KNOT EXTERIORS WITH ALL COMPACT SURFACES OF POSITIVE GENUS ESSENTIALLY EMBEDDED

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ABSTRACT. It is well known the existence of knots with Seifert surfaces of arbitrarily high genus. In this paper we show the existence of infinitely many knot exteriors each of which having longitudinal essential surfaces of any positive genus and number of boundary components.

1. Introduction

Essential surfaces have an important role on understanding 3-manifold topology since the second half of the last century. One particularly interesting property is the existence of essential surfaces of arbitrarily large Euler characteristics in some 3-manifolds. For knot exteriors in particular, it is well known since the work of Lyon [8] that a knot exterior can have closed essential surfaces and also Seifert surfaces of arbitrarily high genus. Many more examples of knot exteriors with these properties have been published throughout the years. For instance, besides the result of Lyon, several other collections of knots have been given for which there are Seifert surfaces of arbitrarily high genus as in work of Parris [15] (see also [19]), Gustafson [5], Ozawa and Tsutsumi [17] or Tsutsumi [18]. We will show further that a knot exterior can have a collection of longitudinal essential surfaces with arbitrarily large Euler characteristics not only because of large genus, as a collection of Seifert surfaces can have, but also from the number of boundary components. In fact, the collection of longitudinal essential surfaces can be with every number of boundary components and all positive genus.

Theorem 1. *There are infinitely many knots in the 3-sphere each of which having in its exterior a longitudinal essential surface of any positive genus and any number of boundary components. That is, all compact surfaces of positive genus with boundary have an essential embedding into each of the knot exteriors in this collection.*

With respect to surfaces of genus zero, we know from work of Gabai [4] when proving the Property R and the Poenaru conjectures, also related to the Cabling conjecture, that there are no longitudinal essential planar surfaces in knot exteriors besides the disk bounded by the unknot. However, it is known by work of Hatcher and Thurston, Proposition 1(3) of [6], the existence of a collection of (non-meridional) essential surfaces with arbitrarily large number of boundary components in some 2-bridge knot exteriors. As it is not clear in Proposition 1(3) of [6] if the surfaces are orientable, the number of boundary components for orientable

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surfaces can only be claimed to be even, by taking the boundary of a regular neighborhood of a non-orientable surface if that is the case. Also, as it is well known, composite knots do not have genus one Seifert surfaces, and the boundary slope of a Seifert surface in the knot exterior is 0 (longitudinal). Hence we cannot have a statement as in this theorem for composite knots or for a different boundary slope. This theorem contrasts with other results in the literature. For instance, Wilson [19] proved that a small knot in S^3, *i.e.* without closed essential surfaces in its complement, cannot have an infinite number of Seifert surfaces. From work of Oertel in [14], on a theorem of Jaco and Sedgwick, it is finite the number of compact essential surfaces of uniformly bounded genus (closed or with boundary) in knot exteriors without essential genus one surfaces (closed or with boundary). In [2], in work related to the Kervaire conjecture, Eudave-Muñoz proved that any odd number can be realized as the number of boundary components of an essential orientable connected surface properly embedded in a knot exterior. However, not necessarily the same knot exterior and the genus not arbitrarily large for the same number of boundary components.
Besides closed or longitudinal boundary slope, a similar result exists for meridional surfaces in knot exteriors. In fact, in [9] the author proved a prime knot exterior can have a collection of meridional essential surfaces with two boundary components and any positive genus. However, a collection of essential surfaces in a knot exterior can be of arbitrarily large Euler characteristics not only because of large genus but also from the number of boundary components. This was shown first by the author in [10] with a collection of essential planar surfaces with arbitrarily large number of boundary components, and, even further, in [11] and in [12] with a collection of essential surfaces with independently large genus and number of boundary components in a satellite knot exterior and hyperbolic knot exterior, respectively.

The paper is organized as follows: In section 2 we define a class of satellite knots that we will use throughout the paper. In sections 3 and 4 we construct branched surfaces and use branched surface theory to prove that the surfaces they carry as in the statement of Theorem 1 are essential in the corresponding knot exterior. Throughout this paper all manifolds are orientable, all submanifolds are assumed to be in general position and we work in the smooth category.

2. Square double of a knot

In this section we define a class of satellite knots, which we refer to as square double of a knot, that we will use to prove Theorem 1.

First we consider the 2-string tangle $T_1 = (B_1; s_1 \cup s_2)$ as in Figure 1(a) which we refer to as the square tangle, following nomenclature in the literature [16].

Let T_2' denote the $n/1$ rational tangle, where n is an integer, with $T_2' = (B_2; p_1 \cup p_2)$. The arcs p_1 and p_2 co-bound a disk D in B_2 with two arcs in ∂B_2, say a_1 and a_2. Consider the operation on T_2' where we assume the core of a regular neighborhood of D in B_2 follows the pattern of a knot J. Hence, the image of the arcs p_i, $i = 1, 2$, in B_2, follow this pattern. That is, through the closure of p_i with an arc in ∂B_2 we obtain the knot J. We denote the resulting tangle as $T_2(n; J)$, or only as T_2, and we assume that the arcs a_i, of $\partial D \cap \partial B_2$, are in the boundary circle of the diagram disk of T_2. (See Figure 1(b)).

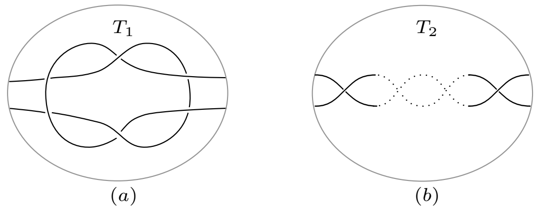

FIGURE 1: In (a) we have a diagram of the square tangle, denoted T_1; in (b) we have represented a tangle of two twisted parallel arcs following the pattern of a knot J, denoted T_2.

Definition 1. For each integer n, consider now the knot $K(n; J)$ obtained as the closure of T_1 with $T_2(n; J)$, such that the boundary circle of their diagram disks, as in Figure 1, are identified with opposite orientation and, for each $i = 1, 2$, the end points of a_i are connected to s_i. The knot $K(n; J)$ is referred to as a *square double of* J.

Note that the square knot is a square double of the unknot, more exactly $K(0; \text{unknot})$. In case the knot J is non-trivial a square double of J is a satellite knot with companion J. In Figure 2, we have diagrams of two examples of square doubles, the square knot and a square double of a trefoil. In Figure 3, we have a schematic diagram illustration of a square double of J.

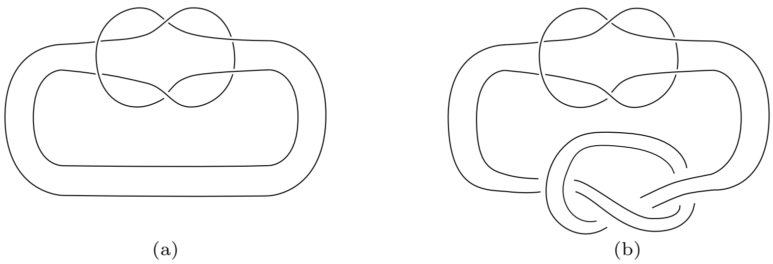

FIGURE 2: (a) A diagram of the square knot, which is a square double of the unknot; (b) a diagram of a square double of the trefoil knot.

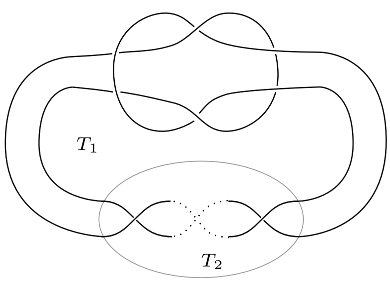

FIGURE 3: Square double of a knot J.

Lemma 1. *A square double of a knot is a ribbon 2 knot.*

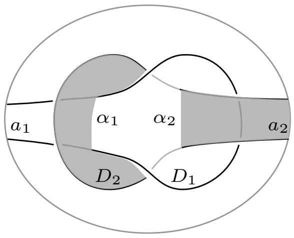

FIGURE 4:

Proof. Each arc s_i cobounds with a_i a disk D_i in B_1, $i = 1, 2$. In general position, D_1 and D_2 intersect each other in two arcs, α_1 and α_2, as illustrated schematically in Figure 4.

Let E be the immersed disk obtained by connecting D_1, D and D_2 along a_1 and a_2. The immersed disk E has boundary K and self-intersects in two disjoint arcs which are ribbon singularities in E, that is the preimage of each of these arcs is two disjoint arcs in the preimage of E. Hence, as there are no other singularities, E is a ribbon disk of K with two ribbon singularities. □

Lemma 2. *A square double of a non-trivial knot is prime. The only square double of the unknot that is composite is the square knot.*

Proof. The first part of this theorem is a consequence of a result of Lickorish [7] stating that a knot with a 2-string prime decomposition is a prime knot. We recall that a tangle is said prime if it is essential and there are no local knots, that is no ball intersects the strings of the tangle in a single non-trivial arc. As it is observed also in [7] the tangle T_1 is a 2-string prime tangle. The tangle T_2 is defined by two parallel arcs with a pattern of a non-trivial knot. Hence, there are no local knots in T_2, otherwise the strings could not be parallel, and T_2 is an essential tangle, otherwise the pattern J would be of the unknot. Therefore, the square double of a non-trivial knot has a 2-string prime tangle decomposition. Hence, from [7] it is prime.

One other way to prove this statement is by using the wrapping number. The wrapping number of a square double of a non-trivial knot is 2, as it can be seen by taking the meridian of the companion solid torus that is a regular neighborhood of a_i. If there was a decomposing sphere, by taking its intersection with the companion torus and an innermost curve argument on the intersecting curves on the sphere, we would obtain a compressing disk for the companion torus or a meridian disk for the companion solid torus intersecting K once, contradicting the existence of a meridian disk intersecting K at two points.

For the second part of the theorem note first that if the pattern of the parallel arcs in T_2 is the unknot then the tangle is an integral rational tangle. We know that if T_2 is the 0 rational tangle we obtain the square knot, which is composite. From Eudave-Muñoz work [1], if r/s and p/q are two rational tangles whose closure of T_1 return a composite knot then $|ps - qr| \leq 1$. Then, the only other possibilities for T_2 when closing T_1 to return a composite knot is if T_2 is a 1 or -1 rational tangle. However, these knots have genus 1, as it can easily seen by an adaptation of the diagram of Figure 7 for the knots $K(2m+1, J)$, where in this case $2m + 1 = \pm 1$ and J is the unknot. As genus is additive under connected sum it cannot be 1 if the knot is composite. Therefore, these knots are also prime. Hence, the only square double of the unknot that is not prime is the square knot. □

3. Longitudinal branched surface in a square double exterior

For the proof of Theorem 1 we use branched surface theory based on work of Oertel in [13] and of Floyd and Oertel in [3]. First, we revise the definition of branched surface and of surface carried by a branched surface in the following paragraphs.

A *branched surface* R with generic branched locus is a compact space locally modeled on Figure 5(a). Hence, a union of finitely many compact smooth surfaces in a 3-manifold M, glued together to form a compact subspace of M respecting the local model, is a branched surface. We denote by $N = N(R)$ a fibered regular neighborhood of R (embedded) in M, locally modeled on Figure 5(b). The boundary of N is the union of three compact surfaces $\partial_h N$, $\partial_v N$ and $\partial M \cap \partial N$, where a fiber of N meets $\partial_h N$ transversely at its endpoints and either is disjoint from $\partial_v N$ or meets $\partial_v N$ in a closed interval in its interior. We say that a surface S is *carried* by R if it can be isotoped into N so that it is transverse to the fibers. Furthermore, S is carried by R with *positive weights* if S intersects every fiber of N. If we associate a weight $w_i \geq 0$ to each component on the complement of the branch locus in R we say that we have an *invariant measure* provided that the weights satisfy *branch equations* as in Figure 5(c). Given an invariant measure on R we can define a surface carried by R, with respect to the number of intersections between the fibers and the surface. We also note that if all weights are positive then the surface carried can be isotoped to be transverse to all fibers of N, and hence is carried with positive weights by R.

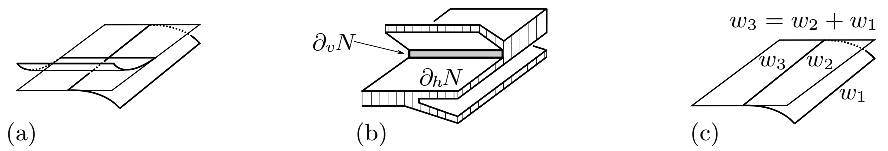

Figure 5: Local model for a branched surface, in (a), its regular neighborhood, in (b), and weights with branch equations, in (c).

In this section we will construct a branched surface in square double exteriors of composite knots which we use on the proof of Theorem 1.

Let $K = K(2m + 1; J)$, with m an integer, be a square double of J, a composite knot with summands J_1 and J_2, where we assume J_1 is as in [9]. That is, J_1 has in its exterior meridional essential surfaces of any genus with two boundary components, which we denote X_g for genus $g - 1$. Denote by Q_i the ball intersecting J in a single arc with pattern J_i. We assume, after an ambient isotopy if needed, that Q_i is in B_2, and the intersection of Q_i with $p_1 \cup p_2$ is two parallel arcs with pattern J_i. Let S_i be the boundary sphere of Q_i. The sphere S_i intersects A at two circles. These two circles cobound an annulus U_i in A, and an annulus V_i in S_i. With U_1 as the boundary of a tubular neighborhood of an arc with pattern J_1, we can assume, after an isotopy if needed, that the boundary components of X_g are also the two boundary circles of U_1.

Without loss of generality we assume that a path in A from $\partial_1 A$ must first intersect S_1 before S_2 and S_2 before $\partial_2 A$. Let A_1 (resp., A_2) be the annulus in A between $\partial_1 A$ and U_1 (resp., $\partial_2 A$ and U_2). At last we denote by A' the annulus in A from U_1 and U_2. (See Figure 6.)

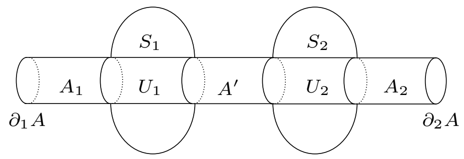

FIGURE 6: Partition of the annulus A with respect to the intersection with S_1 and S_2.

Consider the union of P, A_1, U_1, X_g, V_1, A', U_2, V_2 and A_2 denoted by R_g. We smooth the space R_g on the intersection of these union parts as follows: At $\partial_i A$, $i = 1, 2$, there is no singularity in R_g, hence it is smoothed with P. The annulus U_i at $A_i \cap U_i$, $i = 1, 2$ respectively, is smoothed towards A_i. The annulus V_i at $V_i \cap A_i$ is smoothed towards A_i, and at $V_i \cap A'$ is smoothed towards U_i, $i = 1, 2$. The surface X_g at $X_g \cap A_1$ is smoothed towards A_1, and at $X_g \cap A'$ is smoothed towards A'. The annulus A' at $A' \cap U_2$ is smoothed towards V_2 and at $A' \cap U_1$ it is smoothed towards U_1 (and X_g). In Figure 7, we have an illustration of R_g.

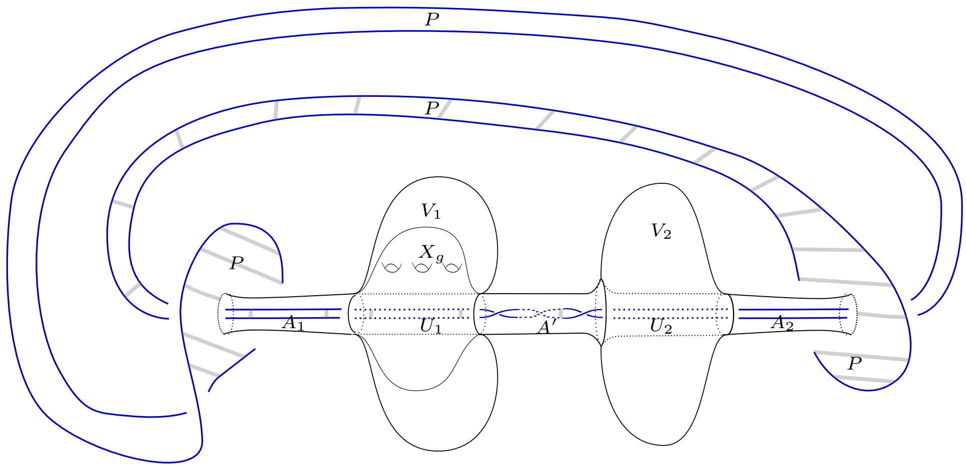

FIGURE 7: An illustration of the branched surface R_g.

Note that some sections of R_g branch into three sections, as A_1 with V_1, X_g, U_1, or two branches into two branches, as X_g, U_1 and V_1, A', as illustrated in Figure 7, which is not in conformity with the local model of branched surface. However, a small isotopy corrects this for the purpose of the local model. For convenience on the number of sections involved, we continue with the current choice of sections up to a small isotopy correction. So, from this construction and up to a small isotopy, the space R_g is a branched surface with sections denoted by P, A_1, U_1, X_g, V_1, A', U_2, V_2 and A_2.

4. Longitudinal essential surfaces from a branched surface

In this section we prove Theorem 1. We will do so by proving that the surfaces carried by R_g are essential.

We recall that the components on the complement of the branched locus of R_g are P, A_1, U_1, X_g, V_1, A', U_2, V_2 and A_2. We denote the weights of each component on an invariant measure for R_g as W_P, W_{A_1}, W_{U_1}, W_{X_g}, W_{V_1}, $W_{A'}$, W_{U_2}, W_{V_2} and W_{A_2}, respectively.

Now we define surfaces F_g^n carried by R_g with genus g and any number n of longitudinal boundary components. We define F_g^1 as the surface carried by R_g with invariant measure $W_P = 1$, $W_{A_1} = 1$, $W_{U_1} = 0$, $W_{X_g} = 1$, $W_{V_1} = 0$, $W_{A'} = 1$, $W_{U_2} = 1$, $W_{V_2} = 0$ and $W_{A_2} = 1$. Let us define F_g^2 as the surface carried by R_g with invariant measure $W_P = 2$, $W_{A_1} = 2$, $W_{U_1} = 0$, $W_{X_g} = 1$, $W_{V_1} = 1$, $W_{A'} = 0$, $W_{U_2} = 1$, $W_{V_2} = 1$ and $W_{A_2} = 2$. For $n \geq 3$, we define F_g^n as the surface carried by R_g with invariant measure $W_P = n$, $W_{A_1} = n$, $W_{U_1} = n-2$, $W_{X_g} = 1$, $W_{V_1} = 1$, $W_{A'} = n-2$, $W_{U_2} = 1$, $W_{V_2} = n-1$ and $W_{A_2} = n$. In Figure 8 we have a schematic illustration of the surfaces F_g^n for $n \geq 3$.

Lemma 3. *The surface F_g^n is orientable and connected.*

Proof. For convenience of notation, without loss of generality, we consider one side of P to be the positive "+" side and the other side to be the negative "−" side. If $n = 1$, we connect the non-peripheral boundary components of P respecting the orientation of P. If $n = 2$ we connect the non-peripheral boundary components of P_1 to the ones of P_2 always from the + or − side of P_1 to the corresponding + or − side of P_2. Hence, we obtain an orientable connected surface.
Suppose now that $n \geq 3$. As it can be observed from the schematic representation of F_g^n in Figure 8, all the odd indexed P_i's are connected in linear order, and similarly the even indexed are connected in linear order. Here, we respect the positive and negative sides of the P_i's. The positive side of P_1 is connected to the negative side of P_2, and the positive side of P_{n-1} is connected to the negative side of P_n. So, we connect the odd indexed to the even indexed sequences of P_i's to opposite sides at the beginning and at the end of the odd and even sequences, forming a loop. Hence, F_g^n is connected and orientable. □

Lemma 4. *The surface F_g^n has genus g and n boundary components.*

Proof. From the construction of F_g^n, its boundary components come from the longitudinal boundary component of each copy of P. As for F_g^n we use n copies of P, and all the other components have their boundaries connected to each other, we have that F_g^n has n boundary components. In the definition of F_g^n, the components other than copies of P and X_g are always annuli. The Euler characteristic, χ, of an annulus is 0 and χ is additive under gluing components along their boundaries. Hence, the Euler characteristic of F_g^n is $n \times \chi(P) + \chi(X_g)$, which is $-n + (2 - 2(g-1) - 2) = 2 - 2g - n$. As F_g^n is orientable and connected (Lemma 3) with n boundary components, it comes from the Euler characteristics formula of a n punctured orientable connected surface that it has genus g, completing the proof of the lemma. □

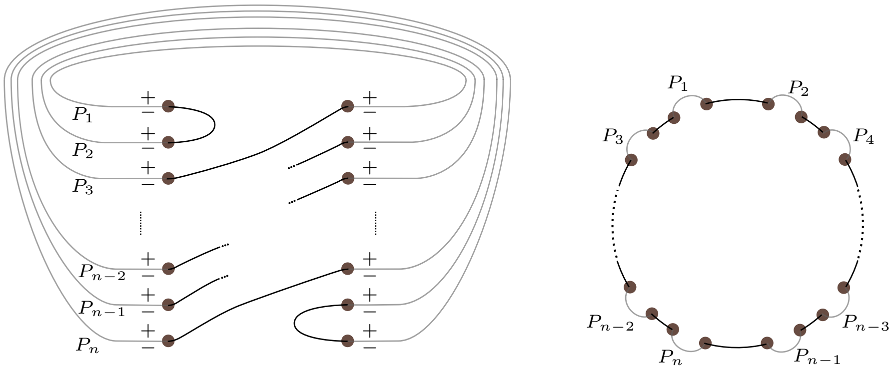

FIGURE 8: On the left, there is a schematic representation of F_g^n, $n \geq 3$, on how the components corresponding to sections of R_g connect. On the right, there is a schematic representation on how the copies of P are connected in F_g^n, for $n \geq 3$ odd.

The following concepts are relevant for the definition of incompressible branched surface. Let R denote a branched surface in a 3-manifold M, with regular neighbood N in M. A *disc of contact* is a disc O embedded in N transverse to fibers and with $\partial O \subset \partial_v N$. A *half-disc of contact* is a disc O embedded in N transverse to fibers with ∂O being the union of an arc in $\partial M \cap \partial N$ and an arc in $\partial_v N$. A *monogon* in the closure of $M - N$ is a disc O with $O \cap N = \partial O$ which intersects $\partial_v N$ in a single fiber. (See Figure 9.) We say a branched surface R in M contains a *Reeb component* if R carries a compressible torus or properly embedded annulus, transverse to the fibers of N, bounding a solid torus in M. (This is a weaker version of the definition of Reeb component in [13] by Oertel.)

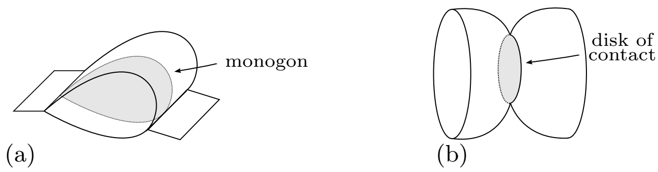

FIGURE 9: Illustration of a monogon and a disk of contact on a branched surface.

We recall that a branched surface R embedded in a 3-manifold M is said to be incompressible if it satisfies the following three properties:

(i) R has no disk of contact or half-disks of contact;
(ii) $\partial_h N$ is incompressible and boundary incompressible in the closure of $M - N$, where a boundary compressing disk is assumed to have boundary defined by an arc in ∂M and an arc in $\partial_h N$;
(iii) There are no monogons in the closure of $M - N$.

and *without Reeb components* if it satisfies the following property:

(iv) R doesn't carry a Reeb component.

The following theorem, proved by Oertel in [13], let us infer if a surface carried by a branched surface is essential. Note that condition (iv), R not carrying a torus or an annulus cutting a solid torus from M, implies the non-existence of Reeb components in the sense of Oertel [13].

Theorem 2 (Oertel, [13])**.** *If R is an incompressible branched surface without Reeb components (i.e. satisfies (i)-(iv)) and R carries some surface with positive weights then any surface carried by R is essential.*

Lemma 5. *The branched surface R_g, $g \geq 1$, is incompressible and without Reeb components.*

Proof. First we observe that R_g doesn't carry a Reeb component. In fact, if R_g would carry a torus T, this torus couldn't be transverse to the regular neighborhood of sections of R_g with boundary components, or of sections of R_g of genus bigger than 1. In this case, it could only be transverse to regular neighborhoods of sections A_1, A', X_g (in case $g = 1$), U_1, V_1, U_2, V_2 and A_2. Only one boundary component of A_1 and of A_2 is connected to these other sections. Hence, A_1 and of A_2 cannot contribute for R_g to carry a torus. The annulli U_2 and V_2 are smoothed towards A_2. As $W_{A_2} = 0$ in an invariant measure for a torus carried by R_g, we have that necessarily $W_{U_2} = W_{V_2} = 0$ as well. Similarly, U_1, V_1 and X_g are smoothed towards A_1. As $W_{A_1} = 0$ in an invariant measure for a torus carried by R_g, we also have that $W_{U_1} = W_{V_1} = W_{X_g} = 0$. Then A' is the only section left whose regular neighborhood could be transverse to a torus. As A' is smoothed towards U_1, X_g and V_2, similarly we have that $W_{A'} = 0$ in an invariant measure of a torus carried by R_g. Therefore, R_g does not carry a torus.

If R_g would carry a properly embedded annulus in the exterior of K, $E(K)$, this annulus would have to be transverse to the regular neighborhood of sections of R_g with boundary components, that is P. However, P has negative Euler characteristics, and the remaining sections of R_g have non-positive Euler characteristics. Hence, any surface carried by R_g with positive weight on P has negative Euler characteristics, which cannot be an annulus. Therefore, R_g does not carry an annulus.

Now we prove that R_g is incompressible in $E(K)$. First observe that there are no (half) disks of contact as no circle on the branched locus of R bounds a disk in $\partial_h N(R_g)$ and there are no properly embedded arcs on the branched locus of R_g. The space $N(R_g)$ decomposes $E(K)$ into four components: a component cut from $E(K)$ by X_g and V_1, denoted M_1; a component cut from $E(K)$ by X_g and U_1, denoted M_1'; a component cut from $E(K)$ by U_2, and V_2, denoted M_2; a component cut from $E(K)$ by all sections but X_g denoted M.

There are six components in $\partial_v N(R_g)$: one in M_1, two in M_1', one in M_2 and two in M. The component of $\partial_v N(R_g)$ in M_1 corresponds to a non-separating circle in $X_g \cup V_1$. Hence, a monogon in M_1 is a compressing disk for $X_g \cup V_1$ in M_1, but this contradicts X_g being essential in the exterior of J_1. A monogon in M_1' is impossible as the boundary of a monogon disk in M_1' would have to intersect one boundary component of the annulus U_1 but be disjoint from the other boundary component, as both correspond to components of $\partial_v N(R_g)$. Therefore, there are no monogons in M_1 and in M_1'. Similarly, as $\partial_v N(R_g)$ in M_2 corresponds to a non-separating circle in $U_2 \cup V_2$ and $U_2 \cup V_2$ is essential in the exterior of J_2, there are no monogons

in M_2.
Each curve of the branched locus corresponding to the two components of $\partial_v N(R_g)$ in M are non-separating in ∂M. Hence, a monogon in M corresponds to a compressing disk for ∂M in M. Let c_1 be the curve $V_1 \cap A'$ and c_2 be the curve $V_2 \cap A'$. Suppose there is a monogon O_1 corresponding to c_1. Then the surface Y_1 defined by P, A_1, V_1, A', V_2 and A_2 has a compressing disk O_1. But Y_1 is a genus one surface bounded by K, which is incompressible because K is non-trivial and hence we have a contradiction. Suppose there is a monogon O_2 corresponding to c_2. Then the surface defined by P, A_1, U_1, A', U_2 and A_2 has compressing disk O_2. But Y_2 is a genus one surface bounded by K, so it is a minimal genus Seifert surface and hence it is essential, contradicting O_2 being a compressing disk for O_2. Therefore, there are no monogons in M.

We proceed to prove that $\partial_h N(R_g)$ is (boundary) incompressible in the exterior of K. In M_1, $\partial_h N(R_g)$ corresponds to the complement of $A_1 \cap X_g \cap V_1$ in $X_g \cup V_1$. Hence, as X_g is incompressible in M_1 we have that $\partial_h N(R_g) \cap M_1$ is incompressible. In M_1', we argue similarly that $\partial_h N(R_g) \cap M_1'$ is incompressible. In M_2, $\partial_h N(R_g) \cap M_2$ corresponds to the complement of $A_2 \cap U_2 \cap V_2$ in $U_2 \cup V_2$, which corresponds to the exterior in S^3 of the knot J_2. Hence, $\partial_h N(R_g) \cap M_2$ is incompressible in M_2. At last, in M, $\partial_h N(R_g) \cap M$ corresponds to the complement of $\partial A'$ in ∂M, which is represented by A' and P, after an ambient isotopy. An essential simple closed curve in A' corresponds to a meridian of the satellite torus of K (with pattern $J_1 \# J_2$), hence there is no compressing disk for A' in M. The twice punctured disk P is essential in M, otherwise a (boundary) compressing disk would also be a (boundary) compressing disk for the Seifert surface defined by $P \cup A$ (see also the end of proof for Lemma 2).

Hence, $\partial_h N(R_g)$ is incompressible and boundary incompressible in $E(K)$, and there are no Reeb components, monogons or disks of contact in R_g. Therefore, R_g, $g \geq 1$, is an incompressible branched surface without Reeb components. □

Proof of Theorem 1. Let $K = K(2m+1; J)$, with m an integer, be a square double of a composite knot J and R_g, $g \geq 1$, as defined in section 3. The collection of such knots is infinite. In fact, the choice of companion J can be for arbitrarily large bridge number, and by work of Schubert, so is the bridge number of a corresponding satellite K.
Now we prove that the knots K are as in the statement of the theorem. From Proposition 5 we have that R_g, $g \geq 1$, is an incompressible branched surface without Reeb components. We also have that the surfaces F_g^n, $n \geq 3$, are carried with positive weights by R_g. Hence, we are under the conditions of Theorem 2. It then follows that all surfaces carried by R_g are incompressible and boundary incompressible in $E(K)$. Therefore, as F_g^n, $n \geq 1$, is carried by R_g, $g \geq 1$, and is orientable (Lemma 3), it is essential in $E(K)$. From Lemma 4, the surfaces F_g^n, $g \geq 1$, $n \geq 1$, have genus g and n boundary components. Then, as we wanted to prove, each compact orientable connected surface of positive genus with boundary has an essential proper embedding into the exterior of K. □

If we apply and follow the proof of Theorem 2 of [11] with the collection of knot exteriors and essential surfaces of Theorem 1 of this paper, we can conclude

that there are hyperbolic 3-manifolds with torus boundary each with a collection of longitudinal essential surfaces of independently unbounded genus and any number of boundary components.
As a final remark, note that the knots J_1 or J_2 can be chosen with closed essential surfaces in their exteriors of every genus or meridional essential surfaces of every genus and (even) number of boundary components. (See [11] for instance.) So, in this case, the exterior of K, as in the proof of Theorem 1, has (meridional) planar essential surfaces in its exterior but with the number of boundary components divisible by 4. Furthermore, in this case, the exterior of K has in fact all compact surfaces of positive genus essentially embedded, closed and with boundary.

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