Th'\cdot Th \arrow[dl, "l\cdot\phi'\cdot Th"'] \arrow[dr, "\chi\cdot Th'\cdot Th"] & \\
 & l\cdot h'\cdot b\cdot Th \arrow[dl, "l\cdot h'\cdot\phi"'] \arrow[d, "\sigma\cdot b\cdot Th"] & & c'\cdot Tl\cdot Th'\cdot Th \arrow[d, "c'\cdot\sigma^T\cdot Th"] \\
l\cdot h'\cdot h\cdot a \arrow[d, "\sigma\cdot h\cdot a"'] & k'\cdot g\cdot b\cdot Th \arrow[dl, "k'\cdot g\cdot\phi"] \arrow[dr, "k'\cdot\psi\cdot Th"'] & & c'\cdot Tk'\cdot Tg\cdot Th \arrow[dl, "\xi'\cdot Tg\cdot Th"'] \arrow[d, "c'\cdot Tk'\cdot\omega^T"] \\
k'\cdot g\cdot h\cdot a \arrow[d, "k'\cdot\omega\cdot a"'] & & k'\cdot d\cdot Tg\cdot Th \arrow[d, "k'\cdot d\cdot\omega^T"'] & c'\cdot Tk'\cdot Tk\cdot Tf \arrow[dl, "\xi'\cdot Tk\cdot Tf"] \\
k'\cdot k\cdot f\cdot a \arrow[dr, "k'\cdot k\cdot\zeta"'] & & k'\cdot d\cdot Tk\cdot Tf \arrow[dl, "k'\cdot\psi\cdot Tf"] & \\
 & k'\cdot k\cdot c\cdot Tf & &
\end{tikzcd}$$

with analogous definitions for θ^T and σ^T, plus a couple of omitted coherence conditions which confirm that $(\theta^T \cdot Tf) \circ (Tg' \cdot \omega^T) = (\theta \circ \omega)^T$ and $(Tk' \cdot \omega^T) \circ (\sigma^T \cdot Th) = (\sigma \cdot \omega)^T$.

2. Spans versus matrices

Let $\mathcal{V}$ be a distributive, cartesian monoidal category with finite limits. Our starting point is the adjunction

$$\begin{tikzcd} \mathsf{Set} \arrow[r, bend left, "- \cdot 1"] \arrow[r, phantom, "\perp"] & \mathcal{V} \arrow[l, bend left, "\mathcal{V}(1,-)"] \end{tikzcd} \tag{2.1}$$

whose unit and counit we denote by $\hat{\eta}, \hat{\varepsilon}$ respectively; here, $- \cdot 1$ is the copower with the terminal object 1.

After fixing some notation regarding $\mathcal{V}$-Mat and $\mathsf{Span}(\mathcal{V})$, we confirm that (2.1) induces an adjunction of internal Cat-graph morphisms

$$- \cdot 1 \colon \mathcal{V}\text{-}\mathsf{Mat} \to \mathsf{Span}(\mathcal{V}) \qquad \mathcal{V}(1,-) \colon \mathsf{Span}(\mathcal{V}) \to \mathcal{V}\text{-}\mathsf{Mat}$$

and we will furthermore confirm that $- \cdot 1$ defines an oplax functor.

Together with the tools and terminology provided in Sections 3 and 4, we will be able to deduce that $\mathcal{V}(1,-)$ is a lax functor, and that we have a conjunction in the double category $\mathsf{PsDbCat}$. The unit and counit may be depicted as follows

$$\begin{tikzcd} \mathcal{V}\text{-}\mathsf{Mat} \arrow[r, equal] \arrow[d, "- \cdot 1"'] \arrow[dr, phantom, "\hat{\eta}"] & \mathcal{V}\text{-}\mathsf{Mat} \arrow[d, equal] \\ \mathsf{Span}(\mathcal{V}) \arrow[r, "\mathcal{V}(1,-)"'] & \mathcal{V}\text{-}\mathsf{Mat} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Span}(\mathcal{V}) \arrow[r, "\mathcal{V}(1,-)"] \arrow[d, equal] \arrow[dr, phantom, "\hat{\varepsilon}"] & \mathcal{V}\text{-}\mathsf{Mat} \arrow[d, "- \cdot 1"] \\ \mathsf{Span}(\mathcal{V}) \arrow[r, equal] & \mathsf{Span}(\mathcal{V}) \end{tikzcd}$$

alluding to the fact that these are generalized vertical transformations in $\mathsf{PsDbCat}$.

Notation for $\mathsf{Span}(\mathcal{V})$: The Cat-graph $\mathsf{Span}(\mathcal{V})$ is succintly defined as $[l \leftarrow m \to r, \mathcal{V}] \rightrightarrows \mathcal{V}$, whose underlying functors are the evaluations at l and r. Throughout this work, we opt to denote spans $p \colon X \nrightarrow Y$ in $\mathcal{V}$ as the following diagram

$$\begin{tikzcd} X & M_p \arrow[l, "l_p"'] \arrow[r, "r_p"] & Y \end{tikzcd}$$

and a 2-cell θ will be denoted as a morphism $M_p \to M_q$ making both of the following squares commute:

$$\begin{tikzcd} X \arrow[d, "f"'] & M_p \arrow[l, "l_p"'] \arrow[r, "r_p"] \arrow[d, "\theta"] & Y \arrow[d, "g"] \\ W & M_q \arrow[l, "l_q"] \arrow[r, "r_q"'] & Z \end{tikzcd}$$

The unit span $1_X \colon X \nrightarrow X$ is defined on objects by $M_{1_X} = X$ and $l_{1_x} = r_{1_x} = \mathsf{id}_X$, and on morphisms $f \colon X \to Y$ by $l_f = r_f = f$.

Let $q \colon Y \nrightarrow Z$ be another span in $\mathcal{V}$. We write the pullback which defines $q \cdot p$ as

$$\begin{tikzcd} M_{q \cdot p} \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & M_q \arrow[d, "l_q"] \\ M_p \arrow[r, "r_p"'] & Y \end{tikzcd}$$

so that we have $l_{q \cdot p} = l_p \circ \pi_1$ and $r_{q \cdot p} = r_q \circ \pi_0$. By abuse of notation, we may refer to instances of such pullback diagrams as $M_{q \cdot p}$.

The unitors $\lambda \colon 1 \cdot p \to p$ and $\rho \colon p \cdot 1 \to p$ in $\mathsf{Span}(\mathcal{V})$ are given by the pullback projections $\pi_1 \colon M_{1 \cdot p} \to M_p$ and $\pi_0 \colon M_{p \cdot 1} \to M_p$, respectively.

Given a third span $r\colon Z \nrightarrow W$, note that the universal property of the pullback $M_{q\cdot p}$ guarantees the existence of a unique map $\pi_2\colon M_{(r\cdot q)\cdot p} \to M_{q\cdot p}$ such that $\pi_1 \circ \pi_2 = \pi_1$ and $\pi_1 \circ \pi_0 = \pi_0 \circ \pi_2$:

$$\begin{tikzcd}
M_{(r\cdot q)\cdot p} \arrow[r, "\pi_0"] \arrow[dr, dashed, "\pi_2"'] \arrow[ddr, bend right, "\pi_1"'] & M_{r\cdot q} \arrow[dr, "\pi_1"] & \\
& M_{q\cdot p} \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & M_q \arrow[d, "l_q"] \\
& M_p \arrow[r, "r_p"'] & Y
\end{tikzcd}$$

With this, the associator $\alpha\colon (r\cdot q)\cdot p \to r\cdot(q\cdot p)$ may be defined as the unique map such that $\pi_1 \circ \alpha = \pi_2$ and $\pi_0 \circ \alpha = \pi_0 \circ \pi_0$, via the universal property of the pullback $M_{r\cdot(q\cdot p)}$:

$$\begin{tikzcd}
M_{(r\cdot q)\cdot p} \arrow[drr, bend left, "\pi_0\circ\pi_0"] \arrow[dr, dashed, "\alpha"'] \arrow[ddr, bend right, "\pi_2"'] & & \\
& M_{r\cdot(q\cdot p)} \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & M_r \arrow[d, "l_r"] \\
& M_{q\cdot p} \arrow[r, "r_q\circ\pi_0"'] & Z
\end{tikzcd}$$

Notation for $\mathcal{V}$-Mat: Let $p\colon U \nrightarrow V$ be a $\mathcal{V}$-matrix. We denote by $p(u,v) \in \mathcal{V}$ the value of p at the pair $(u,v) \in U \times V$. A 2-cell of $\mathcal{V}$-matrices

$$\begin{tikzcd}
U \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\theta"] & V \arrow[d, "g"] \\
W \arrow[r, "q"'] & X
\end{tikzcd}$$

consists of a family of morphisms $\theta_{u,v}\colon p(u,v) \to q(fu,gv)$ in $\mathcal{V}$, for $u \in U$ and $v \in V$. Given another 2-cell

$$\begin{tikzcd}
W \arrow[r, "q"] \arrow[d, "h"] \arrow[dr, phantom, "\omega"] & X \arrow[d, "k"] \\
Y \arrow[r, "r"'] & Z
\end{tikzcd}$$

the composite $\omega \circ \theta$ is given at u,v by the composite of

$$p(u,v) \xrightarrow{\theta_{u,v}} q(fu,gv) \xrightarrow{\omega_{fu,gv}} r(hfu,kgv),$$

exhibiting the structure of $\mathcal{V}$-Mat as an internal Cat-graph.

Given $u, u' \in U$, we write $[u = u']$ for the set that is a singleton if $u = u'$ and empty otherwise. Note that if we have a function $f\colon U \to V$, then there is a unique morphism $[u = u'] \to [fu = fu']$. With this, the unit $\mathcal{V}$-matrix $1_U\colon U \nrightarrow U$ is defined by $1_U(u,u') = [u = u'] \cdot 1$ for a set U, and 1_f is given by $1_f(u,u')\colon [u = u'] \cdot 1 \to [fu = fu'] \cdot 1$ for a function $f\colon U \to V$.

Recall that if $t\colon V \nrightarrow W$ is another $\mathcal{V}$-matrix, we have

$$(t \cdot s)(u,w) = \sum_{v\in V} t(v,w) \times s(u,v)$$

which is the composition of $\mathcal{V}$-matrices. This is likewise defined for 2-cells.

The unitors and associators are then given by taking coproducts over the unitors and associators for the cartesian monoidal structure of $\mathcal{V}$.

Lifting the adjunction to Grph(Cat): For a $\mathcal{V}$-matrix $p\colon X \nrightarrow Y$, we define $M_{p\cdot 1} = \sum_{x,y} p(x,y)$, and we define $p \cdot 1\colon X \cdot 1 \nrightarrow Y \cdot 1$ to be the span given by taking the coproduct of $p(x,y) \to 1$ indexed by $X \times Y$; this gives a morphism $M_{p\cdot 1} \to X \cdot 1 \times Y \cdot 1$ (see Diagram (2.2) below, which is commutative by the universal property of the coproduct), whose composite with the projections determine $l_{p\cdot 1}$ and $r_{p\cdot 1}$.

(2.2)
$$\begin{tikzcd}
p(x,y) \arrow[r] \arrow[d] & 1 \arrow[d, "\hat{\eta}x{,}\hat{\eta}y"] \\
M_{p\cdot 1} \arrow[r, "l_{p\cdot 1}{,}\, r_{p\cdot 1}"'] & X\cdot 1 \times Y \cdot 1
\end{tikzcd}$$

We write $\hat{\pi}_0\colon (t\cdot s)\cdot 1 \to t\cdot 1$ (respectively, $\hat{\pi}_1\colon (t\cdot s)\cdot 1 \to s\cdot 1$) for the coproducts of the projections $t(v,w)\times s(u,v) \to t(v,w)$ indexed by $U\times V\times W \to V\times W$ (respectively, $t(v,w)\times s(u,v)\to s(u,v)$ indexed by $U\times V\times W\to U\times V$).

For a span $p\colon V \nrightarrow W$ in $\mathcal{V}$, we define the $\mathcal{V}$-matrix $\mathcal{V}(1,p)\colon \mathcal{V}(1,V)\nrightarrow \mathcal{V}(1,W)$ to be given at v,w by the following pullback:

(2.3)
$$\begin{tikzcd}
\mathcal{V}(1,p)(v,w) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & 1 \arrow[d, "v{,}w"] \\
M_p \arrow[r, "l_p{,}r_p"'] & V\times W
\end{tikzcd}$$

and if we have a 2-cell of spans θ:

$$\begin{tikzcd}
V & M_p \arrow[l, "l_p"'] \arrow[r, "r_p"] \arrow[d, "\theta"] & W \arrow[d, "g"] \\
X \arrow[from=u, "f"'] & M_q \arrow[l, "l_q"] \arrow[r, "r_q"'] & Y
\end{tikzcd}$$

then $\mathcal{V}(1,\theta)$ is the 2-cell uniquely determined by pullback as follows:

$$\begin{tikzcd}
\mathcal{V}(1,p)(v,w) \arrow[r, dashed, "\mathcal{V}(1,\theta)(v,w)"] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathcal{V}(1,q)(fv,gw) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & 1 \arrow[d, "fv{,}gw"] \\
M_p \arrow[r, "\theta"'] & M_q \arrow[r, "l_q{,}r_q"'] & X\times Y
\end{tikzcd}$$

We observe that $l_q\circ\theta = f\circ l_p$ and $r_q\circ\theta = f\circ r_p$.

We extend $\hat{\eta}$, $\hat{\varepsilon}$ to $\mathcal{V}$-Mat and $\mathsf{Span}(\mathcal{V})$: for a $\mathcal{V}$-matrix $p\colon X\nrightarrow Y$, we define $\hat{\eta}_p\colon p\to\mathcal{V}(1,p\cdot 1)$ at x,y to be given by the dashed arrow

(2.4)
$$\begin{tikzcd}
p(x,y) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{V}(1,p\cdot 1)(\hat{\eta}x,\, \hat{\eta}y) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & 1 \arrow[d, "\hat{\eta}x{,}\, \hat{\eta}y"] \\
& M_{p\cdot 1} \arrow[r, "l_{p\cdot 1}{,}\, r_{p\cdot 1}"'] & X\cdot 1\times Y\cdot 1
\end{tikzcd}$$

For a span $p\colon V\nrightarrow W$, we let $\hat{\varepsilon}_p\colon \mathcal{V}(1,p)\cdot 1\to p$ to be given by taking the coproduct of (2.3) indexed by

$$\begin{tikzcd}
\mathcal{V}(1,V)\times\mathcal{V}(1,W) \arrow[r, equal] \arrow[d] & \mathcal{V}(1,V)\times\mathcal{V}(1,W) \arrow[d] \\
1 \arrow[r, equal] & 1
\end{tikzcd}$$

which yields a commutative square

(2.5)
$$\begin{tikzcd}
M_{\mathcal{V}(1,p)\cdot 1} \arrow[r] \arrow[d, "\hat{\varepsilon}_p"'] & \mathcal{V}(1,V)\cdot 1\times\mathcal{V}(1,W)\cdot 1 \arrow[d, "\hat{\varepsilon}_V{,}\, \hat{\varepsilon}_W"] \\
M_a \arrow[r, "l_p{,}\, r_p"'] & V\times W
\end{tikzcd}$$

By taking the coproduct of (2.4) over the diagram

$$\begin{tikzcd}
X \times Y \arrow[dr, equal] \arrow[drr, equal, bend left] \arrow[ddr, bend right] & & \\
& X \times Y \arrow[r, equal] \arrow[d] & X \times Y \arrow[d] \\
& 1 \arrow[r, equal] & 1
\end{tikzcd}$$

we conclude that $\hat{\varepsilon}_{p\bullet 1} \circ \hat{\eta}_p \bullet 1 = \mathsf{id}_{p\bullet 1}$. Moreover, by considering the following diagram

$$\begin{tikzcd}
\mathcal{V}(1,s)(v,w) \arrow[r] \arrow[dr, bend right] & \mathcal{V}(1,\mathcal{V}(1,s)\cdot 1)(\hat{\eta}v,\hat{\eta}w) \arrow[r, "\mathcal{V}(1,\hat{\varepsilon}_s)_{\hat{\eta}v,\hat{\eta}w}"] \arrow[d] & \mathcal{V}(1,s)(v,w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, "v,w"] \\
& M_{\mathcal{V}(1,s)\bullet 1} \arrow[r, "\hat{\varepsilon}_s"] & M_s \arrow[r, "l_s{,}r_s"] & V \times W
\end{tikzcd}$$

we conclude $\mathcal{V}(1,\hat{\varepsilon}_s) \circ \eta_{\mathcal{V}(1,s)} = \mathsf{id}_{\mathcal{V}(1,s)}$. Hence, we have confirmed that

Proposition 2.1. *We have an adjunction* $- \cdot 1 \dashv \mathcal{V}(1,-)\colon \mathsf{Span}(\mathcal{V}) \to \mathcal{V}\text{-}\mathsf{Mat}$ *in* $\mathsf{Grph}(\mathsf{Cat})$.

Coherence for $- \bullet 1$: Now, we check that $- \bullet 1$ is a normal oplax functor: for a set X, consider the pullback diagram

(2.6)
$$\begin{tikzcd}
{[x = y]} \arrow[r] \arrow[d, "\delta_{x,y}"'] & 1 \arrow[d, "x{,}\, y"] \\
X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

so that $\delta_{x,x} = x$ and $\delta_{x,y}$ is uniquely determined when $x \neq y$. Now, we consider the image of the diagram (2.6) under $- \cdot 1$, and take its coproduct indexed by:

(2.7)
$$\begin{tikzcd}
X \times X \arrow[r, equal] \arrow[d] & X \times X \arrow[d] \\
1 \arrow[r] & 1
\end{tikzcd}$$

This yields us $\mathsf{e}_X^{-\bullet 1}$; since $[x = y] \bullet 1 \cong 0$ for $x \neq y$, we conclude that this 2-cell is invertible.

Moreover, given a function $f\colon X \to Y$, we observe that the following diagram

$$\begin{tikzcd}
{[x = y] \bullet 1} \arrow[r, "1_{f,x,y}"] \arrow[d] & {[fx = fy] \bullet 1} \arrow[d] \\
X \cdot 1 \arrow[r, "f \cdot 1"] & Y \cdot 1
\end{tikzcd}$$

commutes, as it is the image via $- \bullet 1$ of a commutative diagram in Set. Taking the coproduct over (2.7) confirms naturality of $\mathsf{e}^{-\bullet 1}$.

For $\mathcal{V}$-matrices $p\colon X \nrightarrow Y$, $q\colon Y \nrightarrow Z$, $\mathsf{m}^{-\bullet 1}$ is depicted in Diagram (2.8) below by a dashed arrow, and is uniquely determined by the universal property of the pullback square:

(2.8)
$$\begin{tikzcd}
M_{(p\cdot q)\bullet 1} \arrow[dr, dashed, "\mathsf{m}^{-\bullet 1}"] \arrow[drr, bend left, "\hat{\pi}_0"] \arrow[ddr, bend right, "\hat{\pi}_1"'] & & \\
& M_{(p\bullet 1)\cdot(q\bullet 1)} \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & M_{q\bullet 1} \arrow[d, "l_{q\bullet 1}"] \\
& M_{p\bullet 1} \arrow[r, "r_{p\bullet 1}"'] & Y \bullet 1
\end{tikzcd}$$

We consider the following horizontally composable 2-cells of $\mathcal{V}$-Mat:

$$\begin{tikzcd}
U \arrow[r, "p_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\zeta_0"] & V \arrow[r, "p_1"] \arrow[d, dashed, "g" description] \arrow[dr, phantom, "\zeta_1"] & W \arrow[d, "h"] \\
X \arrow[r, "q_0"'] & Y \arrow[r, "q_1"'] & Z
\end{tikzcd}$$

We have

$$\begin{aligned}\pi_j \circ \mathsf{m}^{-\cdot 1}_{q_0,q_1} \circ ((\zeta_1 \cdot \zeta_0) \cdot 1) &= (\pi_j \cdot 1) \circ ((\zeta_1 \cdot \zeta_0) \cdot 1)\\ &= (\zeta_j \cdot 1) \circ (\pi_j \cdot 1)\\ &= (\zeta_j \cdot 1) \circ \pi_j \circ \mathsf{m}^{-\cdot 1}_{p_0,p_1}\\ &= \pi_j \circ ((\zeta_1 \cdot 1) \cdot (\zeta_0 \cdot 1)) \circ \mathsf{m}^{-\cdot 1}_{p_0,p_1}\end{aligned}$$

for $j = 0, 1$. We obtain naturality via the universal property of the pullback $M_{(q_1 \cdot 1)\cdot(q_0 \cdot 1)}$.

To verify the unit comparison coherence of $- \cdot 1$, we let $p\colon X \nrightarrow Y$ be a span in $\mathcal{V}$, and we consider the composite

$$\textstyle\sum_{x,y,z} 1_Y(y,z) \times p(x,y) \xrightarrow{\mathsf{m}^{-\cdot 1}} M_{(1_Y \cdot 1)\cdot(p \cdot 1)} \xrightarrow{\mathsf{e}^{-\cdot 1} \cdot \mathsf{id}} M_{1_{Y \cdot 1} \cdot (p \cdot 1)} \xrightarrow{\lambda} \sum_{x,y} p(x,y).$$

By definition, $\lambda\colon M_{1_{Y\cdot 1}\cdot(p\cdot 1)} \to M_{p \cdot 1}$ is simply the pullback projection, thus $\lambda \circ (\mathsf{e}^{-\cdot 1} \cdot \mathsf{id}) = \pi_1$ is the pullback projection $M_{(1_Y \cdot 1)\cdot(p\cdot 1)} \to M_{p \cdot 1}$, and therefore $\lambda \circ (\mathsf{e}^{-\cdot 1} \cdot \mathsf{id}) \circ \mathsf{m}^{-\cdot 1} = \pi_1 \circ \mathsf{m}^{-\cdot 1} = \hat{\pi}_1$ by (2.8). But $\hat{\pi}_1$ itself is the coproduct of $1_Y(y,z) \times p(x,y) \to p(x,y)$ indexed by the projection $X \times Y \times Z \to Y \times Z$, which is just $\lambda \cdot 1$. A similar argument confirms the right unitor case.

Now, we're left with verifying the composition comparison coherence of $- \cdot 1$. For the remainder of the section, we will denote horizontal composition simply by concatenation. For an easier understanding of the calculations, we provide the following diagram:

$$\begin{tikzcd}
((ts)r)\cdot 1 \arrow[r] \arrow[dddd] & ((ts)\cdot 1)(r\cdot 1) \arrow[r] \arrow[d] & ((t\cdot 1)(s\cdot 1))(r\cdot 1) \arrow[rr] \arrow[dddd] \arrow[dddr] & & r\cdot 1 \arrow[dr] & \\
& (ts)\cdot 1 \arrow[ddrr] & & (s\cdot 1)(r\cdot 1) \arrow[ur] \arrow[dr] & & X\cdot 1 \\
& & & & s\cdot 1 \arrow[ur] \arrow[dr] & \\
& (sr)\cdot 1 \arrow[uurr] & & (t\cdot 1)(s\cdot 1) \arrow[ur] \arrow[dr] & & Y\cdot 1 \\
(t(sr))\cdot 1 \arrow[r] & (t\cdot 1)((sr)\cdot 1) \arrow[r] \arrow[u] & (t\cdot 1)((s\cdot 1)(r\cdot 1)) \arrow[rr] \arrow[uuur] & & t\cdot 1 \arrow[ur] &
\end{tikzcd}$$

First, we verify that $\mathsf{m}^{-\cdot 1} \circ (\pi_1 \cdot 1) \circ (\alpha \cdot 1) = \pi_1 \circ \alpha \circ (\mathsf{m}^{-\cdot 1} \cdot \mathsf{id}) \circ \mathsf{m}^{-\cdot 1}$ as 2-cells $((ts)r)\cdot 1 \to (s \cdot 1)(r \cdot 1)$. We have

$$\begin{aligned}\pi_0 \circ \mathsf{m}^{-\cdot 1} \circ \hat{\pi}_1 \circ (\alpha \cdot 1) &= \hat{\pi}_0 \circ \hat{\pi}_1 \circ (\alpha \cdot 1)\\ &= \hat{\pi}_1 \circ \hat{\pi}_0\\ &= \pi_1 \circ \mathsf{m}^{-\cdot 1} \circ \pi_0 \circ \mathsf{m}^{-\cdot 1}\\ &= \pi_1 \circ \pi_0 \circ (\mathsf{m}^{-\cdot 1} \cdot \mathsf{id}) \circ \mathsf{m}^{-\cdot 1}\\ &= \pi_0 \circ \pi_1 \circ \alpha \circ (\mathsf{m}^{-\cdot 1} \cdot \mathsf{id}) \circ \mathsf{m}^{-\cdot 1}\\ \pi_1 \circ \mathsf{m}^{-\cdot 1} \circ \hat{\pi}_1 \circ (\alpha \cdot 1) &= \hat{\pi}_1 \circ \hat{\pi}_1 \circ (\alpha \cdot 1)\\ &= \hat{\pi}_1\\ &= \pi_1 \circ \mathsf{m}^{-\cdot 1}\\ &= \pi_1 \circ (\mathsf{m}^{-\cdot 1} \cdot \mathsf{id}) \circ \mathsf{m}^{-\cdot 1}\\ &= \pi_1 \circ \pi_1 \circ \alpha \circ (\mathsf{m}^{-\cdot 1} \cdot \mathsf{id}) \circ \mathsf{m}^{-\cdot 1},\end{aligned}$$

and then we apply the universal property of the pullback $M_{(s\bullet 1)(r\bullet 1)}$. With this, we finish our proof: note that

$$\begin{aligned}
\pi_0 \circ \alpha \circ (\mathsf{m}^{-\bullet 1} \cdot \mathsf{id}) \circ \mathsf{m}^{-\bullet 1} &= \pi_0 \circ \pi_0 \circ (\mathsf{m}^{-\bullet 1} \cdot \mathsf{id}) \circ \mathsf{m}^{-\bullet 1} \\
&= \pi_0 \circ \mathsf{m}^{-\bullet 1} \circ \pi_0 \circ \mathsf{m}^{-\bullet 1} \\
&= (\pi_0 \bullet 1) \circ (\pi_0 \bullet 1) \\
&= (\pi_0 \bullet 1) \circ (\alpha \bullet 1) \\
&= \pi_0 \circ \mathsf{m}^{-\bullet 1} \circ (\alpha \bullet 1) \\
&= \pi_0 \circ (\mathsf{id} \cdot \mathsf{m}^{-\bullet 1}) \circ \mathsf{m}^{-\bullet 1} \circ (\alpha \bullet 1) \\
\pi_1 \circ (\mathsf{id} \cdot \mathsf{m}^{-\bullet 1}) \circ \mathsf{m}^{-\bullet 1} \circ (\alpha \bullet 1) &= \mathsf{m}^{-\bullet 1} \circ \pi_1 \circ \mathsf{m}^{-\bullet 1} \circ (\alpha \bullet 1) \\
&= \mathsf{m}^{-\bullet 1} \circ (\pi_1 \bullet 1) \circ (\alpha \bullet 1) \\
&= \pi_1 \circ \alpha \circ (\mathsf{m}^{-\bullet 1} \cdot \mathsf{id}) \circ \mathsf{m}^{-\bullet 1}
\end{aligned}$$

then we apply the universal property of $M_{(t\bullet 1)((s\bullet 1)(r\bullet 1))}$. This concludes the proof of

Proposition 2.2. $- \bullet 1 \colon \mathcal{V}\text{-}\mathsf{Mat} \to \mathsf{Span}(\mathcal{V})$ *is a normal oplax functor.*

3. Conjoints and companions

As introduced in [19], and studied in [42, 15, 16, 41], there exist two notions of "adjunction" between vertical and horizontal 1-cells in a pseudodouble category; these were introduced as *orthogonal adjunctions.*

To be precise, let $\mathbb{D}$ be a pseudodouble category, and let $f \colon a \to b$ be a vertical 1-cell, $r \colon b \nrightarrow a$ be a horizontal 1-cell. Following the terminology from [42, 41], we say that r is the *conjoint* of f if there exist 2-cells

$$\begin{tikzcd}
a \arrow[r, "1"] \arrow[d, "f"'] \arrow[dr, phantom, "\eta"] & a \arrow[d, equal] & b \arrow[r, "r"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & a \arrow[d, "f"] \\
b \arrow[r, "r"'] & a & b \arrow[r, "1"'] & b
\end{tikzcd}$$

such that $\varepsilon \circ \eta = 1_f$ and $\eta \cdot \varepsilon = \rho^{-1} \circ \lambda$. We say η, ε are the unit, counit of the conjoint, respectively. Also denote by *companion* the horizontally dual notion of conjoint; we denote the unit and counit 2-cells of a companion as $\nu \colon 1 \to r$ and $\delta \colon r \to 1$, respectively.

In any pseudodouble category $\mathbb{D}$, the identity vertical 1-cell on any 0-cell x always has both a companion and a conjoint; in both cases, it is given by the horizontal unit 1_x, with unit and counit given by $\mathsf{id}_{1_x} = 1_{\mathsf{id}_x}$, which trivially satisfies all four conditions. Unless otherwise specified, 1_x will be our fixed choice of companion/conjoint for id_x.

We say that $\mathbb{D}$ is *conjoint (companion) closed* if every vertical 1-cell of $\mathbb{D}$ has a conjoint (companion). For instance, *equipments* may be defined as the pseudodouble categories which are both conjoint and companion closed (see [42, Theorem A.2]), of which $\mathsf{Span}(\mathcal{V})$ and $\mathcal{V}\text{-}\mathsf{Mat}$ are examples.

Let T be a pseudomonad on a 2-category $\mathbb{B}$, and consider the double category of lax T-algebras as described in Section 1. Our next result, Proposition 3.1, given in [36, Theorems 1.4.11 and 1.4.14], originally stated for strict T-algebras in [28], may be used to characterize conjoints and companions in $\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$. Since this is just a restatement of the results of [36, Chapter 1], we omit the argument.

Proposition 3.1 (Doctrinal adjunction)**.** *Let $(f, g, \eta, \varepsilon)$ be an adjunction in a 2-category $\mathbb{B}$. There is a bijection between 2-cells ζ making (f, ζ) into an lax T-algebra oplax morphism and 2-cells ξ making (g, ξ) into a lax T-algebra lax morphism.*

Moreover, (f, ζ) is the conjoint of (g, ξ) in $\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$ if and only if ζ and ξ correspond to each other via the aforementioned bijection, and f has a companion if and only if ζ is invertible; in which case, its companion is (f, ζ^{-1}).

As is the case with ordinary adjunctions in a 2-category, there is also a notion of *mate theory* for conjoints (and dually, companions), which we present in Lemma 3.2. Results along these lines were already present in [19, 1.6], as well as [15, Corollary 7.21] and [41, Propositions 5.13 and 5.19]. We have decided to provide a slightly different statement and proof: our goal is to provide explicit formulas as an aid for calculations involving conjoints and companions, abundant in this work.

Lemma 3.2. *Let $(f, f^*, \eta, \varepsilon)$ and $(g, g^*, \eta, \varepsilon)$ be conjoints, and consider 2-cells*

$$\begin{tikzcd} u \arrow[r, "r"] \arrow[d, "k\circ f"'] \arrow[dr, phantom, "\zeta"] & x \arrow[d, "g\circ h"] \\ w \arrow[r, "s"'] & z \end{tikzcd} \qquad \begin{tikzcd} v \arrow[r, "r\cdot f^*"] \arrow[d, "k"'] \arrow[dr, phantom, "\xi"] & x \arrow[d, "h"] \\ w \arrow[r, "g^*\cdot s"'] & y \end{tikzcd} \tag{3.1}$$

Then the following are equivalent:

(a) $\xi = \rho \circ \Big(\big((\eta \circ 1_h) \cdot \zeta \big) \cdot (1_k \circ \varepsilon) \Big) \circ \alpha^{-1} \circ \lambda^{-1}$,

(b) $\zeta = \lambda \circ (\varepsilon \cdot \mathsf{id}_s) \circ \xi \circ (\mathsf{id}_r \cdot \eta) \circ \rho^{-1}$,

(c) $\lambda \circ (\varepsilon \cdot \mathsf{id}_s) \circ \xi = \rho \circ \big(\zeta \cdot (1_k \circ \varepsilon) \big)$,

(d) $\xi \circ (\mathsf{id}_r \cdot \eta) \circ \rho^{-1} = \big((\eta \circ 1_h) \cdot \zeta \big) \circ \lambda^{-1}$

In particular, the sets of 2-cells as given in (3.1) *are in pairwise correspondence, explicitly given by the formulas* (a) *and* (b)*. Pairs of such 2-cells are said to be mates or under mate correspondence.*

Proof. We will prove that (c) $\to_{\mathrm{i}}$ (b) $\to_{\mathrm{ii}}$ (d) $\to_{\mathrm{iii}}$ (a) $\to_{\mathrm{iv}}$ (c).

(i) Since $\varepsilon \circ \eta = 1_f$, we have

$$\begin{aligned} \lambda \circ (\varepsilon \cdot \mathsf{id}_q) \circ \xi \circ (\mathsf{id}_p \cdot \eta) \circ \rho^{-1} &= \rho \circ \big(\zeta \cdot (1 \circ \varepsilon) \big) \circ (\mathsf{id}_p \cdot \eta) \circ \rho^{-1} \\ &= \rho \circ \big((\zeta \circ \mathsf{id}_p) \cdot (1 \circ \varepsilon \circ \eta) \big) \circ \rho^{-1} = \zeta \end{aligned}$$

(ii) Since $\eta \cdot \varepsilon = \rho^{-1} \circ \lambda$, we have

$$\begin{aligned} \big((\eta \circ 1) \cdot \zeta \big) \circ \lambda^{-1} &= \Big((\eta \circ 1) \cdot \big(\lambda \circ (\varepsilon \cdot \mathsf{id}_q) \circ \xi \circ (\mathsf{id}_q \cdot \eta) \circ \rho^{-1}) \big) \Big) \circ \lambda^{-1} \\ &= (\mathsf{id}_s \cdot \lambda) \circ (\eta \cdot (\varepsilon \cdot \mathsf{id}_q)) \circ \big(1 \cdot (\xi \circ (\mathsf{id}_p \cdot \eta) \circ \rho^{-1}) \big) \circ \lambda^{-1} \\ &= (\mathsf{id}_s \cdot \lambda) \circ \alpha \circ (\rho^{-1} \cdot \mathsf{id}_q) \circ (\lambda \cdot \mathsf{id}) \circ \alpha^{-1} \circ \lambda^{-1} \circ \xi \circ (\mathsf{id}_p \cdot \eta) \circ \rho^{-1} \end{aligned}$$

and coherence guarantees

$$(\mathsf{id} \cdot \lambda) \circ \alpha \circ (\rho^{-1} \cdot \mathsf{id}) \circ (\lambda \cdot \mathsf{id}) \circ \alpha^{-1} \circ \lambda^{-1} = \mathsf{id},$$

as desired.

(iii) Since $\eta \cdot \varepsilon = \rho^{-1} \circ \lambda$, we have

$$\begin{aligned} \big((\eta \circ 1) \cdot \zeta \big) \cdot (1 \circ \varepsilon) &= \big(\xi \circ (\mathsf{id}_p \cdot \eta) \circ \rho^{-1} \circ \lambda \big) \cdot \varepsilon \\ &= (\xi \cdot 1) \circ ((\mathsf{id}_p \cdot \eta) \cdot \varepsilon) \circ ((\rho^{-1} \circ \lambda) \cdot \mathsf{id}_g) \\ &= (\xi \cdot 1) \circ \alpha^{-1} \circ (\mathsf{id}_p \cdot (\rho^{-1} \circ \lambda)) \circ \alpha \circ ((\rho^{-1} \circ \lambda) \cdot \mathsf{id}_{g^*}) \end{aligned}$$

and coherence guarantees

$$\rho \circ \alpha^{-1} \circ (\mathsf{id}_p \cdot (\rho^{-1} \circ \lambda)) \circ \alpha \circ ((\rho^{-1} \circ \lambda) \cdot \mathsf{id}_{g^*}) \circ \alpha^{-1} \circ \lambda^{-1} = \mathsf{id},$$

as desired.

(iv) Since $\varepsilon \circ \eta = 1_g$, we have

$$\begin{aligned} (\varepsilon \cdot \mathsf{id}_q) \circ \xi &= (\varepsilon \cdot \mathsf{id}_q) \circ \rho \circ \big(((\eta \circ 1) \cdot \zeta) \cdot (1 \circ \varepsilon) \big) \circ \alpha^{-1} \circ \lambda^{-1} \\ &= \rho \circ ((\varepsilon \cdot \mathsf{id}_q) \cdot \mathsf{id}_1) \circ (((\eta \circ 1) \cdot \zeta) \cdot (1 \circ \varepsilon)) \circ \alpha^{-1} \circ \lambda^{-1} \\ &= \rho \circ ((1 \cdot \zeta) \cdot \varepsilon) \circ \alpha^{-1} \circ \lambda^{-1} \\ &= \rho \circ \alpha^{-1} \circ \lambda^{-1} \circ (\zeta \cdot \varepsilon) \end{aligned}$$

and coherence guarantees

$$\rho \circ \alpha^{-1} \circ \lambda^{-1} = \lambda^{-1} \circ \rho,$$

as desired.

□

Remark 3.3. Once more, we consider the pair of 2-cells ζ, ξ given in (3.1). We will consider the following specialized instances of the mate correspondence:

(i) For $k = \mathsf{id}$, (c) becomes $(\varepsilon \cdot \mathsf{id}) \circ \xi = \gamma^{-1} \circ (\zeta \cdot \varepsilon)$.

(ii) For $h = \mathsf{id}$, (d) becomes $\xi \circ (\mathsf{id} \cdot \eta) = (\eta \cdot \zeta) \circ \gamma^{-1}$.
(iii) For $s = 1$, (c) becomes $\varepsilon \circ \theta = \rho \circ (\zeta \cdot (1_k \circ \varepsilon))$, where $\theta = \rho \circ \xi$.
(iv) For $r = 1$, (d) becomes $\theta \circ \eta = ((\eta \circ 1_h) \cdot \zeta) \circ \lambda^{-1}$, where $\theta = \xi \circ \lambda^{-1}$.
(v) For $f = \mathsf{id}$, both (b) and (c) become $\zeta = \lambda \circ (\varepsilon \cdot \mathsf{id}) \circ \theta$, where $\theta = \xi \circ \rho^{-1}$.
(vi) For $g = \mathsf{id}$, both (b) and (d) become $\zeta = \theta \circ (\mathsf{id} \cdot \eta) \circ \rho^{-1}$, where $\theta = \lambda \circ \xi$.

And by combining these, we may obtain simpler forms. For example, (v) and (iii) (respectively, (vi) and (iv))) provide the result that the counit (unit) of a conjunction is a cartesian (opcartesian) 2-cell in the sense of [42, 15].

The combination of (iii) and (iv) is mainly used under the hypothesis that we have a commutative square $k \circ f = g \circ h$ of vertical 1-cells, that is, $\zeta = \mathsf{id}$. In this case, the unit $1_{g \circ h} = 1_{k \circ f}$ has two mates; they are said to be the *mates of the commutative square* $k \circ f = g \circ h$, and are the unique 2-cells θ, ω, respectively satisfying

$$\varepsilon \circ \theta = 1_k \circ \varepsilon \quad \text{and} \quad \theta \circ \eta = \eta \circ 1_h$$

$$\varepsilon \circ \omega = 1_g \circ \varepsilon \quad \text{and} \quad \omega \circ \eta = \eta \circ 1_f$$

In practice, we will consider "the" mate of a commutative square $k \circ f = g \circ h$, and we let context determine which mate is being considered.

We proceed to review well-known [15, 19, 16, 42, 41], yet fundamental results about companions and conjoints. Our aim is to demonstrate the applications of their mate theory, while fixing notation to use for later reference in Sections 5 and 6.

Let $F \colon \mathbb{D} \to \mathbb{E}$ be a lax functor of conjoint closed pseudodouble categories, and let f is a vertical 1-cell in $\mathbb{D}$. We denote the mate of $F\eta \circ \mathsf{e}^F$ obtained via (vi) by $\sigma^F_f \colon (Ff)^* \to F(f^*)$:

$$\begin{tikzcd}
\cdot \arrow[r, "1"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\eta"] & \cdot \arrow[d, equal] & & \cdot \arrow[r, "1"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{e}^F"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "(Ff)^*" description] \arrow[d, equal] \arrow[dr, phantom, "\sigma^F_f"] & \cdot \arrow[d, equal] & = & \cdot \arrow[r, "F1" description] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\eta"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "F(f^*)"] & \cdot & & \cdot \arrow[r, "F(f^*)"'] & \cdot
\end{tikzcd}$$

We say that F *preserves the conjoint* of f if σ^F_f is an invertible 2-cell; we say F *preserves conjoints* if σ^F_f is invertible for all vertical 1-cells f. We can show that:

Lemma 3.4. *Let $F \colon \mathbb{D} \to \mathbb{E}$ be a lax functor of conjoint closed pseudodouble categories. The following are equivalent:*

(a) F preserves conjoints of identities,
(b) F preserves all conjoints,
(c) F is normal.

Proof. We begin by showing that any lax functor F satisfies the identity $F\varepsilon \circ \sigma^F = \mathsf{e}^F \circ \varepsilon$, for we have

$$F\varepsilon \circ \sigma^F \circ \eta = F\varepsilon \circ F\eta \circ \mathsf{e}^F = F1_f \circ \mathsf{e}^F = \mathsf{e}^F \circ 1_{Ff} = \mathsf{e}^F \circ \varepsilon \circ \eta,$$

so the desired equation follows by (vi).

Moreover, whenever σ^F_f is invertible, the following relations hold:

$$\varepsilon \circ (\sigma^F)^{-1} \circ F\eta \circ \mathsf{e}^F = \varepsilon \circ \eta = 1_f,$$
$$\mathsf{e}^F \circ \varepsilon \circ (\sigma^F)^{-1} \circ F\eta = F\varepsilon \circ F\eta = F1_f.$$

Hence, if σ^F_{id} is invertible for all 0-cells, we conclude that e^F is invertible; this confirms (a) → (c).

Now, if we assume F is normal, we let χ^F be the unique 2-cell such that $\varepsilon \circ \chi^F = (\mathsf{e}^F)^{-1} \circ F\varepsilon$, obtained via (v). From this, it is clear that $\chi^F \circ \sigma^F = \mathsf{id}$, since

$$\varepsilon \circ \chi^F \circ \sigma^F = (\mathsf{e}^F)^{-1} \circ F\varepsilon \circ \sigma^F = \varepsilon,$$

and

$$\begin{aligned}
\rho^{-1}\circ\sigma^F\circ\chi^F\circ\lambda &= (\sigma^F\cdot\mathsf{id})\circ\rho^{-1}\circ\lambda\circ(\mathsf{id}\cdot\chi^F)\\
&= (\sigma^F\cdot\mathsf{id})\circ(\eta\cdot\varepsilon)\circ(\mathsf{id}\cdot\chi^F)\\
&= (\sigma^F\circ\eta)\cdot(\varepsilon\circ\chi^F)\\
&= (F\eta\circ\mathsf{e}^F)\cdot((\mathsf{e}^F)^{-1}\circ F\varepsilon)\\
&= (\mathsf{id}\cdot(\mathsf{e}^F)^{-1})\circ(F\eta\cdot F\varepsilon)\circ(\mathsf{e}^F\cdot\mathsf{id})\\
&= \rho^{-1}\circ F\rho\circ\mathsf{m}^F\circ(F\eta\cdot F\varepsilon)\circ(\mathsf{e}^F\cdot\mathsf{id})\\
&= \rho^{-1}\circ F\rho\circ F(\eta\cdot\varepsilon)\circ\mathsf{m}^F\circ(\mathsf{e}^F\cdot\mathsf{id})\\
&= \rho^{-1}\circ\lambda
\end{aligned}$$

confirms that χ^F is the inverse of σ^F. We have shown that (c) → (b), and of course, (a) is a particular case of (b). ☐

For the case of companions, we write $\tau^F_f\colon (Ff)_! \to F(f_!)$ for the mate of $F\nu\circ\mathsf{e}^F$, and we say that F *preserves the companion* of f if τ^F is invertible. The horizontally dual result states that F preserves companions iff F is normal. Thus, we obtain the result that these three notions are equivalent for lax functors between pseudodouble categories ([16, Proposition 3.8]).

Lemma 3.5. *Let $F\colon \mathbb{D}\to\mathbb{E}$ be a lax functor between conjoint closed pseudodouble categories, and let $r\colon x \twoheadrightarrow y$, $f\colon z\to y$ be horizontal, vertical 1-cells respectively. Then the 2-cell*

$$(Ff)^*\cdot Fr \xrightarrow{\sigma^F\cdot\mathsf{id}} F(f^*)\cdot Fr \xrightarrow{\mathsf{m}^F} F(f^*\cdot r)$$

is invertible. In particular, $\mathsf{m}^F\colon F(f^*)\cdot Fr\to F(f^*\cdot r)$ is invertible for all such r, f if and only if F is normal.

Proof. We claim the inverse l^F is given by the mate of $F\theta$ via

$$l^F = \begin{tikzcd}
\cdot \arrow[rr, "F(f^*\cdot r)"] \arrow[d, equal] \arrow[drr, phantom, "\lambda^{-1}"] & & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "F(f^*\cdot r)" description] \arrow[d, equal] \arrow[dr, phantom, "F\theta"] & \cdot \arrow[r, "1" description] \arrow[d, "Ff"] \arrow[dr, phantom, "\eta"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "Fr"'] & \cdot \arrow[r, "(Ff)^*"'] & \cdot
\end{tikzcd}, \quad \text{where} \quad \theta = \begin{tikzcd}
\cdot \arrow[r, "r"] \arrow[d, equal] \arrow[dr, phantom, "="] & \cdot \arrow[r, "f^*"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "r" description] \arrow[d, equal] \arrow[drr, phantom, "\lambda"] & \cdot \arrow[r, "1" description] & \cdot \arrow[d, equal] \\
\cdot \arrow[rr, "r"'] & & \cdot
\end{tikzcd}.$$

Note that l^F is the mate of $F\theta$, and θ is the mate of $\mathsf{id}_{f^*\cdot r}$, via (iv) and (v), respectively. Now, note that

$$\begin{aligned}
(\varepsilon\cdot\mathsf{id})\circ l^F\circ\mathsf{m}^F\circ(\sigma^F\cdot\mathsf{id}) &= \lambda^{-1}\circ F\theta\circ\mathsf{m}^F\circ(\sigma^F\cdot\mathsf{id})\\
&= \lambda^{-1}\circ F\lambda\circ\mathsf{m}^F\circ(F\varepsilon\cdot\mathsf{id})\circ(\sigma^F\cdot\mathsf{id})\\
&= \lambda^{-1}\circ F\lambda\circ\mathsf{m}^F\circ(\mathsf{e}^F\cdot\mathsf{id})\circ(\varepsilon\cdot\mathsf{id}) = \varepsilon\cdot\mathsf{id}\\
\mathsf{m}^F\circ(\sigma^F\cdot\mathsf{id})\circ l^F &= \mathsf{m}^F\circ(F\eta\cdot F\theta)\circ(\mathsf{e}^F\cdot\mathsf{id})\circ\lambda^{-1}\\
&= F(\eta\cdot\theta)\circ\mathsf{m}^F\circ(\mathsf{e}^F\cdot\mathsf{id})\circ\lambda^{-1} = \mathsf{id}
\end{aligned}$$

So, the result follows by the mate correspondence. ☐

In a conjoint closed pseudodouble category $\mathbb{D}$, let f, g be composable vertical 1-cells with conjoints f^* and g^*, and let $\pi\colon f^*\cdot g^*\to(g\circ f)^*$ be the mate of $1_g\circ\varepsilon\colon f^*\to 1$. Via (i) and (iii), we obtain:

$$\begin{tikzcd}
\cdot \arrow[r, "f^*\cdot g^*"] \arrow[d, equal] \arrow[dr, phantom, "\pi"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "(g\circ f)^*" description] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \cdot \arrow[d, "g\circ f"] \\
\cdot \arrow[r, "1"'] & \cdot
\end{tikzcd} = \begin{tikzcd}
\cdot \arrow[r, "g^*"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \cdot \arrow[r, "f^*"] \arrow[d, "g"] \arrow[dr, phantom, "1_g\circ\varepsilon"] & \cdot \arrow[d, "g\circ f"] \\
\cdot \arrow[r, "1" description] \arrow[d, equal] \arrow[drr, phantom, "\rho"] & \cdot \arrow[r, "1" description] & \cdot \arrow[d, equal] \\
\cdot \arrow[rr, "1"'] & & \cdot
\end{tikzcd} \tag{3.2}$$

We can also define a 2-cell $(g \circ f)^* \to f^* \cdot g^*$ as the mate of $\eta \circ 1_f$, which can be shown to be the inverse of π, using a method similar to the proof of Lemma 3.5; as this is not needed, we omit the details.

Now that we have fixed the notation we will need for the rest of the paper, we end the section with Theorem 3.6, to justify the utility of conjoints and companions.

Theorem 3.6. *Let $\mathbb{D}, \mathbb{E}$ be pseudodouble categories. If $\mathbb{E}$ is conjoint closed, then so is $\mathsf{Lax}_{lax}(\mathbb{D}, \mathbb{E})$. Dually, if $\mathbb{E}$ is companion closed, then so is $\mathsf{Lax}_{opl}(\mathbb{D}, \mathbb{E})$.*

Proof. Fix a vertical transformation $\phi \colon F \to G$ where $F, G \colon \mathbb{D} \to \mathbb{E}$ are lax functors. For each 0-cell x, we write

$$\begin{tikzcd}
Fx \arrow[r, "1"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\eta_x"] & Fx \arrow[d, equal] & Gx \arrow[r, "\phi_x^*"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon_x"] & Fx \arrow[d, "\phi_x"] \\
Gx \arrow[r, "\phi_x^*"'] & Fx & Gx \arrow[r, "1"'] & Gx
\end{tikzcd}$$

for the 2-cells satisfying $\varepsilon_x \circ \eta_x = 1_{\phi_x}$ and $\eta_x \cdot \varepsilon_x = \rho^{-1} \circ \lambda$, so that ϕ_x^* is the conjoint of ϕ_x for all x.

Define $\phi_f^* \colon \phi_x^* \to \phi_y^*$ to be the mate of 1_{Ff} via (ii), so that $\phi_f^* \circ \eta_x = \eta_y \circ 1_{Ff}$ and $\varepsilon_y \circ \phi_f^* = 1_{Gf} \circ \varepsilon_x$. Moreover, note that

$$\phi_g^* \circ \phi_f^* \circ \eta_x = \phi_g^* \circ \eta_y \circ 1_{Ff} = \eta_z \circ 1_{Gf} \circ 1_{Ff} = \eta_z \circ 1_{G(g \circ f)},$$

so we conclude that $\phi_{g \circ f}^* = \phi_g^* \circ \phi_f^*$ by mate correspondence. Similarly, we have $\phi_{\mathsf{id}_x}^* = \mathsf{id}_{\phi_x}^*$.

Next, we consider the map $r \mapsto Fr \cdot \phi_x^*$, where $r \colon x \nrightarrow y$ is a horizontal 1-cell. It is functorial: for 2-cells

$$\begin{tikzcd}
x \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\theta"] & y \arrow[d, "g"] & u \arrow[r, "q"] \arrow[d, "h"'] \arrow[dr, phantom, "\xi"] & v \arrow[d, "k"] \\
w \arrow[r, "s"'] & z & x \arrow[r, "r"'] & y
\end{tikzcd} \tag{3.3}$$

we have

$$F(\theta \circ \chi) \cdot \phi_{h \circ f}^* = (F\theta \circ F\chi) \cdot (\phi_h^* \circ \phi_f^*) = (F\theta \cdot \phi_h^*) \circ (F\chi \cdot \phi_f^*),$$

and $F(\mathsf{id}) \cdot \phi_{\mathsf{id}}^* = \mathsf{id}$, as desired. Analogously, $r \mapsto \phi_y^* \cdot Gr$ is also functorial.

We define $\mathsf{n}_r^{\phi^*} \colon Fr \cdot \phi_x^* \to \phi_y^* \cdot Gf$ to be the mate of

$$\begin{tikzcd}
Fx \arrow[r, "Fr"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_r"] & Fy \arrow[d, "\phi_y"] \\
Gx \arrow[r, "Gr"'] & Gy
\end{tikzcd}$$

via (i). We claim this data makes ϕ^* into a lax horizontal transformation $G \nrightarrow F$.

Given a 2-cell θ as in the left diagram of (3.3), we have $\phi_s \circ F\theta = G\theta \circ \phi_r$, since ϕ is a vertical transformation. The following pairs

$$G\theta \circ \phi_r \quad \text{and} \quad \mathsf{n}_s^{\phi^*} \circ (F\theta \cdot \phi_f^*),$$
$$\phi_s \circ F\theta \quad \text{and} \quad (\phi_g^* \cdot G\theta) \circ \mathsf{n}_r^{\phi^*}$$

are mates, so that we have

$$\mathsf{n}_s^{\phi^*} \circ (F\theta \cdot \phi_f^*) = (\phi_g^* \cdot G\theta) \circ \mathsf{n}_r^{\phi^*},$$

giving naturality. To confirm this,

$$\begin{aligned}
\lambda \circ (\varepsilon \cdot \mathsf{id}) \circ \mathsf{n}_s^{\phi^*} \circ (F\theta \cdot \phi_f^*) &= \rho \circ (\phi_s \cdot \varepsilon) \circ \circ (F\theta \cdot \phi_f^*) \\
&= \rho \circ \big((\phi_s \circ F\theta) \cdot (\varepsilon \circ \phi_f^*) \big) \\
&= \rho \circ \big((\phi_s \circ F\theta) \cdot (1 \circ \varepsilon) \big), \\
(\phi_g^* \cdot G\theta) \circ \mathsf{n}_r^{\phi^*} \circ (\mathsf{id} \cdot \eta) \circ \rho^{-1} &= (\phi_g^* \cdot G\theta) \circ (\eta \cdot \phi_r) \circ \lambda^{-1} \\
&= \big((\phi_g^* \circ \eta) \cdot (G\theta \circ \phi_r) \big) \circ \lambda^{-1} \\
&= \big((\eta \circ 1) \cdot (G\theta \circ \phi_r) \big) \circ \lambda^{-1}
\end{aligned}$$

Now, we note that $\phi_{1_x} \circ \mathsf{e}^F_x = \mathsf{e}^G_x \circ 1_{\phi_x}$ and $\phi_{s\cdot r} \circ \mathsf{m}^F = \mathsf{m}^G \circ (\phi_s \cdot \phi_r)$. We shall deduce that the coherence diagrams for ϕ^* commute by taking the mates of these commutative squares, thereby confirming that ϕ^* is a lax horizontal transformation. Via (i), we will prove that the following pairs

$$\begin{aligned}
\mathsf{m}^G \circ (\phi_s \cdot \phi_r) \quad &\text{and} \quad (\mathsf{id} \cdot \mathsf{m}^G) \circ \alpha \circ (\phi^*_s \cdot \mathsf{id}) \circ \alpha^{-1} \circ (\mathsf{id} \cdot \phi^*_r) \circ \alpha \\
\phi_{s\cdot r} \circ \mathsf{m}^F \quad &\text{and} \quad \mathsf{n}^{\phi^*}_{s\cdot r} \circ (\mathsf{m}^F \cdot \mathsf{id}) \\
\phi_{1_x} \circ \mathsf{e}^F \quad &\text{and} \quad \mathsf{n}^{\phi^*}_{1_x} \circ (\mathsf{e}^F \cdot \mathsf{id}) \\
\mathsf{e}^G \circ 1_{\phi_x} \quad &\text{and} \quad (\mathsf{id} \cdot \mathsf{e}^G) \circ \rho^{-1} \circ \lambda
\end{aligned}$$

are under mate correspondence. The last three are one-liners, respectively:

$$\begin{aligned}
\lambda \circ (\varepsilon \cdot \mathsf{id}) \circ \mathsf{n}^{\phi^*}_{s\cdot r} \circ (\mathsf{m}^F \cdot \mathsf{id}) &= \rho \circ (\phi_{s\cdot r} \cdot \varepsilon) \circ (\mathsf{m}^F \cdot \mathsf{id}) = \rho \circ ((\phi_{s\cdot r} \circ \mathsf{m}^F) \cdot \varepsilon), \\
\lambda \circ (\varepsilon \cdot \mathsf{id}) \circ \mathsf{n}^{\phi^*}_{1_x} \circ (\mathsf{e}^F \cdot \mathsf{id}) &= \rho \circ (\phi_{1_x} \cdot \varepsilon) \circ (\mathsf{e}^F \cdot \mathsf{id}) = \rho \circ ((\phi_{1_x} \circ \mathsf{e}^F) \cdot \mathsf{id}) \\
\lambda \circ (\varepsilon \cdot \mathsf{id}) \circ (\mathsf{id} \cdot \mathsf{e}^G) \circ \rho^{-1} \circ \lambda \circ (\mathsf{id} \cdot \eta) \circ \rho^{-1} &= \mathsf{e}^G \circ \lambda \circ (\varepsilon \cdot \mathsf{id}) \circ \rho^{-1} \circ \eta = \mathsf{e}^G \circ 1_{\phi_x}.
\end{aligned}$$

For the first pair, observe that

$$\begin{aligned}
\lambda \circ (\varepsilon \cdot \mathsf{id}) \circ (\mathsf{id} \cdot \mathsf{m}^G) \circ \alpha &= \lambda \circ (1 \cdot \mathsf{m}^G) \circ (\varepsilon \cdot \mathsf{id}) \circ \alpha \\
&= \mathsf{m}^G \circ \lambda \circ \alpha \circ ((\varepsilon \cdot \mathsf{id}) \cdot \mathsf{id}) \\
&= \mathsf{m}^G \circ (\lambda \cdot \mathsf{id}) \circ ((\varepsilon \cdot \mathsf{id}) \cdot \mathsf{id}) \\
((\varepsilon \cdot \mathsf{id}) \cdot \mathsf{id}) \circ (\mathsf{n}^{\phi^*}_s \cdot \mathsf{id}) \circ \alpha^{-1} &= (((\varepsilon \cdot \mathsf{id}) \circ \mathsf{n}^{\phi^*}_s) \cdot \mathsf{id}) \circ \alpha^{-1} \\
&= ((\gamma^{-1} \circ (\phi_s \cdot \varepsilon)) \cdot \mathsf{id}) \circ \alpha^{-1} \\
&= (\gamma^{-1} \cdot \mathsf{id}) \circ ((\phi_s \cdot \varepsilon) \cdot \mathsf{id}) \circ \alpha^{-1} \\
&= (\lambda^{-1} \cdot \lambda) \circ (\phi_s \cdot (\varepsilon \cdot \mathsf{id})) \\
(\phi_s \cdot (\varepsilon \cdot \mathsf{id})) \circ (\mathsf{id} \cdot \mathsf{n}^{\phi^*}_r) \circ \alpha &= (\phi_s \cdot ((\varepsilon \cdot \mathsf{id}) \circ \mathsf{n}^{\phi^*}_r)) \circ \alpha \\
&= (\phi_s \cdot (\gamma^{-1} \circ (\phi_r \cdot \varepsilon))) \circ \alpha \\
&= (\mathsf{id} \cdot \gamma^{-1}) \circ (\phi_s \cdot (\phi_r \cdot \varepsilon)) \circ \alpha \\
&= (\mathsf{id} \cdot \gamma^{-1}) \circ \alpha \circ ((\phi_s \cdot \phi_r) \cdot \varepsilon)
\end{aligned}$$

and pasting the expressions above together verifies the claim.

Finally, note that

$$\begin{aligned}
\mathsf{n}^{\phi^*}_r \circ (\mathsf{id}_{Fr} \cdot \eta_x) &= (\eta_y \cdot \phi_r) \circ \gamma^{-1} \\
(\varepsilon_y \cdot \mathsf{id}_{Gr}) \circ \mathsf{n}^{\phi^*}_r &= \gamma \circ (\phi_r \cdot \varepsilon_x)
\end{aligned}$$

are immediate consequences of mate correspondence. Thus, η and ε define modifications, and by calculating pointwise, we conclude that ϕ^* is the conjoint of ϕ. □

We say that a vertical transformation ϕ *has a strong conjoint (companion)* if its conjoint (companion) in the appropriate pseudodouble category is a strong horizontal transfomation; that is, if n^{ϕ^*} ($\mathsf{n}^{\phi_!}$) is an invertible natural transformation. The notion of a vertical transformation ϕ having a strong companion (conjoint) is present in [15, A.4]; therein, the terminology is *(co)horizontally strong.*

To provide a class of examples, recall from [15, A.6] that for a natural transformation $\phi\colon F \to G$ between pullback-preserving functors $F\colon \mathcal{B} \to \mathcal{C}$ on categories with pullbacks, the induced vertical transformation $\hat{\phi}\colon \hat{F} \to \hat{G}$ between the induced strong functors $\hat{F}, \hat{G}\colon \mathsf{Span}(\mathcal{B}) \to \mathsf{Span}(\mathcal{C})$ has a strong conjoint if and only if it has a strong companion, if and only if ϕ is a cartesian natural transformation.

We also have the following the result:

Lemma 3.7. *Let $\phi\colon F \to G$ be a vertical transformation of lax functors $F, G\colon \mathbb{D} \to \mathbb{E}$, let $H\colon \mathbb{E} \to \mathbb{F}$ be another lax functor. We assume $\mathbb{E}$ is conjoint closed, and that ϕ has a strong conjoint.*

$H\phi$ has a strong conjoint if and only if $\mathsf{m}^H \circ (\mathsf{id} \cdot \sigma^H)\colon HFr \cdot (H\phi_x)^* \to H(Fr \cdot \phi^*_x)$ is invertible for all x and all r.

Proof. We shall verify that

(3.4)

$$\begin{tikzcd}
\cdot \arrow[r, "(H\phi_x)_*"] \arrow[d, equal] \arrow[dr, phantom, shift left=6ex, "\mathsf{n}_r^{(H\phi)^*}"] & \cdot \arrow[r, "HFr"] & \cdot \arrow[d, equal] & & \cdot \arrow[r, "(H\phi_x)_*"] \arrow[d, equal] \arrow[dr, phantom, shift right=3ex, "\sigma^H"] & \cdot \arrow[r, "HFr"] \arrow[d, equal] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "HGr" description] \arrow[d, equal] & \cdot \arrow[r, "(H\phi_y)^*" description, "|" marker] \arrow[d, equal] \arrow[dr, phantom, "\sigma^H"] & \cdot \arrow[d, equal] & {} \arrow[d, phantom, "="] & \cdot \arrow[r, "H(\phi_x^*)" description, "|" marker] \arrow[d, equal] \arrow[dr, phantom, shift left=6ex, "\mathsf{m}^H"] & \cdot \arrow[r, "HFr"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "HGr" description] \arrow[d, equal] \arrow[dr, phantom, shift left=6ex, "\mathsf{m}^H"] & \cdot \arrow[r, "H(\phi_y^*)" description, "|" marker] & \cdot \arrow[d, equal] & {} & \cdot \arrow[rr, "H(Fr\cdot\phi_x^*)" description] \arrow[d, equal] \arrow[dr, phantom, shift left=6ex, "H\,\mathsf{n}_r^{\phi^*}"] & & \cdot \arrow[d, equal] \\
\cdot \arrow[rr, "H(\phi_y^*\cdot Gr)"'] & & \cdot & & \cdot \arrow[rr, "H(\phi_y\cdot HGr)"'] & & \cdot
\end{tikzcd}$$

for all r and x, from which our result follows as a consequence of Lemma 3.5. Note that

$$\begin{aligned}
\mathsf{m}^H \circ(\sigma^H \cdot \mathsf{id}) \circ \mathsf{n}_r^{(H\phi)^*} \circ(\mathsf{id}\cdot\eta) &= \mathsf{m}^H \circ(\sigma^H \cdot \mathsf{id}) \circ (\eta\cdot H\phi_r) \circ \gamma^{-1} \\
&= \mathsf{m}^H \circ(H\eta\cdot H\phi_r) \circ (\mathsf{e}^H \cdot\mathsf{id}) \circ \gamma^{-1} \\
&= H(\eta\cdot\phi_r) \circ \mathsf{m}^H \circ(\mathsf{e}^H \cdot\mathsf{id}) \circ \gamma^{-1} \\
&= H(\eta\cdot\phi_r) \circ H\lambda^{-1} \circ \rho \\
H\,\mathsf{n}_r^{\phi^*} \circ \mathsf{m}^H \circ(\mathsf{id}\cdot\sigma^H) \circ (\mathsf{id}\cdot\eta) &= H\,\mathsf{n}_r^{\phi^*} \circ \mathsf{m}^H \circ(\mathsf{id}\cdot H\eta) \circ (\mathsf{id}\cdot\mathsf{e}^H) \\
&= H\,\mathsf{n}_r^{\phi^*} \circ H(\mathsf{id}\cdot\eta) \circ \mathsf{m}^H \circ(\mathsf{id}\cdot\mathsf{e}^H) \\
&= H(\eta\cdot\phi_r) \circ H\gamma^{-1} \circ \mathsf{m}^H \circ(\mathsf{id}\cdot\mathsf{e}^H) \\
&= H(\eta\cdot\phi_r) \circ H\lambda^{-1} \circ \rho
\end{aligned}$$

so (3.4) holds by mate correspondence. □

This invertibility condition is satisfied, for instance, by Barr extensions of monads on Set; see [24, 1.10.2(2)], and by strong functors.

4. Double categories as pseudo-algebras

This section is devoted to proving the following result:

Proposition 4.1. *We have an equivalence of double categories* PsDbCat $\simeq$ Ps-$\mathfrak{F}$-Alg*, where* $\mathfrak{F} = (\mathfrak{F}, m, e)$ *is the free internal category 2-monad on* Grph(Cat)*, and* Ps-$\mathfrak{F}$-Alg *is the sub-double category of* Lax-$\mathfrak{F}$-Alg *consisting of the pseudo-*$\mathfrak{F}$*-algebras.*

The proof is laid out as follows:

- We recall the definition of $\mathfrak{F}$, verifying it is a 2-monad.
- We provide a construction of a pseudo-$\mathfrak{F}$-algebra from a given pseudodouble category.
- We provide a construction of (op)lax morphisms of pseudo-$\mathfrak{F}$-algebras from given (op)lax functors of pseudodouble categories. Moreover, we verify this construction defines a functor $\mathsf{PsDbCat}_{\mathrm{lax}} \to \mathsf{Ps}\text{-}\mathfrak{F}\text{-}\mathsf{Alg}_{\mathrm{lax}}$ (and dually, $\mathsf{PsDbCat}_{\mathrm{opl}} \to \mathsf{Ps}\text{-}\mathfrak{F}\text{-}\mathsf{Alg}_{\mathrm{opl}}$).
- We prove the aforementioned functor is fully faithful and essentially surjective.
- Let $H\colon \mathbb{A} \to \mathbb{B}$ and $K\colon \mathbb{C} \to \mathbb{D}$ be lax functors, and let $F\colon \mathbb{A} \to \mathbb{C}$ and $G\colon \mathbb{B} \to \mathbb{D}$ be oplax functors, and consider the induced lax and oplax $\mathfrak{F}$-algebra morphisms (as in (4.3)). Given a 2-cell $\omega\colon GH \to KF$ of internal Cat-graphs, we prove that ω is a generalized vertical transformation if and only if ω is a generalized 2-cell of pseudo $\mathfrak{F}$-algebras.

We begin by recalling that Grph(Cat) is the functor 2-category $[\cdot_1 \rightrightarrows \cdot_0, \mathsf{Cat}]$, whose 2-cells $\theta\colon F \to G$ are pairs of natural transformations $\theta_i\colon F_i \to G_i$ for $i = 0$, 1 such that $d_j \cdot \theta_1 = \theta_0 \cdot d_j$ for $j = 0$, 1.

Since Cat is an extensive category with pullbacks, we can define the free internal category monad $\mathfrak{F} = (\mathfrak{F}, m, e)$ on the underlying category of Grph(Cat).

To extend $\mathfrak{F}$ to a 2-monad, let $\theta\colon F \to G$ be a 2-cell in Grph(Cat). We define $\mathfrak{F}\theta$ by letting $(\mathfrak{F}\theta)_0 = \theta_0$ and $(\mathfrak{F}\theta)_1\colon (\mathfrak{F}F)_1 \to (\mathfrak{F}G)_1$ is given at a composable string of horizontal arrows $r_1, \ldots, r_n$ by

$$(\mathfrak{F}\theta)_{r_1,\ldots,r_n} = (\theta_{r_1}, \ldots, \theta_{r_n}),$$

which is a horizontally composable string of 2-cells, and $(\mathfrak{F}\theta)_{()} = \theta_0$.

We must check $(\mathfrak{F}\theta)_1$ is natural; indeed, if $\phi_i\colon r_i \to s_i$ is a horizontally composable string of 2-cells, then $\theta_{s_i} \circ F\phi_i = G\phi_i \circ \theta_{r_i}$ for all i, so

$$\begin{aligned}(\mathfrak{F}\theta)_{s_1,\dots,s_n} \circ (\mathfrak{F}F)(\phi_1,\dots,\phi_n) &= (\theta_{s_1},\dots,\theta_{s_n}) \circ (F\phi_1,\dots,F\phi_n)\\ &= (G\phi_1,\dots,G\phi_n) \circ (\theta_{r_1},\dots,\theta_{r_n})\\ &= (\mathfrak{F}G)(\phi_1,\dots,\phi_n) \circ (\mathfrak{F}\theta)_{r_1,\dots,r_n},\end{aligned}$$

and since θ_0 is already natural, there is nothing to check for $n = 0$.

Finally, note that $d_1 \cdot (\mathfrak{F}\theta)_{r_1,\dots,r_n} = d_1(\theta_{r_1},\dots,\theta_{r_n}) = d_1(\theta_{r_1}) = \theta_{d_1 r_1} = \theta_{d_1(r_1,\dots,r_n)}$, and likewise $d_0 \cdot (\mathfrak{F}\theta)_1 = \theta_0 \cdot d_0$.

To verify $\mathfrak{F}$ is a 2-functor, we must prove we have strict preservation of vertical and horizontal composition of 2-cells. Therefore, let $\omega\colon G \to H$ and $\xi\colon H \to K$ be 2-cells, with H, K composable with F, G respectively. We have $\mathfrak{F}(\omega \circ \theta)_0 = \mathfrak{F}(\omega)_0 \circ \mathfrak{F}(\theta)_0$ and $\mathfrak{F}(\xi \cdot \theta)_0 = \mathfrak{F}(\xi)_0 \cdot \mathfrak{F}(\theta)_0$. Moreover, given a composable string of horizontal arrows $r_1,\dots,r_n$, we have

$$\begin{aligned}\mathfrak{F}(\omega \circ \theta)_{r_1,\dots,r_n} &= ((\omega \circ \theta)_{r_1},\dots,(\omega \circ \theta)_{r_n})\\ &= (\omega_{r_1},\dots,\omega_{r_n}) \circ (\theta_{r_1},\dots,\theta_{r_n})\\ &= \mathfrak{F}(\omega)_{r_1,\dots,r_n} \circ \mathfrak{F}(\theta)_{r_1,\dots,r_n},\\ \mathfrak{F}(\xi \cdot \theta)_{r_1,\dots,r_n} &= ((\xi \cdot \theta)_{r_1},\dots,(\xi \cdot \theta)_{r_n})\\ &= (\xi_{r_1},\dots,\xi_{r_n}) \cdot (\theta_{r_1},\dots,\theta_{r_n})\\ &= \mathfrak{F}(\xi)_{r_1,\dots,r_n} \cdot \mathfrak{F}(\theta)_{r_1,\dots,r_n},\end{aligned}$$

as desired. Nothing needs to be done to verify that m, e are 2-natural transformations.

A pseudodouble category consists of a graph of categories $\mathbb{D} = (\mathbb{D}_1 \rightrightarrows \mathbb{D}_0)$, with vertical domain and codomain functors. The algebra structure $a\colon \mathfrak{F}\mathbb{D} \to \mathbb{D}$ is the identity on 0-cells and vertical 1-cells. We define $a() = 1$ (at 0-cells), and if a is defined for $\mathbb{D}^{(n)}$, we define

$$a(r_1,\dots,r_{n+1}) = r_{n+1} \cdot a(r_1,\dots,r_n).$$

We let $\eta\colon \mathsf{id} \to a \cdot e$ be the identity on $\mathsf{id}_{\mathbb{D}_0}$, and $\eta_r\colon r \to a(r)$ is given by

$$\rho^{-1}\colon r \to r \cdot 1 = r \cdot a() = a(r).$$

We define $\mu\colon a \cdot \mathfrak{F}a \to a \cdot m$ to be the identity on $\mathsf{id}_{\mathbb{D}_0}$, and on $\mathfrak{F}\mathfrak{F}\mathbb{D}_1 \to \mathfrak{F}\mathbb{D}_1$ by double induction:

$$\begin{aligned}\mu_{()} &= \mathsf{id},\\ \mu_{k_1,\dots,k_n,0} &= \mu_{k_1,\dots,k_n} \circ \lambda,\\ \mu_{k_1,\dots,k_{n+1}+1} &= (\mathsf{id} \cdot \mu_{k_1,\dots,k_{n+1}}) \circ \alpha\end{aligned}$$

where

$$\mu_{k_1,\dots,k_n}\colon a(a(r_{1,1},\dots,r_{1,k_1}),\dots,a(r_{n,1},\dots,r_{n,k_n})) \to a(r_{1,1},\dots,r_{n,k_n}).$$

To prove that $(\mathbb{D}, a, \eta, \mu)$ is a pseudo-$\mathfrak{F}$-algebra, we must verify that

$$\begin{aligned}\mu_m \circ \eta_{a(r_1,\dots,r_m)} &= \mathsf{id},\\ \mu_{1,\dots,1} \circ a(\eta_{r_1},\dots,\eta_{r_m}) &= \mathsf{id},\\ \mu_{j_{1,1},\dots,j_{n,k_n}} \circ \mu_{k_1,\dots,k_n} &= \mu_{\hat{j}_{1,k_1},\dots,\hat{j}_{n,k_n}} \circ a(\sigma_1,\dots,\sigma_n)\end{aligned}$$

where we use the following abbreviations:

$$\begin{aligned}\hat{j}_{i,k_i} &= \sum_{p=1}^{k_i} j_{i,p}\\ \sigma_p &= \mu_{j_{p,1},\dots,j_{p,k_p}}\end{aligned}$$

For the first and second, we argue by induction. When $m = 0$, the first becomes $\lambda_1 \circ \rho_1^{-1} = \mathsf{id}$, and the second trivializes. If we assume the equations hold for some m, then

$$\mu_{m+1} \circ \rho^{-1} = (\mathsf{id} \cdot \mu_m) \circ \alpha \circ \rho^{-1} = (\mathsf{id} \cdot \mu_m) \circ (\mathsf{id} \cdot \rho^{-1}) = \mathsf{id},$$

and

$$\begin{aligned}\mu_{1,\dots,1,1} \circ a(\rho^{-1}, \dots, \rho^{-1}, \rho^{-1}) &= (\mathsf{id} \cdot \mu_{1,\dots,1}) \circ (\mathsf{id} \cdot \lambda) \circ \alpha \circ (\rho^{-1} \cdot \mathsf{id}) \circ (\mathsf{id} \cdot a(\rho^{-1}, \dots, \rho^{-1}))\\ &= (\mathsf{id} \cdot \mu_{1,\dots,1}) \circ (\mathsf{id} \cdot a(\rho^{-1}, \dots, \rho^{-1}))\\ &= \mathsf{id}\end{aligned}$$

For the third, we use triple induction. If $n = 0$, it trivializes, so we assume it holds for some n. If $k_{n+1} = 0$, we have

$$\begin{aligned}\mu_{j_{1,1},\dots,j_{n,k_n}} \circ \mu_{k_1,\dots,k_n,k_{n+1}} &= \mu_{j_{1,1},\dots,j_{n,k_n}} \circ \mu_{k_1,\dots,k_n} \circ \lambda\\ &= \mu_{\hat{j}_{1,k_1},\dots,\hat{j}_{n,k_n}} \circ a(\sigma_1, \dots, \sigma_n) \circ \lambda\\ &= \mu_{\hat{j}_{1,k_1},\dots,\hat{j}_{n,k_n},0} \circ a(\sigma_1, \dots, \sigma_n, \mathsf{id})\\ &= \mu_{\hat{j}_{1,k_1},\dots,\hat{j}_{n,k_n},\hat{j}_{n+1,k_{n+1}}} \circ a(\sigma_1, \dots, \sigma_n, \sigma_{n+1}).\end{aligned}$$

Now, we assume it holds for some k_{n+1}. If $j_{n+1,k_{n+1}+1} = 0$, we have

$$\begin{aligned}\mu_{j_{1,1},\dots,j_{n+1,k_{n+1}},0} \circ \mu_{k_1,\dots,k_n,k_{n+1}+1} &= \mu_{j_{1,1},\dots,j_{n+1,k_{n+1}}} \circ \lambda \circ (\mathsf{id} \cdot \mu_{k_1,\dots,k_{n+1}}) \circ \alpha\\ &= \mu_{j_{1,1},\dots,j_{n+1,k_{n+1}}} \circ \mu_{k_1,\dots,k_{n+1}} \circ \lambda \circ \alpha\\ &= \mu_{\hat{j}_{1,k_1},\dots,\hat{j}_{n+1,k_{n+1}}} \circ a(\sigma_1, \dots, \sigma_{n+1}) \circ (\lambda \cdot \mathsf{id})\\ &= \mu_{\hat{j}_{1,k_1},\dots,\hat{j}_{n+1,k_{n+1}+1}} \circ a(\sigma_1, \dots, (\sigma_{n+1} \circ \lambda)),\end{aligned}$$

and finally, if we assume it holds for some $j_{n+1,k_{n+1}+1}$, then we have

$$\begin{aligned}\mu_{j_{1,1},\dots,j_{n+1,k_{n+1}+1}+1} \circ \mu_{k_1,\dots,k_{n+1}+1} &= (\mathsf{id} \cdot \mu_{j_{1,1},\dots,j_{n+1,k_{n+1}+1}}) \circ \alpha \circ (\mathsf{id} \cdot \mu_{k_1,\dots,k_{n+1}}) \circ \alpha\\ &= (\mathsf{id} \cdot \mu_{j_{1,1},\dots,j_{n+1,k_{n+1}+1}}) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \mu_{k_1,\dots,k_{n+1}})) \circ \alpha \circ \alpha\\ &= (\mathsf{id} \cdot \mu_{j_{1,1},\dots,j_{n+1,k_{n+1}+1}}) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \mu_{k_1,\dots,k_{n+1}})) \circ (\mathsf{id} \cdot \alpha) \circ \alpha \circ (\alpha \cdot \mathsf{id})\\ &= (\mathsf{id} \cdot \mu_{j_{1,1},\dots,j_{n+1,k_{n+1}+1}}) \circ (\mathsf{id} \cdot \mu_{k_1,\dots,k_{n+1}+1}) \circ \alpha \circ (\alpha \cdot \mathsf{id})\\ &= (\mathsf{id} \cdot \mu_{\hat{j}_{1,k_1},\dots,\hat{j}_{n+1,k_{n+1}}}) \circ (\mathsf{id} \cdot a(\sigma_1, \dots, \sigma_{n+1})) \circ \alpha \circ (\alpha \cdot \mathsf{id})\\ &= (\mathsf{id} \cdot \mu_{\hat{j}_{1,k_1},\dots,\hat{j}_{n+1,k_{n+1}}}) \circ \alpha \circ ((\mathsf{id} \cdot \sigma_{n+1}) \cdot a(\sigma_1, \dots, \sigma_n)) \circ (\alpha \cdot \mathsf{id})\\ &= \mu_{\hat{j}_{1,k_1},\dots,\hat{j}_{n+1,k_{n+1}}+1} \circ a(\sigma_1, \dots, \sigma_n, (\mathsf{id} \cdot \sigma_{n+1}) \circ \alpha),\end{aligned}$$

so the result holds by induction. It should be noted that the proof (so far) remains unchanged if we consider left-biased double categories, in which case $(\mathbb{D}, a, \eta, \mu)$ is a lax $\mathfrak{F}$-algebra instead. Respectively, right-biased double categories → oplax $\mathfrak{F}$-algebras.

If we have a lax functor $F\colon \mathbb{D} \to \mathbb{E}$ between ordinary double categories, we define a pseudo $\mathfrak{F}$-algebra lax morphism $(F, \gamma^F)\colon (\mathbb{D}, a, \eta, \mu) \to (\mathbb{E}, b, \eta, \mu)$, taking F to be the same underlying graph morphism, and we define $\gamma^F : b \circ \mathfrak{F}F \to F \circ a$ inductively as follows:

$$\begin{aligned}\gamma^F_x &= \mathsf{e}^F_x,\\ \gamma^F_{r_1,\dots,r_{n+1}} &= \mathsf{m}^F \circ (\mathsf{id} \cdot \gamma^F_{r_1,\dots,r_n}).\end{aligned}$$

to confirm (F, γ^F) is indeed a lax morphism, we will prove that

$$F\eta = \gamma_r \circ \eta$$

and

$$\gamma_{r_{1,1},\dots,r_{n,k_n}} \circ \mu_{k_1,\dots,k_n} = F\mu_{k_1,\dots,k_n} \circ \gamma_{s_1,\dots,s_n} \circ b(\sigma_1, \dots, \sigma_n),$$

where

$$\begin{aligned}s_i &= a(r_{i,1}, \dots, r_{i,k_i}),\\ \sigma_i &= \gamma_{r_{i,1},\dots,r_{i,k_i}}.\end{aligned}$$

The first is just a restatement of the coherence diagram for the right unitor. For the second, when $n = 0$, the equation is trivial; $\mathsf{e}^F = \mathsf{e}^F$. Now, we assume the equation holds for some n. If $k_{n+1} = 0$, then

$$\begin{aligned}
\gamma_{r_{1,1},\dots,r_{n,k_n}} \circ \mu_{k_1,\dots,k_n,0} &= \gamma_{r_{1,1},\dots,r_{n,k_n}} \circ \mu_{k_1,\dots,k_n} \circ \lambda \\
&= F\mu_{k_1,\dots,k_n} \circ \gamma_{s_1,\dots,s_n} \circ b(\sigma_1,\dots,\sigma_n) \circ \lambda \\
&= \lambda \circ (\mathsf{id} \cdot F\mu_{k_1,\dots,k_n}) \circ (\mathsf{id} \cdot \gamma_{s_1,\dots,s_n}) \circ (\mathsf{id} \cdot b(\sigma_1,\dots,\sigma_n)) \\
&= F\lambda \circ \mathsf{m}^F \circ (\mathsf{e}^F \cdot \mathsf{id}) \circ (\mathsf{id} \cdot F\mu_{k_1,\dots,k_n}) \circ (\mathsf{id} \cdot \gamma_{s_1,\dots,s_n}) \circ (\mathsf{id} \cdot b(\sigma_1,\dots,\sigma_n)) \\
&= F\lambda \circ F(\mathsf{id} \cdot \mu_{k_1,\dots,k_n}) \circ \mathsf{m}^F \circ (\mathsf{id} \cdot \gamma_{s_1,\dots,s_n}) \circ (\mathsf{e}^F \cdot b(\sigma_1,\dots,\sigma_n)) \\
&= F\mu_{k_1,\dots,k_n,0} \circ \gamma_{s_1,\dots,s_n,s_{n+1}} \circ b(\sigma_1,\dots,\sigma_n,\sigma_{n+1}),
\end{aligned}$$

so, we assume the identity holds for some k_{n+1}. We have

$$\begin{aligned}
&\gamma_{r_{1,1},\dots,r_{n+1,k_{n+1}},r_{n+1,k_{n+1}+1}} \circ \mu_{k_1,\dots,k_n,k_{n+1}+1} \\
&\quad= \mathsf{m}^F \circ (\mathsf{id} \cdot \gamma_{r_{1,1},\dots,r_{n+1,k_{n+1}}}) \circ (\mathsf{id} \cdot \mu_{k_1,\dots,k_{n+1}}) \circ \alpha \\
&\quad= \mathsf{m}^F \circ (\mathsf{id} \cdot F\mu_{k_1,\dots,k_{n+1}}) \circ (\mathsf{id} \cdot \gamma_{s_1,\dots,s_n,s_{n+1}}) \circ (\mathsf{id} \cdot b(\sigma_1,\dots,\sigma_n,\sigma_{n+1})) \circ \alpha \\
&\quad= F(\mathsf{id} \cdot \mu_{k_1,\dots,k_{n+1}}) \circ \mathsf{m}^F \circ (\mathsf{id} \cdot \mathsf{m}^F) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \gamma_{s_1,\dots,s_n})) \circ \alpha \circ ((\mathsf{id} \cdot \sigma_{n+1}) \cdot b(\sigma_1,\dots,\sigma_n)) \\
&\quad= F(\mathsf{id} \cdot \mu_{k_1,\dots,k_{n+1}}) \circ F\alpha \circ \mathsf{m}^F \circ (\mathsf{m}^F \cdot \mathsf{id}) \circ (\mathsf{id} \cdot \gamma_{s_1,\dots,s_n}) \circ ((\mathsf{id} \cdot \sigma_{n+1}) \cdot b(\sigma_1,\dots,\sigma_n)) \\
&\quad= F\mu_{k_1,\dots,k_{n+1}+1} \circ \gamma_{s_1,\dots,s_{n+1}} \circ b(\sigma_1,\dots,\sigma_n,\mathsf{m}^F \circ (\mathsf{id} \cdot \sigma_{n+1})),
\end{aligned}$$

so, the result follows by induction.

This assignment preserves maps identities to identities (trivially), and preserves composition; that is, this defines a functor $\mathsf{PsDbCat}_{\mathrm{lax}} \to \mathsf{Ps}\text{-}\mathfrak{F}\text{-}\mathsf{Alg}_{\mathrm{lax}}$. To see this, let $G\colon \mathbb{E} \to \mathbb{C}$ be another lax functor. We have

$$\begin{aligned}
(G\gamma^F \circ \gamma^G)_{()} &= G\,\mathsf{e}^F \circ e^G, \\
(G\gamma^F \circ \gamma^G)_{r_1,\dots,r_{n+1}} &= \mathsf{m}^{GF} \circ (\mathsf{id} \cdot G\gamma^F_{r_1,\dots,r_n}) \circ (\mathsf{id} \cdot \gamma^G_{Fr_1,\dots,Fr_n}) \\
&= G\,\mathsf{m}^F \circ \mathsf{m}^G \circ (\mathsf{id} \cdot G\gamma^F_{r_1,\dots,r_n}) \circ (\mathsf{id} \cdot \gamma^G_{Fr_1,\dots,Fr_n}) \\
&= G\,\mathsf{m}^F \circ G(\mathsf{id} \cdot \gamma^F_{r_1,\dots,r_n}) \circ \mathsf{m}^G \circ (\mathsf{id} \cdot \gamma^G_{Fr_1,\dots,Fr_n}) \\
&= G(\gamma^F_{r_1,\dots,r_{n+1}}) \circ \gamma^G_{Fr_1,\dots,Fr_{n+1}}.
\end{aligned}$$

We claim the functor $\mathsf{PsDbCat}_{\mathrm{lax}} \to \mathsf{Ps}\text{-}\mathfrak{F}\text{-}\mathsf{Alg}_{\mathrm{lax}}$ is fully faithful; if $(F,\gamma^F)\colon (\mathbb{D},a,\eta,\mu) \to (\mathbb{E},b,\eta,\mu)$ is a lax morphism between (the image of) double categories, we define

$$\begin{aligned}
\mathsf{e}^F_x &= \gamma^F_x \\
\mathsf{m}^F_{r,s} &= F(\mathsf{id} \cdot \rho) \circ \gamma^F_{r,s} \circ (\mathsf{id} \cdot \rho^{-1})
\end{aligned}$$

We must confirm these satisfy the coherence conditions. First, we observe that

$$\begin{aligned}
\mathsf{m}^F_{1,s} \circ (\mathsf{id} \cdot \mathsf{e}^F) &= F(\mathsf{id} \cdot \rho) \circ \gamma^F_{1,s} \circ (\mathsf{id} \cdot \rho^{-1}) \circ (\mathsf{id} \cdot \mathsf{e}^F) \\
&= F(\mathsf{id} \cdot \lambda) \circ F(\rho \cdot \mathsf{id}) \circ \gamma^F_{1,a(s)} \circ (F\rho^{-1} \cdot \mathsf{id}) \circ (\mathsf{id} \cdot (\mathsf{e}^F \cdot \mathsf{id})) \circ (\mathsf{id} \cdot \lambda^{-1}) \\
&= F\mu_{0,1} \circ \gamma^F_{1,a(s)} \circ b(\gamma^F_{()},\gamma^F_s) \circ \mu^{-1}_{0,1} = \gamma^F_s, \\
\mathsf{m}^F_{r,1} \circ (\mathsf{e}^F \cdot \mathsf{id}) &= F(\mathsf{id} \cdot \rho) \circ \gamma^F_{r,1} \circ (\mathsf{id} \cdot \rho^{-1}) \circ (\mathsf{e}^F \cdot \mathsf{id}) \\
&= F(\mathsf{id} \cdot \rho) \circ F(\mathsf{id} \cdot (\rho \cdot \mathsf{id})) \circ \gamma^F_{a(r),1} \circ (\mathsf{id} \cdot (F\rho^{-1} \cdot \mathsf{id})) \circ (\mathsf{e}^F \cdot \mathsf{id}) \circ (\mathsf{id} \cdot \rho^{-1}) \\
&= F(\mathsf{id} \cdot \rho) \circ F(\mathsf{id} \cdot \rho) \circ \gamma^F_{a(r),1} \circ b(\gamma^F_r,\gamma^F_{()}) \circ (\mathsf{id} \cdot \rho^{-1}) \circ (\mathsf{id} \cdot \rho^{-1}) \\
&= F(\mathsf{id} \cdot \rho) \circ F(\mathsf{id} \cdot \rho) \circ F\mu^{-1}_{1,0} \circ \gamma^F_r \circ \mu_{1,0} \circ (\mathsf{id} \cdot \rho^{-1}) \circ (\mathsf{id} \cdot \rho^{-1}) \\
&= F\lambda^{-1} \circ F\rho \circ \gamma^F_r \circ \rho^{-1} \circ \lambda \\
&= F\lambda^{-1} \circ \lambda
\end{aligned}$$

which gives the unit comparsion coherences for F, and after calculating

$$\begin{aligned}
\mu_{1,2} &= \alpha \circ ((\mathsf{id} \cdot \rho) \cdot (\rho \cdot \mathsf{id})) \\
\mu_{2,1} &= (\mathsf{id} \cdot \alpha) \circ (\mathsf{id} \cdot ((\mathsf{id} \cdot \rho) \cdot \mathsf{id})) \circ (\rho \cdot \mathsf{id})
\end{aligned}$$

we verify that

$$\begin{aligned}
&F\alpha \circ \mathsf{m}^F_{r,t\cdot s} \circ (\mathsf{m}^F_{s,t} \cdot \mathsf{id}) \\
&\quad = F\alpha \circ F(\mathsf{id} \cdot \rho) \circ \gamma^F_{r,t\cdot s} \circ (\mathsf{id} \cdot \rho^{-1}) \circ (F(\mathsf{id} \cdot \rho) \cdot \mathsf{id}) \circ (\gamma^F_{s,t} \cdot \mathsf{id}) \circ ((\mathsf{id} \cdot \rho^{-1}) \cdot \mathsf{id}) \\
&\quad = F\alpha \circ F(\mathsf{id} \cdot \rho) \circ F((\mathsf{id} \cdot \rho) \cdot (\rho \cdot \mathsf{id})) \circ \gamma^F_{a(r),a(s,t)} \circ (\gamma^F_{s,t} \cdot (F\rho^{-1} \cdot \mathsf{id})) \circ ((\mathsf{id} \cdot \rho^{-1}) \cdot \rho^{-1}) \\
&\quad = F(\mathsf{id} \cdot (\mathsf{id} \cdot \rho)) \circ F\alpha \circ F((\mathsf{id} \cdot \rho) \cdot (\rho \cdot \mathsf{id})) \circ \gamma^F_{a(r),a(s,t)} \circ b(\gamma^F_r, \gamma^F_{s,t}) \circ ((\mathsf{id} \cdot \rho^{-1}) \cdot (\rho^{-1} \cdot \mathsf{id})) \circ (\mathsf{id} \cdot \rho^{-1}) \\
&\quad = F(\mathsf{id} \cdot (\mathsf{id} \cdot \rho)) \circ \gamma^F_{r,s,t} \circ \alpha \circ (\mathsf{id} \cdot \rho^{-1})
\end{aligned}$$

$$\begin{aligned}
&\mathsf{m}^F_{s\cdot r,t} \circ (\mathsf{id} \cdot \mathsf{m}^F_{r,s}) \circ \alpha \\
&\quad = F(\mathsf{id} \cdot \rho) \circ \gamma^F_{s\cdot r,t} \circ (\mathsf{id} \cdot \rho^{-1}) \circ (\mathsf{id} \cdot F(\mathsf{id} \cdot \rho)) \circ (\mathsf{id} \cdot \gamma^F_{r,s}) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \rho^{-1})) \circ \alpha \\
&\quad = F(\mathsf{id} \cdot \rho) \circ F(\rho \cdot ((\mathsf{id} \cdot \rho) \cdot \mathsf{id})) \circ \gamma^F_{a(r,s),a(t)} \circ (F\rho^{-1} \cdot (F(\mathsf{id} \cdot \rho^{-1}) \cdot \mathsf{id})) \circ (\mathsf{id} \cdot \rho^{-1}) \circ (\mathsf{id} \cdot F(\mathsf{id} \cdot \rho)) \\
&\qquad\qquad \circ (\mathsf{id} \cdot \gamma^F_{r,s}) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \rho^{-1})) \circ \alpha \\
&\quad = F(\mathsf{id} \cdot (\mathsf{id} \cdot \rho)) \circ F(\rho \cdot \rho) \circ \gamma^F_{a(r,s),a(t)} \circ b(\gamma^F_{r,s}, \gamma^F_t) \circ (\rho^{-1} \cdot \rho^{-1}) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \rho^{-1})) \circ \alpha \\
&\quad = F(\mathsf{id} \cdot (\mathsf{id} \cdot \rho)) \circ \gamma^F_{r,s,t} \circ \alpha \circ (\mathsf{id} \cdot \rho^{-1})
\end{aligned}$$

which confirms coherence for the associator comparison. We further verify that, by induction, $\mu_{n,1} \circ (\rho^{-1} \cdot \rho^{-1}) = \mathsf{id}$ (pattern matching), so that

$$\begin{aligned}
\mathsf{m}^F_{a(r_1,\ldots,r_n),r_{n+1}} \circ (\mathsf{id} \cdot \gamma^F_{r_1,\ldots,r_n}) &= F(\mathsf{id} \cdot \rho) \circ \gamma^F_{a(r_1,\ldots,r_n),r_{n+1}} \circ (\mathsf{id} \cdot \rho^{-1}) \circ (\mathsf{id} \cdot \gamma^F_{r_1,\ldots,r_n}) \\
&= F(\rho \cdot \rho) \circ \gamma^F_{a(r_1,\ldots,r_n),a(r_{n+1})} \circ b(\gamma^F_{r_1,\ldots,r_n}, \gamma^F_{r_{n+1}}) \circ (\rho^{-1} \cdot \rho^{-1}) \\
&= \gamma^F_{r_1,\ldots,r_{n+1}},
\end{aligned}$$

confirming that the functor $\mathsf{PsDbCat}_{\mathrm{lax}} \to \mathsf{Ps}\text{-}\mathfrak{F}\text{-}\mathsf{Alg}_{\mathrm{lax}}$ is fully faithful.

We claim the functor $\mathsf{PsDbCat}_{\mathrm{lax}} \to \mathsf{Ps}\text{-}\mathfrak{F}\text{-}\mathsf{Alg}_{\mathrm{lax}}$ is essentially surjective; let $(\mathbb{D}, a, \eta, \mu)$ be a pseudo-$\mathfrak{F}$-algebra. We define

$$\begin{aligned}
1 &= a(), \\
s \cdot r &= a(r, s), \\
\lambda_r &= \eta_r^{-1} \circ \mu_{r,-} \circ a(\eta_r, \mathsf{id}), \\
\rho_r &= \eta_r^{-1} \circ \mu_{-,r} \circ a(\mathsf{id}, \eta_r), \\
\alpha_{r,s,t} &= a(\mathsf{id}, \eta_t^{-1}) \circ \mu^{-1}_{rs,t} \circ \mu_{r,st} \circ a(\eta_r, \mathsf{id}).
\end{aligned}$$

These endow $\mathbb{D}$ with the structure of a double category; to see this, we must verify the coherence conditions hold. First, we have

$$\begin{aligned}
(\mathsf{id} \cdot \lambda_r) \circ \alpha_{r,1,s} &= a(\eta_r^{-1}, \mathsf{id}) \circ a(\mu_{r,-}, \mathsf{id}) \circ a(a(\eta_r, \mathsf{id}), \mathsf{id}) \circ a(\mathsf{id}, \eta_s^{-1}) \circ \mu^{-1}_{r1,s} \circ \mu_{r,1s} \circ a(\eta_r, \mathsf{id}) \\
&= a(\eta_r^{-1}, \eta_s^{-1}) \circ a(\mu_{r,-}, \mathsf{id}) \circ a(a(\eta_r, \mathsf{id}), \mathsf{id}) \circ \mu^{-1}_{r1,s} \circ \mu_{r,1s} \circ a(\eta_r, \mathsf{id}) \\
&= \mu_{r,s} \circ a(\mu_{r,-}, \mathsf{id}) \circ \mu^{-1}_{a(r)1,s} \circ a(\eta_r, \mathsf{id}, \mathsf{id}) \circ \mu_{r,1s} \circ a(\eta_r, \mathsf{id}) \\
&= \mu_{r,s} \circ a(\mu_{r,-}, \mu_s) \circ \mu^{-1}_{a(r)1,a(s)} \circ a(\eta_r, \mathsf{id}, \eta_s) \circ \mu_{r,1s} \circ a(\eta_r, \mathsf{id}) \\
&= \mu_{r,-,s} \circ \mu_{a(r),1a(s)} \circ a(a(\eta_r), a(\mathsf{id}, \eta_s)) \circ a(\eta_r, \mathsf{id}) \\
&= \mu_{r,s} \circ a(\mu_r, \mu_{-,s}) \circ a(a(\eta_r), a(\mathsf{id}, \eta_s)) \circ a(\eta_r, \mathsf{id}) \\
&= a(\eta_r^{-1}, \eta_s^{-1}) \circ a(\mathsf{id}, \mu_{-,s}) \circ a(\mathsf{id}, a(\mathsf{id}, \eta_s)) \circ a(\eta_r, \mathsf{id}) \\
&= a(\mathsf{id}, \eta_s^{-1}) \circ a(\mathsf{id}, \mu_{-,s}) \circ a(\mathsf{id}, a(\mathsf{id}, \eta_s)) = \rho_s \cdot \mathsf{id}.
\end{aligned}$$

and for the associator pentagon, we have, on one hand

$$\begin{aligned}
&(\mathsf{id}\cdot \alpha_{r,s,t}) \circ \alpha_{q,a(r,s),t} \circ (\alpha_{q,r,s}\cdot \mathsf{id}) \\
&\quad = a(a(\mathsf{id},\eta_s^{-1}),\mathsf{id}) \circ a(\mu_{qr,s}^{-1},\mathsf{id}) \circ a(\mu_{q,rs},\mathsf{id}) \circ a(a(\eta_q,\mathsf{id}),\mathsf{id}) \\
&\qquad \circ a(\mathsf{id},\eta_t^{-1}) \circ \mu_{qa(r,s),t}^{-1} \circ \mu_{q,a(r,s)t} \circ a(\eta_q,\mathsf{id}) \\
&\qquad \circ a(\mathsf{id},a(\mathsf{id},\eta_t^{-1})) \circ a(\mathsf{id},\mu_{rs,t}^{-1}) \circ a(\mathsf{id},\mu_{r,st}) \circ a(\mathsf{id},a(\eta_r,\mathsf{id})) \\
&\quad = a(a(\mathsf{id},\eta_s^{-1}),\eta_t^{-1}) \circ a(\mu_{qr,s}^{-1},\mathsf{id}) \circ a(\mu_{q,rs},\mathsf{id}) \\
&\qquad \circ a(a(\eta_q,\mathsf{id}),\mathsf{id}) \circ \mu_{qa(r,s),t}^{-1} \circ \mu_{q,a(r,s)t} \circ a(\mathsf{id},a(\mathsf{id},\eta_t^{-1})) \\
&\qquad \circ a(\mathsf{id},\mu_{rs,t}^{-1}) \circ a(\mathsf{id},\mu_{r,st}) \circ a(\eta_q,a(\eta_r,\mathsf{id})) \\
&\quad = a(a(\mathsf{id},\eta_s^{-1}),\eta_t^{-1}) \circ a(\mu_{qr,s}^{-1},\mathsf{id}) \circ a(\mu_{q,rs},\mathsf{id}) \\
&\qquad \circ a(\mathsf{id},a(\eta_t^{-1})) \circ \mu_{a(q)a(r,s),a(t)}^{-1} \circ \mu_{a(q),a(r,s)a(t)} \circ a(a(\eta_q),\mathsf{id}) \\
&\qquad \circ a(\mathsf{id},\mu_{rs,t}^{-1}) \circ a(\mathsf{id},\mu_{r,st}) \circ a(\eta_q,a(\eta_r,\mathsf{id})) \\
&\quad = a(a(\mathsf{id},\eta_s^{-1}),\eta_t^{-1}) \circ a(\mu_{qr,s}^{-1},\mathsf{id}) \circ a(\mu_{q,rs},\mu_t) \\
&\qquad \circ \mu_{a(q)a(r,s),a(t)}^{-1} \circ \mu_{a(q),a(r,s)a(t)} \\
&\qquad \circ a(\mu_q^{-1},\mu_{rs,t}^{-1}) \circ a(\mathsf{id},\mu_{r,st}) \circ a(\eta_q,a(\eta_r,\mathsf{id})) \\
&\quad = a(a(\mathsf{id},\eta_s^{-1}),\eta_t^{-1}) \circ a(\mu_{qr,s}^{-1},\mathsf{id}) \circ \mu_{qrs,t}^{-1} \\
&\qquad \circ \mu_{q,rst} \circ a(\mathsf{id},\mu_{r,st}) \circ a(\eta_q,a(\eta_r,\mathsf{id})),
\end{aligned}$$

while on the other, we have

$$\begin{aligned}
\alpha_{a(q,r),s,t} \circ \alpha_{q,r,a(s,t)} &= a(\mathsf{id},\eta_t^{-1}) \circ \mu_{a(q,r)s,t}^{-1} \circ \mu_{a(q,r),st} \circ a(\eta_{a(q,r)},\mathsf{id}) \\
&\quad \circ a(\mathsf{id},\eta_{a(s,t)}^{-1}) \circ \mu_{qr,a(s,t)}^{-1} \circ \mu_{q,ra(s,t)} \circ a(\eta_q,\mathsf{id}) \\
&= a(\mathsf{id},\eta_t^{-1}) \circ \mu_{a(q,r)s,t}^{-1} \circ a(\mathsf{id},\eta_s^{-1},\eta_r^{-1}) \circ \mu_{a(q,r),a(s)a(t)} \\
&\quad \circ a(\mathsf{id},a(\eta_s,\eta_t)) \circ a(\mathsf{id},\eta_{a(s,t)}^{-1}) \circ a(\eta_{a(q,r)},\mathsf{id}) \circ a(a(\eta_q,\eta_r),\mathsf{id}) \\
&\quad \circ \mu_{a(q)a(r),a(s,t)}^{-1} \circ a(\eta_q^{-1},\eta_r^{-1},\mathsf{id}) \circ \mu_{q,ra(s,t)} \circ a(\eta_q,\mathsf{id}) \\
&= a(\mathsf{id},\eta_t^{-1}) \circ \mu_{a(q,r)s,t}^{-1} \circ a(\mathsf{id},\eta_s^{-1},\eta_r^{-1}) \\
&\quad \circ \mu_{a(q,r),a(s)a(t)} \circ a(\eta_{a(q,r)},a(\eta_s,\eta_t)) \\
&\quad \circ a(a(\eta_q^{-1},\eta_r^{-1}),\eta_{a(s,t)}^{-1}) \circ \mu_{a(q)a(r),a(s,t)} \\
&\quad \circ a(\eta_q,\eta_r,\mathsf{id}) \circ \mu_{q,ra(s,t)} \circ a(\eta_q,\mathsf{id}) \\
&= a(\mathsf{id},\eta_t^{-1}) \circ \mu_{a(q,r)s,t}^{-1} \circ a(\mathsf{id},\eta_s^{-1},\eta_r^{-1}) \circ \mu_{qr,s,t}^{-1} \\
&\quad \circ \mu_{q,r,st} \circ a(\eta_q,\eta_r,\mathsf{id}) \circ \mu_{q,ra(s,t)} \circ a(\eta_q,\mathsf{id}) \\
&= a(\mathsf{id},\eta_t^{-1}) \circ \mu_{a(q,r)s,t}^{-1} \circ a(\mathsf{id},\eta_s^{-1},\eta_r^{-1}) \circ \mu_{qr,s,t}^{-1} \\
&\quad \circ \mu_{q,r,st} \circ a(\eta_q,\eta_r,\mathsf{id}) \circ \mu_{q,ra(s,t)} \circ a(\eta_q,\mathsf{id}) \\
&= a(\mathsf{id},\eta_t^{-1}) \circ \mu_{a(q,r)s,t}^{-1} \circ a(\mathsf{id},\eta_s^{-1},\eta_r^{-1}) \\
&\quad \circ a(\mu_{q,r},\mu_s,\mu_t) \circ \mu_{a(q)a(r),a(s),a(t)}^{-1} \\
&\quad \circ \mu_{a(q),a(r),a(s)a(t)} \circ a(\mu_q^{-1},\mu_r^{-1},\mu_{s,t}^{-1}) \\
&\quad \circ a(\eta_q,\eta_r,\mathsf{id}) \circ \mu_{q,ra(s,t)} \circ a(\eta_q,\mathsf{id}) \\
&= a(a(\mathsf{id},\eta_s^{-1}),\eta_t^{-1}) \circ a(\mathsf{id},\mu_t) \circ \mu_{a(q,r)a(s),a(t)}^{-1} \\
&\quad \circ a(\mu_{q,r},\mu_s,\mu_t) \circ \mu_{a(q)a(r),a(s),a(t)}^{-1} \\
&\quad \circ \mu_{a(q),a(r),a(s)a(t)} \circ a(\mu_q^{-1},\mu_r^{-1},\mu_{s,t}^{-1}) \\
&\quad \circ \mu_{a(q),a(r)a(s,t)} \circ a(\mu_q,\mathsf{id}) \circ a(\eta_q,a(\eta_r,\mathsf{id})),
\end{aligned}$$

so, our goal is to prove that

$$\begin{aligned} a(\mu_{qr,s}^{-1}, \mu_t^{-1}) \circ \mu_{qrs,t}^{-1} \circ \mu_{q,rst} \circ a(\mu_q, \mu_{r,st}) = {} & \mu_{a(q,r)a(s),a(t)}^{-1} \circ a(\mu_{q,r}, \mu_s, \mu_t) \\ & \circ \mu_{a(q)a(r),a(s),a(t)}^{-1} \circ \mu_{a(q),a(r),a(s)a(t)} \\ & \circ a(\mu_q^{-1}, \mu_r^{-1}, \mu_{s,t}^{-1}) \circ \mu_{a(q),a(r)a(s,t)}. \end{aligned} \tag{4.1}$$

And to do so, we observe that the following diagrams

$$\begin{tikzcd} a(a(a(q)),a(a(r),a(s,t))) \arrow[r, "a(\mu_q{,}\mu_{r,st})"] \arrow[d, "\mu_{a(q),a(r)a(s,t)}"'] & a(a(q),a(r,s,t)) \arrow[d, "\mu_{q,rst}"] \\ a(a(q),a(r),a(s,t)) \arrow[r, "\mu_{q,r,st}" description] \arrow[d, "a(\mu_q^{-1}{,}\mu_r^{-1}{,}\mu_{s,t}^{-1})"'] & a(q,r,s,t) \arrow[d, "\mu_{q,r,s,t}^{-1}"] \\ a(a(a(q)),a(a(r)),a(a(s),a(t))) \arrow[r, "\mu_{a(q),a(r),a(s)a(t)}"'] & a(a(q),a(r),a(s),a(t)) \end{tikzcd}$$

$$\begin{tikzcd} a(a(q,r,s),a(t)) \arrow[r, "a(\mu_{qr,s}^{-1}{,}\mu_t^{-1})"] & a(a(a(q,r),a(s)),a(a(t))) \\ a(q,r,s,t) \arrow[u, "\mu_{qrs,t}^{-1}"] \arrow[r, "\mu_{qr,s,t}^{-1}" description] & a(a(q,r),a(s),a(t)) \arrow[u, "\mu_{a(q,r)a(s),a(t)}^{-1}"'] \\ a(a(q),a(r),a(s),a(t)) \arrow[u, "\mu_{q,r,s,t}"] \arrow[r, "\mu_{a(q)a(r),a(s),a(t)}^{-1}"'] & a(a(a(q),a(r)),a(a(s)),a(a(t))) \arrow[u, "a(\mu_{q,r}{,}\mu_s{,}\mu_t)"'] \end{tikzcd}$$

are pastings of associativity squares for μ, and are therefore commutative. Pasting these diagrams along $\mu_{q,r,s,t}$ will confirm (4.1), and we conclude that $\mathbb{D}$ has the structure of a pseudodouble category.

Now, write $(\mathbb{D}, \overline{a}, \overline{\eta}, \overline{\mu})$ for the pseudo-$\mathfrak{F}$-algebra induced by the above double category. We define $\gamma \colon \overline{a} \to a$ to be the natural transformation inductively given by

$$\begin{aligned} \gamma_{()} &= \mathsf{id}, \\ \gamma_{r_1,\dots,r_n,r_{n+1}} &= \mu_{r_1 \cdots r_n, r_{n+1}} \circ a(\gamma_{r_1,\dots,r_n}, \eta_{r_{n+1}}) \end{aligned}$$

We claim that $(\mathsf{id}, \gamma) \colon (\mathbb{D}, a, \eta, \mu) \to (\mathbb{D}, \overline{a}, \overline{\eta}, \overline{\mu})$ is an invertible lax morphism of pseudo-$\mathfrak{F}$-algebras. First, note that

$$\gamma_r \circ \overline{\eta}_r = \mu_{-,r} \circ a(\mathsf{id}, \eta_r) \circ a(\mathsf{id}, \eta_r^{-1}) \circ \mu_{-,r}^{-1} \circ \eta_r = \eta_r,$$

and we shall prove that

$$\gamma_{r_{1,1},\dots,r_{n,k_n}} \circ \overline{\mu}_{k_1,\dots,k_n} = \mu_{(r_{1,i})_{i=1}^{k_1},\dots,(r_{n,i})_{i=1}^{k_n}} \circ \gamma_{a(r_{1,i})_{i=1}^{k_1},\dots,(r_{n,i})_{i=1}^{k_n}} \circ \overline{a}(\gamma_{r_{1,1},\dots,r_{1,k_1}}, \dots, \gamma_{r_{n,1},\dots,r_{n,k_n}}) \tag{4.2}$$

by double induction. When $n = 0$, the above reduces to

$$\mathsf{id} = \mu_{()} \circ a(\mathsf{id}),$$

which holds, so assume the above is true for some n. If $k_{n+1} = 0$, the left-hand side of (4.2) becomes

$$\begin{aligned} \gamma_{r_{1,1},\dots,r_{n,k_n}} \circ \overline{\mu}_{k_1,\dots,k_n,0} &= \gamma_{r_{1,1},\dots,r_{n,k_n}} \circ \overline{\mu}_{k_1,\dots,k_n} \circ \lambda \\ &= \lambda \circ a(\gamma_{r_{1,1},\dots,r_{n,k_n}}, \mathsf{id}) \circ a(\overline{\mu}_{k_1,\dots,k_n}, \mathsf{id}) \end{aligned}$$

while the right-hand side of (4.2) becomes

$$\begin{aligned} & \mu_{(r_{1,i}),\dots,(r_{n,i}),()} \circ \gamma_{a(r_{1,i}),\dots,a(r_{n,i})} \circ \overline{a}(\gamma_{r_{1,i}}, \dots, \gamma_{r_{n,i}}, \mathsf{id}) \\ & \quad = \mu_{(r_{1,i}),\dots,(r_{n,i}),()} \circ \mu_{a(r_{1,i}) \cdots a(r_{n,i}),a()} \circ a(\gamma_{a(r_{1,i}),\dots,a(r_{n,i})}, \eta_{a()}) \circ a(\overline{a}(\gamma_{r_{1,i}}, \dots, \gamma_{r_{n,i}}), \mathsf{id}) \\ & \quad = \mu_{r_{1,1} \cdots r_{n,i},()} \circ a(\mu_{(r_{1,i}),\dots,(r_{n,i})}, \mu_{(())}) \circ a(\gamma_{a(r_{1,i}),\dots,a(r_{n,i})}, \eta_{a()}) \circ a(\overline{a}(\gamma_{r_{1,i}}, \dots, \gamma_{r_{n,i}}), \mathsf{id}) \\ & \quad = \mu_{r_{1,1} \cdots r_{n,i},()} \circ a(\gamma_{r_{1,1},\dots,r_{n,k_n}}, \mathsf{id}) \circ a(\overline{\mu}_{k_1,\dots,k_n}, \mathsf{id}) \end{aligned}$$

so the equality holds by verifying that

$$\mu_{r_1 \cdots r_n,()} = \lambda,$$

which is equivalent to proving that the following diagram commutes

$$\begin{tikzcd}
a(a(a(r_1,\dots,r_n)),a()) \arrow[r, "\mu_{a(r_1,\dots,r_n),()}"] \arrow[d, "{a(\mu_{r_1\cdots r_n},\mathsf{id})}"'] & a(a(r_1,\dots,r_n)) \arrow[d, "\mu_{r_1\cdots r_n}"] \\
a(a(r_1,\dots,r_n),a()) \arrow[r, "\mu_{r_1\cdots r_n,()}"'] & a(r_1,\dots,r_n)
\end{tikzcd}$$

which is given by the coherence condition $\mu\circ\mathfrak{F}\mu=\mu\circ\mu_{\mathfrak{F}}$; recall that $\mu_{()}=\mathsf{id}$.

Hence, for the final induction step, we suppose (4.2) holds for some k_{n+1}. For $k_{n+1}+1$, the left-hand side of (4.2) is given by

$$\begin{aligned}
&\gamma_{r_{1,1},\dots,r_{n+1,k_{n+1}+1}}\circ\overline{\mu}_{k_1,\dots,k_{n+1}+1}\\
&\quad=\mu_{r_{1,1}\cdots r_{n+1,k_{n+1}},r_{n+1,k_{n+1}+1}}\circ a(\gamma_{r_{1,1},\dots,r_{n+1,k_{n+1}}},\eta_{r_{n+1,k_{n+1}+1}})\\
&\qquad\qquad\circ a(\overline{\mu}_{k_1,\dots,k_{n+1}},\mathsf{id})\circ\alpha\\
&\quad=\mu_{r_{1,1}\cdots r_{n+1,k_{n+1}},r_{n+1,k_{n+1}+1}}\circ a(\mu_{(r_{1,i}),\dots,(r_{n+1,i})},\mathsf{id})\\
&\qquad\qquad\circ a(\gamma_{a(r_{1,i}),\dots,a(r_{n+1,i})}\eta_{r_{n+1,k_{n+1}+1}})\circ\alpha\\
&\quad=\mu_{r_{1,1}\cdots r_{n+1,k_{n+1}},r_{n+1,k_{n+1}+1}}\circ a(\mu_{(r_{1,i}),\dots,(r_{n+1,i})},\mathsf{id})\\
&\qquad\qquad\circ a(\mu_{a(r_{1,i}\cdots r_{n,i}),a(r_{n+1,i})},\mathsf{id})\\
&\qquad\qquad\circ a(a(\gamma_{a(r_{1,i}),\dots,a(r_{n,i})},\eta_{a(r_{n+1,i})}),\mathsf{id})\\
&\qquad\qquad\circ a(a(\overline{a}(\gamma_{r_{1,i}},\dots,\gamma_{r_{n,i}}),\gamma_{r_{n+1},i}),id)\\
&\qquad\qquad\circ a(\mathsf{id},\eta_{r_{n+1,k_{n+1}+1}})\circ\alpha\\
&\quad=\mu_{r_{1,1}\cdots r_{n+1,k_{n+1}},r_{n+1,k_{n+1}+1}}\circ a(\mu_{(r_{1,i}\cdots r_{n,k_n}),(r_{n+1,i})},\mathsf{id})\\
&\qquad\qquad\circ a(a(\mu_{(r_{1,i}),\dots(r_{n,i})},\mu_{(r_{n+1,i})}),\mathsf{id})\circ\alpha\\
&\qquad\qquad\circ a(\gamma_{a(r_{1,i}),\dots,a(r_{n,i})},a(\eta_{a(r_{n+1,i})},\mathsf{id}))\\
&\qquad\qquad\circ a(\overline{a}(\gamma_{r_{1,i}},\dots,\gamma_{r_{n,i}}),a(\gamma_{r_{n+1},i},\eta_{r_{n+1,k_{n+1}+1}}))\\
&\quad=\mu_{r_{1,1}\cdots r_{n+1,k_{n+1}},r_{n+1,k_{n+1}+1}}\circ a(\mu_{(r_{1,i}\cdots r_{n,k_n}),(r_{n+1,i})},\mathsf{id})\circ\alpha\\
&\qquad\qquad\circ a(\gamma_{r_{1,1},\dots,r_{n,k_n}},\mathsf{id})\\
&\qquad\qquad\circ a(\overline{\mu}_{k_1,\dots,k_n},\mathsf{id})\\
&\qquad\qquad\circ a(\mathsf{id},a(\gamma_{r_{n+1},i},\eta_{r_{n+1,k_{n+1}+1}})),
\end{aligned}$$

while the right-hand side is given by

$$\begin{aligned}
&\mu_{(r_{1,i}),\dots,(r_{n,i}),(r_{n+1,i},r_{n+1,k_{n+1}+1})}\circ\gamma_{a(r_{1,i}),\dots,a(r_{n,i}),a(r_{n+1,i},r_{n+1,k_{n+1}+1})}\circ\overline{a}(\gamma_{r_{1,i}},\dots,\gamma_{r_{n,i}},\gamma_{r_{n+1,i},r_{n+1,k_{n+1}+1}})\\
&=\mu_{(r_{1,i}),\dots,(r_{n,i}),(r_{n+1,i},r_{n+1,k_{n+1}+1})}\circ\mu_{a(r_{1,i})\cdots a(r_{n,i}),a(r_{n+1,i},r_{n+1,k_{n+1}+1})}\\
&\qquad\qquad\circ a(\gamma_{a(r_{1,i}),\dots,a(r_{n,i})},\eta_{a(r_{n+1,i},r_{n+1,k_{n+1}+1})})\\
&\qquad\qquad\circ a(\overline{a}(\gamma_{r_{1,i}},\dots,\gamma_{r_{n,i}}),\gamma_{r_{n+1,i},r_{n+1,k_{n+1}+1}})\\
&=\mu_{(r_{1,1},\dots,r_{n,k_n}),(r_{n+1,i},r_{n+1,k_{n+1}+1})}\circ a(\mu_{(r_{1,i}),\dots,(r_{n,i})},\mu_{(r_{n+1,i}r_{n+1,k_{n+1}+1})})\\
&\qquad\qquad\circ a(\gamma_{a(r_{1,i}),\dots,a(r_{n,i})},\eta_{a(r_{n+1,i},r_{n+1,k_{n+1}+1})})\\
&\qquad\qquad\circ a(\overline{a}(\gamma_{r_{1,i}},\dots,\gamma_{r_{n,i}}),\mathsf{id})\\
&\qquad\qquad\circ a(\mathsf{id},\mu_{(r_{n+1,i}),r_{n+1,k_{n+1}+1}})\\
&\qquad\qquad\circ a(\mathsf{id},a(\gamma_{r_{n+1,i}},\eta_{r_{n+1,k_{n+1}+1}}))\\
&=\mu_{(r_{1,1},\dots,r_{n,k_n}),(r_{n+1,i},r_{n+1,k_{n+1}+1})}\circ a(\gamma_{r_{1,1},\dots,r_{n,k_n}},\mathsf{id})\\
&\qquad\qquad\circ a(\overline{\mu}_{k_1,\dots,k_n},\mathsf{id})\\
&\qquad\qquad\circ a(\mathsf{id},\mu_{(r_{n+1,i}),r_{n+1,k_{n+1}+1}})\\
&\qquad\qquad\circ a(\mathsf{id},a(\gamma_{r_{n+1,i}},\eta_{r_{n+1,k_{n+1}+1}})),
\end{aligned}$$

and therefore, proving (4.2) reduces to verifying that

$$\begin{aligned}&\mu_{r_{1,1}\cdots r_{n+1,k_{n+1}},r_{n+1,k_{n+1}+1}} \circ a(\mu_{(r_{1,i}\cdots r_{n,k_n}),(r_{n+1,i})}, \mathsf{id}) \circ \alpha \\ &\quad = \mu_{(r_{1,1},\ldots,r_{n,k_n}),(r_{n+1,i},r_{n+1,k_{n+1}+1})} \circ a(\mathsf{id}, \mu_{(r_{n+1,i}),r_{n+1,k_{n+1}+1}}).\end{aligned}$$

Here, we have

$$\begin{aligned}\alpha = {} & a(\mathsf{id}, \mu_{r_{n+1,k_{n+1}+1}}) \circ \mu^{-1}_{a(r_{1,1},\ldots,r_{n,k_n})a(r_{n+1,i}),a(r_{n+1,k_{n+1}+1})} \\ & \circ \mu_{a(r_{1,1},\ldots,r_{n,k_n}),a(r_{n+1,i})a(r_{n+1,k_{n+1}+1})} \\ & \circ a(\mu^{-1}_{r_{1,1}\cdots r_{n,k_n}}, \mathsf{id}),\end{aligned}$$

so we just need to verify that

$$\begin{aligned}&\mu_{r_{1,1}\cdots r_{n+1,k_{n+1}},r_{n+1,k_{n+1}+1}} \circ a(\mu_{(r_{1,i}\cdots r_{n,k_n}),(r_{n+1,i})}, \mu_{r_{n+1,k_{n+1}+1}}) \circ \mu^{-1}_{a(r_{1,1},\ldots,r_{n,k_n})a(r_{n+1,i}),a(r_{n+1,k_{n+1}+1})} \\ &\quad = \mu_{(r_{1,1},\ldots,r_{n,k_n}),(r_{n+1,i},r_{n+1,k_{n+1}+1})} \circ a(\mu_{r_{1,1}\cdots r_{n,k_n}}\mu_{(r_{n+1,i}),r_{n+1,k_{n+1}+1}}) \circ \mu^{-1}_{a(r_{1,1},\ldots,r_{n,k_n}),a(r_{n+1,i})a(r_{n+1,k_{n+1}+1})},\end{aligned}$$

which holds, since both sides of the above expression equal

$$\mu_{r_{1,1}\cdots r_{n,k_n},(r_{n+1,i}),r_{n+1,k_{n+1}+1}},$$

confirming that (id, γ) is an invertible pseudo-morphism of pseudo-$\mathfrak{F}$-algebras.

We consider double categories, lax (horizontal) and oplax (vertical) functors as in the following diagram, and the respective diagram in the double category $\mathsf{Ps}\text{-}\mathfrak{F}\text{-}\mathsf{Alg}$.

(4.3)
$$\begin{tikzcd}
\mathbb{A} \arrow[r, "H"] \arrow[d, "F"'] & \mathbb{B} \arrow[d, "G"] & (\mathbb{A}, a, \eta, \mu) \arrow[r, "{(H,\gamma^H)}"] \arrow[d, "{(F,\delta^F)}"'] & (\mathbb{B}, b, \eta, \mu) \arrow[d, "{(G,\delta^G)}"] \\
\mathbb{C} \arrow[r, "K"'] & \mathbb{D} & (\mathbb{C}, c, \eta, \mu) \arrow[r, "{(K,\gamma^K)}"'] & (\mathbb{D}, d, \eta, \mu)
\end{tikzcd}$$

Let ω be a 2-cell $GH \to KF$ of internal Cat-graphs. The claim is that ω is a generalized vertical transformation if and only if ω is a generalized 2-cell of pseudo-$\mathfrak{F}$-algebras.

If ω is a generalized vertical transformation, we wish to prove that the following diagram commutes

$$\begin{tikzcd}
& Gb(Hr_1, \ldots, Hr_n) \arrow[dl, "\delta^G_{Hr_1,\ldots,Hr_n}"'] \arrow[dr, "G\gamma^H_{r_1,\ldots,r_n}"] & \\
d(GHr_1, \ldots, GHr_n) \arrow[d, "{d(\omega_{r_1},\ldots,\omega_{r_n})}"'] & & GHa(r_1, \ldots, r_n) \arrow[d, "\omega_{a(r_1,\ldots,r_n)}"] \\
d(KFr_1, \ldots, KFr_n) \arrow[dr, "\gamma^K_{Fr_1,\ldots,Fr_n}"'] & & KFa(r_1, \ldots, r_n) \arrow[dl, "K\delta^F_{r_1,\ldots,r_n}"] \\
& Kc(Fr_1, \ldots, Fr_n) &
\end{tikzcd}$$

For all n and all horizontal 1-cells $r_1, \ldots, r_n$. We proceed by induction: when $n = 0$, the above is just coherence of ω for the unit comparsion. If true for some n, then

$$\begin{aligned}K\delta^F_{r_1,\ldots,r_{n+1}} & \circ \omega_{a(r_1,\ldots,r_{n+1})} \circ G\gamma^H_{r_1,\ldots,r_{n+1}} \\ &= K(\mathsf{id} \cdot \delta^F_{r_1,\ldots,r_n}) \circ K\,\mathsf{m}^F \circ \omega_{r_{n+1}\cdot a(r_1,\ldots,r_n)} \circ G\,\mathsf{m}^H \circ G(\mathsf{id} \cdot \gamma^H_{r_1,\ldots,r_n}) \\ &= K(\mathsf{id} \cdot \delta^F_{r_1,\ldots r_n}) \circ \mathsf{m}^K \circ (\omega_{r_{n+1}} \cdot \omega_{a(r_1,\ldots,r_n)}) \circ \mathsf{m}^G \circ G(\mathsf{id} \cdot \gamma^H_{r_1,\ldots,r_n}) \\ &= \mathsf{m}^K \circ (\mathsf{id} \cdot K\delta^F_{r_1,\ldots,r_n}) \circ (\omega_{r_{n+1}} \cdot \omega_{a(r_1,\ldots,r_n)}) \circ (\mathsf{id} \cdot G\gamma^H_{r_1,\ldots,r_n}) \circ \mathsf{m}^G \\ &= \mathsf{m}^K \circ (\mathsf{id} \cdot \gamma^K_{Fr_1,\ldots,Fr_n}) \circ (\omega_{r_{n+1}} \cdot d(\omega_{r_1}, \ldots, \omega_{r_n})) \circ (\mathsf{id} \cdot \delta^G_{Hr_1,\ldots,Hr_n}) \circ \mathsf{m}^G \\ &= \gamma^K_{Fr_1,\ldots,Fr_{n+1}} \circ d(\omega_{r_1}, \ldots, \omega_{r_{n+1}}) \circ \delta^G_{Hr_1,\ldots,Hr_{n+1}},\end{aligned}$$

so ω is a pseudo-$\mathfrak{F}$-algebra 2-cell as well.

Now, if ω is a pseudo-$\mathfrak{F}$-algebra 2-cell, the coherence for the unit comparison holds by definition, and

$$\begin{aligned}
&\mathsf{m}^K_{Fr,Fs} \circ (\omega_s \cdot \omega_r) \circ \mathsf{m}^G_{Hr,Hs} \\
&\quad = K(\mathsf{id} \cdot \rho) \circ \gamma^K_{Fr,Fs} \circ (\mathsf{id} \cdot \rho^{-1}) \circ (\omega_s \cdot \omega_r) \circ (\mathsf{id} \cdot \rho) \circ \delta^G_{Hr,Hs} \circ G(\mathsf{id} \cdot \rho^{-1}) \\
&\quad = K(\mathsf{id} \cdot \rho) \circ \gamma^K_{Fr,Fs} \circ (\omega_s \cdot (\omega_r \cdot \mathsf{id})) \circ \delta^G_{Hr,Hs} \circ G(\mathsf{id} \cdot \rho^{-1}) \\
&\quad = K(\mathsf{id} \cdot \rho) \circ \gamma^K_{Fr,Fs} \circ d(\omega_r, \omega_s) \circ \delta^G_{Hr,Hs} \circ G(\mathsf{id} \cdot \rho^{-1}) \\
&\quad = K(\mathsf{id} \cdot \rho) \circ K\delta^F_{r,s} \circ \omega_{a(r,s)} \circ G\gamma^H_{r,s} \circ G(\mathsf{id} \cdot \rho^{-1}) \\
&\quad = K(\mathsf{id} \cdot \rho) \circ K\delta^F_{r,s} \circ KF(\mathsf{id} \cdot \rho^{-1}) \circ \omega_{s \cdot r} \circ GH(\mathsf{id} \cdot \rho) \circ G(\mathsf{id} \cdot \rho^{-1}) \\
&\quad = K(\mathsf{m}^F) \circ \omega_{s \cdot r} \circ G(\mathsf{m}^H),
\end{aligned}$$

verifies coherence for composition comparsion, completing our proof.

Now, as promised at the start of Section 2, we obtain:

Proposition 4.2. *We have a conjunction*

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathsf{Mat} \arrow[r, bend left, "- \bullet 1"] \arrow[r, phantom, "\perp"] & \mathsf{Span}(\mathcal{V}) \arrow[l, bend left, "\mathcal{V}(1,-)"]
\end{tikzcd}$$

in the double category $\mathsf{PsDbCat}$.

Proof. Via the equivalence $\mathsf{PsDbCat} \simeq \mathsf{Ps}\text{-}\mathfrak{F}\text{-}\mathsf{Alg}$, we simply apply Proposition 3.1 to the adjunction $- \bullet 1 \dashv \mathcal{V}(1,-)$ in $\mathsf{Grph}(\mathsf{Cat})$, with the oplax functor structure of $- \bullet 1 \colon \mathsf{Span}(\mathcal{V}) \to \mathcal{V}\text{-}\mathsf{Mat}$, all of which were described in Section 2. ☐

5. HORIZONTAL LAX ALGEBRAS AND CHANGE OF BASE

We will review the notion of categories of *horizontal lax algebras* introduced in [15], and we define the *change-of-base* functors between such categories, induced by an appropriate notion of monad morphism. We begin by fixing monads $S = (\mathbb{D}, S, m, e)$ and $T = (\mathbb{E}, T, m, e)$ in the 2-category $\mathsf{PsDbCat}_{\mathrm{lax}}$.

We define the category $\mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$ of *horizontal lax T-algebras*, as follows:

- Objects are given by 4-tuples (x, a, υ, μ) where x is a 0-cell, $a \colon Tx \nrightarrow x$ is a horizontal 1-cell, and υ, μ are 2-cells

$$\begin{tikzcd}
x \arrow[r, "1"] \arrow[d, "e"'] \arrow[dr, phantom, "\upsilon"] & x \arrow[d, equal] \\
Tx \arrow[r, "a"'] & x
\end{tikzcd}
\qquad
\begin{tikzcd}
TTx \arrow[r, "Ta"] \arrow[d, "m"'] & Tx \arrow[r, "a"] \arrow[d, phantom, "\mu"] & a \arrow[d, equal] \\
Tx \arrow[rr, "a"'] & & x
\end{tikzcd}$$

satisfying

$$\begin{gathered}
\mu \circ (\upsilon \cdot e_a) = \lambda \\
\mu \circ (\mathsf{id} \cdot (T\upsilon \circ \mathsf{e}^T)) = \rho \\
\mu \circ (\mathsf{id} \cdot (T\mu \circ \mathsf{m}^T)) = \mu \circ (\mu \cdot m_a) \circ \alpha^{-1}
\end{gathered}$$

- A morphism $(x, a, \upsilon, \mu) \to (y, b, \upsilon, \mu)$ is a pair (f, ζ) where $f \colon x \to y$ is a vertical 1-cell and ζ is a 2-cell

$$\begin{tikzcd}
Tx \arrow[r, "a"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\zeta"] & x \arrow[d, "f"] \\
Ty \arrow[r, "b"'] & y
\end{tikzcd}$$

  satisfying $\zeta \circ \upsilon = \upsilon \circ 1_f$ and $\zeta \circ \mu = \mu \circ (\zeta \cdot T\zeta)$.

It should be noted that $\mathsf{id} = (\mathsf{id}, \mathsf{id}) \colon (x, a, \upsilon, \mu) \to (x, a, \upsilon, \mu)$ is a horizontal lax T-algebra morphism, and if (f, ζ), (g, ξ) are composable horizontal lax T-algebra morphisms, then so is $(g, \xi) \circ (f, \zeta) = (g \circ f, \xi \circ \zeta)$. Associativity and identity properties are inherited from $\mathbb{E}_0$ and $\mathbb{E}_1$, making $\mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$ into a category.

Our work focuses on the cases $\mathbb{E} = \mathsf{Span}(\mathcal{V})$, with T induced by a cartesian monad (also denoted by T) on $\mathcal{V}$, and $\mathbb{D} = \mathcal{V}\text{-}\mathsf{Mat}$ with S a lax monad. Then, $\mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg} = \mathsf{Cat}(T,\, \mathcal{V})$ is the category of *internal* T*-categories* of [22], while $\mathbb{H}\,\mathsf{Lax}\text{-}S\text{-}\mathsf{Alg} = (S,\, \mathcal{V})\text{-}\mathsf{Cat}$ is a generalization of the category of *enriched* S*-categories* introduced by [12], by not requiring S to be normal.[3]

Let $(F, \phi)\colon S \to T$ be a monad oplax morphism, and we assume $\mathbb{E}$ is conjoint closed. By Theorem 3.6, ϕ has a conjoint, given by a lax horizontal transformation $\phi^*\colon TF \to FS$. We define a 2-cell $\mathsf{e}^{\phi^*_x}$ for each 0-cell x given by

$$\begin{tikzcd}
Fx \arrow[r, "1"] \arrow[d, "e_{Fx}"'] \arrow[dr, phantom, "\mathsf{e}^{\phi^*_x}"] & Fx \arrow[d, "Fe_x"] \\
TFx \arrow[r, "\phi^*_x"'] & FSx
\end{tikzcd}$$

as the mate of the commutative square $\phi_x \circ Fe_x = e_{Fx} \circ \mathsf{id}$, and a 2-cell $\mathsf{m}^{\phi^*_x}$ given by

$$\begin{tikzcd}
TTFx \arrow[r, "(T\phi_x)^*"] \arrow[d, equal] & TFSx \arrow[r, "\phi^*_{Sx}"] \arrow[d, phantom, "\pi"] & FSSx \arrow[d, equal] \\
TTFx \arrow[rr, "(T\phi_x \circ \phi_{Sx})^*" description] \arrow[d, "m_{Fx}"'] & \arrow[d, phantom, "1^\vee"] & FSSx \arrow[d, "Fm_x"] \\
TFx \arrow[rr, "\phi^*_x"'] & & FSx
\end{tikzcd}$$

where π is given as in (3.2), and $1^\vee$ is the mate of the commutative square $\phi_x \circ Fm_x = m_{Fx} \circ (T\phi_x \circ \phi_{Sx})$. To be explicit, via mate correspondence we have

$$\varepsilon \circ \mathsf{e}^{\phi^*_x} = 1_{e_{Fx}}, \quad \text{and} \quad \varepsilon \circ \mathsf{m}^{\phi^*_x} = 1_{m_{Fx}} \circ \rho \circ ((1_{T\phi_x} \circ \varepsilon) \cdot \varepsilon). \tag{5.1}$$

Analogously, when $\mathbb{D}$ is companion closed, we define 2-cells $\mathsf{e}^{\psi_!}_y$ and $\mathsf{m}^{\psi_!}_y$ for a monad lax morphism $(G, \psi)\colon T \to S$.

Lemma 5.1. *If* $(F, \phi)\colon S \to T$ *is a monad oplax morphism and* $\mathbb{E}$ *is conjoint closed, then* e^{ϕ^*} *and* m^{ϕ^*} *are modifications, and the following relations hold:*

- *(a)* $\mathsf{m}^{\phi^*}_x \circ (\mathsf{e}^{\phi^*}_{Sx} \cdot e^\vee_{\phi_x}) = \lambda$,
- *(b)* $\mathsf{m}^{\phi^*}_x \circ (\phi^*_e \cdot \mathsf{e}^{(T\phi)^*}_x) = \rho$,
- *(c)* $\mathsf{m}^{\phi^*}_x \circ (\mathsf{m}^{\phi^*}_{Sx} \cdot m^\vee_{\phi_x}) = \mathsf{m}^{\phi^*_x} \circ (\phi^*_m \cdot \mathsf{m}^{(T\phi)^*}_x) \circ \alpha$,

where $e^\vee_{\phi_x}$ *and* $m^\vee_{\phi_x}$ *are the mates of the naturality squares of* e *and* m *at* ϕ_x*, and* $\mathsf{e}^{(T\phi)^*}_x$, $\mathsf{m}^{(T\phi)^*_x}$ *are respectively given by the mate of the commutative square* $T\phi_x \circ TFe_x = Te_{Fx} \circ \mathsf{id}$*, and the mate of the commutative square* $T\phi_x \circ TFm_x = Tm_{Fx} \circ (TT\phi_x \circ T\phi_{Sx})$ *composed with* π*, satisfying properties similar to* (5.1)*.*

Proof. Note that π is given as a 2-cell (modification) in $\mathsf{Lax}(\mathbb{D}, \mathbb{E})_{\text{lax}}$, and e^{ϕ^*} and $1^\vee$ are mates of equations of vertical 1-cells. It follows that e^{ϕ^*} and m^{ϕ^*} are modifications.

We have

$$\begin{aligned}
\varepsilon \circ \mathsf{m}^{\phi^*}_x \circ (\mathsf{e}^{\phi^*}_{Sx} \cdot e^\vee_{\phi_x}) &= 1_{m_{Fx}} \circ \lambda \circ ((1_{T\phi_x} \circ \varepsilon) \cdot \varepsilon) \circ (\mathsf{e}^{\phi^*}_{Sx} \cdot e^\vee_{\phi_x}) \\
&= 1_{m_{Fx}} \circ \lambda \circ (1_{T\phi_x \circ e_{FSx}} \cdot (1_{e_{TFx}} \circ \varepsilon)) \\
&= 1_{m_{Fx}} \circ 1_{e_{TFx}} \circ \varepsilon \circ \lambda = \varepsilon \circ \lambda,
\end{aligned}$$

$$\begin{aligned}
\varepsilon \circ \mathsf{m}^{\phi^*}_x \circ (\phi^*_e \cdot \mathsf{e}^{(T\phi)^*}_x) &= 1_{m_{Fx}} \circ \rho \circ ((1_{T\phi_x} \circ \varepsilon) \cdot \varepsilon) \circ (\phi^*_e \cdot \mathsf{e}^{(T\phi)^*}_x) \\
&= 1_{m_{Fx}} \circ \rho \circ ((1_{T\phi_x \circ TFe_x} \circ \varepsilon) \cdot 1_{Te_{Fx}}) \\
&= 1_{m_{Fx}} \circ 1_{Te_{Fx}} \circ \varepsilon \circ \rho = \varepsilon \circ \rho,
\end{aligned}$$

Now, we note that

$$\varepsilon \circ \mathsf{m}^{\phi^*}_x \circ (\mathsf{m}^{\phi^*}_{Sx} \cdot m^\vee_{\phi_x}) = \rho \circ \big((1_{m_{Fx} \circ T\phi_x} \circ \varepsilon \circ \mathsf{m}^{\phi^*}_{Sx}) \cdot (1_{m_{Fx}} \circ \varepsilon \circ m^\vee_{\phi_x})\big), \tag{5.2}$$

and we note that

$$\varepsilon \circ \mathsf{m}^{\phi^*}_{Sx} = \rho \circ \big((1_{m_{FSx} \circ T\phi_{Sx}} \circ \varepsilon) \cdot (1_{m_{FSx}} \circ \varepsilon)\big),$$

---

[3] When $\mathcal{V}$ is a quantale, this generalization is already present in [40].

and

$$\varepsilon \circ m^{\vee}_{\phi_x} = 1_{m_{FSx}} \circ \varepsilon,$$

so that (5.2) becomes

$$\varepsilon \circ \mathsf{m}^{\phi^*}_x \circ (\mathsf{m}^{\phi^*}_{Sx} \cdot m^{\vee}_{\phi_x}) = \rho \circ (\rho \cdot \mathsf{id}) \circ ((A \cdot B) \cdot C)$$

where $A = 1_{\hat{A}} \circ \varepsilon$, $B = 1_{\hat{B}} \circ \varepsilon$, $C = 1_{\hat{C}} \circ \varepsilon$, and

- $\hat{A} = m_{Fx} \circ T\phi_x \circ m_{FSx} \circ T\phi_{Sx}$,
- $\hat{B} = m_{Fx} \circ T\phi_x \circ m_{FSx}$,
- $\hat{C} = m_{Fx} \circ m_{TFx}$.

On the other hand, we have

$$\varepsilon \circ \mathsf{m}^{\phi^*_x} \circ (\phi^*_m \cdot \mathsf{m}^{(T\phi)^*}_x) \circ \alpha = \rho \circ \big((1_{m_{Fx} \circ T\phi_x} \circ \varepsilon \circ \phi^*_m) \cdot (1_{m_{Fx}} \circ \varepsilon \circ \mathsf{m}^{(T\phi)^*}_x)\big), \tag{5.3}$$

and we note that

$$\varepsilon \circ \phi^*_m = 1_{TFm_x} \circ \varepsilon,$$

and

$$\varepsilon \circ \mathsf{m}^{(T\phi)^*}_x = \lambda \circ \big((1_{Tm_{Fx} \circ TT\phi_x} \circ \varepsilon) \cdot (1_{Tm_{Fx}} \circ \varepsilon)\big),$$

so that (5.3) becomes

$$\varepsilon \circ \mathsf{m}^{\phi^*_x} \circ (\phi^*_m \cdot \mathsf{m}^{(T\phi)^*}_x) \circ \alpha = \rho \circ (\mathsf{id} \cdot \lambda) \circ \alpha \circ ((X \cdot Y) \cdot Z),$$

where $X = 1_{\hat{X}} \circ \varepsilon$, $Y = 1_{\hat{Y}} \circ \varepsilon$, $Z = 1_{\hat{Z}} \circ \varepsilon$, and

- $\hat{X} = m_{Fx} \circ T\phi_x \circ TFm_x$,
- $\hat{Y} = m_{Fx} \circ Tm_{Fx} \circ TT\phi_x$,
- $\hat{Z} = m_{Fx} \circ Tm_{Fx}$.

We conclude the proof by observing that $A = X$, $B = Y$ and $C = Z$. □

Theorem 5.2. *We suppose that $(F, \phi)\colon S \to T$ is an monad oplax morphism and that $\mathbb{E}$ is conjoint closed. If ϕ and $T\phi$ have strong conjoints (see Lemma 3.7), then (F, ϕ) induces a functor $F_!\colon \mathbb{H}\,\mathsf{Lax}\text{-}S\text{-}\mathsf{Alg} \to \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$.*

Analogously, if we suppose that $(G, \psi)\colon T \to S$ is a monad lax morphism and that $\mathbb{D}$ is companion closed, then (G, ψ) induces a functor $G_!\colon \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg} \to \mathbb{H}\,\mathsf{Lax}\text{-}S\text{-}\mathsf{Alg}$.

Proof. The functors $F_!$ and $G_!$ are given on objects by

$$F_!(x, a, \upsilon, \mu) = (Fx, Fa \cdot \phi^*_x, F_!\upsilon, F_!\mu) \quad \text{and} \quad G_!(y, b, \upsilon, \mu) = (Gy, Gb \cdot \psi_{y!}, G_!\upsilon, G_!\mu),$$

where $F_!\upsilon$, $F_!\mu$ are respectively given by the following 2-cells:

$$\begin{tikzcd}
Fx \arrow[r, "1"] \arrow[dd, "e_{Fx}"'] & Fx \arrow[r, "1"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{e}^F"] & Fx \arrow[d, equal] \\
\arrow[r, phantom, "\mathsf{e}^{\phi^*}_x"] & Fx \arrow[r, "F1" description] \arrow[d, "Fe_x" description] \arrow[dr, phantom, "F\upsilon"] & Fx \arrow[d, equal] \\
TFx \arrow[r, "\phi^*_x"'] & FSx \arrow[r, "Fa"'] & Fx
\end{tikzcd}$$

$$\begin{tikzcd}
TTFx \arrow[rr, "T(Fa \cdot \phi^*_x)"] \arrow[d, equal] \arrow[drr, phantom, "\theta^T_\phi"] & & TFx \arrow[r, "\phi^*_x" description] \arrow[d, equal] & FSx \arrow[r, "Fa" description] \arrow[d, equal] & Fx \arrow[dd, equal] \\
TTFx \arrow[r, "(T\phi_x)^*" description] \arrow[d, equal] & TFSx \arrow[r, "TFa" description] \arrow[d, equal] \arrow[dr, phantom, "(\mathsf{n}^{\phi^*}_a)^{-1}"] & TFx \arrow[r, "\phi^*_x" description] & FSx \arrow[d, equal] & \\
TTFx \arrow[r, "(T\phi_x)^*" description] \arrow[dd, "m_{Fx}" description] & TFSx \arrow[r, "\phi^*_{Sx}" description] \arrow[d, phantom, "\mathsf{m}^{\phi^*}_x"] & FSSx \arrow[r, "FSa" description] \arrow[d, equal] & FSx \arrow[r, "Fa" description] \arrow[d, phantom, "\mathsf{m}^F"] & Fx \arrow[d, equal] \\
 & & FSSx \arrow[rr, "F(a \cdot Sa)" description] \arrow[d, "Fm_x" description] & \arrow[d, phantom, "F\mu"] & Fx \arrow[d, equal] \\
TFx \arrow[rr, "\phi^*_x"'] & & FSx \arrow[rr, "Fa"'] & & Fx
\end{tikzcd}$$

where θ^T_ϕ is the inverse of $\mathsf{m}^T \circ(\mathsf{id} \cdot \sigma^T)$, and $G_!\upsilon$, $G_!\mu$ are respectively given by the following 2-cells:

$$\begin{tikzcd}
Gy \arrow[r, "1"] \arrow[dd, "e_{Gy}"'] & Gy \arrow[r, "1"] \arrow[d, equal] \arrow[rd, phantom, "\mathsf{e}^G"] & Gy \arrow[d, equal] \\
{} \arrow[r, phantom, "\mathsf{e}^{\psi_!}_y"] & Gy \arrow[r, "G1" description] \arrow[d, "Ge_y" description] \arrow[rd, phantom, "G\upsilon"] & Gy \arrow[d, equal] \\
SGy \arrow[r, "\psi_{y!}"'] & GTy \arrow[r, "Gb"'] & Gy
\end{tikzcd}$$

$$\begin{tikzcd}
SSGy \arrow[rr, "S(Gb\cdot\psi_{y!})"] \arrow[d, equal] \arrow[rrd, phantom, "\theta^S_\psi"] & & SGy \arrow[r, "\psi_{y!}" description] \arrow[d, equal] & SGy \arrow[r, "Gb" description] \arrow[d, equal] & Fy \arrow[dd, equal] \\
SSGy \arrow[r, "(S\psi_y)_!" description] \arrow[d, equal] & SGTy \arrow[r, "SGb" description] \arrow[d, equal] & SGy \arrow[r, "\psi_{y!}" description] \arrow[d, phantom, "\mathsf{n}^{\psi_!}_b"] & GTy \arrow[d, equal] & \\
SSGy \arrow[r, "(S\psi_t)_!" description] \arrow[dd, "m_{Gx}" description] & SGTy \arrow[r, "\psi_{Tx!}" description] \arrow[d, phantom, "\mathsf{m}^{\psi_!}_y"] & GTTy \arrow[r, "GTb" description] \arrow[d, equal] & GTy \arrow[r, "Gb" description] \arrow[d, phantom, "\mathsf{m}^G"] & Gy \arrow[d, equal] \\
& {} & GTTy \arrow[rr, "G(b\cdot Tb)" description] \arrow[d, "Gm_y" description] & {} \arrow[d, phantom, "G\mu"] & Fy \arrow[d, equal] \\
SGy \arrow[rr, "\psi_{y!}"'] & & GTy \arrow[rr, "Gb"'] & {} & Gy
\end{tikzcd}$$

where θ^S_ψ is the inverse of $\mathsf{m}^S \circ(\mathsf{id} \cdot \tau^S)$, given by Lemma 3.5.

If $(f, \zeta)\colon (w, a, \upsilon, \mu) \to (x, b, \upsilon, \mu)$ is a horizontal lax S-algebra morphism, and if $(g, \xi)\colon (y, c, \upsilon, \mu) \to (z, d, \upsilon, \mu)$ is a horizontal lax T-algebra morphism, then we have

$$F_!(f, \zeta) = (Ff, F\zeta \cdot \phi^*_f) \quad \text{and} \quad G_!(g, \xi) = (Gg, G\xi \cdot \psi_{f!}).$$

We observe that ϕ and $T\phi$ are required to have strong conjoints only to guarantee the existence of $F_!\mu$. All other things being equal, up to a letter substitution, we conclude it is enough to verify that one of $F_!(x, a, \upsilon, \mu)$, $G_!(y, b, \upsilon, \mu)$ is a horizontal lax algebra, and likewise for the morphisms.

Throughout the calculations, we will use the following abbreviations:

- $\upsilon^F = F\upsilon \circ \mathsf{e}^F$,
- $\mu^F = F\mu \circ \mathsf{m}^F$,
- $\hat{\alpha} = (\mathsf{id} \cdot \alpha^{-1}) \circ \alpha$,
- $\tilde{\theta} = \mathsf{id} \cdot (\theta \cdot \mathsf{id})$ for a 2-cell θ.
- $N^\omega_p = \hat{\alpha}^{-1} \circ (\tilde{\mathsf{n}}^\omega_p)^{-1} \circ \hat{\alpha}$ for a strong (lax) horizontal transformation ω.

We begin by verifying that the following equalities hold:

$$\theta^T \circ e_{Fa\cdot\phi^*_x} = e_{Fa} \cdot e^\vee_{\phi_x}, \tag{5.4}$$

$$\theta^T \circ T(\upsilon^F \cdot \mathsf{e}^{\phi^*}_x) \circ T\rho^{-1} \circ \mathsf{e}^T = (\upsilon^{TF} \cdot \mathsf{e}^{(T\phi)^*}_x) \circ \rho^{-1}, \tag{5.5}$$

We obtain (5.4), via mate correspondence, by noting that

$$\begin{aligned}
\mathsf{m}^T \circ(\mathsf{id} \cdot \sigma^T) \circ (e_{Fa} \cdot e^\vee_{\phi_x}) \circ (\mathsf{id} \cdot \eta) \circ \rho^{-1} &= \mathsf{m}^T \circ(\mathsf{id} \cdot \sigma^T) \circ (\mathsf{id} \cdot \eta) \circ (e_{Fa} \cdot 1_{e_{FSx}}) \circ \rho^{-1} \\
&= \mathsf{m}^T \circ(\mathsf{id} \cdot T\eta) \circ (\mathsf{id} \cdot \mathsf{e}^T) \circ \rho^{-1} \circ e_{Fa} \\
&= T(\mathsf{id} \cdot \eta) \circ \mathsf{m}^T \circ(\mathsf{id} \cdot \mathsf{e}^T) \circ \rho^{-1} \circ e_{Fa} \\
&= T(\mathsf{id} \cdot \eta) \circ T\rho^{-1} \circ e_{Fa} \\
&= e_{Fa\cdot\phi^*_x} \circ (\mathsf{id} \cdot \eta) \circ \rho^{-1},
\end{aligned}$$

and (5.5), directly, since

$$\begin{aligned}\mathsf{m}^T \circ(\mathsf{id} \cdot \sigma^T) \circ (\upsilon^{TF} \cdot \mathsf{e}_x^{(T\phi)^*}) \circ \rho^{-1} &= \mathsf{m}^T \circ(\mathsf{id} \cdot T\eta) \circ (\mathsf{id} \cdot \mathsf{e}^T) \circ (\upsilon^{TF} \cdot 1_{TFe_x}) \circ \rho^{-1}\\ &= T(\mathsf{id} \cdot \eta) \circ \mathsf{m}^T \circ(\mathsf{id} \cdot \mathsf{e}^T) \circ \rho^{-1} \circ \upsilon^{TF}\\ &= T(\mathsf{id} \cdot \eta) \circ T\rho^{-1} \circ \upsilon^{TF}\\ &= T(\mathsf{id} \cdot \eta) \circ T(\upsilon^F \cdot 1_{Fe_x}) \circ T\rho^{-1} \circ \mathsf{e}^T\\ &= T(\upsilon^F \cdot \mathsf{e}_x^{\phi^*}) \circ T\rho^{-1} \circ \mathsf{e}^T .\end{aligned}$$

Furthermore, since e^{ϕ^*} is a modification and n^{ϕ^*} is natural, we respectively obtain

$$(\mathsf{n}_a^{\phi^*})^{-1} \circ (\mathsf{e}_x^{\phi^*} \circ e_{Fa}) = (Fe_a \cdot \mathsf{e}_{Sx}^{\phi^*}) \circ \gamma, \tag{5.6}$$

$$(\mathsf{n}_a^{\phi^*})^{-1} \circ (\mathsf{id} \cdot \upsilon^{TF}) = (\upsilon^{FS} \cdot \phi_e^*) \circ \gamma. \tag{5.7}$$

And lastly, we note that the following diagrams commute

$$\begin{tikzcd}
1 \cdot Fa \arrow[rrr, "\lambda"] \arrow[dr, "\mathsf{e}^F \cdot \mathsf{id}"'] \arrow[ddr, bend right, "\upsilon^F \cdot Fe_a"'] & & & Fa \\
& F1 \cdot Fa \arrow[r, "\mathsf{m}^F"] \arrow[d, "Fv \cdot Fe_a" description] & F(1 \cdot a) \arrow[ur, "F\lambda"'] \arrow[d, "F(\upsilon \cdot e_a)" description] & \\
& Fa \cdot FSa \arrow[r, "\mathsf{m}^F"'] & F(a \cdot Sa) \arrow[uur, bend right, "F\mu"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
Fa \cdot 1 \arrow[rrr, "\rho"] \arrow[dr, "\mathsf{id} \cdot \mathsf{e}^F"'] \arrow[ddr, bend right, "\mathsf{id} \cdot \upsilon^{FS}"'] & & & Fa \\
& Fa \cdot F1 \arrow[r, "\mathsf{m}^F"] \arrow[d, "\mathsf{id} \cdot F\upsilon^S" description] & F(a \cdot 1) \arrow[ur, "F\rho"'] \arrow[d, "F(\mathsf{id} \cdot \upsilon^S)" description] & \\
& Fa \cdot FSa \arrow[r, "\mathsf{m}^F"'] & F(a \cdot Sa) \arrow[uur, bend right, "F\mu"'] &
\end{tikzcd}$$

which respectively confirm that

$$\mu^F \circ (\upsilon^F \cdot Fe_a) = \lambda, \tag{5.8}$$

$$\mu^F \circ (\mathsf{id} \cdot \upsilon^{FS}) = \rho. \tag{5.9}$$

By applying (5.4), (5.6), (5.8), and (a) from Lemma 5.1, we obtain

$$\begin{aligned}&(\mu^F \cdot \mathsf{m}_x^{\phi^*}) \circ N_a^{\phi^*} \circ (\mathsf{id} \cdot \theta^T) \circ \big((\upsilon^F \cdot \mathsf{e}_x^{\phi^*}) \cdot e_{Fa \cdot \phi_x^*}\big) \circ (\lambda^{-1} \cdot \mathsf{id})\\ &\qquad = (\mu^F \cdot \mathsf{m}_x^{\phi^*}) \circ N_a^{\phi^*} \circ \big((\upsilon^F \cdot \mathsf{e}_x^{\phi^*}) \cdot (e_{Fa} \cdot e_{\phi_x}^{\vee})\big) \circ (\lambda^{-1} \cdot \mathsf{id})\\ &\qquad = (\mu^F \cdot \mathsf{m}_x^{\phi^*}) \circ \big((\upsilon^F \cdot Fe_a) \cdot (\mathsf{e}_{Sx}^{\phi^*} \cdot e_{\phi_x}^{\vee})\big) \circ N_a^1 \circ (\lambda^{-1} \cdot \mathsf{id})\\ &\qquad = (\lambda \cdot \lambda) \circ \hat{\alpha}^{-1} \circ \tilde{\gamma} \circ \hat{\alpha} \circ (\lambda^{-1} \cdot \mathsf{id}) = \lambda,\end{aligned}$$

verifying the left identity law, and by applying (5.5), (5.7), (5.9) and (b) from Lemma 5.1, we obtain

$$\begin{aligned}&(\mu^F \cdot \mathsf{m}_x^{\phi^*}) \circ N_a^{\phi^*} \circ (\mathsf{id} \cdot \theta^T) \circ \big(\mathsf{id} \cdot (T(\upsilon^F \cdot \mathsf{e}^{\phi^*}) \circ T\lambda^{-1} \circ \mathsf{e}^T)\big)\\ &\qquad = (\mu^F \cdot \mathsf{m}_x^{\phi^*}) \circ N_a^{\phi^*} \circ \big(\mathsf{id} \cdot (\upsilon^{TF} \cdot \mathsf{e}_x^{(T\phi)^*})\big) \circ (\mathsf{id} \cdot \rho^{-1})\\ &\qquad = (\mu^F \cdot \mathsf{m}_x^{\phi^*}) \circ \big((\mathsf{id} \cdot \upsilon^{FS}) \cdot (\phi_e^* \cdot \mathsf{e}_x^{(T\phi)^*})\big) \circ N_a^1 \circ (\mathsf{id} \cdot \rho^{-1})\\ &\qquad = (\rho \cdot \rho) \circ \hat{\alpha}^{-1} \circ \tilde{\gamma} \circ \hat{\alpha} \circ (\mathsf{id} \cdot \rho^{-1}) = \rho,\end{aligned}$$

verifying the right identity law.

Now, note that

$$\theta^T \circ m_{Fa \cdot \phi_x^*} = (m_{Fa} \cdot m_{\phi_x}^{\vee}) \circ \theta^{TT} \tag{5.10}$$

holds via mate correspondence, since we have

$$\begin{aligned}
m_{Fa\cdot\phi_x^*}\circ\mathsf{m}^{TT}\circ(\mathsf{id}\cdot\sigma^{TT})\circ(\mathsf{id}\cdot\eta)\circ\rho^{-1} &= m_{Fa\cdot\phi_x^*}\circ\mathsf{m}^{TT}\circ(\mathsf{id}\cdot TT\eta)\circ(\mathsf{id}\cdot\mathsf{e}^{TT})\circ\rho^{-1}\\
&= m_{Fa\cdot\phi_x^*}\circ TT(\mathsf{id}\cdot\eta)\circ\mathsf{m}^{TT}\circ(\mathsf{id}\cdot\mathsf{e}^{TT})\circ\rho^{-1}\\
&= T(\mathsf{id}\cdot\eta)\circ m_{Fa\cdot 1}\circ TT\rho^{-1}\\
&= T(\mathsf{id}\cdot\eta)\circ T\rho^{-1}\circ m_{Fa},\\
\mathsf{m}^{T}\circ(\mathsf{id}\cdot\sigma^{T})\circ(m_{Fa}\cdot m^{\vee}_{\phi_x})\circ(\mathsf{id}\cdot\eta)\circ\rho^{-1} &= \mathsf{m}^{T}\circ(\mathsf{id}\cdot\sigma^{T})\circ(\mathsf{id}\cdot\eta)\circ(m_{Fa}\cdot 1)\circ\rho^{-1}\\
&= \mathsf{m}^{T}\circ(\mathsf{id}\cdot T\eta)\circ(\mathsf{id}\cdot\mathsf{e}^{T})\circ\rho^{-1}\circ m_{Fa}\\
&= T(\mathsf{id}\cdot\eta)\circ\mathsf{m}^{T}\circ(\mathsf{id}\cdot\mathsf{e}^{T})\circ\rho^{-1}\circ m_{Fa}\\
&= T(\mathsf{id}\cdot\eta)\circ T\rho^{-1}\circ m_{Fa},
\end{aligned}$$

and since m^{ϕ^*} is a modification, we get

$$(\mathsf{n}_a^{\phi^*})^{-1}\circ(\mathsf{m}_x^{\phi^*}\cdot m_{Fa}) = (Fm_a\cdot\mathsf{m}_{Sx}^{\phi^*})\circ(\mathsf{n}_a^{\phi_S^*\cdot(T\phi)^*})^{-1}. \tag{5.11}$$

Now, note that the following diagram commutes

$$\begin{tikzcd}
(Fa\cdot FSa)\cdot FSSa \arrow[r, "\alpha"] \arrow[d, "\mathsf{m}^F\cdot\mathsf{id}"'] & Fa\cdot(FSa\cdot FSSa) \arrow[r, "\mathsf{id}\cdot\mathsf{m}^F"] & Fa\cdot F(Sa\cdot SSa) \arrow[r, "\mathsf{id}\cdot F\mu^S"] \arrow[d, "\mathsf{m}^F"] & Fa\cdot FSa \arrow[d, "\mathsf{m}^F"] \\
F(a\cdot Sa)\cdot FSSa \arrow[r, "\mathsf{m}^F"] \arrow[d, "F\mu\cdot Fm_a"'] & F((a\cdot Sa)\cdot SSa) \arrow[r, "F\alpha"] \arrow[d, "F(\mu\cdot m_a)"] & F(a\cdot(Sa\cdot SSa)) \arrow[r, "F(\mathsf{id}\cdot\mu^S)"] & F(a\cdot Sa) \arrow[d, "F\mu"] \\
Fa\cdot FSa \arrow[r, "\mathsf{m}^F"'] & F(a\cdot Sa) \arrow[rr, "F\mu"'] & & Fa
\end{tikzcd}$$

which confirms

$$\mu^F\circ(\mu^F\cdot Fm_a) = \mu^F\circ(\mathsf{id}\cdot\mu^{FS})\circ\alpha. \tag{5.12}$$

Our next step is to confirm that

$$((\mathsf{id}\cdot\mathsf{m}^{FS})\cdot\mathsf{id})\circ(\alpha\cdot\alpha)\circ N_a^{\phi_S^*\cdot(T\phi)^*}\circ(N_a^{\phi^*}\cdot\mathsf{id}) = N_{a\cdot Sa}^{\phi^*}\circ(\mathsf{id}\cdot(\mathsf{m}^{TF}\cdot\mathsf{id}))\circ(\mathsf{id}\cdot N_a^{(T\phi)^*})\circ\alpha. \tag{5.13}$$

First, we recall that

$$\mathsf{n}_a^{\phi_S^*\cdot(T\phi)^*} = \alpha^{-1}\circ(\mathsf{id}\cdot\mathsf{n}_a^{(T\phi)^*})\circ\alpha\circ(\mathsf{n}_{Sa}^{\phi^*}\cdot\mathsf{id})\circ\alpha^{-1},$$

and

$$\mathsf{n}_{a\cdot Sa}^{\phi^*}\circ(\mathsf{m}^{FS}\cdot\mathsf{id}) = (\mathsf{id}\cdot\mathsf{m}^{TF})\circ\alpha\circ(\mathsf{n}_a^{\phi^*}\cdot\mathsf{id})\circ\alpha^{-1}\circ(\mathsf{id}\cdot\mathsf{n}_{Sa}^{\phi^*})\circ\alpha,$$

and note that by coherence, we have

$$\begin{aligned}
\tilde{\alpha}\circ\hat{\alpha}\circ(\hat{\alpha}^{-1}\cdot\mathsf{id}) &= \alpha^{-1}\circ(\mathsf{id}\cdot\hat{\alpha})\circ(\mathsf{id}\cdot\hat{\alpha})\circ\alpha,\\
(\alpha\cdot\alpha)\circ\hat{\alpha}^{-1}\circ\tilde{\alpha} &= \hat{\alpha}^{-1}\circ\tilde{\alpha}^{-1}\circ\hat{\alpha}\circ(\mathsf{id}\cdot\alpha),\\
\hat{\alpha}\circ(\mathsf{id}\cdot\alpha)\circ\tilde{\alpha}^{-1}\circ\alpha^{-1}\circ(\mathsf{id}\cdot\hat{\alpha}) &= \tilde{\alpha}\circ(\mathsf{id}\cdot\hat{\alpha}^{-1}),\\
\hat{\alpha}^{-1}\circ\tilde{\alpha}\circ(\mathsf{id}\cdot\hat{\alpha})^{-1} &= (\mathsf{id}\cdot\hat{\alpha}^{-1})\circ\alpha^{-1}\circ(\mathsf{id}\cdot\alpha)\\
\alpha^{-1}\circ(\mathsf{id}\cdot\alpha)\circ(\mathsf{id}\cdot\hat{\alpha})\circ\alpha\circ(\hat{\alpha}\cdot\mathsf{id}) &= (\mathsf{id}\cdot\hat{\alpha})\circ\alpha,
\end{aligned}$$

so that

$$\begin{aligned}
&((\mathsf{id}\cdot\mathsf{m}^{FS})\cdot\mathsf{id})\circ(\alpha\cdot\alpha)\circ N_a^{\phi^*_S\cdot(T\phi)^*}\circ(N_a^{\phi^*}\cdot\mathsf{id})\\
&=((\mathsf{id}\cdot\mathsf{m}^{FS})\cdot\mathsf{id})\circ(\alpha\cdot\alpha)\circ\hat{\alpha}^{-1}\circ\tilde{\alpha}\circ(\mathsf{id}\cdot((\mathsf{n}_{Sa}^{\phi^*}\cdot\mathsf{id})\cdot\mathsf{id}))^{-1}\circ\tilde{\alpha}^{-1}\\
&\quad\circ(\mathsf{id}\cdot((\mathsf{id}\cdot\mathsf{n}_a^{(T\phi)^*})\cdot\mathsf{id}))^{-1}\circ\tilde{\alpha}\circ\hat{\alpha}\circ(\hat{\alpha}^{-1}\cdot\mathsf{id})\circ((\mathsf{id}\cdot(\mathsf{n}_a^{\phi^*}\cdot\mathsf{id}))\cdot\mathsf{id})^{-1}\circ(\hat{\alpha}\cdot\mathsf{id})\\
&=\hat{\alpha}^{-1}\circ(\mathsf{id}\cdot((\mathsf{m}^{FS}\cdot\mathsf{id})\cdot\mathsf{id}))\circ\tilde{\alpha}^{-1}\circ(\mathsf{id}\cdot((\mathsf{id}\cdot\mathsf{n}_{Sa}^{\phi^*})\cdot\mathsf{id}))^{-1}\\
&\quad\circ\hat{\alpha}\circ(\mathsf{id}\cdot\alpha)\circ\tilde{\alpha}^{-1}\circ\alpha^{-1}\circ(\mathsf{id}\cdot\hat{\alpha})\circ(\mathsf{id}\cdot(\mathsf{n}_a^{\phi^*}\cdot\mathsf{id}))^{-1}\\
&\quad\circ(\mathsf{id}\cdot(\mathsf{id}\cdot(\mathsf{n}_a^{(T\phi)^*}\cdot\mathsf{id})))^{-1}\circ(\mathsf{id}\cdot\hat{\alpha})\circ\alpha\circ(\hat{\alpha}\cdot\mathsf{id})\\
&=\hat{\alpha}^{-1}\circ(\mathsf{id}\cdot((\mathsf{m}^{FS}\cdot\mathsf{id})\cdot\mathsf{id}))\circ\tilde{\alpha}^{-1}\circ(\mathsf{id}\cdot((\mathsf{id}\cdot\mathsf{n}_{Sa}^{\phi^*})\cdot\mathsf{id}))^{-1}\circ\tilde{\alpha}\circ(\mathsf{id}\cdot((\mathsf{n}_a^{\phi^*}\cdot\mathsf{id})\cdot\mathsf{id}))^{-1}\\
&\quad\circ(\mathsf{id}\cdot\hat{\alpha})^{-1}\circ(\mathsf{id}\cdot(\mathsf{id}\cdot(\mathsf{n}_a^{(T\phi)^*}\cdot\mathsf{id})))^{-1}\circ(\mathsf{id}\cdot\hat{\alpha})\circ\alpha\circ(\hat{\alpha}\cdot\mathsf{id})\\
&=\hat{\alpha}^{-1}\circ(\mathsf{id}\cdot(\mathsf{n}_{a\cdot Sa}^{\phi^*}\cdot\mathsf{id}))^{-1}\circ(\mathsf{id}\cdot((\mathsf{id}\cdot\mathsf{m}^{TF})\cdot\mathsf{id}))\circ\tilde{\alpha}\circ(\mathsf{id}\cdot N_a^{(T\phi)^*})\circ\alpha\circ(\hat{\alpha}\cdot\mathsf{id})\\
&=N_{a\cdot Sa}^{\phi^*}\circ(\mathsf{id}\cdot(\mathsf{m}^{TF}\cdot\mathsf{id}))\circ\hat{\alpha}^{-1}\circ\tilde{\alpha}\circ(\mathsf{id}\cdot N_a^{(T\phi)^*})\circ\alpha\circ(\hat{\alpha}\cdot\mathsf{id})\\
&=N_{a\cdot Sa}^{\phi^*}\circ(\mathsf{id}\cdot(\mathsf{m}^{TF}\cdot\mathsf{id}))\circ(\mathsf{id}\cdot\hat{\alpha}^{-1})\circ(\mathsf{id}\cdot\tilde{\mathsf{n}}_a^{(T\phi)^*})^{-1}\circ\alpha^{-1}\circ(\mathsf{id}\cdot\alpha)\circ(\mathsf{id}\cdot\hat{\alpha})\circ\alpha\circ(\hat{\alpha}\cdot\mathsf{id})\\
&=N_{a\cdot Sa}^{\phi^*}\circ(\mathsf{id}\cdot(\mathsf{m}^{TF}\cdot\mathsf{id}))\circ(\mathsf{id}\cdot N_a^{(T\phi)^*})\circ\alpha.
\end{aligned}$$

Next, we observe that

$$(FS\mu\cdot\phi^*_m)\circ(\mathsf{n}_{a\cdot Sa}^{\phi^*})^{-1}=(\mathsf{n}_a^{\phi^*})^{-1}\circ(\mathsf{id}\cdot TF\mu)\tag{5.14}$$

holds by naturality of n^{ϕ^*}, and lastly we must confirm that

$$(\mu^{TF}\cdot\mathsf{m}_a^{(T\phi)^*})\circ N_a^{(T\phi)^*}\circ(\theta^T\cdot\theta^{TT})=\theta^T\circ T(\mu^F\cdot\mathsf{m}_x^{\phi^*})\circ TN_a^{\phi^*}\circ T(\mathsf{id}\cdot\theta^T)\circ\mathsf{m}^T,\tag{5.15}$$

which we reduce to proving the following relations:

$$TN_a^{\phi^*}\circ T(\mathsf{id}\cdot\theta^T)\circ\mathsf{m}^T\circ(\mathsf{m}^T\cdot\mathsf{m}^{TT})\circ((\mathsf{id}\cdot\sigma^T)\cdot(\mathsf{id}\cdot\sigma^{TT}))=\mathsf{m}^T\circ(\mathsf{m}^T\cdot\mathsf{m}^T)\circ(\mathsf{id}\cdot(\sigma^T\cdot\sigma^T))\circ N_a^{(T\phi)^*}$$

$$\sigma^T\circ\mathsf{m}_x^{(T\phi)^*}=T\,\mathsf{m}_x^{\phi^*}\circ\mathsf{m}^T\circ(\sigma^T\cdot\sigma^T)$$

For the first, we have the commutativity of the following diagram, omitting horizontal 1-cells

$$\begin{tikzcd}
\cdot \arrow[r, "\mathsf{id}\cdot(\mathsf{m}^T\cdot\mathsf{id})"] \arrow[d, "\mathsf{id}\cdot\alpha"'] & \cdot \arrow[r, "\mathsf{id}\cdot\mathsf{m}^T"] & \cdot \arrow[r, "\mathsf{m}^T"] \arrow[d, "\mathsf{id}\cdot T\alpha" description] & \cdot \arrow[d, "T(\mathsf{id}\cdot\alpha)"] \\
\cdot \arrow[r, "\mathsf{id}\cdot(\mathsf{id}\cdot\mathsf{m}^T)" description] \arrow[d, "\alpha^{-1}"'] & \cdot \arrow[r, "\mathsf{id}\cdot\mathsf{m}^T" description] \arrow[d, "\alpha^{-1}" description] & \cdot \arrow[r, "\mathsf{m}^T"] & \cdot \arrow[d, "T\alpha^{-1}"] \\
\cdot \arrow[r, "\mathsf{id}\cdot\mathsf{m}^T"'] & \cdot \arrow[r, "\mathsf{m}^T\cdot\mathsf{id}"'] & \cdot \arrow[r, "\mathsf{m}^T"'] & \cdot
\end{tikzcd}$$

then we observe that

$$\begin{aligned}
&T(\mathsf{id}\cdot\theta^T)\circ\mathsf{m}^T\circ(\mathsf{m}^T\cdot\mathsf{m}^{TT})\circ((\mathsf{id}\cdot\sigma^T)\cdot(\mathsf{id}\cdot\sigma^{TT}))\\
&\quad=\mathsf{m}^T\circ(\mathsf{id}\cdot T\theta^T)\circ(\mathsf{m}^T\cdot\mathsf{m}^{TT})\circ((\mathsf{id}\cdot\sigma^T)\cdot(\mathsf{id}\cdot\sigma^{TT}))\\
&\quad=\mathsf{m}^T\circ(\mathsf{m}^T\cdot\mathsf{m}^T)\circ((\mathsf{id}\cdot\sigma^T)\cdot(\mathsf{id}\cdot\sigma^T)),
\end{aligned}$$

and finally recall from (3.4) that

$$\mathsf{m}^T\circ(\sigma^T\cdot\mathsf{id})\circ\mathsf{n}_a^{(T\phi)^*}=T\,\mathsf{n}_a^{\phi^*}\circ\mathsf{m}^T\circ(\mathsf{id}\cdot\sigma^T),$$

so that we may calculate

$$\begin{aligned}
T(\mathsf{id}\cdot\theta^T)\circ\mathsf{m}^T\circ(\mathsf{m}^T\cdot\mathsf{m}^{TT})\circ((\mathsf{id}\cdot\sigma^T)\cdot(\mathsf{id}\cdot\sigma^{TT}))\circ\hat{\alpha}^{-1}\circ\tilde{\mathsf{n}}^{(T\phi)^*}\circ\hat{\alpha}&\\
=\mathsf{m}^T\circ(\mathsf{m}^T\cdot\mathsf{m}^T)\circ((\mathsf{id}\cdot\sigma^T)\cdot(\mathsf{id}\cdot\sigma^T))\circ\hat{\alpha}^{-1}\circ\tilde{\mathsf{n}}^{(T\phi)^*}\circ\hat{\alpha}&\\
=\mathsf{m}^T\circ(\mathsf{m}^T\cdot\mathsf{m}^T)\circ\hat{\alpha}^{-1}\circ(\mathsf{id}\cdot((\sigma^T\cdot\mathsf{id})\cdot\sigma^T))\circ\tilde{\mathsf{n}}^{(T\phi)^*}\circ\hat{\alpha}&\\
=T\hat{\alpha}^{-1}\circ\mathsf{m}^T\circ(\mathsf{id}\cdot\mathsf{m}^T)\circ\tilde{\mathsf{m}}^T\circ(\mathsf{id}\cdot((\sigma^T\cdot\mathsf{id})\cdot\sigma^T))\circ\tilde{\mathsf{n}}^{(T\phi)^*}\circ\hat{\alpha}&\\
=T\hat{\alpha}^{-1}\circ\mathsf{m}^T\circ(\mathsf{id}\cdot\mathsf{m}^T)\circ(\mathsf{id}\cdot(T\,\mathsf{n}_a^{\phi^*}\cdot\mathsf{id}))\circ\tilde{\mathsf{m}}^T\circ(\mathsf{id}\cdot((\mathsf{id}\cdot\sigma^T)\cdot\sigma^T))\circ\hat{\alpha}&\\
=T\hat{\alpha}^{-1}\circ T\tilde{\mathsf{n}}_a^{\phi^*}\circ\mathsf{m}^T\circ(\mathsf{id}\cdot\mathsf{m}^T)\circ\tilde{\mathsf{m}}^T\circ\hat{\alpha}\circ(\mathsf{id}\cdot(\sigma^T\cdot\sigma^T))&\\
=T\hat{\alpha}^{-1}\circ T\tilde{\mathsf{n}}_a^{\phi^*}\circ T\hat{\alpha}\circ\mathsf{m}^T\circ(\mathsf{m}^T\cdot\mathsf{m}^T)\circ(\mathsf{id}\cdot(\sigma^T\cdot\sigma^T))&
\end{aligned}$$

The second follows by applying the mate correspondence twice: we have

$$\begin{aligned}
\sigma^T\circ\mathsf{m}_x^{(T\phi)^*}\circ(\eta\cdot(\eta\circ 1))\circ\rho^{-1}&=\sigma^T\circ 1^\vee\circ\eta\\
&=\sigma^T\circ\eta\circ 1\\
&=T\eta\circ\mathsf{e}^T\circ 1\\
&=T(\eta\circ 1)\circ\mathsf{e}^T,
\end{aligned}$$

and

$$\begin{aligned}
T\,\mathsf{m}_x^{\phi^*}\circ\mathsf{m}^T\circ(\sigma^T\cdot\sigma^T)\circ(\eta\cdot(\eta\circ 1))\circ\rho^{-1}&=T\,\mathsf{m}_x^{\phi^*}\circ\mathsf{m}^T\circ(T\eta\cdot T(\eta\circ 1))\circ(\mathsf{e}^T\cdot\mathsf{e}^T)\circ\rho^{-1}\\
&=T\,\mathsf{m}_x^{\phi^*}\circ T(\eta\cdot(\eta\circ 1))\circ\mathsf{m}^T\circ(\mathsf{id}\cdot\mathsf{e}^T)\circ\rho^{-1}\circ\mathsf{e}^T\\
&=T(1^\vee\circ\eta)\circ\mathsf{e}^T\\
&=T(\eta\circ 1)\circ\mathsf{e}^T\,.
\end{aligned}$$

We obtain (5.15) via the following calculation:

$$\begin{aligned}
T(\mu^F\cdot\mathsf{m}_x^{\phi^*})\circ TN\phi_a^*\circ T(\mathsf{id}\cdot\theta^T)\circ\mathsf{m}^T\circ(\mathsf{m}^T\cdot\mathsf{m}^{TT})\circ((\mathsf{id}\cdot\sigma^T)\cdot(\mathsf{id}\cdot\sigma^{TT}))\circ(N_a^{(T\phi)^*})^{-1}&\\
=T(\mu^F\cdot\mathsf{m}_x^{\phi^*})\circ\mathsf{m}^T\circ(\mathsf{m}^T\cdot\mathsf{m}^T)\circ(\mathsf{id}\cdot(\sigma^T\cdot\sigma^T))&\\
=\mathsf{m}^T\circ(\mu^{TF}\cdot\mathsf{m}_x^{\phi^*})\circ(\mathsf{id}\cdot\mathsf{m}^T)\circ(\mathsf{id}\cdot(\sigma^T\cdot\sigma^T))&\\
=\mathsf{m}^T\circ(\mathsf{id}\cdot\sigma^T)\circ(\mu^{TF}\cdot\mathsf{m}_x^{(T\phi)^*}).&
\end{aligned}$$

Now, we apply (5.10), (5.11), (5.12), (c) from Lemma 5.1, (5.13), (5.14), (5.15) in sucession, to obtain

$$\begin{aligned}
&(\mu^F\cdot\mathsf{m}_x^{\phi^*})\circ N_a^{\phi^*}\circ(\mathsf{id}\cdot\theta^T)\circ((\mu^F\cdot\mathsf{m}_x^{\phi^*})\cdot m_{Fa\cdot\phi^*})\circ(N_a^{\phi^*}\cdot\mathsf{id})\circ((\mathsf{id}\cdot\theta^T)\cdot\mathsf{id})\\
&\quad=(\mu^F\cdot\mathsf{m}_x^{\phi^*})\circ N_a^{\phi^*}\circ((\mu^F\cdot\mathsf{m}_x^{\phi^*})\cdot(m_{Fa}\cdot m_{\phi^*}^\vee))\circ(N_a^{\phi^*}\cdot\mathsf{id})\circ((\mathsf{id}\cdot\theta^T)\cdot\theta^{TT})\\
&\quad=(\mu^F\cdot\mathsf{m}_x^{\phi^*})\circ((\mu^F\cdot Fm_a)\cdot(\mathsf{m}_{Sx}^{\phi^*}\cdot m_{\phi^*}^\vee))\circ N_a^{\phi_S^*\cdot(T\phi)^*}\circ(N_a^{\phi^*}\cdot\mathsf{id})\circ((\mathsf{id}\cdot\theta^T)\cdot\theta^{TT})\\
&\quad=(\mu^F\cdot\mathsf{m}_x^{\phi^*})\circ((\mathsf{id}\cdot\mu^{FS})\cdot(\phi_m^*\cdot m_{\phi^*}^\vee))\circ(\alpha\cdot\alpha)\circ N_a^{\phi_S^*\cdot(T\phi)^*}\circ(N_a^{\phi^*}\cdot\mathsf{id})\circ((\mathsf{id}\cdot\theta^T)\cdot\theta^{TT})\\
&\quad=(\mu^F\cdot\mathsf{m}_x^{\phi^*})\circ((\mathsf{id}\cdot FS\mu)\cdot(\phi_m^*\cdot m_{\phi^*}^\vee))\circ N_{a\cdot Sa}^{\phi^*}\circ(\mathsf{id}\cdot(\mathsf{m}^{TF}\cdot\mathsf{id}))\circ(\mathsf{id}\cdot N_a^{(T\phi)^*})\circ(\mathsf{id}\cdot(\theta^T\cdot\theta^{TT}))\circ\alpha\\
&\quad=(\mu^F\cdot\mathsf{m}_x^{\phi^*})\circ N_a^{\phi^*}\circ(\mathsf{id}\cdot(\mu^{TF}\cdot\mathsf{m}_x^{(T\phi)^*}))\circ(\mathsf{id}\cdot N_a^{(T\phi)^*})\circ(\mathsf{id}\cdot(\theta^T\cdot\theta^{TT}))\circ\alpha\\
&\quad=(\mu^F\cdot\mathsf{m}_x^{\phi^*})\circ N_a^{\phi^*}\circ(\mathsf{id}\cdot\theta^T)\circ(\mathsf{id}\cdot T(\mu^F\cdot\mathsf{m}_x^{\phi^*}))\circ(\mathsf{id}\cdot TN_a^{\phi^*})\circ(\mathsf{id}\cdot T(\mathsf{id}\cdot\theta^T))\circ(\mathsf{id}\cdot\mathsf{m}^T)\circ\alpha
\end{aligned}$$

which confirms the associativity law.

We conclude the proof by confirming that $(Ff, F\zeta\cdot\phi_f^*)$ is a horizontal lax T-algebra morphism, for any given horizontal lax S-algebra morphism (f,ζ): indeed, note that the following diagrams commute

$$\begin{tikzcd}
1_{Fx} \arrow[r, "\lambda"] \arrow[d, "1_{Ff}"'] & 1_{Fx}\cdot 1_{Fx} \arrow[r, "\upsilon^F\cdot\mathsf{e}_x^{\phi^*}"] \arrow[d, "1_{Ff}\cdot 1_{Ff}"] & Fa\cdot\phi_x^* \arrow[d, "F\zeta\cdot\phi_f^*"] \\
1_{Fy} \arrow[r, "\lambda"'] & 1_{Fy}\cdot 1_{Fy} \arrow[r, "\upsilon^F\cdot\mathsf{e}_y^{\phi^*}"'] & Fb\cdot\phi_y^*
\end{tikzcd}$$

$$\begin{tikzcd}
(Fa \cdot \phi^*_x) \cdot T(Fa \cdot \phi^*_x) \arrow[r, "(F\zeta\cdot\phi^*_f)\cdot T(F\zeta\cdot\phi^*_f)"] \arrow[d, "\mathsf{id}\cdot\theta^T"'] & (Fb \cdot \phi^*_y) \cdot T(Fb \cdot \phi^*_y) \arrow[d, "\mathsf{id}\cdot\theta^T"] \\
(Fa \cdot \phi^*_x) \cdot (TFa \cdot (T\phi)^*_x) \arrow[r, "(F\zeta\cdot\phi^*_f)\cdot(TF\zeta\cdot(T\phi)^*_f)" description] \arrow[d, "N^{\phi^*}_a"'] & (Fb \cdot \phi^*_y) \cdot (TFb \cdot (T\phi)^*_y) \arrow[d, "N^{\phi^*}_b"] \\
(Fa \cdot FSa) \cdot (\phi^*_{Sx} \cdot (T\phi)^*_x) \arrow[r, "(F\zeta\cdot FS\zeta)\cdot(\phi^*_{Sf}\cdot(T\phi)^*_f)" description] \arrow[d, "\mu^F\cdot\mathsf{m}^{\phi^*}_x"'] & (Fb \cdot FSb) \cdot (\phi^*_{Sy} \cdot (T\phi)^*_y) \arrow[d, "\mu^F\cdot\mathsf{m}^{\phi^*}_y"] \\
Fa \cdot \phi^*_x \arrow[r, "F\zeta\cdot\phi^*_f"'] & Fb \cdot \phi^*_y
\end{tikzcd}$$

via pasting of naturality and modification squares.

Functoriality is confirmed componentwise. □

We close this section with a comparative analysis of Theorem 5.2 with the notions of change-of-base for internal T-categories described in [33], and with the notions of change-of-base for enriched T-categories described in [12, Sections 5 and 6]; we confirm all of these generalize to our setting. The description of our main object of study, the functor $(\overline{T},\, \mathcal{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T,\, \mathcal{V})$ induced by $- \bullet 1 \colon \mathcal{V}\text{-}\mathsf{Mat} \to \mathsf{Span}(\mathcal{V})$, must be postponed to Section 8.

5.1. **Internal T-categories:** Let $\mathcal{D}, \mathcal{E}$ be categories with pullbacks, with respective cartesian monads S, T on $\mathcal{D}$, $\mathcal{E}$. We consider the equipments $\mathbb{D} = \mathsf{Span}(\mathcal{D})$ and $\mathbb{E} = \mathsf{Span}(\mathcal{E})$, and, abusing notation, we denote by S and T the induced strong monads on $\mathbb{D}$ and $\mathbb{E}$.

Using the terminology of [33], we consider a cartesian monad oplax morphism $(P, \phi) \colon S \to T$ and a cartesian monad lax morphism $(Q, \psi) \colon T \to S$. Their underlying data is given by

- pullback-preserving functors $P, Q \colon \mathcal{D} \to \mathcal{E}$,
- a cartesian natural transformation $\phi \colon PS \to TP$,
- a natural transformation $\psi \colon SQ \to QT$.

We note P and Q induce strong functors $\hat{P} \colon \mathbb{D} \to \mathbb{E}$, $\hat{Q} \colon \mathbb{E} \to \mathbb{D}$, and ϕ, ψ induce vertical transformations $\hat{\phi}$, $\hat{\psi}$, which, in turn, define a monad oplax morphism $(\hat{P}, \hat{\phi})$ and a monad lax morphism $(\hat{Q}, \hat{\psi})$.

We conclude, by Theorem 5.2 that $(\hat{Q}, \hat{\psi})$ defines a functor $\hat{Q}_! \colon \mathsf{Cat}(T,\, \mathcal{V}) \to \mathsf{Cat}(S,\, \mathcal{V})$. Moreover, since ϕ is cartesian, $\hat{\phi}$ has a strong conjoint, and since $\hat{P}$ is strong, $\hat{P}\hat{\phi}$ also has a strong conjoint; therefore we may also conclude that $(\hat{P}, \hat{\phi})$ induces a functor $\hat{P}_! \colon \mathsf{Cat}(S,\, \mathcal{V}) \to \mathsf{Cat}(T,\, \mathcal{V})$.

In fact, this notion of change-of-base can easily be extended to include Burroni's T-categories [8]. This would require a notion of horizontal lax T-algebra for T an *oplax* monad (possible with merely a couple of adjustments), and a replacement of lax functors with oplax functors in Theorem 5.2. We leave a pursuit of these results and possible applications for future work.

5.2. **Enriched T-categories:** Two instances of change-of-base are constructed in [12]; we begin by fixing a distributive monoidal category $\mathcal{V}$, and let $\mathbb{D} = \mathcal{V}\text{-}\mathsf{Mat}$. Therein, a *lax extension* of a Set-monad T to $\mathcal{V}\text{-}\mathsf{Mat}$ is a normal lax monad on $\mathbb{D}$ with underlying Set-monad T.

First, we suppose we have two normal lax monads S and T on $\mathbb{D}$, and let $\phi \colon T \to S$ be a vertical transformation, so that $(\mathsf{id}, \phi) \colon S \to T$ defines a monad lax morphism. This is precisely the data described in [12, Section 5], restated in a double categorical language. Theorem 5.2 produces a functor $(S,\, \mathcal{V})\text{-}\mathsf{Cat} \to (T,\, \mathcal{V})\text{-}\mathsf{Cat}$, which coincides with the *algebraic functor* construction in the aforementioned work.

Now, let $\mathcal{W}$ be another distributive monoidal category, let $\mathbb{E} = \mathcal{W}\text{-}\mathsf{Mat}$, and let $F \colon \mathcal{V} \to \mathcal{W}$ be a normal lax monoidal functor, preserving the initial object. F induces a normal lax functor $\hat{F} \colon \mathbb{D} \to \mathbb{E}$ with $\hat{F}_0 = \mathsf{id}_{\mathsf{Set}}$.

We let T and S be a lax monads on $\mathbb{D}$ and on $\mathbb{E}$, respectively, with the same underlying monad on Set; in other words, S and T are *lax extensions* of the same Set-monad.

Given a vertical transformation $\phi \colon T\hat{F} \to \hat{F}S$, such that ϕ_0 is the identity and such that $(\hat{F}, \phi) \colon S \to T$ is a monad lax morphism, we may apply Theorem 5.2 to produce a functor $\hat{F}_! \colon (S,\, \mathcal{V})\text{-}\mathsf{Cat} \to (T,\, \mathcal{W})\text{-}\mathsf{Cat}$; this is precisely the functor constructed in [12, Section 6].

We should highlight that all normality conditions can be omitted, as well as the preservation of the initial object by F, and still obtain change-of-base functors. This normality-free setting for both instances of change-of-base was already studied in [24, Sections 3.4, 3.5], for thin categories $\mathcal{V}$.

6. Induced adjunction

As observed in [33, Section 6.7], an adequate notion of adjunction between cartesian monads will induce an adjunction on the categories of internal T-categories, which has proven fruitful in their study. Moreover, in [24, Section 3], several change-of-base adjunctions between categories of (monad, quantale)-categories are studied as well. Our aim is to extend these ideas to arbitrary horizontal lax algebras, aiming to compare the enriched and the internal notions of generalized multicategory.

Throughout this section, our setting is a conjunction

$$\begin{tikzcd}(S,\mathbb{D}) \arrow[r, bend left, "{(F,\phi)}"] \arrow[r, phantom, "\perp"] & (T,\mathbb{E}) \arrow[l, bend left, "{(G,\psi)}"]\end{tikzcd}$$

in the double category $\mathsf{Mnd}(\mathsf{PsDbCat}_{\mathrm{lax}})$, such that $\mathbb{D}$ and $\mathbb{E}$ are equipments, and ϕ, $T\phi$ have strong conjoints. We denote the unit and counit by $\hat{\eta}, \hat{\varepsilon}$, respectively.

We recall that

- $(F,\phi)\colon (S,\mathbb{D}) \to (T,\mathbb{E})$ is a monad oplax morphism,
- $(G,\psi)\colon (T,\mathbb{E}) \to (S,\mathbb{D})$ is a monad lax morphism,
- we have an adjunction $F \dashv G$ in $\mathsf{PsDbCat}_{\mathrm{lax}}$ with unit $\hat{\eta}$ and counit $\hat{\varepsilon}$,
- and by doctrinal adjunction, F is strong, and ϕ, ψ are mates,

so that by Theorem 5.2, we may construct functors

$$F_!\colon \mathbb{H}\,\mathsf{Lax}\text{-}S\text{-}\,\mathsf{Alg} \to \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\,\mathsf{Alg} \quad \text{and} \quad G_!\colon \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\,\mathsf{Alg} \to \mathbb{H}\,\mathsf{Lax}\text{-}S\text{-}\,\mathsf{Alg}$$

induced by (F,ϕ^*) and $(G,\psi_!)$, respectively.

Theorem 6.1. *We have an adjunction $F_! \dashv G_!$.*

Proof. To fix notation, for $f\colon x \to Gy$ and $g\colon Fx \to y$, we let $f^\sharp = \hat{\varepsilon}_y \circ Ff$ and $g^\flat = Gg \circ \hat{\eta}_x$. This is similarly defined for 2-cells.

We claim that the hom-isomorphism and its inverse are given by

$$(f,\zeta) \mapsto (f^\sharp, \zeta^{\vee\sharp\wedge}), \quad \text{and} \quad (g,\xi) \mapsto (g^\flat, \xi^{\vee\flat\wedge}) \tag{6.1}$$

where we use $(-)^\vee$ and $(-)^\wedge$ as short-hand for mates. To be explicit, these are respectively given by

$$\zeta^\vee = \rho \circ (\mathsf{id} \cdot \delta) \circ \zeta \quad \text{and} \quad \xi^\vee = \xi \circ (\mathsf{id} \cdot \eta) \circ \rho^{-1},$$
$$\theta^\wedge = (\theta \cdot (\nu \circ 1)) \circ \rho^{-1} \quad \text{and} \quad \chi^\wedge = \rho \circ (\chi \cdot (1 \circ \varepsilon))$$

for adequate 2-cells ζ, θ in $\mathbb{D}$ and ξ, χ are 2-cells in $\mathbb{E}$.

To make sure that the horizontal composition for these mates of 2-cells is defined, note that

$$\hat{\varepsilon}_{Ty} \circ F\psi_y \circ FSf = T\hat{\varepsilon}_y \circ \phi_{Gy} \circ FSf = T\hat{\varepsilon}_y \circ TFf \circ \phi_x = Tf^\sharp \circ \phi_x,$$

$$GTg \circ G\phi_x \circ \hat{\eta}_{Sx} = GTg \circ \psi_{Gx} \circ S\hat{\eta}_x = \psi_y \circ SFg \circ S\hat{\eta}_x = \psi_y \circ Sg^\flat.$$

Since $\vee$, $\wedge$ and $\flat$, $\sharp$ are pairs of inverse operators, we find that the maps in (6.1) are each others' inverse.

To check these maps are natural, let $(k,\omega)\colon (w,d,\upsilon,\mu) \to (x,a,\upsilon,\mu)$ be a horizontal lax S algebra morphism, and $(h,\chi)\colon (y,b,\upsilon,\mu) \to (z,c,\upsilon,\mu)$ a horizontal lax T-algebra morphism. We have

$$\zeta^{\vee\sharp\wedge} \circ (F\omega \cdot \phi_k^*) \circ (\mathsf{id} \cdot \eta) \circ \rho^{-1} = \zeta^{\vee\sharp} \circ F\omega = (\zeta \circ \omega)^{\vee\sharp},$$

$$((G\chi \cdot \psi_{!h}) \circ \zeta)^\vee = G\chi \circ \zeta^\vee, \quad (G\chi \circ \zeta^\vee)^\sharp = \hat{\varepsilon}_c \circ FG\chi \circ F\zeta^\vee = \chi \circ \zeta^{\vee\sharp}$$

and reach our desired conclusion via mate correspondence.

So we're left with proving that $(f^\sharp, \zeta^{\vee\sharp\wedge})$ and $(g^\flat, \xi^{\vee\flat\wedge})$ are horizontal lax algebra morphisms. Since the proofs are similar, we omit the second one.

We must check the following identities hold:

$$\zeta^{\vee\sharp\wedge} \circ F_!\upsilon = \upsilon \circ 1_{f^\sharp} \quad \text{and} \quad \zeta^{\vee\sharp\wedge} \circ F_!\mu = \mu \circ (\zeta^{\vee\sharp\wedge} \cdot T\zeta^{\vee\sharp\wedge})$$

Since (f, ζ) is a lax S-algebra morphism, we have

$$\zeta \circ \upsilon = G_! \upsilon \circ 1_f,$$

so that

$$\begin{aligned}
\zeta^{\vee\sharp\wedge} \circ F_! \upsilon &= \zeta^{\vee\sharp} \circ F\upsilon \circ \mathsf{e}^F \\
&= (\zeta \circ \upsilon)^{\vee\sharp} \circ \mathsf{e}^F \\
&= \hat{\varepsilon}_b \circ FG\upsilon \circ F\,\mathsf{e}^G \circ F1_f \circ \mathsf{e}^F \\
&= \upsilon \circ \hat{\varepsilon}_1 \circ \mathsf{e}^{FG} \circ 1_{Ff} \\
&= \upsilon \circ 1_{f^\sharp},
\end{aligned}$$

which gives the unit law for $(f^\sharp, \zeta^{\vee\sharp\wedge})$.

For the multiplication law, we will confirm that

$$\zeta^{\vee\sharp\wedge} \circ (\mu^F \cdot \mathsf{m}^{\phi^*}_x) = \mu \circ (\zeta^{\vee\sharp\wedge} \cdot T\zeta^{\vee\sharp\wedge}) \circ (\mathsf{id} \cdot \mathsf{m}^T) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \sigma^T)) \circ \hat{\alpha}^{-1} \circ \tilde{\mathsf{n}}^{\phi^*}_a \circ \hat{\alpha}$$

via mate correspondence.

On one hand, we have

$$\begin{aligned}
\zeta^{\vee\sharp\wedge} \circ (\mu^F \cdot \mathsf{m}^{\phi^*}_x) \circ (\mathsf{id} \cdot (\eta \cdot (\eta \circ 1))) \circ (\mathsf{id} \cdot \rho^{-1}) \circ \rho^{-1} &= \zeta^{\vee\sharp\wedge} \circ (\mu^F \cdot (\eta \circ 1)) \circ \rho^{-1} \\
&= \zeta^{\vee\sharp} \circ \mu^F \\
&= (\zeta \circ \mu)^{\vee\sharp} \circ \mathsf{m}^F,
\end{aligned}$$

while on the other, we begin by noting that

$$\begin{aligned}
&(\mathsf{id} \cdot \mathsf{m}^T) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \sigma^T)) \circ \hat{\alpha}^{-1} \circ \tilde{\mathsf{n}}^{\phi^*}_a \circ \hat{\alpha} \circ (\mathsf{id} \cdot (\eta \cdot (\eta \circ 1))) \circ (\mathsf{id} \cdot \rho^{-1}) \circ \rho^{-1} \\
&\quad = (\mathsf{id} \cdot \mathsf{m}^T) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \sigma^T)) \circ ((\mathsf{id} \cdot \eta) \cdot (\phi_a \cdot (\eta \circ 1))) \circ \hat{\alpha}^{-1} \circ (\mathsf{id} \cdot (\gamma^{-1} \cdot \mathsf{id})) \circ \hat{\alpha} \circ (\mathsf{id} \cdot \rho^{-1}) \circ \rho^{-1} \\
&\quad = (\mathsf{id} \cdot \mathsf{m}^T) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \sigma^T)) \circ ((\mathsf{id} \cdot \eta) \cdot (\phi_a \cdot (\eta \circ 1))) \circ (\rho^{-1} \cdot \rho^{-1}) \\
&\quad = (\mathsf{id} \cdot \mathsf{m}^T) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \sigma^T)) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \eta)) \circ (\mathsf{id} \cdot \rho^{-1}) \circ ((\mathsf{id} \cdot \eta) \cdot \phi_a) \circ (\rho^{-1} \cdot \mathsf{id}) \\
&\quad = (\mathsf{id} \cdot \mathsf{m}^T) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot T\eta)) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \mathsf{e}^T)) \circ (\mathsf{id} \cdot \rho^{-1}) \circ ((\mathsf{id} \cdot \eta) \cdot \phi_a) \circ (\rho^{-1} \cdot \mathsf{id}) \\
&\quad = (\mathsf{id} \cdot T(\mathsf{id} \cdot \eta)) \circ (\mathsf{id} \cdot \mathsf{m}^T) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \mathsf{e}^T)) \circ (\mathsf{id} \cdot \rho^{-1}) \circ ((\mathsf{id} \cdot \eta) \cdot \phi_a) \circ (\rho^{-1} \cdot \mathsf{id}) \\
&\quad = (\mathsf{id} \cdot T(\mathsf{id} \cdot \eta)) \circ (\mathsf{id} \cdot T\rho^{-1}) \circ ((\mathsf{id} \cdot \eta) \cdot \phi_a) \circ (\rho^{-1} \cdot \mathsf{id}),
\end{aligned}$$

hence, if we write

$$Y = (\mathsf{id} \cdot \mathsf{m}^T) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \sigma^T)) \circ \hat{\alpha}^{-1} \circ \tilde{\mathsf{n}}^{\phi^*}_a \circ \hat{\alpha}, \quad Z = (\mathsf{id} \cdot (\eta \cdot (\eta \circ 1))) \circ (\mathsf{id} \cdot \rho^{-1}) \circ \rho^{-1}$$

we deduce that

$$\begin{aligned}
&\mu \circ (\zeta^{\vee\sharp\wedge} \cdot T\zeta^{\vee\sharp\wedge}) \circ Y \circ Z \\
&\quad = \mu \circ (\zeta^{\vee\sharp\wedge} \cdot T\zeta^{\vee\sharp\wedge}) \circ (\mathsf{id} \cdot T(\mathsf{id} \cdot \eta)) \circ (\mathsf{id} \cdot T\rho^{-1}) \circ ((\mathsf{id} \cdot \eta) \cdot \phi_a)) \circ (\rho^{-1} \cdot \mathsf{id}) \\
&\quad = \mu \circ \big(\zeta^{\vee\sharp} \cdot (T\zeta^{\vee\sharp} \circ \phi_a)\big) \\
&\quad = \mu \circ \big((\varepsilon_b \circ F\zeta^\vee) \cdot (T\varepsilon_b \circ TF\zeta^\vee \circ \phi_a)\big) \\
&\quad = \mu \circ \big((\varepsilon_b \circ F\zeta^\vee) \cdot (T\varepsilon_b \circ \phi_{Gb} \circ FS\zeta^\vee)\big) \\
&\quad = \mu \circ \big((\varepsilon_b \circ F\zeta^\vee) \cdot (\varepsilon_{Tb} \circ F\psi_b \circ FS\zeta^\vee)\big) \\
&\quad = \mu \circ (\varepsilon_b \cdot \varepsilon_{Tb}) \circ \big(F\zeta^\vee \cdot (F\psi_b \circ FS\zeta^\vee)\big) \\
&\quad = \varepsilon_b \circ F\mu^G \circ \mathsf{m}^F \circ \big(F\zeta^\vee \cdot (F\psi_b \circ FS\zeta^\vee)\big) \\
&\quad = \Big(\mu^G \circ \big(\zeta^\vee \cdot (\psi_b \circ S\zeta^\vee)\big)\Big)^\sharp \circ \mathsf{m}^F,
\end{aligned}$$

so we conclude it is sufficient to show that

$$(\zeta \circ \mu)^\vee = \mu^G \circ \big(\zeta^\vee \cdot (\psi_b \circ S\zeta^\vee)),$$

and indeed, we have

$$\begin{aligned}(\zeta \circ \mu)^{\vee} &= \rho \circ (\mathsf{id} \cdot \delta) \circ G_! \mu \circ (\zeta \cdot S\zeta) \\ &= \rho \circ (\mathsf{id} \cdot \delta) \circ (\mu^G \cdot \mathsf{m}_b^{\psi_!}) \circ \hat{\alpha}^{-1} \circ \tilde{\mathsf{n}}_b^{\psi_!} \circ \hat{\alpha} \circ (\mathsf{id} \cdot \chi^S) \circ (\zeta \cdot S\zeta) \\ &= \mu^G \circ \rho \circ (\mathsf{id} \cdot \rho) \circ \hat{\alpha}^{-1} \circ (\mathsf{id} \cdot (\gamma \cdot \mathsf{id})) \circ \hat{\alpha} \circ ((\mathsf{id} \cdot \delta) \cdot (\psi_b \cdot (1 \circ \delta))) \circ (\mathsf{id} \cdot \chi^S) \circ (\zeta \cdot S\zeta) \\ &= \mu^G \circ (\rho \cdot \rho) \circ ((\mathsf{id} \cdot \delta) \cdot (\psi_b \cdot (1 \circ \delta))) \circ (\mathsf{id} \cdot \chi^S) \circ (\zeta \cdot S\zeta) \\ &= \mu^G \circ (\rho \cdot \mathsf{id}) \circ ((\mathsf{id} \cdot \delta) \cdot \psi_b) \circ (\mathsf{id} \cdot \rho) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \delta)) \circ (\mathsf{id} \cdot \chi^S) \circ (\zeta \cdot S\zeta) \\ &= \mu^G \circ (\rho \cdot \mathsf{id}) \circ ((\mathsf{id} \cdot \delta) \cdot \psi_b) \circ (\mathsf{id} \cdot S\rho) \circ (\mathsf{id} \cdot S(\mathsf{id} \cdot \delta)) \circ (\zeta \cdot S\zeta) \\ &= \mu^G \circ \big(\zeta^{\vee} \cdot (\psi_b \circ S\zeta^{\vee})\big),\end{aligned}$$

which concludes the proof. □

For the purposes of applying this adjunction to the study of full faithfulness of $F_!$ and subsequent applications to descent theory, it is useful to establish criteria for the invertibility of the unit and counit of the adjunction $F_! \dashv G_!$, which are provided by the following results:

Lemma 6.2. *Let (y, b, υ, μ) be a horizontal lax T-algebra. If $\hat{\varepsilon}_y$ and $F\psi_y$ are invertible, then $\hat{\varepsilon}_{(y,b,\upsilon,\mu)}$ is invertible if and only if $\mathsf{n}_b^{\hat{\varepsilon}^*}$ is invertible.*

Proof. We have $\hat{\varepsilon}_{(y,b,\upsilon,\mu)} = (\hat{\varepsilon}_y, \mathsf{id}^{\vee\sharp\wedge})$. We first observe that, up to coherence isomorphisms, we have $\mathsf{id}^{\vee\sharp\wedge} = \Omega$, where

$$\Omega = \begin{tikzcd} TFGy \arrow[r, "\phi^*_{Gy}"] \arrow[d, equal] & FSGy \arrow[rr, "F(Gb\cdot\psi_{y!})"] \arrow[d, equal] & \arrow[d, phantom, "\chi^F"] & FGy \arrow[d, equal] \\ TFGy \arrow[r, dashed, "\phi^*_{Gy}" description] \arrow[d, "T\hat{\varepsilon}_y"'] & FSGy \arrow[r, phantom, "(F\psi_y)_!"] \arrow[d, phantom, "\omega"] & FGTy \arrow[r, dashed, "FGb" description] \arrow[d, equal] & FGy \arrow[d, equal] \\ Ty \arrow[rr, "\hat{\varepsilon}^*_{Ty}" description] \arrow[d, equal] & \arrow[d, phantom, "\mathsf{n}_b^{\hat{\varepsilon}^*}"] & FGTy \arrow[r, dashed, "FGb" description] & FGy \arrow[d, equal] \\ Ty \arrow[rr, "b" description] \arrow[d, equal] & & y \arrow[r, "\hat{\varepsilon}^*_y" description] \arrow[d] \arrow[dr, phantom, "\varepsilon"] & FGy \arrow[d, "\hat{\varepsilon}_y"] \\ Ty \arrow[rr, "b"'] & & y \arrow[r, "1"'] & y \end{tikzcd}$$

and $\omega = \lambda \circ (\delta \cdot \iota)$ is the mate of ι, which is in turn the mate of $\hat{\varepsilon}_{Ty} \circ F\psi_y = T\hat{\varepsilon}_y \circ \phi_{Gx}$.

Indeed, we note that

$$\begin{aligned}\Omega^{\vee} &= \lambda \circ (\varepsilon \cdot \mathsf{id}) \circ \mathsf{n}_b^{\hat{\varepsilon}^*} \circ (\mathsf{id} \cdot \omega) \circ \alpha \circ (\chi^F \cdot \mathsf{id}) \circ (\mathsf{id} \cdot \eta) \circ \rho^{-1} \\ &= \rho \circ (\hat{\varepsilon}_b \cdot \varepsilon) \circ (\mathsf{id} \cdot \omega) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \eta)) \circ \alpha \circ (\chi^F \cdot \mathsf{id}) \circ \rho^{-1} \\ &= \rho \circ (\hat{\varepsilon}_b \cdot \varepsilon) \circ (\mathsf{id} \cdot \omega) \circ (\mathsf{id} \cdot (\mathsf{id} \cdot \eta)) \circ (\mathsf{id} \cdot \rho^{-1}) \circ \chi^F,\end{aligned}$$

and since

$$\varepsilon \circ \omega \circ (\mathsf{id} \cdot \eta) \circ \rho^{-1} = 1 \circ \delta,$$

we obtain

$$\Omega^{\vee} = \rho \circ (\hat{\varepsilon}_b \cdot 1) \circ (\mathsf{id} \cdot \delta) \circ \chi^F = \hat{\varepsilon}_b \circ \rho \circ (\mathsf{id} \cdot \delta) \circ \chi^F,$$

and of course, $\rho \circ (\mathsf{id} \cdot \delta) \circ \chi^F$ is precisely $F(\rho \circ (\mathsf{id} \cdot \delta))$, so we obtain $\Omega^{\vee} = \mathsf{id}^{\vee\sharp}$, as desired.

Our claim follows by noting that if $\hat{\varepsilon}_y$ and $F\psi_y$ are invertible, then so is ι, and since $\delta \colon (F\psi_y)_! \to 1$ is invertible, so is ω.

The inverse of ι is given by the mate of $\phi_{Gy} \circ (F\psi_y)^{-1} = T\hat{\varepsilon}_y^{-1} \circ \varepsilon_{Ty}$, which we denote by θ. We have

$$\varepsilon \circ \iota \circ \theta = 1_{T\varepsilon_y} \circ \varepsilon \circ \theta = 1_{\mathsf{id}} \circ \varepsilon = \varepsilon \quad \text{and} \quad \varepsilon \circ \theta \circ \iota = 1_{T\varepsilon_y^{-1}} \circ \varepsilon \circ \iota = 1_{\mathsf{id}} \circ \varepsilon = \varepsilon,$$

finishing the proof. □

The analogous characterization for the unit is not quite the dual of Lemma 6.2; it requires one more verification.

Lemma 6.3. *Let (x, a, υ, μ) be a horizontal lax S-algebra. If η_x, $G\phi_x$ and $\mathsf{n}_a^{(G\phi)^*}$ are invertible, then $\hat{\eta}_{(x,a,\upsilon,\mu)}$ is invertible if and only if $\mathsf{n}_a^{\eta_!}$ is invertible.*

Proof. The only missing detail is that, if $\mathsf{n}_a^{(G\phi)^*}$ is invertible, then so is

$$\mathsf{m}^G \circ (\mathsf{id} \cdot \sigma^G) \colon GFa \cdot (G\phi)_x^* \to G(Fa \cdot \phi_x^*).$$

To see this, we take (3.4), with $H = G$ and $r = a$, and we recall that ϕ has a strong conjoint, by hypothesis. □

As an immediate corollary, we obtain:

Corollary 6.4. *$F_!\colon \mathbb{H}\,\mathsf{Lax}\text{-}S\text{-}\mathsf{Alg} \to \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$ is fully faithful whenever $F\colon \mathbb{D} \to \mathbb{E}$ is fully faithful and $G\phi$ is invertible.*

Proof. If $F\colon \mathbb{D} \to \mathbb{E}$ is fully faithful, then $\hat{\eta}$ is invertible, and therefore has a strong companion. Likewise, $G\phi$ has a strong conjoint. Thus, η_x, $\mathsf{n}_a^{\eta_!}$ and $\mathsf{n}_a^{(G\phi)^*}$ are invertible for all x and all a, so we apply Lemma 6.3. □

For the remainder of this section, we will compare Theorem 6.1 with [33, Section 6.7] and [24, Section 3], confirming we have a common generalization of these results. Furthermore, we provide some comments comparing our approach with the pseudofunctoriality ideas stated in [15, 4.4].

6.1. **Internal T-categories:** We recall the setting described in Subsection 5.1. If $P \dashv Q$ and ϕ and ψ are mates, we can immediately apply Theorem 6.1, to obtain an adjunction $P_! \dashv Q_!$ as claimed in [33, Section 6.7].

Likewise, with an adequate restatement of Theorem 6.1 for oplax monads and functors, we can also obtain adjunctions between categories of Burroni's T-categories.

6.2. **Enriched T-categories:** We note that Theorem 6.1 is a generalization of [24, Proposition 3.5.1], however, we cannot obtain the adjunction studied in [24, Subsection 3.4], using our result in the current form.

We will work out the same argument in our more general setting, to emphasize what goes wrong. Given a monad $T = (T, m, e)$ in $\mathbb{E}$, note that $e\colon \mathsf{id} \to T$ defines a monad lax morphism $(\mathsf{id}, e)\colon T \to \mathsf{id}$, which, by Theorem 5.2, gives a functor

$$e_!\colon \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg} \to \mathbb{H}\,\mathsf{Lax}\text{-}\mathsf{id}\text{-}\mathsf{Alg},$$

meaning every horizontal lax T-algebra has an underlying horizontal lax id-algebra (monad!). Moreover, e also defines a monad oplax morphism $(\mathsf{id}, e)\colon \mathsf{id} \to T$, but unless e and Te have strong conjoints, Theorem 5.2 cannot be applied to construct a functor $\mathbb{H}\,\mathsf{Lax}\text{-}\mathsf{id}\text{-}\mathsf{Alg} \to \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$, which would guarantee $e_!$ has a left adjoint.

However, it is possible to expand our notion of change-of-base to rectify the above problem: an analogous version of Theorem 5.2 can be obtained for a monad oplax morphism $(F, \phi)\colon \mathsf{id} \to T$, without requiring strong conjoints for either ϕ or $T\phi$, by defining $F_!(x, a, \eta, \mu)$ so that $F_!a = \phi_x^* \cdot TFa$; note that this is precisely $a_\sharp$ of [24, Subsection 3.4] when $F = \mathsf{id}$, and is isomorphic to the construction of Theorem 5.2 when ϕ and $T\phi$ do have strong conjoints.

This would also require an analogous version of Theorem 6.1 for this specialized change-of-base construction, but since such results are outside of our scope, we leave them for future work.

6.3. **Pseudofunctoriality:** Theorems 5.2 and 6.1 prompt one to view $\mathbb{H}\,\mathsf{Lax}\text{-}(-)\text{-}\mathsf{Alg}$ as a double pseudofunctor $\mathbb{M} \to \mathsf{CAT}$ (see [41, Section 6]), for a suitable sub-double category $\mathbb{M}$ of $\mathsf{Mnd}(\mathsf{PsDbCat}_{\mathrm{lax}})$. Since double pseudofunctors preserve conjoints, we would obtain the conclusion of Theorem 6.1 as an immediate corollary, for those conjunctions which are in $\mathbb{M}$.

We haven't pursued this line of reasoning, as obtaining a suitable choice of $\mathbb{M}$ which includes our main examples has proved to be elusive, as we briefly explain below.

We observe that the hypotheses required for Theorem 5.2 restrict us to a setting where the vertical 1-cells $(F, \phi)\colon S \to T$ (monad oplax morphisms) of $\mathbb{M}$ are those such that ϕ and $T\phi$ have strong conjoints. Unfortunately, this property on its own doesn't determine a sub-double category, as it is not closed under vertical composition: if $(G, \psi)\colon T \to U$ is another vertical 1-cell, there is no reason for $\omega = \psi_F \circ G\phi$ nor $U\omega$ to have strong conjoints, so this property doesn't define a sub-double category.

This obstacle could be overcome, provided we can guarantee that $G\phi$ and $UG\phi$ have strong conjoints. The first condition can be guaranteed if we require that the underlying functor of every monad oplax morphism (H, χ) is such that

$$Hr \cdot (Hf)^* \xrightarrow{\mathsf{id}\cdot\sigma^H} Hr \cdot H(f^*) \xrightarrow{\mathsf{m}^H} H(r \cdot f^*) \tag{6.2}$$

is invertible for all horizontal 1-cells r and vertical 1-cells f; note that this implies that $H\phi$ has a strong conjoint whenever ϕ has a strong conjoint. This property is satisfied, for instance, when H is strong. Therefore, this extra requirement is still within the setting of Theorem 6.1, as the underlying functors of the left adjoints are necessarily strong.

The problem lies in guaranteeing that $UG\phi$ has a strong conjoint; we would need to guarantee that the underlying lax functors of the monads make (6.2) invertible. However, it can be shown that this does not hold for our applications.

Lacking an alternative method to overcome this obstacle, we opted for the current *ad-hoc*, yet more general, approach for obtaining an adjunction of change-of-base functors, instead of going for the more attractive pseudofunctoriality argument.

7. EXTENSIVE CATEGORIES

Extensivity of $\mathcal{V}$ is a crucial hypothesis to construct and study the comparison functor $\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathcal{V})$ (see [35] and [13]), and therefore we shall devote this section to the study of extensive categories.

Let $\mathcal{C}$ be a category with coproducts. We say $\mathcal{C}$ is *extensive* if the coproduct functor

$$\prod_{i\in I} \mathcal{C} \downarrow X_i \to \mathcal{C} \downarrow \sum_{i\in I} X_i \tag{7.1}$$

is an equivalence for all families $(X_i)_{i\in I}$ of objects in $\mathcal{C}$. We refer to [9] for a comprehensive introduction to these categories. Extensive categories to keep in mind are Set, Top, Cat, any Grothendieck topos such as Grph, and any free coproduct completion $\mathsf{Fam}(\mathcal{B})$ of a category $\mathcal{B}$.

The following characterization of extensivity in terms of Artin glueing [21, p. 465] is quite important: an immediate corollary is that $\sum\colon \mathsf{Fam}(\mathcal{C}) \to \mathcal{C}$ preserves limits when $\mathcal{C}$ has all finite limits (that is, when $\mathcal{C}$ is *lextensive*). The converse was also shown to hold in [10, Section 4.3].

Lemma 7.1. *Let $\mathcal{C}$ be a category with coproducts and a terminal object. Then Diagram* (7.2)

(7.2)
$$\begin{tikzcd}
\mathsf{Fam}(\mathcal{C}) \arrow[r, "\sum"] \arrow[d] & \mathcal{C} \arrow[d, equal] \arrow[dl, Rightarrow, shorten=4mm, "\sigma"'] \\
\mathsf{Set} \arrow[r, "-\cdot 1"'] & \mathcal{C}
\end{tikzcd}$$

is a comma diagram if and only if $\mathcal{C}$ is extensive, where $\sigma_{(c_x)_{x\in X}}\colon \sum_{x\in X} c_x \to X \cdot 1$ is the coproduct over X of the morphisms $c_x \to 1$.

Proof. If $\mathcal{C}$ is extensive, for a morphism $f\colon c \to X \cdot 1$, we consider the family $(c_x)_{x\in X}$ given by the following family of pullbacks:

$$\begin{tikzcd}
c_x \arrow[r] \arrow[d] & 1 \arrow[d, "\hat{\eta}x"] \\
c \arrow[r, "f"'] & X \cdot 1
\end{tikzcd}$$

The family is, by definition, indexed over X, and by extensivity, we have an isomorphism $\sum_{x\in X} c_x \cong c$, whose composition with f equals $\sigma_{(c_x)_{x\in X}}$.

Let $(c_x)_{x\in X}$, $(d_y)_{y\in Y}$ be families of objects, and let $\hat{f}\colon \sum_{x\in X} c_x \to \sum_{y\in Y} d_y$ be a morphism and let $f\colon X \to Y$ be a function such that $\sigma \circ \hat{f} = (f \cdot 1) \circ \sigma$. For each $y \in Y$, we consider the following diagram:

$$\begin{tikzcd}
\sum_{x\in f^*y} c_x \arrow[rrr] \arrow[dr] \arrow[ddd] & & & d_y \arrow[dl] \arrow[ddd] \\
& f^*y\cdot 1 \arrow[r] \arrow[d] & 1 \arrow[d, "\hat{\eta}y"] & \\
& X\cdot 1 \arrow[r, "f\cdot 1"'] & Y\cdot 1 & \\
\sum_{x\in X} c_x \arrow[ur] \arrow[rrr, "\hat{f}"'] & & & \sum_{y\in Y} d_y \arrow[ul]
\end{tikzcd}$$

The left, right and inside squares are pullbacks, hence the outer square must be a pullback; let $\hat{f}|_x\colon c_x \to d_{fx}$ be the top morphism composed with the inclusion $c_x \to \sum_{x\in f^*fx} c_x$, and consider the morphism $(f, \hat{f}|_x)\colon (c_x)_{x\in X} \to (d_y)_{y\in Y}$ in $\mathsf{Fam}(\mathcal{C})$. It is the unique morphism $\psi\colon (c_x)_{x\in X} \to (d_y)_{y\in Y}$ indexed by f such that $\sum \psi = \hat{f}$, by extensivity. With this, we conclude that (7.2) is a comma diagram.

Now, given that (7.2) is a comma diagram, we aim to confirm (7.1) is an equivalence. First, full faithfulness: given a commutative triangle in $\mathcal{C}$

$$\begin{tikzcd}
\sum_{i\in I} Y_i \arrow[rr, "\phi"] \arrow[dr, "\sum_i f_i"'] & & \sum_{i\in I} Z_i \arrow[dl, "\sum_i g_i"] \\
& \sum_{i\in I} X_i &
\end{tikzcd}$$

we have

$$\sigma_{(X_i)} \circ \sum_i f_i = \sigma_{(Y_i)} \quad \text{and} \quad \sigma_{(X_i)} \circ \sum_i g_i = \sigma_{(Z_i)},$$

from which we obtain $\sigma_{(Y_i)} = \sigma_{(Z_i)} \circ \phi$. This implies unique existence of a morphism $(\mathsf{id}, \phi_i)\colon (Y_i)_{i\in I} \to (Z_i)_{i\in I}$ in $\mathsf{Fam}(\mathcal{C})$ such that $\sum_i \phi_i = \phi$, by the 2-dimensional universal property of comma squares.

Now, if we have a morphism $\omega\colon S \to \sum_{i\in I} X_i$, we consider its composite with $\sigma_{(X_i)}$. This yields, by the 2-dimensional universal property, a family $(S_i)_{i\in I}$ and an isomorphism $\nu\colon \sum_{i\in I} S_i \cong S$. From full faithfulness above we obtain a family $\omega_i\colon S_i \to X_i$ such that $\sum_i \omega_i \circ \nu = \omega$. ☐

The following instance of limit preservation is extensively used:

Theorem 7.2. *Let $\mathcal{C}$ be an extensive category. If we have a commutative square in $\mathsf{Fam}(\mathcal{C})$*

$$\begin{tikzcd}
(a_w)_{w\in W} \arrow[r, "(f,\hat{f})"] \arrow[d, "(g,\hat{g})"'] & (b_x)_{x\in X} \arrow[d, "(h,\hat{h})"] \\
(c_y)_{y\in Y} \arrow[r, "(k,\hat{k})"'] & (d_z)_{z\in Z}
\end{tikzcd} \tag{7.3}$$

such that

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "h"] \\
Y \arrow[r, "k"'] & Z
\end{tikzcd} \tag{7.4}$$

is a pullback diagram, as well as

$$\begin{tikzcd}
a_w \arrow[r, "\hat{f}_w"] \arrow[d, "\hat{g}_w"'] & b_{fw} \arrow[d, "\hat{h}_{fw}"] \\
c_{gw} \arrow[r, "\hat{k}_{gw}"'] & d_{kgw} = d_{hfw}
\end{tikzcd} \tag{7.5}$$

for each $w \in W$. Then

$$\begin{tikzcd}
\sum_w a_w \arrow[r, "\sum_f \hat{f}"] \arrow[d, "\sum_g \hat{g}"'] & \sum_x b_x \arrow[d, "\sum_h \hat{h}"] \\
\sum_y c_y \arrow[r, "\sum_k \hat{k}"'] & \sum_z d_z
\end{tikzcd} \tag{7.6}$$

is a pullback diagram.

Proof. The hypotheses (7.4) and (7.5) guarantee that (7.3) is a pullback in $\mathsf{Fam}(\mathcal{C})$, by [23, Definition 4.7 and Corollary 4.9]. Since $\sum$ preserves limits, (7.6) must be a pullback diagram, as desired. ☐

As corollaries, we obtain succinct proofs of a couple of results from [35] and [13], which clarify the role of the extensivity condition.

Lemma 7.3. *If $\mathcal{V}$ is extensive with finite limits, $- \bullet 1\colon \mathcal{V}\text{-}\mathsf{Mat} \to \mathsf{Span}(\mathcal{V})$ is strong.*

Proof. Since $- \bullet 1$ is normal, it is enough to prove that $\mathsf{m}^{-\bullet 1}$ is invertible. Indeed, we have the following pullback diagrams:

$$\begin{tikzcd}
X \times Y \times Z \arrow[r] \arrow[d] & Y \times Z \arrow[d] & s(y,z) \times r(x,y) \arrow[r] \arrow[d] & s(y,z) \arrow[d] \\
X \times Y \arrow[r] & Y & r(x,y) \arrow[r] & 1
\end{tikzcd}$$

thus, applying Theorem 7.2, we conclude that the outer square of diagram (2.8) is a pullback, verifying our claim. ☐

Remark 7.4. We observe that the above lemma can be restated in terms of a Beck-Chevalley condition; see [36, Definition 1.4.13]. To wit, the lax functor $\mathcal{V}(1,-)\colon \mathsf{Span}(\mathcal{V}) \to \mathcal{V}\text{-}\mathsf{Mat}$ satisfies the Beck-Chevalley condition if $\mathcal{V}$ is extensive. Then, by [36, Theorem 1.4.14], we conclude that $- \bullet 1 \dashv \mathcal{V}(1,-)$ is an *adjunction* in the 2-category $\mathsf{PsDbCat}_{\mathrm{lax}}$.

We can also give a short proof that a considerable class of monads are cartesian:

Lemma 7.5. *Let $\mathcal{V}$ be a lextensive, monoidal category, whose tensor product $\otimes$ preserves coproducts and pullbacks. Then the free $\otimes$-monoid monad on $\mathcal{V}$ is cartesian.*

Proof. We let $X^0 = I$ be the unit object, and $X^{n+1} = X^n \otimes X$. Recall that the underlying functor of the free $\otimes$-monoid monad may be given by $X \mapsto X^* = \sum_{n \in N} X^n$ (see, for instance, [26, Theorem 23.4]), and note that since pullbacks are preserved by $\otimes$ (by hypothesis) and by coproducts (as a corollary of Lemma 7.1), we conclude the free $\otimes$-monoid monad preserves pullbacks. Moreover, note that

$$\begin{tikzcd}
X^n \arrow[r, "\iota_n"] \arrow[d, "f^n"'] & X^* \arrow[d, "f^*"] \\
Y^n \arrow[r, "\iota_n"'] & Y^*
\end{tikzcd}$$

is a pullback diagram for all $n \in \mathbb{N}$, due to extensivity. Taking $n = 1$ confirms η is a cartesian natural transformation.

Now, we consider the following pullback diagrams

$$\begin{tikzcd}
\sum_{k \in \mathbb{N}} \mathbb{N}^k \arrow[r, "\mathsf{sum}"] \arrow[d, equal] & \mathbb{N} \arrow[d, equal] & X^{n_1} \otimes \ldots \otimes X^{n_k} \arrow[r, "\cong"] \arrow[d, "f^{n_1} \otimes \ldots \otimes f^{n_k}"'] & X^{n_1 + \ldots + n_k} \arrow[d, "f^{n_1 + \ldots + n_k}"] \\
\sum_{k \in \mathbb{N}} \mathbb{N}^k \arrow[r, "\mathsf{sum}"'] & \mathbb{N} & Y^{n_1} \otimes \ldots \otimes Y^{n_k} \arrow[r, "\cong"'] & Y^{n_1 + \ldots + n_k}
\end{tikzcd}$$

to which we may apply Theorem 7.2, allowing us to conclude that μ is a cartesian natural transformation as well. ☐

Connected categories. An object x in a category $\mathcal{V}$ is said to be *connected* if the hom-functor $\mathcal{V}(x,-)$ preserves coproducts, and, borrowing terminology from topos theory, $\mathcal{V}$ is said to be *connected* if the terminal object is connected.

Under the hypothesis that $\mathcal{V}$ is lextensive, understading this condition turns out to be helpful in our work on the enriched → internal embedding, particularly regarding the study of certain monads on $\mathcal{V}$; see Lemma 8.7.

Lemma 7.6. *If $\mathcal{V}$ is lextensive, then it is connected if and only if $-\cdot 1\colon \mathsf{Set}\to\mathcal{V}$ is fully faithful.*

Proof. Given a morphism $p\colon 1\to\sum_{i\in I}X_i$, we consider the following diagram

$$\begin{tikzcd}
u \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, "i"] \\
1 \arrow[r, "p"'] & \sum_{i\in I} X_i \arrow[r, "\sigma"] & I\cdot 1
\end{tikzcd}$$

which is a pasting of pullback squares.

It is clear that if $-\cdot 1$ is fully faithful, then $u\cong[\sigma\circ p=i]\cdot 1$. Thus, by universality of coproducts, p is uniquely determined by a morphism $1\to X_i$.

Conversely, since 1 is terminal we have $\mathcal{V}(1,X\cdot 1)\cong X\cdot\mathcal{V}(1,1)\cong X$, which implies the unit of $-\cdot 1\dashv\mathcal{V}(1,-)$ must be an isomorphism. ☐

Lemma 7.7. *If $\mathcal{V}$ is a connected, lextensive category, then $-\cdot 1\colon \mathcal{V}\text{-}\mathsf{Mat}\to\mathsf{Span}(\mathcal{V})$ is fully faithful on 2-cells.*

Proof. The outer square of (2.4) is a pullback, due to extensivity. Then, since $\hat{\eta}\colon X\to\mathcal{V}(1,X\cdot 1)$ is an isomorphism for all X, the result follows. ☐

8. Fibrewise discrete morphisms

Let T be a cartesian monad on a lextensive category $\mathcal{V}$, also denoting by T the induced strong monad on $\mathsf{Span}(\mathcal{V})$ [22, 15]. The Set-monad $\overline{T}$ under study (as well as its lax extension to $\mathcal{V}$-Mat, also denoted by $\overline{T}$), is constructed via the following consequence of Proposition 3.1:

Proposition 8.1. *Let $\mathbb{B}$ be a 2-category, let $(l,r,\eta,\varepsilon)\colon b\to c$ be an adjunction in $\mathbb{B}$, and let (t,m,e) be a monad on c. Then $(rtl, r(m\circ t\varepsilon t)l, rel\circ\eta)$ is a monad on b, and we have a conjunction*

$$\begin{tikzcd}
(b,rtl) \arrow[r, bend left, "{(l,\varepsilon tl)}"] \arrow[r, phantom, "\perp"] & (c,t) \arrow[l, bend left, "{(r,rt\varepsilon)}"]
\end{tikzcd}$$

in $\mathsf{Mnd}(\mathbb{B})$.

Indeed, by Remark 7.4, we have an adjunction

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathsf{Mat} \arrow[r, bend left, "-\cdot 1"] \arrow[r, phantom, "\perp"] & \mathsf{Span}(\mathcal{V}) \arrow[l, bend left, "{\mathcal{V}(1,-)}"]
\end{tikzcd} \tag{8.1}$$

in the 2-category $\mathsf{PsDbCat}_{\text{lax}}$, thus, we may apply Proposition 8.1 to (8.1) with the monad T on $\mathsf{Span}(\mathcal{V})$ to obtain a monad $\overline{T}=\mathcal{V}(1,T(-\cdot 1))$ on $\mathcal{V}$-Mat, and a conjunction

$$\begin{tikzcd}
(\overline{T},\mathcal{V}\text{-}\mathsf{Mat}) \arrow[r, bend left, "{(-\cdot 1,\hat{\varepsilon}_{T(-\cdot 1)})}"] \arrow[r, phantom, "\perp"] & (T,\mathsf{Span}(\mathcal{V})) \arrow[l, bend left, "{(\mathcal{V}(1,-),\mathcal{V}(1,T\hat{\varepsilon}))}"]
\end{tikzcd} \tag{8.2}$$

where $(-\cdot 1,\,\hat{\varepsilon}_{T(-\cdot 1)})$ is a monad oplax morphism and $(\mathcal{V}(1,-),\,\mathcal{V}(1,\,T\hat{\varepsilon}))$ is a monad lax morphism.

The only remaining ingredient to place ourselves under the setting of Section 6 and therefore apply Theorem 6.1 to (8.2), is the hypothesis that $\hat{\varepsilon}_{T(-\cdot 1)}$ has a strong conjoint[4]. The study and characterization of this hypothesis (for $\mathcal{V}$ connected) is the central purpose of this Section, culminating in Theorem 8.6.

Once this characterization is obtained, we obtain an adjunction (see Lemma 9.1)

$$\begin{tikzcd}
(\overline{T}, \mathcal{V})\text{-}\mathsf{Cat} \arrow[r, bend left, "(-\cdot 1, \hat{\varepsilon}_{T(-\cdot 1)})"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}(T, \mathcal{V}) \arrow[l, bend left, "(\mathcal{V}(1,-), \mathcal{V}(1,T\hat{\varepsilon}))"]
\end{tikzcd} \tag{8.3}$$

from (8.2), for pairs $(T, \mathcal{V})$ where T is a cartesian monad on a lextensive, connected category $\mathcal{V}$ such that $\hat{\varepsilon}_{T(-\cdot 1)}$ has a strong conjoint.

We begin by establishing the following:

Lemma 8.2. *Let $\omega\colon F \to G$ be a vertical transformation of lax functors $F, G\colon \mathbb{D} \to \mathsf{Span}(\mathcal{V})$. For a horizontal 1-cell $a\colon s \nrightarrow t$ in $\mathbb{D}$, the 2-cell $\mathsf{n}_a^{\omega^*}$ is invertible if and only if*

$$\begin{tikzcd}
M_{Fa} \arrow[r, "r_{Fa}"] \arrow[d, "\omega_a"'] & Ft \arrow[d, "\omega_y"] \\
M_{Ga} \arrow[r, "r_{Ga}"'] & Gt
\end{tikzcd} \tag{8.4}$$

is a pullback diagram.

Proof. We observe that $\mathsf{n}_a^{\omega^*}$ is uniquely determined by the dashed morphism below making the triangles commute

$$\begin{tikzcd}
& & M_{Fa} \arrow[d, dashed] \arrow[ddl, bend right, "\omega_a"'] \arrow[ddr, bend left, "r_{Fa}"] & & \\
& & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge" near start] & & \\
& M_{Ga} \arrow[dl, "l_{Ga}"'] \arrow[dr, "r_{Ga}"] & & Ft \arrow[dl, "\omega_y"'] \arrow[dr, equal] & \\
Gs & & Gt & & Ft
\end{tikzcd}$$

which is invertible if and only if (8.4) is a pullback diagram. ☐

Lemma 8.3. *The following are equivalent:*

(a) $\hat{\varepsilon}_{T(-\cdot 1)}$ has a strong conjoint,
(b) $\hat{\varepsilon}_{T(-\cdot 1)}$ is a cartesian natural transformation.

Proof. Instanciating (8.4) with $\omega = \hat{\varepsilon}_{T(-\cdot 1)}$, we find (a) holds if and only if

$$\begin{tikzcd}
\sum\limits_{\substack{\mathfrak{x}\in\overline{T}s \\ \mathfrak{y}\in\overline{T}t}} (\overline{T}a)(\mathfrak{x},\mathfrak{y}) \arrow[r] \arrow[d, "\hat{\varepsilon}_{T(a\cdot 1)}"'] & \overline{T}t \cdot 1 \arrow[d, "\hat{\varepsilon}_{T(t\cdot 1)}"] \\
T\sum\limits_{\substack{x\in s \\ y\in t}} a(x,y) \arrow[r, "r_{T(a\cdot 1)}"'] & T(t\cdot 1)
\end{tikzcd} \tag{8.5}$$

is a pullback diagram for all $\mathcal{V}$-matrices a.

---

[4] Note that, since T is strong, it follows that $T\hat{\varepsilon}_{T(-\cdot 1)}$ has a strong conjoint as well.

If we take $a = f_!$ for a function $f\colon s \to t$, (8.5) becomes

$$\begin{tikzcd}
\overline{T}s \cdot 1 \arrow[r, "\overline{T}f \cdot 1"] \arrow[d, "\hat{\varepsilon}_{T(s\cdot 1)}"'] & \overline{T}t \cdot 1 \arrow[d, "\hat{\varepsilon}_{T(t\cdot 1)}"] \\
T(s \cdot 1) \arrow[r, "T(f\cdot 1)"'] & T(t \cdot 1)
\end{tikzcd}$$

thereby verifying (a) → (b).

Now, we assume that the outer square of Diagram (8.6) below commutes:

$$\begin{tikzcd}
v \arrow[rrr, "r_v"] \arrow[dr, dashed, "l_v"] \arrow[ddd, "\omega"'] & & & \overline{T}t \cdot 1 \arrow[dl] \arrow[ddd, "\hat{\varepsilon}_{T(t\cdot 1)}"] \\
& \overline{T}s \cdot 1 \arrow[r] \arrow[d, "\hat{\varepsilon}_{T(s\cdot 1)}"'] & \overline{T}1 \cdot 1 \arrow[d] & \\
& T(s \cdot 1) \arrow[r] & T1 & \\
T\sum\limits_{\substack{x\in s\\ y\in t}} a(x,y) \arrow[ur, "l_{T(a\cdot 1)}"] \arrow[rrr, "r_{T(a\cdot 1)}"'] & & & T(t \cdot 1) \arrow[ul]
\end{tikzcd}\tag{8.6}$$

An immediate calculation shows the entire diagram commutes. Since the square in the middle is a pullback, there exists a unique morphism $l_v\colon v \to \overline{T}s \cdot 1$ such that the left and top squares commute.

Thus, we conclude that the following diagram

$$\begin{tikzcd}
v \arrow[r, "l_v{,}r_v"] \arrow[d, "\omega"'] & \overline{T}s \cdot 1 \times \overline{T}t \cdot 1 \arrow[d, "\hat{\varepsilon}_{T(s\cdot 1)} \times \hat{\varepsilon}_{T(t\cdot 1)}"] \\
T\sum\limits_{\substack{x\in s\\ y\in t}} a(x,y) \arrow[r, "l_{T(a\cdot 1)}{,}r_{T(a\cdot 1)}"'] & T(s \cdot 1) \times T(t \cdot 1)
\end{tikzcd}$$

commutes. We observe that Diagram (2.5) is a pullback square when $\mathcal{V}$ is extensive, by Theorem 7.2, so there exists a unique morphism

$$\omega^\sharp\colon v \to \sum_{\substack{\mathfrak{x}\in\overline{T}s\\ \mathfrak{y}\in\overline{T}t}} (\overline{T}a)(\mathfrak{x},\mathfrak{y})$$

such that $\hat{\varepsilon}_{T(a\cdot 1)} \circ \omega^\sharp = \omega$, $r_{\overline{T}a\cdot 1} \circ \omega^\sharp = r_v$ and $l_{\overline{T}a\cdot 1} \circ \omega^\sharp = l_v$, which, in particular, confirms that Diagram (8.5) is a pullback square. □

Fibrewise discrete monads. The search for a more concrete notion of what it means for $\hat{\varepsilon}_{T(-\cdot 1)}$ to have a strong conjoint led us to the notion of fibrewise discreteness.

Let $\mathcal{V}$ be a lextensive category, and let $f\colon x \to y$ be a morphism in $\mathcal{V}$. We say f is *fibrewise discrete* if for every pullback diagram

$$\begin{tikzcd}
f^*p \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & x \arrow[d, "f"] \\
1 \arrow[r, "p"'] & y
\end{tikzcd}$$

the object f^*p is *discrete*; that is, $\hat{\varepsilon}_{f^*p}$ is an isomorphism. For instance, in $\mathcal{V} = \mathsf{Top}$, local homeomorphisms are fibrewise discrete.

We say an endofunctor F on $\mathcal{V}$ is *fibrewise discrete* if for all sets X, the morphism $F!\colon F(X\cdot 1) \to F1$ is fibrewise discrete.

Lemma 8.4. *Let F be an endofunctor on a lextensive category $\mathcal{V}$. If $\hat{\varepsilon}_{F(-\cdot 1)}$ is cartesian, then F is fibrewise discrete. The converse holds when $\mathcal{V}$ is connected.*

Proof. Let $\overline{F} = \mathcal{V}(1, F(- \cdot 1))$. We consider Diagram (8.7)

(8.7)
$$\begin{tikzcd}
\overline{F}X \cdot 1 \arrow[dr, dashed, "\theta_X"] \arrow[drr, bend left, "\hat{\varepsilon}_{F(X\cdot 1)}"] \arrow[ddr, bend right, "\overline{F}!\cdot 1"'] & & \\
& \tau_X \arrow[r, "\omega_X"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & F(X \cdot 1) \arrow[d, "F!"] \\
& \overline{F}1 \cdot 1 \arrow[r, "\hat{\varepsilon}_{F1}"'] & F1
\end{tikzcd}$$

where the square in the lower right corner is a pullback. The outer square commutes by naturality, so there exists a unique θ_X, depicted by a dashed arrow, making both incident triangles commute. Note that $\hat{\varepsilon}_{F(-\cdot 1)}$ is cartesian if and only if θ_X is an isomorphism for all X.

Now, consider the image of (8.7) via $\mathcal{V}(1, -)$, which preserves pullbacks. Note that since $\mathcal{V}$ connected, $\mathcal{V}(1, \hat{\varepsilon}_{F1}) = \eta^{-1}_{\mathcal{V}(1,F1)}$ is an isomorphism, so we conclude that $\mathcal{V}(1, \omega_x)$ is an isomorphism as well.

Hence, we consider the following naturality square

$$\begin{tikzcd}
\mathcal{V}(1, \tau_X) \cdot 1 \arrow[r, "\mathcal{V}(1,\omega_X)\cdot 1"] \arrow[d, "\hat{\varepsilon}_{\tau_X}"'] & \overline{F}X \cdot 1 \arrow[d, "\hat{\varepsilon}_{F(X\cdot 1)}"] \\
\tau_X \arrow[r, "\omega_X"'] & F(X \cdot 1)
\end{tikzcd}$$

and we observe that $\theta_X \circ \mathcal{V}(1, \omega_X) \cdot 1 = \hat{\varepsilon}_{\tau_X}$ holds, by the universal property. Thus, θ_X is invertible if and only if $\hat{\varepsilon}_{\tau_x}$ is invertible; that is, if and only if τ_x is discrete. ☐

Remark 8.5. In Diagram (8.7), we have a morphism $\tau_x \to \overline{F}1 \cdot 1$, which corresponds to a family $(\tau_{x,p})_{p \in \overline{F}1}$, by extensivity (see Theorem 7.1); these are given via pullback

(8.8)
$$\begin{tikzcd}
\tau_{x,p} \arrow[r] \arrow[d] & F(x \cdot 1) \arrow[d, "F!"] \\
1 \arrow[r, "p"'] & F1
\end{tikzcd}$$

and we also have $\sum_{p \in \overline{F}1} \tau_{x,p} \cong \tau_x$. Thus, τ_x is discrete if and only if $\tau_{x,p}$ is discrete for all $p \in \overline{F}1$.

With this, we obtain the following characterization:

Theorem 8.6. *If $\mathcal{V}$ is connected, the following are equivalent for an endofunctor $F \colon \mathcal{V} \to \mathcal{V}$:*

(i) $\hat{\varepsilon}_{F(-\cdot 1)}$ has a strong conjoint.
(ii) $\hat{\varepsilon}_{F(-\cdot 1)}$ is a cartesian natural transformation.
(iii) F is fibrewise discrete.
(iv) $\tau_{x,p}$, as given in (8.8)*, is discrete for all x and all $p \in \overline{F}1$.*

Proof. The equivalence (i) $\iff$ (ii) is given by Lemma 8.3, we have (ii) $\iff$ (iii) by Lemma 8.4, and Remark 8.5 confirms (iii) $\iff$ (iv). ☐

Naturally, we are most concerned with cartesian monads T such that T is fibrewise discrete, and, armed with Theorem 8.6, we can promptly verify that many familiar examples of cartesian monads are fibrewise discrete. We begin with the following:

Lemma 8.7. *Let $\mathcal{V}$ be a connected, distributive monoidal category. The free $\otimes$-monoid monad on $\mathcal{V}$ is fibrewise discrete.*

Proof. Let X be a set, and let $p \colon 1 \to (X \cdot 1)^*$ be a morphism. Since $\mathcal{V}$ is connected, we may apply Lemma 7.6, to confirm p factors uniquely through $q \colon 1 \to (X \cdot 1)^n$ for some $n \in \mathbb{N}$. Now, note that $(X \cdot 1)^n \cong X^n \cdot 1$ if $n > 0$, $(X \cdot 1)^0 = I$, and that we have pullback diagrams

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & I \arrow[d] \\
1 \arrow[r] & I
\end{tikzcd}
\quad \text{and, for } n > 0, \quad
\begin{tikzcd}
X^n \cdot 1 \arrow[r] \arrow[d] & X^n \cdot 1 \arrow[d] \\
1 \arrow[r] & 1
\end{tikzcd}$$

whence $\tau_{X,p} \cong X^n \cdot 1$ for some $n \in \mathbb{N}$; this concludes the proof, by Theorem 8.6. ☐

Lemma 8.8. *Let S, T be endofunctors on $\mathcal{V}$, and let $\alpha\colon S \to T$ be a cartesian natural transformation. If T is fibrewise discrete, then so is S.*

Proof. Consider the following composite of pullbacks:

$$\begin{tikzcd}
\sigma_{x,p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S(x\cdot 1) \arrow[r, "\alpha_{x\cdot 1}"] \arrow[d] & T(x\cdot 1) \arrow[d, "T!"] \\
1 \arrow[r, "p"'] & S1 \arrow[r, "\alpha_1"'] & T1
\end{tikzcd}$$

We have $\tau_{x,\alpha_1\circ p} \cong \sigma_{x,p}$, which is discrete for all x, p. □

8.1. **Free monoid monad** $\mathsf{Set}\times\mathsf{Set}$**:** We will confirm this monad is not fibrewise discrete. Indeed, we have the following pullback diagram

$$\begin{tikzcd}
(X^m, X^n) \arrow[r] \arrow[d] & (X^*, X^*) \arrow[d] \\
(1,1) \arrow[r, "{m,n}"'] & (\mathbb{N},\mathbb{N})
\end{tikzcd}$$

for each $m, n \in \mathbb{N}$ and each set X. However, (X^m, X^n) is not discrete in general, so we cannot obtain a functor $-\cdot 1\colon (\overline{T},\ \mathcal{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T,\ \mathcal{V})$ via Theorem 5.2.

8.2. **Cartesian monads on slice categories:** If we have a pair $(T, \mathcal{V})$ where T is a cartesian monad on a category $\mathcal{V}$ with finite limits, and $\mathscr{C}$ is an internal T-category, we may construct [33, Proposition 6.2.1] a cartesian monad $T_{\mathscr{C}}$ on $\mathcal{V} \downarrow \mathscr{C}_0$, and we obtain an equivalence [33, Corollary 6.2.5] of categories

$$\mathsf{Cat}(T_{\mathscr{C}}, \mathcal{V} \downarrow \mathscr{C}_0) \simeq \mathsf{Cat}(T, \mathcal{V}) \downarrow \mathscr{C},$$

which raises the question: can we obtain (8.3) for the pair $(T_{\mathscr{C}}, \mathcal{V} \downarrow \mathscr{C}_0)$?

Already when $T = \mathsf{id}$, $\mathcal{V} = \mathsf{Set}$, we cannot generally guarantee an affirmative answer. Indeed, let $\mathcal{C}$ be an ordinary small category. In this case, $T_{\mathcal{C}}$ is the cartesian monad induced by the monadic adjunction

$$\begin{tikzcd}
{[\mathcal{C}, \mathsf{Set}]} \arrow[r, bend right, "\iota_{\mathcal{C}}^*"'] \arrow[r, phantom, "\perp"] & \mathsf{Set} \downarrow \mathsf{ob}\,\mathcal{C} \arrow[l, bend right, "\mathsf{Lan}_{\iota_{\mathcal{C}}}"']
\end{tikzcd}$$

However, $\mathsf{Set} \downarrow \mathsf{ob}\,\mathcal{C} \cong [(\mathsf{ob}\,\mathcal{C})\cdot 1, \mathsf{Set}]$ is connected precisely when $\mathsf{ob}\,\mathcal{C} \cong 1$ or $\mathsf{ob}\,\mathcal{C} \cong 0$; in fact, we shall confirm that while it is true that $T_{\mathcal{C}}$ is fibrewise discrete, $\hat{\varepsilon}_{T_{\mathcal{C}}(-\cdot 1)}$ is not cartesian, and thus we cannot obtain (8.3) for general $\mathcal{C}$.

Let $X = \mathsf{ob}\,\mathcal{C}$. $T_{\mathcal{C}}$ is defined on objects by

$$(A_x)_{x\in X} \mapsto \big(\sum_{y\in X} A_y \times \mathcal{C}(x,y)\big)_{x\in X},$$

and the terminal object of $\mathsf{Set} \downarrow X$ is precisely the constant family $1 = (1)_{x\in X}$. In this case, $T_{\mathcal{C}}1 = (\mathsf{ob}(x \downarrow \mathcal{C}))_{x\in X}$, while $\overline{T_{\mathcal{C}}}1 = \prod_{x\in X} \mathsf{ob}(x\downarrow\mathcal{C})$.

More generally, for a constant family $A\cdot 1 \cong (A)_{x\in X}$, we have

$$T_{\mathcal{C}}(A\cdot 1) \cong (A \times \mathsf{ob}(x\downarrow\mathcal{C}))_{x\in X},$$

and $\overline{T}_{\mathcal{C}}(A\cdot 1) = A^X \times \prod_{x\in X} \mathsf{ob}(x\downarrow\mathcal{C})$. We have, for each $x \in X$, a pair of pullback diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \times \mathsf{ob}(x\downarrow\mathcal{C}) \arrow[d] & A \times \prod_{x\in X} \mathsf{ob}(x\downarrow\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \times \mathsf{ob}(x\downarrow\mathcal{C}) \arrow[d] \\
1 \arrow[r] & \mathsf{ob}(x\downarrow\mathcal{C}) & \prod_{x\in X} \mathsf{ob}(x\downarrow\mathcal{C}) \arrow[r] & \mathsf{ob}(x\downarrow\mathcal{C})
\end{tikzcd}$$

which confirms that $T_{\mathcal{C}}$ is fibrewise discrete, but, since we cannot guarantee $A \cong A^X$, we cannot guarantee $\hat{\varepsilon}_{T_{\mathcal{C}}(-\cdot 1)}$ to be cartesian as well.

In spite of this, we can obtain the adjunction (8.3) when $\mathscr{C}_0$ is terminal; that is, when $\mathscr{C}$ is a $(T, \mathcal{V})$*-monoid.* We will now treat the case $T = (-)^*$, which are denoted $\mathcal{V}$-operads.

8.3. **$\mathcal{V}$-operadic monads:** An important corollary of Lemmas 8.8 and 8.7 is that for a cartesian monoidal category $\mathcal{V}$, $\mathcal{V}$-operadic monads are fibrewise discrete; note that these are precisely the cartesian monads on $\mathcal{V}$ with a cartesian natural transformation to the free $\times$-monoid monad.

To be explicit, for a $\mathcal{V}$-operad $\mathcal{O}$ [33, p. 44] the monad associated to $\mathcal{O}$ is given on objects by

$$V \mapsto \sum_{n \in \mathbb{N}} \mathcal{O}_n \times V^n,$$

and the projections $\mathcal{O}_n \times V^n \to V^n$ induce a cartesian natural transformation to the free $\times$-monoid monad.

Thus, any pair $(T, \mathcal{V})$ where T an operadic monad over a lextensive, connected category $\mathcal{V}$ induces an adjunction (8.3). Of special interest is the case $T = (-)^*$ is the free monoid monad on $\mathcal{V}$. In this case, the induced Set-monad $\overline{T}$ is precisely the ordinary free monoid monad.

8.4. **Free category monad:** The free category monad $\mathfrak{F}$ on Grph is fibrewise discrete, since we have the following pullback of graphs:

$$\begin{tikzcd}
X \cdot 1 \arrow[rrr] \arrow[ddd] \arrow[dr, Rightarrow] & & & (\mathbb{N} \times X) \cdot 1 \arrow[ddd] \arrow[dl, Rightarrow] \\
& X \cdot 1 \arrow[r] \arrow[d] & X \cdot 1 \arrow[d] & \\
& 1 \arrow[r] & 1 & \\
1 \arrow[rrr] \arrow[ur, Rightarrow] & & & \mathbb{N} \cdot 1 \arrow[ul, Rightarrow]
\end{tikzcd}$$

so, for the pair $(\mathfrak{F}, \mathsf{Grph})$, we also obtain an adjunction (8.3). We note that $\overline{\mathfrak{F}}$ is a lax extension of the $\mathbb{N} \times -$ monad on Set, for the *multiplicative* structure of $\mathbb{N}$.

8.5. **Free finite coproduct completion monad:** For a category $\mathcal{C}$, $\mathsf{Fam}_{\mathrm{fin}}(\mathcal{C})$ is the category of *finite families* of objects of $\mathcal{C}$; it is given on objects by $(\mathsf{ob}\,\mathcal{C})^*$, and a morphism $\mathfrak{x} \to \mathfrak{y}$ is a pair (f, ϕ), consisting of a function $f \colon [m] \to [n]$, where m and n are the lengths of $\mathfrak{x}$ and $\mathfrak{y}$, respectively, and for each $i = 1, \dots, n$, a morphism $\phi_i \colon \mathfrak{x}_i \to \mathfrak{y}_{fi}$.

From [45, 5.16], we learn that $\mathsf{Fam}_{\mathrm{fin}}$ is a cartesian (2-)monad on Cat. We proceed to verify it is fibrewise discrete; first, observe that $\mathsf{ob}\,\mathsf{Fam}_{\mathrm{fin}}(X \cdot 1) = X^*$, and the hom-sets are given by

$$\mathsf{Fam}_{\mathrm{fin}}(X \cdot 1)(\mathfrak{x}, \mathfrak{y}) = \sum_{f \colon [m] \to [n]} \prod_{i=1}^{m} [x_i = y_{fi}],$$

where m, n are the lengths of $\mathfrak{x}$, $\mathfrak{y}$ respectively. Moreover, note that $\mathsf{Fam}_{\mathrm{fin}}(1) \simeq \mathsf{FinSet}$. The fiber of $\mathsf{Fam}_{\mathrm{fin}}(X \cdot 1) \to \mathsf{FinSet}$ at (the identity on) n is given on objects by the set of families of size n, and on morphisms by $[\mathfrak{x} = \mathfrak{y}] \cong \prod_{i=1}^{n} [\mathfrak{x}_i = \mathfrak{y}_i]$, which yields a discrete category; diagrammatically, we have

$$\begin{tikzcd}
{[x = y]} \arrow[r] \arrow[d] & \displaystyle\sum_{f \in [n] \to [n]} \prod_{i=1}^{n} [\mathfrak{x}_i = \mathfrak{y}_{fi}] \arrow[d] \\
1 \arrow[r, "\mathsf{id}"'] & \mathsf{FinSet}([n], [n])
\end{tikzcd} \tag{8.9}$$

as we desired. Thus, the pair $(\mathsf{Fam}_{\mathrm{fin}}, \mathsf{Cat})$ gives an adjunction (8.3) as well.

We note that $\overline{\mathsf{Fam}_{\mathrm{fin}}}$ is a lax extension of the free monoid monad on Set.

8.6. **Free finite product completion monad:** The functor $(-)^{\mathrm{op}} \colon \mathsf{Cat} \to \mathsf{Cat}$ taking each category to its dual is its own, since we have $\mathsf{Cat}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \cong \mathsf{Cat}(\mathcal{C}, \mathcal{D}^{\mathrm{op}})$, so, via 8.1, we can promptly verify that the functor

$$\mathcal{C} \mapsto \mathsf{Fam}^*_{\mathrm{fin}}(\mathcal{C}) = \mathsf{Fam}_{\mathrm{fin}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$$

is a cartesian monad. For a category $\mathcal{C}$, $\mathsf{Fam}^*_{\mathrm{fin}}(\mathcal{C})$ has the same set of objects as $\mathsf{Fam}_{\mathrm{fin}}(\mathcal{C})$, but a morphism $\mathfrak{x} \to \mathfrak{y}$ is a pair (f, ϕ) consisting of a function $f \colon [n] \to [m]$, where m, n is the length of $\mathfrak{x}$, $\mathfrak{y}$ respectively, and $\phi_i \colon \mathfrak{x}_{fi} \to \mathfrak{y}$ is a morphism for each $i = 1, \dots, m$.

This monad is also fibrewise discrete; the only adjustment we need to make to the pullback diagram (8.9) is to replace $[\mathfrak{x}_i = \mathfrak{y}_{fi}]$ with $[\mathfrak{x}_{fi} = \mathfrak{y}_i]$, so the pair $(\mathsf{Fam}^*_{\mathrm{fin}}, \mathsf{Cat})$ induces an adjunction (8.3).

8.7. **Free symmetric strict monoidal category monad:** For a category $\mathcal{C}$, we let $\mathfrak{S}\mathcal{C}$ be a subcategory of $\mathsf{Fam}_{\mathrm{fin}}(\mathcal{C})$ with the same set of objects, and precisely those morphisms $(f, \phi)\colon \mathfrak{x} \to \mathfrak{y}$ such that f is a bijection.

This was shown to be a cartesian monad, for instance, in [33], or in [45, Example 7.5], where it was shown that we have a cartesian (2-)natural transformation $\mathfrak{S} \to \mathsf{Fam}_{\mathrm{fin}}$. For this same reason, it is fibrewise discrete, by Lemma 8.8, giving us another example of an adjunction (8.3), with the pair $(\mathfrak{S}, \mathsf{Cat})$.

Furthermore, note that $\overline{\mathfrak{S}}$ is also a lax extension of the free monoid monad on Set.

9. EMBEDDING

Throughout this section, we fix a lextensive category $\mathcal{V}$, and a cartesian monad $T = (T, m, e)$ on $\mathcal{V}$. Following the notation from Section 8, we denote by $\overline{T}$ the monad on $\mathcal{V}$-Mat induced by T on $\mathsf{Span}(\mathcal{V})$.

Via the tools developed throughout the paper, we shall verify that if $\mathcal{V}$ is connected, and T is fibrewise discrete, then $(\overline{T}, \mathcal{V})$-$\mathsf{Cat} \to \mathsf{Cat}(T, \mathcal{V})$ is a fully faithful, pullback-preserving functor. Moreover, among the pairs $(T, \mathcal{V})$ satisfying these hypotheses at the end of Section 8, we shall provide a description of $(\overline{T}, \mathcal{V})$-$\mathsf{Cat}$ and $\mathsf{Cat}(T, \mathcal{V})$.

Lemma 9.1. *If $\hat{\varepsilon}_{T(-\cdot 1)}$ has a strong conjoint, then we have an adjunction*

$$\begin{tikzcd}(\overline{T}, \mathcal{V})\text{-}\mathsf{Cat} \arrow[r, bend left, "-\cdot 1"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}(T, \mathcal{V}) \arrow[l, bend left, "\mathcal{V}(1,-)"]\end{tikzcd} \tag{9.1}$$

whose unit and counit are also denoted by $\hat{\eta}$ and $\hat{\varepsilon}$, respectively.

Proof. By hypothesis, $- \cdot 1$ is a strong functor, and $\hat{\varepsilon}_{T(-\cdot 1)}$ has a strong conjoint. Since T is a strong functor, we also deduce that $T\hat{\varepsilon}_{T(-\cdot 1)}$ has a strong conjoint as well. This places us in the setting of Section 6, hence, we obtain (9.1) by applying Theorem 6.1 to the conjunction (8.2). ☐

Henceforth, we shall assume that $\mathcal{V}$ is a connected category, and that T is fibrewise discrete.

Theorem 9.2. $- \cdot 1\colon (\overline{T}, \mathcal{V})$-$\mathsf{Cat} \to \mathsf{Cat}(T, \mathcal{V})$ *is fully faithful.*

Proof. By Lemma 7.7, we know $- \cdot 1\colon \mathcal{V}$-$\mathsf{Mat} \to \mathsf{Span}(\mathcal{V})$ is fully faithful, and since $\mathcal{V}(1, \varepsilon_{T(-\cdot 1)})$ is a natural isomorphism, the result follows by Corollary 6.4. ☐

These results can be immediately applied to the last four examples in Section 8; we shall describe both $(\overline{T}, \mathcal{V})$-$\mathsf{Cat}$ and $\mathsf{Cat}(T, \mathcal{V})$ for each such pair $(T, \mathcal{V})$.

9.1. **$\mathcal{V}$-operadic multicategories:** Let $T = T_{\mathfrak{O}}$ be a monad induced by a $\mathcal{V}$-operad $\mathfrak{O}$. When $\mathcal{V}$ is connected, we have shown that T is fibrewise discrete, and therefore $(T, \mathcal{V})$ induces an adjunction (9.1). So, we conclude that $- \cdot 1\colon (\overline{T}, \mathcal{V})$-$\mathsf{Cat} \to \mathsf{Cat}(T, \mathcal{V})$ is fully faithful, by Theorem 9.2.

The induced monad $\overline{T}$ on Set is given on objects by

$$X \mapsto \sum_{n \in \mathbb{N}} \mathcal{V}(1, \mathfrak{O}_n) \times X^n,$$

and note that since $\mathcal{V}(1, -)\colon \mathcal{V} \to \mathsf{Set}$ is a strong monoidal functor (preserves products), it follows that $\mathcal{V}(1, \mathfrak{O})$ a Set-monad, so $\overline{T}$ is an operadic monad as well.

Let $r\colon X \nrightarrow Y$ be a $\mathcal{V}$-matrix, and let $\sigma \in \mathcal{V}(1, \mathfrak{O}_m)$, $\mathfrak{x} \in X^m$, $\tau \in \mathcal{V}(1, \mathfrak{O}_n)$, $\mathfrak{y} \in Y^n$. The $\mathcal{V}$-matrix $\overline{T}r$ is given at $(\sigma, \mathfrak{x}, \tau, \mathfrak{Y})$ by

$$(\overline{T}r)(\sigma, \mathfrak{x}, \tau, \mathfrak{y}) = \begin{cases} 0 & \text{if } \sigma \neq \tau, \\ \prod_{i=1}^n r(x_i, y_i) & \text{otherwise} \end{cases}$$

thus, in practice, we just write $(\overline{T}r)(\sigma, \mathfrak{x}, \mathfrak{y})$ for the possibly non-initial values of $\overline{T}r$.

The objects of $\mathsf{Cat}(T, \mathcal{V})$ are (internal) operadic $\mathcal{V}$-categories, and for this reason, we will consider the objects of $(\overline{T}, \mathcal{V})$-$\mathsf{Cat}$ to be the *enriched* operadic $\mathcal{V}$-categories. Such an object consists of

- a set X of objects,
- a $\mathcal{V}$-matrix $a\colon \overline{T}X \times X \to \mathcal{V}$,
- a $\mathcal{V}$-morphism $1 \to a(ex, x)$ for each $x \in X$,
- a $\mathcal{V}$-morphism $a(\sigma, \mathfrak{x}, x) \times \overline{T}a(\sigma, (\tau_1, \mathfrak{y}_1), \ldots, (\tau_n, \mathfrak{y}_k), \mathfrak{x}) \to a(\sigma(\tau_1, \ldots, \tau_k), \mathfrak{y}_1 \cdots \mathfrak{y}_k, x)$ for $\tau_i \in \mathfrak{O}_{n_i}$, $\mathfrak{y}_i \in X^{n_i}$, $\sigma \in \mathfrak{O}_m$, $\mathfrak{x} \in \mathfrak{X}^m$, where $m = n_1 + \ldots + n_k$.

satisfying suitable identity and associativity conditions.

Of particular interest may be the $T = (-)^*$ free $\times$-monoid monad on $\mathcal{V}$; more generally, monads induced by a *discrete* operad $\mathfrak{O}$ and the $M \times -$ monad for M a $\mathcal{V}$-monoid.

9.2. $(\overline{\mathfrak{F}}, \mathsf{Grph})$**-categories:** As we have verified in Section 8, the pair $(\mathfrak{F}, \mathsf{Grph})$ consists of a fibrewise discrete monad on a connected, lextensive category, so $- \cdot 1\colon (\overline{\mathfrak{F}}, \mathsf{Grph})\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathfrak{F}, \mathsf{Grph})$ is fully faithful.

The object of the category $\mathsf{Cat}(\mathfrak{F}, \mathsf{Grph})$ are precisely the virtual double categories [33, 15]. An enriched $(\overline{\mathfrak{F}}, \mathsf{Grph})$-category X consists of

- A set X_0 of objects,
- A graph $X_1(n, x, y) = (X_{11}(n, x, y) \rightrightarrows X_{10}(n, x, y)$ for each $n \in \mathbb{N}$ and $x, y \in X_0$,
- A loop $1 \to X_1(1, x, x)$ for each $x \in X_0$,
- A graph morphism $X_1(m, y, z) \times \mathfrak{F}_m X_1(n, x, y) \to X_1(m \cdot n, x, z)$ for each $x, y, z \in X$ and $m, n \in \mathbb{N}$, where $\mathfrak{F}_m G$ is the graph of m-chains of G.

satisfying suitable identity and associativity conditions.

Via the induced functor $\mathsf{Grph}(1, -)\colon \mathsf{VDbCat} \to (\overline{\mathfrak{F}}, \mathsf{Grph})\text{-}\mathsf{Cat}$, we can come across examples of $(\overline{\mathfrak{F}}, \mathsf{Grph})$-categories. If $\mathbb{V}$ is a virtual double category, $\mathsf{Grph}(1, \mathbb{V})$ consists of

- a set of objects $\mathsf{Grph}(1, \mathbb{V}_0)$, that is, the set of loops in $\mathbb{V}_0$,
- for each $n \in \mathbb{N}$ and loops r, s of $\mathbb{V}_0$, a graph $\mathbb{V}_1(n, r, s)$ has edges $\theta\colon f \to g$ consisting of the 2-cells of the form

$$\begin{tikzcd}
x \arrow[r, "r"] \arrow[d, "f"'] & x \arrow[r, "r"] & \ldots \arrow[r, "r"] \arrow[d, phantom, "\theta"] & x \arrow[r, "r"] & x \arrow[d, "f"] \\
y \arrow[rrrr, "s"'] & & & & y
\end{tikzcd}$$

whose vertical domain has length n; we may simply write

$$\begin{tikzcd}
x \arrow[r, "r^n"] \arrow[d, "f"'] \arrow[dr, phantom, "\theta"] & x \arrow[d, "g"] \\
y \arrow[r, "s"'] & y
\end{tikzcd}$$

as a shorthand.
- the unit 2-cell at r in $\mathbb{V}_1(1, r, r)$ is given by

$$\begin{tikzcd}
x \arrow[r, "r"] \arrow[d, equal] \arrow[dr, phantom, "="] & x \arrow[d, equal] \\
x \arrow[r, "r"'] & x
\end{tikzcd}$$

for each $r \in \mathsf{Grph}(1, \mathbb{V}_0)$,
- for 2-cells

$$\begin{tikzcd}
y \arrow[r, "s^m"] \arrow[d, "g"'] \arrow[dr, phantom, "\omega"] & y \arrow[d, "h"] & x \arrow[r, "r^n"] \arrow[d, "f_{i-1}"'] \arrow[dr, phantom, "\theta_i"] & x \arrow[d, "f_i"] \\
z \arrow[r, "t"'] & z & y \arrow[r, "s"'] & y
\end{tikzcd}$$

for $i = 1, \ldots, n$, the composite 2-cell is given by

$$\begin{tikzcd}
x \arrow[rr, "r^{m \cdot n}"] \arrow[d, "g \circ f_0"'] & \omega(\theta_1 \cdots \theta_n) & x \arrow[d, "h \circ f_n"] \\
z \arrow[rr, "t"'] & & z
\end{tikzcd}$$

9.3. **Clubs:** We begin by considering the pair $(\mathfrak{S}, \mathsf{Cat})$. The category $\mathsf{Cat}(\mathfrak{S}, \mathsf{Cat})$ is the so-called category of *enhanced symmetric multicategories* in [33, p. 212], first defined by [1] (therein, these are called opetopes).

By analogy, we let

- $\mathsf{Cat}(\mathsf{Fam}_{\mathrm{fin}}, \mathsf{Cat})$ be the category of *enhanced cocartesian multicategories*, and
- $\mathsf{Cat}(\mathsf{Fam}^*_{\mathrm{fin}}, \mathsf{Cat})$ be the category of *enhanced cartesian multicategories.*

As we shall verify in Section 10, in each case $T = \mathsf{Fam}_{\mathrm{fin}}, \mathsf{Fam}^*_{\mathrm{fin}}, \mathfrak{S}$, the category $(\overline{T}, \mathsf{Cat})\text{-}\mathsf{Cat}$ is the full subcategory of $\mathsf{Cat}(T, \mathsf{Cat})$ with discrete categories of objects. Therefore,

- $(\overline{\mathsf{Fam}_{\mathrm{fin}}}, \mathsf{Cat})\text{-}\mathsf{Cat}$ is the category of *cocartesian multicategories*,
- $(\overline{\mathsf{Fam}^*_{\mathrm{fin}}}, \mathsf{Cat})\text{-}\mathsf{Cat}$ is the category of *cartesian multicategories*,
- $(\overline{\mathfrak{S}}, \mathsf{Cat})\text{-}\mathsf{Cat}$ is the category of *symmetric multicategories.*

10. APPLICATION TO DESCENT THEORY

We shall fix a lextensive, connected category $\mathcal{V}$ with its cartesian monoidal structure, and a fibrewise discrete, cartesian monad T on $\mathcal{V}$. Theorem 9.2 provides us with an embedding $(\overline{T}, \mathcal{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T, \mathcal{V})$. Now, we desire to apply this result to study *effective descent morphisms* in $(\overline{T}, \mathcal{V})\text{-}\mathsf{Cat}$. We promptly review the fundamental aspects of descent theory necessary to draw our desired conclusions. Afterwards, under a suitable hypothesis, we confirm that $(\overline{T}, \mathcal{V})\text{-}\mathsf{Cat}$ is the full subcategory of $\mathsf{Cat}(T, \mathcal{V})$ with a discrete object-of-objects (Theorem 10.3), which we deduce that $(\overline{T}, \mathcal{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T, \mathcal{V})$ reflects effective descent morphisms, generalizing [35, 9.10 Lemma and 9.11 Theorem] to the generalized multicategory setting.

Let $\mathcal{C}$ be a category with finite limits, and let $p \colon x \to y$ be a morphism. We have a change-of-base adjunction

$$\begin{tikzcd} \mathcal{C}/y \arrow[r, bend right, "p^*"'] \arrow[r, phantom, "\perp"] & \mathcal{C}/x . \arrow[l, bend right, "p_!"'] \end{tikzcd}$$

By the Bénabou-Roubaud theorem [5], we obtain $\mathsf{Desc}(p) \cong T^p\text{-}\mathsf{Alg}$, where T^p is the monad induced by the change-of-base adjunction.

Hence, we may consider the Eilenberg-Moore factorization of p^* in the following form:

$$\begin{tikzcd} \mathcal{C}/y \arrow[rr, "\mathcal{K}^p"] \arrow[dr, "p^*"'] & & \mathsf{Desc}(p) \arrow[dl] \\ & \mathcal{C}/x & \end{tikzcd}$$

We say that

- p is an *effective descent morphism* if $\mathcal{K}^p$ is an equivalence,
- p is a *descent morphism* if $\mathcal{K}^p$ is fully faithful,
- p is an *almost descent morphism* if $\mathcal{K}^p$ is faithful.

For categories $\mathcal{C}$ with finite limits, descent morphisms are precisely the pullback-stable regular epimorphisms, and almost descent morphisms are precisely the pullback-stable epimorphisms. If $\mathcal{C}$ is Barr-exact [2] or locally cartesian closed, then effective descent morphisms are precisely the descent morphisms. If $\mathcal{C}$ is a topos, then effective descent morphisms are precisely the epimorphisms [25].

Effective descent morphisms were studied and characterized for $\mathcal{C} = \mathsf{Top}$ in [39, 11], and for $\mathcal{C} = \mathsf{Cat}$ in [32, 25], exhibiting their non-trivial nature.

Having fixed the terminology, we finish our preliminaries by recalling the following result of effective descent morphisms for pseudopullbacks of categories:

Proposition 10.1 ([35, Theorem 1.6]). *If we have pseudopullback diagram of categories with pullbacks and pullback preserving functors*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "H"] \\ \mathcal{C} \arrow[r, "K"'] & \mathcal{D} \end{tikzcd}$$

and a morphism f of $\mathcal{A}$ such that

- *Ff and Gf are effective descent morphisms, and*
- *$KFf \cong HGf$ is a descent morphism,*

then f is an effective descent morphism.

Lemma 10.2. *If $(X\cdot 1, a, \eta, \mu)$ is an internal $(T, \mathcal{V})$-category, then $\hat{\varepsilon}_a$ is a split epimorphism. Moreover, if $\hat{\varepsilon}_{T1}$ is a monomorphism, then $\hat{\varepsilon}_a$ is an isomorphism.*

Proof. We consider the unique morphism $(X \cdot 1, a, \eta, \mu) \to (1, e_1^*, \eta, \mu)$ to the terminal $(T, \mathcal{V})$-category

$$\begin{tikzcd}
 & M_a \arrow[dl, "l_a"'] \arrow[dr, "r_a"] \arrow[dd] & \\
T(X \cdot 1) \arrow[dd, "T!"'] & & X \cdot 1 \arrow[dd, "!"] \\
 & 1 \arrow[dl, "e_1"'] \arrow[dr, equal] & \\
T1 & & 1
\end{tikzcd}$$

and we note that $e_1 = \hat{\varepsilon}_{T1} \circ (\overline{e}_1 \cdot 1)$, so that there there exists a unique $\hat{l}_a \colon M_a \to \overline{T}X \cdot 1$ such that $\hat{\varepsilon}_{T(X\cdot 1)} \circ \hat{l}_a = l_a$ and $\overline{T}! \cdot 1 \circ \hat{l}_a = (\overline{e}_1 \cdot 1) \circ !$:

$$\begin{tikzcd}
M_a \arrow[d, "!"] \arrow[dr, dashed, "\hat{l}_a"] \arrow[drr, bend left, "l_a"] & & \\
1 \arrow[dr, "\overline{e}_1 \cdot 1"'] & \overline{T}X \cdot 1 \arrow[r, "\hat{\varepsilon}_{T(X\cdot 1)}"] \arrow[d, "\overline{T}! \cdot 1"'] \arrow[dr, phantom, "\ulcorner", very near start] & T(X \cdot 1) \arrow[d, "T(! \cdot 1)"] \\
 & \overline{T}1 \cdot 1 \arrow[r, "\hat{\varepsilon}_{T1}"'] & T1
\end{tikzcd} \tag{10.1}$$

It follows that there is a unique $\omega \colon M_a \to M_{\mathcal{V}(1,a)\cdot 1}$ such that $\hat{\varepsilon}_a \circ \omega = \mathsf{id}$ and $(\hat{l}_a, r_a) = (l_a, r_a) \circ \omega$,

$$\begin{tikzcd}
M_a \arrow[dr, dashed, "\omega"] \arrow[drr, bend left, "{\hat{l}_a, r_a}"] \arrow[ddr, equal, bend right] & & \\
 & M_{\mathcal{V}(1,a)\cdot 1} \arrow[r, "{l_{\mathcal{V}(1,a)\cdot 1}, r_{\mathcal{V}(1,a)\cdot 1}}"] \arrow[d, "\hat{\varepsilon}_a"'] \arrow[dr, phantom, "\ulcorner", very near start] & \overline{T}X \cdot 1 \times X \cdot 1 \arrow[d, "\hat{\varepsilon}_{T(X\cdot 1)} \times \mathsf{id}"] \\
 & M_a \arrow[r, "{l_a, r_a}"'] & T(X \cdot 1) \times X \cdot 1
\end{tikzcd} \tag{10.2}$$

thereby confirming $\hat{\varepsilon}_a$ is a split epimorphism.

Moreover, observe that when $\hat{\varepsilon}_{T1}$ is a monomorphism, it follows by the pullback square in (10.1) that $\hat{\varepsilon}_{T(X\cdot 1)}$ is a monomorphism, and by the pullback square in (10.2), we may conclude that $\hat{\varepsilon}_a$ is a monomorphism. Thus, $\hat{\varepsilon}_a$ is an isomorphism. □

As a corollary, we obtain

Theorem 10.3. *If $\hat{\varepsilon}_{T1}$ is a monomorphism, then we have a pseudopullback diagram*

$$\begin{tikzcd}
(\overline{T}, \mathcal{V})\text{-}\mathsf{Cat} \arrow[r, "- \cdot 1"] \arrow[d] & \mathsf{Cat}(T, \mathcal{V}) \arrow[d] \\
\mathsf{Set} \arrow[r, "- \cdot 1"'] & \mathcal{V}
\end{tikzcd} \tag{10.3}$$

of categories with pullbacks and pullback-preserving functors.

Proof. We begin by observing that the objects of the pseudopullback are pairs $(S, (X, a, \eta, \mu), \omega)$ where S is a set, (X, a, η, μ) is an internal $(T, \mathcal{V})$-category, and $\omega \colon S \cdot 1 \to X$ is an isomorphism. Naturally, this implies that $\hat{\varepsilon}_X$ is an isomorphism, since $\hat{\varepsilon}_{S\cdot 1}$ is invertible:

$$\begin{tikzcd}
\mathcal{V}(1, S\cdot 1)\cdot 1 \arrow[r, "\hat{\varepsilon}_{S\cdot 1}"] \arrow[d, "\mathcal{V}(1,\omega)\cdot 1"'] & S\cdot 1 \arrow[d, "\omega"] \\
\mathcal{V}(1, X)\cdot 1 \arrow[r, "\hat{\varepsilon}_X"'] & X
\end{tikzcd}$$

and conversely, for any internal $(T, \mathcal{V})$-category (Y, b, η, μ) such that $\hat{\varepsilon}_Y$ is invertible, the triple

$$(\mathcal{V}(1, Y), (Y, b, \eta, \mu), \hat{\varepsilon}_Y)$$

is an object of the pseudopullback.

Hence, given a $(T, \mathcal{V})$-category (X, a, η, μ) such that $\hat{\varepsilon}_X$ is invertible, we have by Lemma 10.2 that $\hat{\varepsilon}_a$ is invertible, since $\hat{\varepsilon}_{T1}$ is a monomorphism by hypothesis. By Lemma 8.2, it follows that $\mathsf{n}_a^{\hat{\varepsilon}^*}$ is invertible, so that we can apply Lemma 6.2 to conclude that (X, a, η, μ) is isomorphic to an enriched $(\overline{T}, \mathcal{V})$-category, concluding the proof. □

From this, we can now apply Proposition 10.1 to conclude that

Lemma 10.4. *If $\hat{\varepsilon}_{T1}$ is a monomorphism, then $-\cdot 1\colon (\overline{T},\, \mathcal{V})$-$\mathsf{Cat} \to \mathsf{Cat}(T,\, \mathcal{V})$ reflects effective descent morphisms.*

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of enriched $(\overline{T}, \mathcal{V})$-categories such that $F\cdot 1$ is an effective descent morphism. Since $\mathsf{Cat}(T,\, \mathcal{V}) \to \mathcal{V}$ has fully faithful left and right adjoints, we may apply [37, Lemma 2.3] to conclude that it preserves descent morphisms, so that $(F\cdot 1)_0 = F_0\cdot 1$ is a descent morphism.

Since $-\cdot 1\colon \mathsf{Set} \to \mathcal{V}$ reflects epimorphisms, we conclude that F_0 is an epimorphism; hence an effective descent morphism. Now, we apply Proposition 10.1 with the pseudopullback (10.3) to conclude F is effective for descent. □

Via [38, Theorem 5.3], which provides sufficient conditions for effective descent morphisms in $\mathsf{Cat}(T,\, \mathcal{V})$ in terms of effective descent in $\mathcal{V}$, we can now do the same for $(\overline{T},\, \mathcal{V})$-$\mathsf{Cat}$:

Theorem 10.5. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor of $(\overline{T}, \mathcal{V})$-categories. If $\hat{\varepsilon}_{T1}$ is a monomorphism, and*

- *$(p\cdot 1)_1$ is an effective descent morphism,*
- *$(p\cdot 1)_2$ is a descent morphism,*
- *$(p\cdot 1)_3$ is an almost descent morphism,*

then p is an effective descent morphism.

Proof. The three above conditions guarantee that $p\cdot 1$ is an effective descent functor of (internal) $(T, \mathcal{V})$-categories. Since $\hat{\varepsilon}_{T1}$ is a monomorphism, we can apply Lemma 10.4 to obtain the promised conclusion. □

Now, the above work raises (at least) the following two questions:

- For which pairs $(T, \mathcal{V})$ can we guarantee that $\hat{\varepsilon}_{T1}$ is a monomorphism?
- Is the requirement that $\hat{\varepsilon}_{T1}$ be a monomorphism "reasonable"?

To answer the first, we note that this holds when

- the terminal object is a *separator*; that is, when $\mathcal{V}(1, -)$ is faithful, which implies $\hat{\varepsilon}$ is a componentwise monomorphism. This is the case when $\mathcal{V} = \mathsf{Set}, \mathsf{Top}, \mathsf{Cat}$, any hyperconnected Grothendieck topos, but not $\mathcal{V} = \mathsf{Grph}$.
- T is discrete; that is, when $\hat{\varepsilon}_{T1}$ is an isomorphism. This is the case when T is the free $\times$-monoid monad on $\mathcal{V}$, but not when $T = \mathfrak{F}$.

And this, in a sense, answers the second question as well: from a practical perspective, the above conditions are sufficient for nearly all of our examples. And while we haven't confirmed whether the condition "$\hat{\varepsilon}_{T1}$ is a monomorphism" is necessary or not, we can provide a heuristic argument to convey the intuition that this condition correctly captures that $\overline{T}1\cdot 1$ is a "good" discretization of $T1$: a pair which satisfies neither of the above hypotheses is the pair $(\mathfrak{F}, \mathsf{Grph})$, as $\mathsf{ob}\,\hat{\varepsilon}_{\mathfrak{F}1}\colon \mathbb{N} \to 1$; here, $\overline{\mathfrak{F}}1 = \mathsf{Grph}(1, \mathfrak{F}1)$ has too many points to be a "reasonable" discretization.

We now discuss the examples we have worked with so far.

10.1. **$\mathcal{V}$-operadic $\mathcal{V}$-categories:** Let $\mathfrak{O}$ be a $\mathcal{V}$-operad, so that the $\mathcal{V}$-operadic monad $T = T_{\mathfrak{O}}$ induced by $\mathfrak{O}$ is given by

$$X \mapsto \sum_{n\in\mathbb{N}} \mathfrak{O}_n \times X^n.$$

Since $\mathcal{V}(1,-)$ preserves coproducts, we have

$$\mathcal{V}(1, \sum_{n\in\mathbb{N}} \mathfrak{O}_n) \cong \sum_{n\in\mathbb{N}} \mathcal{V}(1, \mathfrak{O}_n),$$

and therefore $\hat{\varepsilon}_{T1} \cong \sum_{n\in\mathbb{N}} \hat{\varepsilon}_{\mathfrak{O}_n}$. It is easy to verify that in an extensive category, a coproduct of morphisms is a monomorphism if and only if every summand is a monomorphism, so we may apply Theorem when $\hat{\varepsilon}_{\mathfrak{O}_n}$ is a monomorphism for all $n \in \mathbb{N}$.

Naturally, the result holds if we consider $\mathcal{V}$-operads for categories $\mathcal{V}$ whose terminal is a separator, such as Cat or Top, or if we consider *discrete* $\mathcal{V}$-operads; that is, $\mathcal{V}$-operads such that $\hat{\varepsilon}_{\mathfrak{O}_n}$ is an isomorphism for all $n \in \mathbb{N}$.

However, the above conditions are not necessary, if, for instance, one considers Grph-operads $\mathfrak{O}$ such that $\mathfrak{O}_n$ has at most one loop at each vertex for all $n \in \mathbb{N}$; this is precisely the case when $\hat{\varepsilon}_{\mathfrak{O}_n}$ is a monomorphism for all $n \in \mathbb{N}$.

If $\mathfrak{O}$ a discrete $\mathcal{V}$-operad, we define $(\overline{T}, \mathcal{V})$-$\mathsf{Cat}$ to be the category of enriched $\mathfrak{O}$-categories.

10.2. **$\mathcal{V}$-multicategories:** An important instance of the previous case is the case $\mathfrak{O} \cong \mathbb{N} \cdot 1$; that is, when T is the free $\times$-monoid monad on $\mathcal{V}$. In this case, $\mathsf{Cat}(T, \mathcal{V})$ is the category of *multicategories internal to* $\mathcal{V}$. We note that T is a discrete monad, and the induced Set-monad $\overline{T}$ is the ordinary free monoid monad.

Thus, we may define the objects of $(\overline{T}, \mathcal{V})$-$\mathsf{Cat}$ to be the *enriched* $\mathcal{V}$-multicategories, and the morphisms are the respective enriched $\mathcal{V}$-functors. An immediate application of Theorem 10.5 provides criteria for such an enriched $\mathcal{V}$-functor to be effective for descent.

10.3. **Clubs:** We consider the pair $(\mathfrak{S}, \mathsf{Cat})$; the free symmetric strict monoidal category monad $\mathfrak{S}$ on Cat. By Theorem 10.3, we recover the categories of (many-object) clubs considered in [29, 27] by taking the fibers of the fibration $(\overline{\mathfrak{S}}, \mathsf{Cat})$-$\mathsf{Cat} \to \mathsf{Set}$ (see [15, 4.19]).

In fact, the above can be carried out for any fibrewise discrete monad T on Cat, as the terminal object of Cat is a separator.

11. Epilogue

We gave a general description of change-of-base functors between horizontal lax algebras induced by monad (op)lax morphisms on the 2-category $\mathsf{PsDbCat}_{\mathrm{lax}}$, and with this description, we made the dichotomy between enriched and internal generalized multicategories explicit. As our main result, we have shown that enriched generalized multicategories are discrete, internal generalized multicategories, under suitable conditions. Moreover, we applied this result to study the effective descent morphisms of $(\overline{T}, \mathcal{V})$-$\mathsf{Cat}$.

There is still a vast amount of open problems left to settle. For the remainder of this section, we will state a couple of these problems, sketch a possible approach to their solution, and highlight possible connections to other work.

11.1. **Object-discreteness.** In [15, Section 8], the authors define and study the full subcategories of *normalized* and *object-discrete* horizontal lax T-algebras. Inspired by our Theorem 9.2, we sketch an argument, for an instance of [15, Theorem 8.7] for the equipment of *modules* of a suitable equipment, via change-of-base.

If $\mathbb{D}$ is an equipment whose hom-categories of the underlying bicategory have all coequalizers, which preserved by horizontal composition, then we have an equipment $\mathsf{Mod}(\mathbb{D})$ whose underlying category of objects is $\mathbb{H}\,\mathsf{Lax}\text{-}\mathsf{id}\text{-}\mathsf{Alg}$, and horizontal 1-cells are *modules*; see [33, Section 5.3], [42, Theorem 11.5]. In fact, Mod defines a 2-functor defined on a suitable full sub-2-category of equipments, hence if T is a monad on $\mathbb{D}$, then $\mathsf{Mod}(T)$ is a monad on $\mathsf{Mod}(\mathbb{D})$. We have an inclusion

$$\mathfrak{I} \colon \mathbb{D} \to \mathsf{Mod}(\mathbb{D}),$$

and this induces a monad oplax morphism $T \to \mathsf{Mod}(T)$ with the unit comparison e^T. When T is normal, we may apply Theorem 5.2 to obtain a change-of-base functor

$$\mathfrak{I}_! \colon \mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg} \to \mathbb{H}\,\mathsf{Lax}\text{-}\mathsf{Mod}(T)\text{-}\mathsf{Alg},$$

which identifies the full subcategory of "object-discrete" horizontal lax $\mathsf{Mod}(T)$-algebras as the category of horizontal lax T-algebras, so we partially obtain [15, Theorem 8.7], when T is normal.

11.2. **Monadicity of horizontal lax algebras.** Let $T = (T, m, e)$ be a monad on an equipment $\mathbb{D}$ in $\mathsf{PsDbCat}_{\mathrm{lax}}$, and let x be an object of $\mathbb{D}$. We define $\mathbb{H}\,\mathsf{Kl}(T, x)$ to be the category whose objects are horizontal 1-cells $a \colon Tx \to x$, and morphisms are the globular 2-cells between them.

If m has a strong conjoint, $\mathbb{H}\,\mathsf{Kl}(T, x)$ has a tensor product defined by

$$b \otimes a = b \cdot (Ta \cdot m_x^*),$$

which makes it into a *skew monoidal category* [43]. If we let $\mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}(x)$ be the category of horizontal lax T-algebras with underlying object x, it can be shown that $\mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}(x)$ is the category of monoids of the skew monoidal category $\mathbb{H}\,\mathsf{Kl}(T, x)$[5]. Therefore, we have a forgetful functor

$$\mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}(x) \to \mathbb{H}\,\mathsf{Kl}(T, x),$$

and we can study its monadicity, by studying free monoids in skew monoidal categories, adapting the work of [17, 26, 30].

11.3. **2-dimensional structure.** As stated in the Introduction, we have not considered the 2-dimensional structure of $\mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$. However, inspired by the fact that $(T, \mathcal{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$ and $\mathsf{Cat}(T, \mathcal{V}) \to \mathcal{V}$ for suitable $\mathcal{V}$ are fibrations, it might be interesting to explore whether $\mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg} \to \mathbb{D}$ is a double fibration (see [14]), as well as possible applications, even in the more general context where $\mathbb{D}$ is a virtual double category.

11.4. **Other notions of change-of-base.** We have already mentioned two notions of change-of-base that are not covered by Theorem 5.2 in Subsections 5.1 and 6.2. In fact, with an adequate notion of "monad morphism" $(F, \phi) \colon S \to T$ for S a lax monad on $\mathbb{D}$, and T an oplax monad on $\mathbb{E}$, we question if it is possible to expand the scope of the dichotomy between enriched and internal multicategories.

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[5] This construction is analogous to the definition of $\mathbb{H}\,\mathsf{Lax}\text{-}T\text{-}\mathsf{Alg}$ in [15] as monoids in $\mathbb{H}\,\mathsf{Kl}(T)$, adapted to the fixed-object case, and with $\mathbb{D}$ an equipment.

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