NON SYMMETRIC CAUCHY KERNELS, DEMAZURE MEASURES AND LPP

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ABSTRACT. We use non symmetric Cauchy kernel identities to get the laws of last passage percolation (LPP) models in terms of Demazure characters. The construction is based on the restrictions of the RSK correspondence to augmented stair (Young) shape matrices and rephrased in a unified way compatible with crystal bases.

1. INTRODUCTION

We introduce the Demazure measure on nonnegative vectors corresponding to the directed last passage percolation (LPP) model on matrices of Young shape, that is, nonnegative integer matrices whose positive entries fit a Young shape. A nonnegative integer vector is always in the Weyl orbit of some partition and therefore all nonnegative vectors in a same Weyl orbit share the size of a largest entry which is the length of a longest row of the unique partition in its orbit. When the Young shape is a rectangle, we recover the Okounkov's Schur measure [4, Chapter 4], on the unique partition of each Weyl orbit, corresponding to the LPP model on nonnegative integer matrices.

Our main contribution is the use of Demazure characters which are non symmetric polynomials in general to study LPP problems: this has only been carried out for models with more symmetries using symmetric polynomials, in particular, Schur polynomials or Weyl characters or geometric analogues as incarnations of Whittaker functions ([6, 7, 17, 13, 19] and references therein). Crystal theory allows the compatibility of Robinson–Schensted–Knuth (RSK) correspondence with non symmetric Cauchy identities by Lascoux [14] and thus, in particular, the Cauchy identity (1). This interpretation was discovered by Choi–Kwon [8] for the non-symmetric case on stair cases (13). We complete the picture with the truncated and augmented stair shape.

This extended abstract is organized in four sections. In §2 we gather relevant definitions on crystals, in §3 present our contributions, and in §4 provide an example for our main result. The full version [3] (to which we refer the reader for details and proofs) containing the results hereby presented has been submitted for publication elsewhere.

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1.1. **LPP, rectangle shape and Schur measure.** Given two sets of indeterminates $x = \{x_1, \dots, x_m\}$ and $y = \{y_1, \dots, y_n\}$ the Cauchy identity asserts that

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j} = \sum_{\lambda\in\mathcal{P}_{\min(m,n)}} s_\lambda(x)s_\lambda(y) \tag{1}$$

where $\mathcal{P}_{\min(m,n)}$ is the set of partitions with at most $\min(m,n)$ parts and, for each such partition λ, $s_\lambda(x)$ and $s_\lambda(y)$ are the Schur polynomials in the indeterminates x and y, respectively.

This identity has several interpretations, applications and generalizations (see [9] and references therein). In particular, one can understand the left hand side as the character of polynomial functions on the space $\mathcal{M}_{m\times n}$ of matrices with m rows, n columns and entries in $\mathbb{Z}_{\geq 0}$ and decompose this space into a direct sum of $\mathfrak{gl}_m \times \mathfrak{gl}_n$ bimodules. The products of Schur functions $s_\lambda(x)$ and $s_\lambda(y)$ on the righthand side show this approach as the characters of the tensor product of irreducibles finite dimensional representations of highest weight λ for the linear Lie algebras $\mathfrak{gl}_m(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{C})$. In fact $\mathcal{M}_{m,n}$ is a realization of the the bicrystal of the symmetric space $S(\mathbb{C}^m \otimes \mathbb{C}^n)$ as a $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-module (see [8] and references therein).

The identity (1) can also be proved using the RSK correspondence [10, 18]. This is a one-to-one map ψ between the set $\mathcal{M}_{m,n}$ and the set

$$\bigsqcup_{\lambda\in\mathcal{P}_{\min(m,n)}} SSYT(\lambda,m)\times SSYT(\lambda,n)$$

of pairs (P,Q) of semistandard tableaux of the same shape λ, and entries in $[m] := \{1,\dots,m\}$ and $[n] := \{1,\dots,n\}$, respectively. (The convention that we use agrees with that of Kashiwara [12] to which we refer for another description of the RSK procedure and the connection with biwords. See §4 and [10] for variations on RSK.) Regarding $SSYT(\lambda,k)$ as the tableau realization for the $\mathfrak{gl}_k$-crystal $B(\lambda,k)$ of highest weight λ, then

$$\begin{aligned}\psi : \mathcal{M}_{m,n} \to & \bigsqcup_{\lambda\in\mathcal{P}_{\min(m,n)}} B(\lambda,m)\times B(\lambda,n)\\ A \mapsto & \ \psi(A) = (P(A), Q(A))\end{aligned} \tag{2}$$

is a $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-bicrystal isomorphism where the bicrystal structure on $\mathcal{M}_{m,n}$ is afforded from $B(\lambda,m)\times B(\lambda,n)$ by ψ^{-1}, that is, by reverse column Schensted insertion. The RSK correspondence has interesting properties. For each matrix A in $\mathcal{M}_{m,n}$, the greatest integer $p(A)$ obtained by summing up the entries in all the possible paths π starting at position $(1,n)$ and ending at position $(m,1)$ with steps $\longleftarrow$ or $\downarrow$

$$p(A) := \max_{\pi \text{ path in } A} \sum_{(i,j)\in\pi} a_{ij} \tag{3}$$

coincides with the common largest row length of the tableaux $P(A)$ and $Q(A)$ in (2). (We consider the paths which are compatible with the version of RSK that is used here. See §4.) It is then natural to study percolation models based on the RSK correspondence where random matrices whose entries follow independent geometric laws are considered [4]. This type of model, in the case of identical and independent geometric distribution, has been deeply studied by Johansson in [11], who proved that the fluctuations of the previous last passage percolation, once correctly normalized, are controlled by the Tracy-Widom distribution (defined from the study of the largest eigenvalues of random Hermitian matrices). The Schur measure, introduced by Okounkov, based on the Cauchy kernel identity, is an extension of the probability measure on the partitions corresponding to the directed last passage percolation model with the independent and identical geometric distribution of Johansson in [11], [4, Chapter 4].

Let u_i, v_j be real numbers in $[0,1)$, for $1 \leq i \leq m$, $1 \leq j \leq n$. Considering an array $\mathcal{W} = \{W_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ of independent random variables, with values in $\mathbb{Z}_{\geq 0}$, called weights, geometrically distributed as

$$\mathbb{P}(W_{ij} = k) = (1 - u_i v_j)(u_i v_j)^k, \text{ for any } k \in \mathbb{Z}_{\geq 0}, \tag{4}$$

with parameter $u_i v_j$, $\mathcal{W}$ is a random matrix with values in $\mathcal{M}_{m,n}$. We then get

$$\mathbb{P}(\mathcal{W} = A) = \left(\prod_{1 \leq i \leq m, 1 \leq j \leq n} (1 - u_i v_j) \right) (uv)^A$$

where $(uv)^A = \prod_{1 \leq i \leq m, 1 \leq j \leq n} (u_i v_j)^{a_{i,j}}$. The Last Passage Percolation (LPP) time G of $\mathcal{W}$ is defined to be the random variable $G := p \circ \mathcal{W}$. Applying the RSK correspondence, its properties and the Cauchy identity (1), one obtains the law of the random variable G, for any $k \in \mathbb{Z}_{\geq 0}$, in terms of Schur polynomials,

$$\mathbb{P}(G = k) = \prod_{1 \leq i,j \leq n} (1 - u_i v_j) \sum_{\lambda \in \mathcal{P}_{\min(m,n)} | \lambda_1 = k} s_\lambda(u_1, \dots, u_m) s_\lambda(v_1, \dots, v_n)$$

where the sum is over partitions λ with largest part k. Johansson [11] has established this result in the special case of identical geometric distribution, $u_i = v_j = \sqrt{q}$, $1 \leq i, j \leq n$, for a fixed $q \in]0, 1[$, a special case of the Schur measure on partitions (see [4, Chapter 10]). The RSK correspondence admits various generalizations and geometric versions which can also be used to get interesting last passage percolation models involving symmetric polynomials, in particular, characters of representations of Lie algebras other than $\mathfrak{gl}_n$ (symmetric with respect to the Weyl group) and geometric analogues [6, 7, 17, 19].

2. Crystal and Demazure modules

The finite dimensional irreducible polynomial representations of $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$ are parameterized by the partitions λ in $\mathcal{P}_n$. To each partition $\lambda \in \mathcal{P}_n$ corresponds a finite dimensional representation $V(\lambda)$ (or $\mathfrak{gl}_n$-module), and a crystal graph $B(\lambda)$ which can be regarded as the combinatorial skeleton of the simple module $V(\lambda)$. The vertices of $B(\lambda)$ label a distinguished basis of $V(\lambda)$. On the other hand, $B(\lambda)$ has various combinatorial realizations (*i.e.*, vertex labelings) in terms of semistandard tableaux, Littelmann's paths [15] or semiskylines [16].

The (abstract) crystal $B(\lambda)$ is a graph whose set of vertices is endowed with a weight function $wt : B(\lambda) \to \mathbb{Z}^n$ and with the structure of a coloured and oriented graph given by the action of the crystal operators $\tilde{f}_i$ and $\tilde{e}_i$ with $i \in I = [n-1]$. One has an oriented arrow $b \xrightarrow{i} b'$ between two vertices b and b' in $B(\lambda)$ if and only if $b' = \tilde{f}_i(b) \Leftrightarrow b = \tilde{e}_i(b')$ in which case $wt(b') = wt(b) - \alpha_i$, with α_i a simple root of $\mathfrak{gl}_n$. The crystal $B(\lambda)$ is generated by the actions of the lowering (resp. raising) operators $\tilde{f}_i$ (respect. $\tilde{e}_i$) on the unique highest (resp. lowest) weight vertex b_λ (resp. $b_{\sigma_0\lambda}$) where one has $wt(b_\lambda) = \lambda$, and σ_0 is the longest element of the Weyl group W here the symmetric group $\mathfrak{S}_n =< s_1, \dots, s_{n-1} >$.

For $\lambda \in \mathcal{P}_n$, W_λ is the stabilizer of λ under the action of W, and W^λ collects the unique minimal length representative of each coset in W/W_λ. Let $\lambda \in \mathcal{P}_n$ and $\sigma \in W$. Up to a scalar in $\mathbb{C}$, there exists a unique vector $v_{\sigma\lambda}$ in $V(\lambda)$ of weight $\sigma\lambda$. Recall the triangular decomposition $\mathfrak{gl}_n = \mathfrak{gl}_n^+ \oplus \mathfrak{h} \oplus \mathfrak{gl}_n^-$ of $\mathfrak{gl}_n$ into its upper, diagonal and lower parts. The Demazure module associated to $v_{\sigma\lambda}$ is the $U(\mathfrak{gl}_n^+)$-module defined by $V_\sigma(\lambda) := U(\mathfrak{gl}_n^+) \cdot v_{\sigma\lambda}$. Demazure introduced the character $\kappa_{\sigma,\lambda}$ of $V_\sigma(\lambda)$ and showed that it can be computed by applying to x^λ a sequence of divided difference operators $D_{i_1} \cdots D_{i_\ell}$ given by any reduced decomposition of $\sigma = s_{i_1} \cdots s_{i_\ell} \in W$ where ℓ is the length of σ. For $i \in I$, D_i is a certain linear operator on $\mathbb{Z}[x_1, \dots, x_n]$ (see [3] and references therein) satisfying the relations

$$
\begin{aligned}
D_i^2 &= D_i \text{ for any } i = 1, \dots, n-1, \\
D_i D_{i+1} D_i &= D_{i+1} D_i D_{i+1} \text{ for any } i = 1, \dots, n-2, \\
D_i D_j &= D_j D_i \text{ for any } i, j = 1, \dots, n-1 \text{ such that } |i-j| > 1.
\end{aligned}
$$

Thus, by Mastumoto's Lemma, the operator $D_\sigma = D_{i_1} \cdots D_{i_\ell}$ only depends on σ and not on the chosen reduced decomposition, and $\kappa_{\sigma,\lambda} = D_\sigma(x^\lambda) \in \mathbb{Z}[x_1, \dots, x_n]$ is the (Demazure) character of $V_\sigma(\lambda)$. In particular, we have $\kappa_{id,\lambda} = x^\lambda$ and $\kappa_{\sigma_{0,\lambda}} = s_\lambda$.

Kashiwara [12] and Littelmann [15] defined a relevant notion of crystals for the Demazure modules. Recall $O(\lambda) = \{\sigma \cdot b_\lambda = b_{\sigma\lambda} \mid \sigma \in W/W_\lambda\}$ the orbit of the highest weight vertex b_λ of $B(\lambda)$. Its elements, uniquely determined by their weight, are called the keys of $B(\lambda)$. (In this sense we may identify $O(\lambda)$ with $W\lambda$.) Given $\sigma, \sigma' \in W/W_\lambda$, we write $\sigma \le \sigma'$ for the Bruhat order on the cosets in W/W_λ to mean that their unique minimal (maximal) coset representatives satisfy the same relation in the strong Bruhat order restricted to W^λ. We also write $b_{\sigma\lambda} \le b_{\sigma'\lambda}$ when $\sigma \le \sigma'$ in W/W_λ.

From the dilatation of crystals [12] each vertex b of $B(\lambda)$ carries a pair of keys $K^+(b) \ge K^-(b)$, right, respectively, left key of b, in $O(\lambda)$. For any $\sigma \in W$, consider the Demazure atom

$$\overline{B}_\sigma(\lambda) = \{b \in B(\lambda) \mid K^+(b) = b_{\sigma\lambda}\}, \text{ where } \overline{B}_{id}(\lambda) = \{b_\lambda\}. \tag{5}$$

For any $\sigma \in W$, the opposite Demazure module, is defined to be $V^\sigma(\lambda) := U_q(\mathfrak{gl}_n^-) \cdot v_{\sigma\lambda}$, for which we define the opposite Demazure atom

$$\overline{B}^\sigma(\lambda) = \{b \in B(\lambda) \mid K^-(b) = b_{\sigma\lambda}\}, \text{ where } \overline{B}^{\sigma_0}(\lambda) = \{b_{\sigma_0\lambda}\}. \tag{6}$$

By definition we have $\overline{B}_\sigma(\lambda) = \overline{B}_{\sigma'}(\lambda)$ and $\overline{B}^\sigma(\lambda) = \overline{B}^{\sigma'}(\lambda)$ whenever σ and σ' belong to the same left coset of W/W_λ. We then get

$$B(\lambda) = \bigsqcup_{\sigma \in W^\lambda} \overline{B}_\sigma(\lambda) = \bigsqcup_{\sigma \in W^\lambda} \overline{B}^\sigma(\lambda).$$

The Demazure crystal $B_\sigma(\lambda)$ and its opposite Demazure crystal $B^\sigma(\lambda)$ are then defined by

$$B_\sigma(\lambda) = \bigsqcup_{\sigma' \in W^\lambda,\, \sigma' W_\lambda \leq \sigma W_\lambda} \overline{B}_{\sigma'}(\lambda) = \{b \in B(\lambda) \mid K^+(b) \leq b_{\sigma\lambda}\},\ B_{id}(\lambda) = \{b_\lambda\} \tag{7}$$

$$B^\sigma(\lambda) = \bigsqcup_{\sigma' \in W^\lambda,\, \sigma W_\lambda \leq \sigma' W_\lambda} \overline{B}^{\sigma'}(\lambda) = \{b \in B(\lambda) \mid K^-(b) \geq b_{\sigma\lambda}\},\ B^{\sigma_0}(\lambda) = \{b_{\sigma_0\lambda}\}. \tag{8}$$

In particular, we have $B_{\sigma_0}(\lambda) = B(\lambda) = B^{id}(\lambda)$. We then note that for a given $\lambda \in \mathcal{P}_n$,

$$\bigsqcup_{\sigma \in W^\lambda} \overline{B}^\sigma(\lambda) \times B_\sigma(\lambda) = \{(b, b') \in B(\lambda) \times B(\lambda) : K^-(b) \geq K^+(b')\} \simeq B(2\lambda). \tag{9}$$

We refer to [8], for the translation of (9) to the crystal of Lakshmibai-Seshadri paths. The Demazure and its opposite, respectively, atoms and its opposite crystals are connected via the Lusztig-Schützenberger involution ι on the crystal $B(\lambda)$, a realization of the action of the longest element of W on finite irreducible representations. The map ι is a set involution on $B(\lambda)$ reversing the arrows, flipping the labels i and $n-i$, and reversing the weight. We then have $K^-(b) = \sigma_0.K^+(\iota(b))$ and we get

$$B^\sigma(\lambda) = \iota(B_{\sigma_0\sigma}(\lambda)),\ \text{or equivalently } B^{\sigma_0\sigma}(\lambda) = \iota B_\sigma(\lambda), \quad \overline{B}^\sigma(\lambda) = \iota(\overline{B}_{\sigma_0\sigma}(\lambda)). \tag{10}$$

Demazure (resp. opposite) crystals can also be generated by the actions of the lowering (resp. raising) operators given by the reduced words in W^λ (resp. $\sigma_0 W^\lambda \sigma_0$) on the highest (resp. lowest) vertex of $B(\lambda)$. The Demazure character $\kappa_{\sigma,\lambda}(x)$ of the Demazure module $V^\sigma(\lambda)$ satisfies $\kappa_{\sigma,\lambda}(x) = \sum_{b \in \mathrm{B}_\sigma(\lambda)} x^{\mathrm{wt}(b)}$, and the opposite Demazure character $\kappa_\lambda^\sigma(x)$ for the opposite Demazure module $V^\sigma(\lambda)$ satisfies

$$\kappa_\lambda^\sigma(x) = \sum_{b \in B^\sigma(\lambda)} x^{wt(b)}.$$

Using the involution ι and (10), we have

$$\kappa_\lambda^\sigma(x_1, \dots, x_n) = \kappa_{\sigma_0\sigma\lambda}(x_n, \dots, x_1)$$

and

$$\overline{\kappa}_\lambda^\sigma(x_1, \dots, x_n) = \overline{\kappa}_{\sigma_0\sigma\lambda}(x_n, \dots, x_1) = \sum_{b \in \overline{B}^\sigma(\lambda)} x^{wt(b)}.$$

Alternatively we may also label the Demazure crystals and the Demazure characters of $B(\lambda)$ directly by the elements in the orbit of λ, $W\lambda$. Given $\mu \in W\lambda$ where $\mu = \sigma\lambda$ and $\sigma \in W^\lambda$, we write B_μ, $B^\mu = \iota B_{\sigma_0\mu}$ instead of $B_\sigma(\lambda)$, $B^\sigma(\lambda)$ respectively, and κ_μ, κ^μ

$=\kappa_{\sigma_0\mu}$, $\overline{\kappa}_\mu, \overline{\kappa}^\mu = \overline{\kappa}_{\sigma_0\mu}$ instead of $\kappa_{\sigma,\lambda}, \kappa_\lambda^\sigma$ and $\overline{\kappa}_{\sigma,\lambda}$, $\overline{\kappa}_\lambda^\sigma$ respectively. The operators D_i act on Demazure characters κ_μ and Demazure atoms $\overline{\kappa}_\mu$ as follows

$$D_i(\kappa_\mu) = \begin{cases} \kappa_{s_i\mu} & \text{if } \mu_i > \mu_{i+1} \\ \kappa_\mu & \text{if } \mu_i \leq \mu_{i+1}, \end{cases} \qquad D_i(\overline{\kappa}_\mu) = \begin{cases} \overline{\kappa}_{s_i\mu} + \overline{\kappa}_\mu & \text{if } \mu_i > \mu_{i+1} \\ \overline{\kappa}_\mu & \text{if } \mu_i = \mu_{i+1} \\ 0, & \text{else.} \end{cases} \tag{11}$$

For $i \in [n-1]$, we define below Δ_i and $\dot{\Delta}_i$ as operators on Demazure respectively Demazure atom crystals to mimic the action of the operator D_i on Demazure respectively on Demazure atom charaters (11), and we then always have $char(\Delta_i(B_\mu)) = D_i(\kappa_\mu)$, and $char(\dot{\Delta}_i(\overline{B}_\mu)) = D_i(\overline{\kappa}_\mu)$,

$$\Delta_i(B_\mu) = \begin{cases} B_{s_i\mu} \text{ if } \mu_i > \mu_{i+1}, \\ B_\mu \text{ otherwise,} \end{cases} \qquad \dot{\Delta}_i(\overline{B}_\mu) = \begin{cases} \Delta_i(\overline{B}_\mu) = \overline{B}_\mu \bigsqcup \overline{B}_{s_i\mu} \text{ if } \mu_i > \mu_{i+1} \\ \Delta_i(\overline{B}_\mu) = \overline{B}_\mu \text{ if } \mu_i = \mu_{i+1}, \\ \emptyset \text{ if } \mu_i < \mu_{i+1}. \end{cases} \tag{12}$$

3. Non symmetric Cauchy kernels, RSK on Young shapes and LPP

We now consider last passage percolation models based on the non symmetric Cauchy kernel (13) as studied by Lascoux in [14] and its extensions to augmented stair shapes. Demazure with its opposite Demazure atom crystals, and certain parabolic subcrystals will describe the image of RSK as a bicrystal isomorphism restricted to stair shape, truncated stair shape and to augmented stair shape matrices. We detach the truncated case from the general Young shape due to its more explicit as well interesting structure.

3.1. LPP, staircase and Demazure measure.

The ordinary Cauchy identity (1) is then replaced by its non-symmetric analogue

$$\prod_{1\leq j\leq i\leq n} \frac{1}{1-x_iy_j} = \sum_{\mu\in\mathbb{Z}^n_{\geq 0}} \overline{\kappa}^\mu(x)\kappa_\mu(y) \tag{13}$$

where $\overline{\kappa}^\mu(x)$ and $\kappa_\mu(y)$ are this time (opposite) Demazure atoms and Demazure characters in the indeterminates x and y (with $m = n$). These polynomials are not symmetric in x and y. They correspond to characters of representations for subalgebras of the enveloping algebra $U(\mathfrak{gl}_n)$. It was proved in [14] that the identity (13) can be obtained by restricting the RSK correspondence ψ to the set of lower triangular matrices. (The convention of our paper differs from that in [14] which considers matrices with nonzero entries in positions (i, j) with $1 \leq i + j \leq n + 1$ rather than lower-triangular matrices.) Since then, other proofs have been proposed using combinatorial objects which explicitly carry the pairs of right and left keys [10]. More precisely, [1, Theorem 3, Corollary 2] uses the combinatorics of Mason's semiskyline augmented fillings [16], and [8] uses the combinatorics of crystal bases, in particular, the combinatorial model of Lakshmibai-Seshadri paths [15]. Recently Assaf-Schilling provided an explicit tableau crystal for Mason's semiskyline augmented fillings [16] by using combinatorially equivalent objects, semistandard key tableaux (see [3] and references therein). Here we stand on the tableau model for $\mathfrak{gl}_n$ crystals where we have effective ways to compute the right key $K^+(T)$ and left key $K^-(T)$ of a semistandard tableau T as Lascoux's jeu de taquin procedure [10].

Let D be any subset of $[n]\times[m]$ and write $\mathcal{M}^D_{m,n}$ for the subset of $\mathcal{M}_{m,n}$ containing the matrices A such that $a_{i,j}\neq 0$ only if $(i,j)\in D$. In general, the set $\psi(\mathcal{M}^D_{m,n})$ is not stable by the $\mathfrak{gl}_m\times\mathfrak{gl}_n$-crystals operators. Nevertheless, when D corresponds to the Young diagram of a fixed partition Λ, see (19), $D=D_\Lambda$ is stable under the action of the crystal raising operators. When $m=n$ and $\varrho=(n,n-1,\dots,1)$, we get in matrix coordinates $D_\varrho=\{(i,j)\mid 1\le j\le i\le n\}$.

Then the bijection ψ (2) restricts to a bijection from the set $\mathcal{M}^{D_\varrho}_{m,n}$ of $n\times n$ lower triangular matrices to the set of pairs (P,Q) of semistandard Young tableaux of the same shape on the alphabet $[n]$ such that $K^-(P)\ge K^+(Q)$ (entrywise comparison). (See also [1, Corollary 2] for the Knuth version of RSK.) This means that the image of this restriction, for a fixed $\lambda\in\mathcal{P}_n$, is $\bigsqcup_{\sigma\in W^\lambda}\overline{B}^\sigma(\lambda)\times B_\sigma(\lambda)$ (9). Thus the restriction of RSK correspondence ψ to D_ϱ gives

$$\psi:\mathcal{M}^{D_\varrho}_{n,n}\to\bigsqcup_{\lambda\in\mathcal{P}_n}\bigsqcup_{\sigma\in W^\lambda}\overline{B}^\sigma(\lambda)\times B_\sigma(\lambda) \tag{14}$$

$$A\mapsto\psi(A)=(P(A),Q(A)):K^+(Q(A))\le K^-(P(A)), \tag{15}$$

where $B_\sigma(\lambda)$ is a Demazure crystal (7) and $\overline{B}^\sigma(\lambda)$ an opposite Demazure atom (6). This time, we only consider independent random variables $W_{i,j}$ when $1\le j\le i\le n$ with geometric distributions as in (4). This defines a lower triangular random square matrix $\mathcal{L}$ with nonnegative integer entries. In this model we consider paths from position $(1,n)$ to position $(n,1)$ where only the entries in the lower part of A contribute to the length of the paths. We define the random variable $L=p\circ\mathcal{L}$ and determine its law. Since (14) gives a bijective correspondence obtained as the restriction of the RSK map ψ (2) to lower triangular matrices, the value of L still corresponds to the length of the largest part of the partitions on the right hand side of (15).

Theorem 1. *For any $k\in\mathbb{Z}_{\ge0}$, we have the law*

$$\mathbb{P}(L=k)=\prod_{1\le j\le i\le n}(1-u_iv_j)\sum_{\mu\in\mathbb{Z}^n_{\ge0}|\max(\mu)=k}\overline{\kappa}^\mu(u_1,\dots,u_n)\kappa_\mu(v_1,\dots,v_n). \tag{16}$$

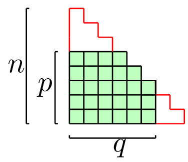
This law was also obtained by Baik-Rains [5, 6, Section 4] when $u_i=v_i$. In this case, (9), and (13) with $x_i=y_i$, together give a refinement of a Littlewood identity:

$$\prod_{1\le j\le i\le n}(1-x_ix_j)^{-1}=\sum_{\mu\in\mathbb{Z}^n_{\ge0}}\overline{\kappa}^\mu(x)\kappa_\mu(x)=\sum_{\lambda\in\mathcal{P}_n}s_{2\lambda}(x).$$

In [6] it is called a law in the point-to-line last passage percolation in zero temperature limit. However this formula is not produced in [6] by the geometric RSK but rather one in terms of a symplectic Cauchy like identity.

3.2. **Main results: LPP on Young shapes and Demazure measure.** Lascoux [14] also established generalizations of the formula (13) where positions with nonzero entries are allowed in the matrices outside their lower triangular part. These augmented staircase formulas below $(*)$ were then obtained just by computations on polynomials and thus not related to the RSK correspondence. This connection was partially done in [2] where certain truncated staircases formulas are proved to be compatible with the RSK correspondence using the combinatorics of semiskyline augmented fillings [16]. More precisely, this applies to the case where nonzero entries are authorized only in positions (i,j) with $n-p \leq i \leq j \leq q$, for p and q two nonnegative integers such that $n \geq q \geq p \geq 1$. We consider the Young diagram $D_{p,q} = \{(i,j) \mid n-p+1 \leq i \leq n, 1 \leq j \leq q\} \cap D_\varrho$ defined by using the matrix coordinates (i,j). It is the intersection of D_ϱ with a quarter of plane defined by the lines $i=p$ and $j=q$ (in Cartesian coordinates). When $n-p+1 \leq q$, we get the Young diagram $D_{p,q} = D_{\Lambda(p,q)}$ with $\Lambda(p,q) = (q^{n-q+1}, q-1, \ldots, n-p+1)$, and $D_{n,n} = D_{\Lambda(n,n)} = D_\varrho$.

Below one illustrates the truncated Young shape $D_{\Lambda(p,q)}$, in green, fitting the p by q rectangle so that the staircase D_ϱ of size n, in red, is the smallest one containing $D_{\Lambda(p,q)}$. If $p \leq q$, $D_{(p,p-1,\ldots,1)}$ is the biggest staircase inside $D_{\Lambda(p,q)}$.



We write $B_p(\lambda)$ for the subcrystal of the $\mathfrak{gl}_n$-crystal $B(\lambda, 0^{n-p})$ with $\lambda \in \mathcal{P}_p$, obtained by keeping only the vertices connected to its highest weight vertex by i-arrows with $i \in [p-1]$. Given $u \in \mathfrak{S}_p$, $B_{p,u}(\lambda)$, $B_p^u(\lambda)$, $\overline{B}_{p,u}(\lambda)$ and $\overline{B}_p^u(\lambda)$ denote the Demazure, its opposite, respectively, atom and its opposite crystals associated to u in the $\mathfrak{gl}_p$-crystal $B_p(\lambda)$. See Example 4 and (21), (22). The restriction of the map ψ from $\mathcal{M}_{n,n}^{D_\varrho}$ (15) to $\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}$ gives

$$\begin{aligned}
\psi(\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}) &= \bigsqcup_{\lambda \in \mathcal{P}_n} \bigsqcup_{\sigma \in W^\lambda} \overline{B}^\sigma(\lambda) \cap B^p(\lambda) \times B_\sigma(\lambda) \cap B_q(\lambda) \\
&= \bigsqcup_{\mu \in \mathbb{Z}^n_{\geq 0}} \overline{B}^\mu \cap B^p(\lambda) \times B_\mu \cap B_q(\lambda).
\end{aligned}$$

By the Borel-Weil theorem, Demazure crystals are in natural correspondence with Schubert varieties. Let $\sigma \in \mathfrak{S}_n$ and $\sigma_0^{[q]}$ be the longest element of $\mathfrak{S}_q$. From the Billey–Fan–Losonczy parabolic map (see [3, Algorithm 3.1, Proposition 3.4] and references therein) the set $\{v \in \mathfrak{S}_q \mid v \leq \sigma\}$ has a unique maximal element σ^{I_q} for the Bruhat order $\leq$ in W. Then the intersections

$$S_{\sigma_0^{[q]}} \cap S_\sigma = S_{\sigma^{I_q}}$$

and

$$B_\sigma(\lambda) \cap B_q(\lambda) = B_\sigma(\lambda) \cap B_{\sigma_0^{[q]}}(\lambda) = \mathrm{B}_{q,\sigma^{I_q}}(\lambda)$$

translate into each other, where $S_\sigma = \cup_{v \leq \sigma} \mathcal{O}_v$; here $\mathcal{O}_v$ is the orbit BvB/B of the Borel subgroup B of the reductive group G with Weyl group W acting on the flag variety G/B, and S_σ is the Schubert variety, that is, the orbit closure of $\mathcal{O}_\sigma$ (we refer to [10, Chapter III] for definitions).

However $\overline{B}^{\sigma}(\lambda)\cap B^p(\lambda)=\emptyset$ unless $\sigma\in\sigma_0\mathfrak{S}_p^{\lambda}$, $\lambda\in\mathcal{P}_p$ and then $\overline{B}^{\sigma}(\lambda)\cap B^p(\lambda)=\iota\overline{B}_{p,\sigma_0\sigma}(\lambda)$ [3]. In this case $B_\sigma(\lambda)\cap B_q(\lambda)=B_{q,\sigma^{I_q}}(\lambda)$. The restriction of the RSK correspondence ψ to $\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}$ then gives a one-to-one correspondence

$$\psi:\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}\to\bigsqcup_{\mu\in\mathbb{Z}_{\geq 0}^p}\iota(\overline{B}_{p,\mu})\times B_{q,\widetilde{\mu}},\ \text{and} \tag{17}$$

$$\prod_{(i,j)\in D_{\Lambda(p,q)}}\frac{1}{1-x_iy_j}=\sum_{(\mu_1,\dots,\mu_p)\in\mathbb{Z}_{\geq 0}^p}\overline{\kappa}_{(\mu_p,\dots,\mu_1)}(x_n,\dots,x_{n-p+1})\kappa_{\widetilde{\mu}}(y_1,\dots,y_q), \tag{18}$$

where for each $\mu\in\mathbb{Z}_{\geq 0}^p$, the vector $\tilde{\mu}=(\sigma_0\tau)^{I_q}(\lambda,0^{q-p},0^{n-q})$ with $\tau\in\mathfrak{S}_p^{\lambda}$ such that $\mu=\tau\lambda$. It can also be explicitly computed by a simple algorithm in [1, 3, Theorem 3.20] (see also examples in [3, Section 3.1]).

One can then similarly use (17) to study the percolation model on random matrices $\mathcal{T}_{p,q}$ with nonnegative random integer coefficients having zero entries in each position (i,j) such that $i\leq n-p$ and $j>q$. Each random variable $W_{i,j}$ with $i\geq n-p+1$ and $j\leq q$ follows a geometric distribution of parameter u_iv_j. Using the same arguments as before, we obtain the law of the random variable $T_{p,q}=p\circ\mathcal{T}_{p,q}$.

Theorem 2. *For any nonnegative integer k, we have for $v=(v_1,\dots,v_q)$*

$$\mathbb{P}(T_{p,q}=k)=\prod_{(i,j)\in D_{\Lambda(p,q)}}(1-u_iv_j)\cdot$$
$$\cdot\sum_{(\mu_1,\dots,\mu_p)\in\mathbb{Z}_{\geq 0}^p|\max(\mu)=k}\overline{\kappa}_{(\mu_p,\dots,\mu_1)}(u_n,\dots,u_{n-p+1})\kappa_{\widetilde{\mu}}(v).$$

In [14] Lascoux gave other non symmetric Cauchy type identities for any partition $\Lambda\in\mathcal{P}_n$. One considers the largest staircase $\rho_\Lambda=(m,m-1,\dots,1)$ contained in the Young diagram of Λ. Then one chooses a box at position (i_0,j_0), in Cartesian coordinates, in the augmented staircase $(m+1,m,\dots,1)$ which is not in Λ. The diagonal $L_{i,j}:j-i=j_0-i_0$, in Cartesian coordinates, cuts Λ in a northwest part and a southeast part corresponding to the boxes above and below $L_{i,j}$, respectively. Now fill the boxes (i,j), in the $n\times n$ matrix convention, of the NW part of Λ by $n-i$ (*i.e.*, by the $n\times n$ matrix reverse row index (equivalently counting rows from bottom to top) minus one), and the boxes (i,j) of the SE part by $j-1$ (*i.e.*, by the index of the column minus one). Let $\sigma(\Lambda,NW)=s_{i_1}\cdots s_{i_a}$ be the element of W where the word $i_1\cdots i_a$ is obtained from right to left column reading of the NW part of Λ, each column being read from top to bottom. Similarly, let $\sigma(\Lambda,SE)=s_{j_1}\cdots s_{j_b}$ be the element of W where the word $j_1\cdots j_b$ is obtained from top to bottom row reading of the SE part of Λ, each row being read from right to left.

For instance, let $n=8$ and $\Lambda=(7,4,2,2,2)$. Take $(i_0,j_0)=(3,3)$ (the box with $\blacktriangle$). Hence $m=4$, $\rho_\Lambda=(4,3,2,1)$, and $\sigma(\Lambda,NW)=s_4s_3s_4$, $\sigma(\Lambda,SE)=s_3s_6s_5s_4$,

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 4 & 4 \\ \cline{1-2} \blacksquare & 3 \\ \cline{1-3} \blacksquare & \blacksquare & \blacktriangle \\ \cline{1-4} \blacksquare & \blacksquare & \blacksquare & 3 \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & 4 & 5 & 6 \\ \hline \end{array} \tag{19}$$

The following identity was established in [14] and reproved for near stair shapes in [2],

$$(*)\quad \prod_{(i,j)\in\Lambda}\frac{1}{1-x_iy_j}= \sum_{(\mu_1,\dots,\mu_m)\in\mathbb{Z}^m} D_{\sigma(\Lambda,NW)}\overline{\kappa}_{(\mu_m,\dots,\mu_1)}(x_n,\dots,x_{n-m+1})D_{\sigma(\Lambda,SE)}\kappa_{(\mu_1,\dots,\mu_m)}(y),$$

where $y=(y_1,\dots,y_m)$, and $D_{\sigma(\Lambda,NW)}=D_{i_1}\cdots D_{i_a}$, $D_{\sigma(\Lambda,SE)}=D_{j_1}\cdots D_{j_b}$ are compositions of Demazure operators (11).

Theorem 3. *The restriction of the RSK correspondence ψ to $\mathcal{M}_{n,n}^{D_\Lambda}$ gives the one-to-one correspondence*

$$\psi:\mathcal{M}_{n,n}^{D_\Lambda}\to \bigsqcup_{(\mu_1,\dots,\mu_m)\in\mathbb{Z}^m_{\geq 0}}\iota\left(\dot{\Delta}_{\sigma(\Lambda,NW)}(\overline{B}_{(\mu_m,\dots,\mu_1)})\right)\times\Delta_{\sigma(\Lambda,SE)}\left(B_{(\mu_1,\dots,\mu_m)}\right) \tag{20}$$

where $\Delta_{\sigma(\Lambda,SE)}=\Delta_{j_1}\cdots\Delta_{j_b}$ and $\dot{\Delta}_{\sigma(\Lambda,NW)}=\dot{\Delta}_{i_1}\cdots\dot{\Delta}_{i_a}$ (12). *(As usual $\emptyset\times U=\emptyset$.)*

Now, for a fixed partition Λ in $\mathcal{P}_n$, we consider random matrices $\mathcal{A}_\Lambda$ with nonnegative random integer coefficients having zero entries in each position (i,j) such that $(i,j)\notin\Lambda$. Here again each random variable $W_{i,j}$ for $(i,j)\in\Lambda$ follows a geometric distribution of parameter u_iv_j. Define the random variable $A_\Lambda=p\circ\mathcal{A}_\Lambda$. Then, by $(*)$ and (20), we get the law of A_Λ.

Theorem 4. *For any nonnegative integer k,*

$$\mathbb{P}(A_\Lambda=k)=\prod_{(i,j)\in D_\Lambda}(1-u_iv_j). \cdot\sum_{(\mu_1,\dots,\mu_m)\in\mathbb{Z}^m\mid\max(\mu)=k} D_{\sigma(\Lambda,NW)}\overline{\kappa}_{(\mu_m,\dots,\mu_1)}(u_n,\dots,u_{n-m+1})D_{\sigma(\Lambda,SE)}\kappa_{(\mu_1,\dots,\mu_m)}(v),$$

where $(v_1,\dots,v_m)$.

4. An example for RSK on augmented stair shapes

Let us resume to the setting of (19) with $n=8$, $\Lambda=(7,4,2,2,2)$, and $\sigma(\Lambda,NW)=s_4s_3s_4$, $\sigma(\Lambda,SE)=s_3s_6s_5s_4$. Let ψ be the RSK applied to $\mathcal{M}_{8,8}^{D_\Lambda}$ the set of 8×8

nonnegative integer matrices whose positive entries fit the shape Λ. Then (20) gives for $m=4$

$$\psi : \mathcal{M}_{8,8}^{D_{(7,4,2,2,2)}} \to \bigsqcup_{(\mu_1,\dots,\mu_4)\in\mathbb{Z}^4_{\geq 0}} \iota\left(\dot\Delta_4\dot\Delta_3\dot\Delta_4(\overline{B}_{(\mu_4,\dots,\mu_1)})\right) \times \Delta_3\Delta_6\Delta_5\Delta_4\left(B_{(\mu_1,\dots,\mu_4)}\right)$$
$$A \mapsto \psi(A) = (P,Q).$$

Let $A=\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \mathbf{0}&\mathbf{1}&0&0&0&0&0&0\\ \mathbf{1}&\mathbf{1}&0&0&0&0&0&0\\ \mathbf{0}&\mathbf{0}&0&0&0&0&0&0\\ \mathbf{2}&\mathbf{0}&\mathbf{1}&\mathbf{1}&0&0&0&0\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{2}&0 \end{pmatrix} \in \mathcal{M}_{8,8}^{D_\Lambda}$ encoded as a tensor product of row tableaux on the alphabet $[8]$ where $a_{i,j}$ gives the number of letters i in the j-th component of the tensor product

$$577 \otimes 45 \otimes 7 \otimes 7 \otimes 8 \otimes \emptyset \otimes 88 \otimes \emptyset.$$

One then applies the column insertion procedure from left to right. This means that we begin by reading the first column (of A) 775 and compute the column insertions $5 \to 7 \to 7$ to get 577 then read the second column 54 and compute the column insertion $4 \to 5 \to 577$ to get 45577, then $7 \to 45577$ to get $\begin{smallmatrix}7\\4\,5\,5\,7\,7\end{smallmatrix}$, and eventually get the tableau P below. The "recording tableau" Q is obtained by filling with letters j the new boxes appearing during the insertion of column j of A,

$$P = \begin{array}{|c|c|c|c|c|}\hline 8&8\\ \hline 7&7&8\\ \hline 4&5&5&7&7\\ \hline\end{array} \qquad K^-(P) = \begin{array}{|c|c|c|c|c|}\hline 8&8\\ \hline 7&7&8\\ \hline 4&4&4&4&4\\ \hline\end{array} = K(0^3,5,0^2,2,3) \tag{21}$$

$$Q = \begin{array}{|c|c|c|c|c|}\hline 5&7\\ \hline 3&4&7\\ \hline 1&1&1&2&2\\ \hline\end{array} \qquad K^+(Q) = \begin{array}{|c|c|c|c|c|}\hline 7&7\\ \hline 4&4&7\\ \hline 2&2&2&2&2\\ \hline\end{array} = K(0,5,0,2,0^2,3,0). \tag{22}$$

We show that there exists $\mu=(\mu_1,\mu_2,\mu_3,\mu_4)\in\mathbb{Z}^4_{\geq 0}$ such that $\psi(A)=(P,Q)\in \iota(\dot\Delta_4\dot\Delta_3\dot\Delta_4\overline{B}_{(\sigma_0\mu,0^4)})\times\Delta_3\Delta_6\Delta_5\Delta_4 B_{(\mu,0^4)}$, where $\sigma_0\in\mathfrak{S}_4$ and ι is the Schützenberger (evacuation) involution on tableaux in the alphabet $[8]$. From (12) one has
$\iota(\dot\Delta_4\dot\Delta_3\dot\Delta_4\overline{B}_{(\mu_4,\dots,\mu_1,0^4)}) =$

$$= \begin{cases} \iota\overline{B}_{(\mu_4,\mu_3,\mu_2,0,0^4)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,0,\mu_2,0^4)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,0^2,\mu_2,0^3)}, \text{ if } \mu_2>\mu_1=0 \qquad (**)\\ \iota\overline{B}_{(\mu_4,\mu_3,0,0,0^4)}, \text{ if } \mu_1=\mu_2=0\\ \iota\overline{B}_{(\mu_4,\mu_3,\mu_2,\mu_2,0^4)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,\mu_2,0,\mu_2,0^3)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,0,\mu_2,\mu_2,0^3)}, \text{ if } \mu_1=\mu_2>0\\ \emptyset, \text{ if } \mu_1>\mu_2\geq 0\\ \iota\overline{B}_{(\mu_4,\dots,\mu_1,0^4)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,\mu_1,\mu_2,0^4)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,\mu_2,0,\mu_1,0^3)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,0,\mu_2,\mu_1,0^3)} \bigsqcup\\ \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,0,\mu_1,\mu_2,0^3)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,\mu_1,0,\mu_2,0^3)}, \text{ if } \mu_2>\mu_1>0. \end{cases}$$

Then, by (6),

$$K^-(P)=K(0^3,5,0^2,2,3) \Leftrightarrow P\in\overline{B}^{(0^3,5,0^2,2,3)} = \iota\overline{B}_{(3,2,0^2,5,0^3)},$$

and we are in case $(**)$, where $\mu_2=5>\mu_1=0$, $\mu_3=2$, $\mu_4=3$. Hence, $\mu=(0,5,2,3)$ and

$$\iota(\dot\Delta_4\dot\Delta_3\dot\Delta_4\overline{B}_{(3,2,5,0,0^4)}) = \iota\overline{B}_{(3,2,5,0,0^4)} \bigsqcup \iota\overline{B}_{(3,2,0,5,0^4)} \bigsqcup \iota\overline{B}_{(3,2,0^2,5,0^3)}.$$

Therefore, by the LHS of (12), $\Delta_3\Delta_6\Delta_5\Delta_4 B_{(\mu,0^4)} = B_{(0,5,0,2,0,0,3,0)}$. Indeed $K_+(Q) \leq K(0,5,0,2,0^2,3,0)$ and from (8), $Q \in B_{(0,5,0,2,0^2,3,0)}$. Hence,

$$(P,Q) \in \overline{B}^{(0^3,5,0^2,2,3)} \times B_{(0,5,0,2,0^2,3,0)}$$

and

$$\psi(A) \in \iota(\dot{\Delta}_4\dot{\Delta}_3\dot{\Delta}_4\overline{B}_{(3,2,5,0,0^4)}) \times \Delta_3\Delta_6\Delta_5\Delta_4 B_{(0,5,2,3,0^4)}.$$

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