THE TYPE C_n BERENSTEIN–KIRILLOV GROUP

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ABSTRACT. We define Bender–Knuth involutions for Kashiwara–Nakashima tableaux of type C_n via partial symplectic Schützenberger–Lusztig (SL) involutions and thereby a symplectic analogue of the Berenstein–Kirillov group. We use the symplectic cactus group as well as the virtualization of Kashiwara–Nakashima tableaux by Baker.

1. Introduction

The type A *Berenstein–Kirillov* or *Gelfand–Tsetlin* group [6] is the free group generated by the Bender–Knuth involutions [3] t_i, $i > 0$, modulo the relations they satisfy on semi-standard Young tableaux of any straight shape. The Bender–Knuth involution t_i, $i \geq 1$, is an operation that acts on a semi-standard tableau T of any straight (or skew) shape as follows. Pairs of letters $(i, i+1)$ within each column of T are considered fixed, and other occurrences of i's or $i+1$'s are considered free. Now, if a row of T has k free i's followed by l free $i+1$'s, then we replace these letters by l free i's followed by k free $i+1$'s. Although the $t_i's$ have many known relations (see (60)-(64) in [1, §10] and references within), it is not known whether these form a complete set of relations [5]. Let $\mathcal{BK}_n$ be the subgroup of $\mathcal{BK}$ generated by $t_1, \ldots, t_{n-1}$, and let

$$q_{[1,i]} := t_1(t_2t_1)\cdots(t_it_{i-1}\cdots t_1), \qquad \text{for } i \geq 1,$$
$$q_{[j,k-1]} := q_{[1,k-1]}q_{[1,k-j]}q_{[1,k-1]}, \qquad \text{for } j < k.$$

Then [6, Remark 1.3] as elements of $\mathcal{BK}$,

$$t_1 = q_{[1,1]}, \qquad t_i = q_{[1,i-1]}q_{[1,i]}q_{[1,i-1]}q_{[1,i-2]}, \text{ for } i \geq 2, \quad q_0 := 1.$$

The elements $q_{[1,1]}, \ldots, q_{[1,n-1]}$ are generators of $\mathcal{BK}_n$ and coincide with the partial Schüt - zenberger involutions restricted to the connected subdiagrams $[1, i], 1 \leq i \leq n-1$, of the Dynkin diagram A_{n-1}. Motivated by the relation between these two sets of involutions in type A_{n-1}, we similarly define a symplectic version of $\mathcal{BK}_n$ by defining Bender–Knuth involutions on the set of type C_n Kashiwara–Nakashima tableaux of straight shape [16], thereby defining a type C_n analogue of the Berenstein–Kirillov group. We also do this by analogy to an interesting interpretation of Bender–Knuth involutions in type A_{n-1} in terms of the *cactus group* J_n. For this we study the symplectic cactus group as defined by Halacheva [13, 11], as well as its action on the set of Kashiwara–Nakashima tableaux in detail, for which we present new explicit algorithms. Along the way, we revisit a virtualization map of Baker which will unwrap our

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symplectic Bender–Knuth involutions while providing aditional structure, and allowing the computation of explicit examples.

This extended abstract is organized in three sections. The full version [1] (to which we refer the reader for details) containing the results hereby presented has been submitted for publication elsewhere.

2. The cactus group, crystals, and virtualization

2.1. **The cactus group.** Although the cactus group was classically and originally introduced by Henriques–Kamnitzer [14] in the context of the commutator and coboundary categories, Halacheva [11, 13] has generalized this concept by defining a cactus group J_X associated to any Dynkin diagram X as follows. The group J_X is defined by generators s_I, for each connected subdiagram $I \subseteq X$, subject to the relations

$$s_I^2 = 1, \tag{1}$$

$$s_I s_J = s_J s_I \text{ if } J \subset X,\ J \sqcup I \text{ is disconnected} \tag{2}$$

$$s_I s_J = s_{\theta_I(J)} s_I \text{ if } J \subseteq I \tag{3}$$

where $\theta : X \to X$ is the Dynkin diagram automorphism defined by $\alpha_{\theta(i)} = -w_0\alpha_i$, where w_0 is the longest element of the Weyl group associated to X, and $\{\alpha_i\}_{i=1}^n$ is a set of simple roots of a reduced root system of type X. For $I \subseteq X$, $\theta_I : I \to I$ is the Dynkin diagram automorphism on the subdiagram I and the associated Weyl group is the parabolic subgroup $W^I \subseteq W$. In [11, 13], Halacheva defines an *internal* action of the cactus group J_X on a normal $\mathfrak{g}_X$-crystal B, where $\mathfrak{g}_X$ is the simple complex Lie algebra associated to X. The generator s_X acts on B via the Schützenberger–Lusztig involution. This action coincides with the Schützenberger evacuation on semi-standard Young tableaux, and has been defined by Santos [18] on Kashiwara–Nakashima tableaux of type C_n, making use of the Sheats–Baker–Lecouvey sliding algorithm [7, 19, 17]. The other generators s_J, for $J \subsetneq X$ act on B by first taking the Levi-branching B_J of B (obtained from B by removing from it all arrows not labelled by elements in J while keeping all the vertices) and applying the Schützenberg–Lusztig involution there.

Example 1. *For $X = A_{n-1}$ the connected subdiagrams are intervals $I = [p, q]$, for $1 \leq p \leq q \leq n-1$. We will denote the generators by $s_{[p,q]}^{A_{n-1}}$ from now on. The relations in this case are well-known and given by:*

- ${s_{[p,q]}^{A_{n-1}}}^2 = 1$
- ${s^{A_{n-1}}}_{[p,q]} s_{[k,l]}^{A_{n-1}} = s_{[p,q]}^{A_{n-1}} s_{[k,l]}^{A_{n-1}}$ *if $[p,q] \cup [k,l], 1 \leq k \leq l \leq n-1$ is disconnected.*
- $s_{[p,q]}^{A_{n-1}} s_{[k,l]}^{A_{n-1}} = s_{[p+q-l,p+q-k]}^{A_{n-1}} s_{[p,q]}^{A_n}$ *for $p \leq k \leq l \leq q \leq n-1$.*

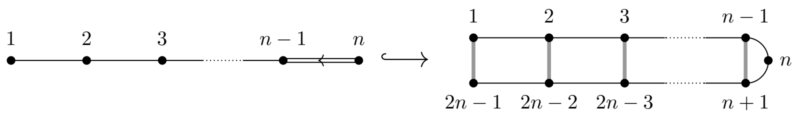
Note that the automorphism $\theta_{[p,q]} : [p,q] \to [p,q]$ is given by $\theta_{[p,q]}(d) = p + q - d$

Example 2. *For $X = C_n$ the connected subdiagrams can also be regarded as intervals $I = [p, q]$, for $1 \leq p \leq q \leq n$. We will denote the generators by $s_{[p,q]}^{C_n}$ from now on. In this case we have two types of intervals: those containing, respectively not containing the node labelled by n. For those subintervals $[p, q], q \neq n$, we have $\theta_{[p,q]}(d) = p+q-d$,* *since these are subdiagrams of type A_{q-p+1}. For $I = [p,n]$ we have $\theta_{[p,n]}(d) = d$. Therefore the relations defining the cactus group J_{C_n} (or* symplectic cactus group*) are:*

- ${s^{C_n}}_{[p,q]}^{\;2} = 1$
- ${s^{C_n}}_{[p,q]} s^{C_n}_{[k,l]} = s^{C_n}_{[p,q]} s^{C_n}_{[k,l]}$ *if $[p,q] \cup [k,l], 1 \leq k \leq l \leq n$, is disconnected.*
- $s^{C_n}_{[p,q]} s^{C_n}_{[k,l]} = s^{C_n}_{[p+q-l,p+q-k]} s^{C_n}_{[p,q]}$ *for $p \leq k \leq l \leq n$.*
- $s^{C_n}_{[p,n]} s^{C_n}_{[k,l]} = s^{C_n}_{[k,l]} s^{C_n}_{[p,n]}$ *for $p \leq k \leq l \leq q < n$.*

Remark 1. *Note that from the above examples we can deduce that J_{A_m} is a subgroup of J_{C_k}, for $m < k$. $J_{A_{n-1}}$ is also generated by the elements $s_{[1,j]}, 1 \leq j \leq n$ [1].*

2.2. **Embedding of J_{C_n} into $J_{A_{2n-1}}$.** We have observed that $J_{A_m} = J_{m+1}$ is a subgroup of J_{C_k}, for $m < k$, in particular $J_{A_{n-1}} = J_n$ is a subgroup of J_{C_n}. We now show that there is a group embedding of J_{C_n} into J_{2n} by folding A_{2n-1} through the middle node n:



Why should such an embedding exist? Let us consider the following elements of J_{2n}:

$$s'_{[p,q]} := s^{A_{2n-1}}_{[p,q]} s^{A_{2n-1}}_{[2n-q,2n-p]} = s^{A_{2n-1}}_{[2n-q,2n-p]} s^{A_{2n-1}}_{[p,q]}, \text{ for all } [p,q] \subseteq [n-1].$$

In Proposition 1 we see that these elements together with the generators $s_{[p,2n-p]}$ for $p \leq n$ generate a subgroup of $J_{A_{2n-1}}$ isomorphic to J_{C_n}. We denote this subgroup of $J_{A_{2n-1}}$ by $\widetilde{J}_{A_{2n-1}}$ and call it virtual symplectic cactus group. Notice the similarity between this and the construction of $\mathfrak{sp}(2n,\mathbb{C})$ as a sub-algebra of $\mathfrak{sl}_{2n}$ by folding [15, Chapter 8, pp. 89 – 102].

Proposition 1. *[1] The following assignment defines a group injection from J_{C_n} to $J_{A_{2n-1}}$:*

$$\begin{array}{llll} \Gamma: & J_{C_n} & \hookrightarrow & J_{A_{2n-1}} \\ & s^{C_n}_{[p,q]} & \mapsto & s'_{[p,q]}, \quad 1 \leq p \leq q < n, \\ & s^{C_n}_{[p,n]} & \mapsto & s^{A_{2n-1}}_{[p,2n-p]}, \quad 1 \leq p \leq n. \end{array}$$

2.3. **Crystals and virtualization.** Let $\mathfrak{g}$ be a simple Lie algebra with Dynkin diagram X, and integral weight lattice Λ. Crystals corresponding to finite-dimensional (quantum group) $U_q(\mathfrak{g})$-representations belong to a family of crystals called *normal crystals* [8]. In classical types, these crystals may be realized by a tableau model [16] and have nice combinatorial properties. Normal crystals arise as the crystals associated to the finite-dimensional representations of a quantum group $U_q(\mathfrak{g})$ [8]. These crystals decompose into connected components, one for each irreducible component to the representation at hand. The Levi restriction of a normal crystal is still a normal crystal, and the union of some connected components of a normal crystal is also a normal crystal [8]. We work with tableau crystals for finite-dimensional representations of $U_q(\mathfrak{sl}(n,\mathbb{C}))$ and $U_q(\mathfrak{sp}(2n,\mathbb{C}))$. Recall the $\mathfrak{sl}(n,\mathbb{C})$ simple roots $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$,

$i \in [n-1]$, and the $\mathfrak{sp}(2n,\mathbb{C})$ simple roots $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, $i \in [n-1]$ and $\alpha_n = 2\mathbf{e}_n$, where $\mathbf{e}_i$, $i \in [n]$, is the $\mathbb{R}^n$ standard basis. The weight lattices are $\Lambda = \mathbb{Z}^n$ for $\mathfrak{sp}(2n,\mathbb{C})$ and $\Lambda = \mathbb{Z}^n / < (1,...,1) >$ for $\mathfrak{sl}(n,\mathbb{C})$. We will often work with representatives in the case of $\mathfrak{sl}(n,\mathbb{C})$. The fundamental weights are $\omega_i = \sum_{j=1}^{i} e_i$, $1 \le i \le n$ and respectively have representatives ω_i, $1 \le i \le n-1$.

A $\mathfrak{g}$-crystal is a finite set B along with maps

$$\mathsf{wt} : B \to \Lambda, \quad e_i, f_i : B \to B \sqcup \{0\},\ \varepsilon_i, \varphi_i : B \to \mathbb{Z},$$

satisfying the following axioms for any $b, b' \in B$ and $i \in X$,

- $b' = e_i(b)$ if and only if $b = f_i(b')$,
- if $f_i(b) \neq 0$ then $\mathsf{wt}(f_i(b)) = \mathsf{wt}(b) - \alpha_i$;
  if $e_i(b) \neq 0$, then $\mathsf{wt}(e_i(b)) = \mathsf{wt}(b) + \alpha_i$, and
- $\varepsilon_i(b) = \max\{a \in \mathbb{Z}_{\ge 0} : e_i^a(b) \neq 0\}$ and $\varphi_i(b) = \max\{a \in \mathbb{Z}_{\ge 0} : f_i^a(b) \neq 0\}$.
- $\varphi_i(b) - \varepsilon_i(b) = \langle \mathsf{wt}(b), \alpha_i^\vee \rangle$,

where $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$ are the coroots.

Remark 2. *Our abstract $\mathfrak{g}$-crystals are defined with the additional condition that they are seminormal [8].*

The *crystal graph* of B is the directed graph with vertices in B and edges labelled by $i \in X$. If $f_i(b) = b'$ for $b, b' \in B$, then we draw an edge $b \to b'$. Given an arbitrary subset $J \subseteq X$, B_J is defined to be the crystal B restricted to the sub-diagram J of X, the Levi branched crystal. The crystal graph of B_J has the same vertices as B, but the arrows are only those labelled in J; that is, we forget the maps e_i, f_i, φ_i, and ε_i, for $i \notin J$ [8]. The weight map, which we denote by wt_J, is $B \overset{\mathsf{wt}}{\to} \Lambda \overset{can}{\to} \Lambda_J$, where wt is the weight map of B, Λ is the weight lattice of $\mathfrak{g}$, a Levi-subalgebra, and $\Lambda_J = \Lambda / < \omega_i : i \notin J >$ is the weight lattice of $\mathfrak{g}_J$, and $\Lambda \overset{can}{\to} \Lambda_J$ is the canonical projection. If $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$ and we restrict to $J = [n-1]$, then we obtain a $\mathfrak{sl}(n,\mathbb{C})$-crystal. Given $b \in B$, $\mathsf{B}(b)$ denotes the connected component of B containing b. A $\mathfrak{g}$-crystal is *normal* if it is isomorphic to a disjoint union of irreducible crystals associated to an irreducible, finite-dimensional $\mathfrak{g}$-representation *of highest weight* λ, where $\lambda \in \Lambda$ is a *dominant weight*. The dominant weights in $\mathbb{Z}^n$, respectively in $\mathbb{Z}^n / < (1,...,1) >$, correspond precisely to *partitions* with at most n parts, that is, weakly decreasing vectors in $\mathbb{Z}^n$ with non-negative entries, respectively to weakly decreasing vectors in $\mathbb{Z}^n$, and each such representative is equivalent to a unique partition in $\mathbb{Z}^{n-1} \hookrightarrow \mathbb{Z}^n$, where the last entry is fixed as zero. An important property of normal crystals B is the existence of a unique highest weight vertex for each connected component of B, that is, an element which is a source in the corresponding crystal graph, whose weight is dominant. Note that we work solely with *highest weight crystals*, namely, crystals B such that for each $b \in B$, there exists a finite sequence $a_1, a_2, \ldots, a_l \in X$ and a highest weight element $u_b \in B(b)$ such that $b = f_{a_l} \cdots f_{a_2} f_{a_1}(u_b)$. For $b, b' \in B$, we have $B(b) = B(b')$ if and only if $u_b = u_{b'}$. From now on, we will refer to $\mathfrak{sp}(2n,\mathbb{C})$-crystals by C_n-crystals, and $\mathfrak{sl}(n,\mathbb{C})$-crystals by A_{n-1}-crystals.

We consider the realization of A_{n-1}-crystals by semi-standard Young tableaux of shape a partition λ of at most $n-1$ parts and filling with letters in the ordered alphabet $[n]$. We will denote this set together with its crystal structure by $SSYT(\lambda, n)$ (we refer the reader to [8] for the definition of the crystal structure). We will also and importantly consider the realization of C_n-crystals given by Kashiwara–Nakashima [16], or, as we will from now on refer to them, KN, tableaux. These are semi-standard tableaux in the ordered alphabet (see also [8, 17])

$$[\pm n] := \left\{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\right\}$$

such that, additionally each one of their columns is *admissible*, and that their *splitting* is a semi-standard Young tableau. A semi-standard column C in the alphabet $[\pm n]$ of length at most n is *admissible* if there exists a set $T = \{t_1 > ... > t_m\}$ of non-barred letters that satisfy $t_1 < z_1$ and is maximal with the property $t_1, \bar{t}_1 \notin C$, and $t_i < \min(t_{i-1}, z_i)$, $t_i, \bar{t}_i \notin C$ and is maximal with these properties, where $Z = \{z_1 > ... > z_m\}$ is the set of non-barred letters z.

The splitting of a column is the two-column tableau $lCrC$ where lC is the column obtained from C by replacing z_i by t_i and possibly re-ordering, and rC is obtained from C by replacing $\bar{z}_i$ by $\bar{t}_i$ and possibly re-ordering. The *splitting* of a semi-standard Young tableau consisting of admissible columns is the concatenation of the splitting of its columns. The weight of a KN tableau is a vector of $\mathbb{Z}^n$ whose i-th entry is the number of occurrences of i minus the number of ocurrences of $\bar{i}$ in the tableau. We will denote the set of KN tableaux of shape λ with entries in $[\pm n]$, together with their crystal structure (see [8, 17]) by $KN(\lambda, n)$.

Example 3. *Let* $n = 2$. *The column* $\begin{array}{|c|}\hline 2 \\ \hline \bar{2} \\ \hline\end{array}$ *is admissible (we have* $Z = \{2\}$ *and* $T = \{1\}$*), thus a KN tableau of weight* $(0, 0)$. *However,* $\begin{array}{|c|}\hline 1 \\ \hline \bar{1} \\ \hline\end{array}$ *is not. Notice that the tableau* $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \bar{2} & \bar{2} \\ \hline\end{array}$*, despite having all admissible columns, is not KN, because its split,* $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 1 & 2 \\ \hline \bar{2} & \bar{1} & \bar{2} & \bar{1} \\ \hline\end{array}$*, is not semi-standard.*

2.4. **Baker's virtualization map.** Let $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n \in \mathbb{Z}^n$ with $\omega_j = \sum_{i=1}^{j} e_i \in \mathbb{Z}^n$ the C_n fundamental weights. Moreover, let $\omega_j^A = \sum_{i=1}^{j} e_i \in \mathbb{Z}^{2n}$ for $1 \le j \le n$, $\omega_{\bar{j}}^A = \omega_{2n-j+1} = \sum_{i=1}^{2n-j+1} e_i \in \mathbb{Z}^{2n}$ for $1 < j \le n$ be the A_{2n-1} fundamental weights, and the $\mathbb{Z}^{2n}$ partition

$$\lambda^A = 2\lambda_n\omega_n^A + \sum_{i=1}^{n-1} \lambda_i(\omega_i^A + \omega_{\overline{i+1}}^A).$$

Let $SSYT(\lambda^A, n, \overline{n})$ be the set, considered with its A_{2n-1}-crystal structure, of semi-standard Young tableaux of shape λ^A with filling in the ordered alphabet $[\pm n]$. We will denote the corresponding crystal operators by f_i^A for $i \in [\pm n]$, and consider for $1 \le i \le n$, the operators $f_i^E = f_i^A f_{\overline{i+1}}^A$ if $i < n$ and $f_n^E = (f_n^A)^2$. Let E denote the virtualization map on type C_n Kashiwara-Nakashima tableaux defined by Baker [2, Proposition 2.2, Proposition 2.3]. More precisely, E is an injective map

$$E : KN(\lambda, n) \hookrightarrow SSYT(\lambda^A, n, \overline{n}) \tag{4}$$

such that $E(f_i(T)) = f_i^E(E(T))$ for $T \in KN(\lambda, n), 1 \leq i \leq n$. We will denote by E^{-1} the restriction of any left inverse of E to the image of $\mathsf{KN}(\lambda, n)$ under E.
Given an admissible column C of shape ω_i, for some $1 \leq i \leq n$, denote by $\psi(C)$ its *Baker virtual split* [2, Proposition 2.2], a two column type A_{2n-1} tableau of shape $\omega_i^A + \omega_{2n-i}^A$. The map ψ is injective and embeds admissible columns of length i into $\mathsf{SSYT}(\omega_i^A + \omega_{2n-i}^A), 1 \leq i \leq n$. Indeed E reduces to ψ on admissible columns and we define ψ^{-1} analogously to E^{-1}. From [2, Proposition 2.3] we know that, if we write $T \in KN(\lambda, n)$ as a concatenation of its columns, that is, $T = C_k \cdots C_1$, then

$$E(T) = [\emptyset \leftarrow w(\psi(C_1)) \leftarrow \cdots \leftarrow w(\psi(C_k))].$$

where the word $w(\psi(C))$ of a two-column $\psi(C)$ is given by the Japanese reading of its two columns (from top to bottom and right to left), and $P \leftarrow w$ is the column insertion of a word w into a semi-standard Young tableau P.

Let $T_\lambda \in KN(\lambda, n)$ be the highest weight element, that is, T_λ is the Yamanouchi tableau of shape and weight λ on the alphabet $[n]$. Then $E(T_\lambda) = T_{\lambda^A}$ is the highest weight element of $SSYT(\lambda^A, n, \overline{n})$, that is, the A_{2n-1} Yamanouchi tableau of shape and weight λ^A in the alphabet $[\pm n]$. The image of $KN(\lambda, n)$ by E in $SSYT(\lambda^A, n, \overline{n})$ is the crystal generated by acting with the lowering operators f_i^E on the highest weight element T_{λ^A} of $SSYT(\lambda^A, n, \overline{n})$. For $T \in KN(\lambda, n)$ a tableau, where $T = C_k \cdots C_1$, let

$$w_T = w(\psi(C_1)) \cdots w(\psi(C_k)).$$

Then w_T is a word in the monoid $\mathcal{C}_n^*$, and $E(T) = \emptyset \leftarrow w_T$. We will call the recording tableau of the column insertion of w_T, the *Baker recording tableau* associated to T.

Proposition 2. *[1] For $T \in KN(\lambda, n)$, the Baker recording tableau $Q(w_T)$ depends only on λ. From now on, we will denote by Q_λ the Baker recording tableau associated to λ.*

2.5. **Internal cactus group action on a normal crystal and virtualization.** Recall first the Schützenberger–Lusztig involution $\xi : B(\lambda) \to B(\lambda)$ acting on an irreducible normal crystal of highest weight λ. This set involution can be seen as flipping the crystal upside down while reversing the edge crystal colours by $i \mapsto \theta(i)$, $i \in X$ and applying the longest element of the Weyl group to the vertex weights. For type A_{n-1}, ξ coincides with the evacuation evac on the set of semi-standard Young tableaux $SSYT(\lambda, n)$ [20], and with $\mathsf{reversal}$ on the set $\mathsf{SSYT}(\lambda/\mu, n)$ of type A_{n-1} tableaux of skew-shape λ/μ in the alphabet $[n]$ [4]. In [18], Santos has introduced a symplectic evacuation algorithm on KN tableaux, which he has proved to coincide with the full Lusztig–Scützenberger involution on a given crystal $KN(\lambda, n)$.

Partial Schützenberger involutions were first studied in the type A_{n-1} case for connected sub-diagrams by Berenstein and Kirillov who have used them alternatively to define Bender–Knuth involutions [6], but the former involutions have been defined by Halacheva in general. Given J any subset of X, the partial SL involution ξ_J is defined to be the SL involution on each connected component of the normal crystal B_J. The partial SL involutions ξ_J for any connected subdiagram J of X satisfy the J_X cactus relations.

Theorem 1 ([11])**.** *The map $s_J \mapsto \xi_J$, J a connected sub-diagram of X, defines an action of the cactus group J_X on the set B, that is, the following is a group homomorphism*

$$\begin{array}{rccc} \Phi_X : & J_X & \to & \mathfrak{S}_B \\ & s_J & \mapsto & \xi_J. \end{array}$$

Moreover $\mathsf{wt}(\xi_J(b)) = w_0^J \mathsf{wt}(b)$, $b \in B$ where w_0^J is the longest element of $W^J \subseteq W$.

That is, s_J acts on each connected component of B_J, as a set, by permuting its vertices via ξ_J and exchanging its highest and lowest weight elements.

Theorem 2. *[1] The map $\tilde{s}_J \mapsto \widetilde{\xi}_J$, J a connected sub-diagram of C_n, below, defines an action of the virtual cactus group $\widetilde{J}_{2n}$ on the set $B = SSYT(\lambda^A, 2n)$, that is, the following is a group homomorphism*

$$\begin{array}{lll} \tilde{\Phi}_{A_{2n-1}} : \widetilde{J}_{2n} & \to \mathfrak{S}_B & \\ (*) \quad \tilde{s}_{[1,q]} & \mapsto \widetilde{\xi}_{[1,q]} = \xi^{A_{2n-1}}_{[1,q]} \xi^{A_{2n-1}}_{[2n-q,2n-1]} & \\ & \quad = \mathsf{evac}_{q+1} \mathsf{evac}_{2n} \mathsf{evac}_{q+1} \mathsf{evac}_{2n}, & 1 \le q < n, \\ \quad \tilde{s}_{[q,n]} & \mapsto \widetilde{\xi}_{[q,n]} = \xi^{A_{2n-1}}_{[q,2n-q]} = \mathsf{reversal}^{A_{2n-1}}_{[q,2n-q]}, & 1 \le q \le n. \end{array}$$

where evac_i is the evacuation on the tableau restricted to the alphabet $[1.i]$, and $\mathsf{reversal}^{A_{2n-1}}_{[q,2n-q]}$ the reversal of the skew-tableau restricted to the alphabet $[q, 2n-q]$. This action of $\widetilde{J}_{2n}$ on $SSYT(\lambda^A, 2n)$ preserves the subset $E(K(\lambda, n))$, and thus it is also an action of $\widetilde{J}_{2n}$ on $E(KN(\lambda, n))$.

We have then the following commutative diagram [1] corresponding to the crystal embedding E and the partial C_n and the virtual SL involutions $\widetilde{\xi}_J$, where $J = [1, q] \subseteq [n-1]$ and $J = [p, n] \subseteq [n]$ are connected subintervals of the Dynkin diagram of C_n,

$$\begin{array}{ccc} KN(\lambda, n) & \xrightarrow{E} & SSYT(\lambda^A, n, \bar{n}) \\ \xi^{C_n}_{[p,n]} \Big\downarrow \xi^{C_n}_{[1,q]} & & \xi^{A_{2n-1}}_{[p,\overline{p+1}]} \Big\downarrow \xi^{A_{2n-1}}_{[1,q]} \xi^{A_{2n-1}}_{[\overline{q+1},\overline{2}]} \\ KN(\lambda, n) & \xrightarrow[E]{} & SSYT(\lambda^A, n, \bar{n}) \end{array}.$$

Example 4. *Consider $n = 5$, J=[1,4] and the KN tableau T of shape $\lambda = \omega_4 + \omega_3$:*

$$T = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \bar{5} \\ \hline \bar{4} & \bar{3} \\ \hline \bar{3} & \\ \cline{1-1} \end{array}, \quad \mathsf{wt}(T) = (2, 0, -1, -1, -1).$$

Labelling the columns of T from left to right as C_2 and C_1, we obtain $E(T)$ with shape $\lambda^A = \omega_7 + \omega_3 + \omega_6 + \omega_4$:

$$\psi(C_2) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \bar{5} \\ \hline 4 & \bar{3} \\ \hline \bar{5} \\ \cline{1-1} \bar{4} \\ \cline{1-1} \bar{3} \\ \cline{1-1} \bar{2} \\ \cline{1-1} \end{array}, \psi(C_1) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 5 & \bar{4} \\ \hline \bar{5} & \bar{2} \\ \hline \bar{4} \\ \cline{1-1} \bar{3} \\ \cline{1-1} \end{array} \Rightarrow E(T) = [\emptyset \leftarrow w(\psi(C_2)) \leftarrow w(\psi(C_1))] = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 4 & \bar{5} \\ \hline 3 & \bar{5} & \bar{4} & \bar{3} \\ \hline 5 & \bar{4} & \bar{2} \\ \cline{1-3} \bar{5} & \bar{3} \\ \cline{1-2} \bar{4} & \bar{2} \\ \cline{1-2} \bar{3} \\ \cline{1-1} \end{array}.$$

Considering the barred and unbarred parts of $E(T)$ separately (denoted by $E(T)^+$, resp. $E(T)^-$), we compute the evacuation, evac, of the unbarred part and the reversal, rev, of the barred part, yielding:

$$\mathsf{evac} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 5 & 5 \\ \cline{1-3} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}, \mathsf{rev} \begin{array}{|c|c|c|c|}\hline * & * & * & * \\ \hline * & * & * & \bar{5} \\ \hline * & \bar{5} & \bar{4} & \bar{3} \\ \hline * & \bar{4} & \bar{2} \\ \cline{1-3} \bar{5} & \bar{3} \\ \cline{1-2} \bar{4} & \bar{2} \\ \cline{1-2} \bar{3} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|}\hline * & * & * & * \\ \hline * & * & * & \bar{3} \\ \hline * & \bar{4} & \bar{2} & \bar{2} \\ \hline * & \bar{3} & \bar{1} \\ \cline{1-3} \bar{4} & \bar{2} \\ \cline{1-2} \bar{3} & \bar{1} \\ \cline{1-2} \bar{1} \\ \cline{1-1} \end{array}.$$

Putting these tableaux together, one obtains from $(*)$ the A_9 tableau $\widetilde{\xi}_{[1,4]}(E(T)) = \xi^{A_9}_{[1,4]}\xi^{A_9}_{[\bar{5},\bar{2}]}(E(T)) = (\mathsf{evac}(E(T)^+), \mathsf{rev}(E(T)^-))$. Using Q_λ to perform the reverse column Schensted insertion on $\widetilde{\xi}_{[1,4]}(E(T))$ provides the image under ψ of two KN columns C_1' and C_2'. Applying ψ^{-1} to each column results in:

$$Q_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 11 & 15 \\ \hline 2 & 5 & 12 & 16 \\ \hline 3 & 6 & 13 & 17 \\ \hline 7 & 14 & 18 \\ \cline{1-3} 8 & 19 \\ \cline{1-2} 9 & 20 \\ \cline{1-2} 10 \\ \cline{1-1} \end{array} \Rightarrow \psi(C_2') = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 & \bar{3} \\ \hline 5 & \bar{2} \\ \hline \bar{4} \\ \cline{1-1} \bar{3} \\ \cline{1-1} \bar{2} \\ \cline{1-1} \bar{1} \\ \cline{1-1} \end{array}, \psi(C_1') = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 5 & \bar{2} \\ \hline \bar{4} & \bar{1} \\ \hline \bar{3} \\ \cline{1-1} \bar{1} \\ \cline{1-1} \end{array} \Rightarrow C_1'C_2' = \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 5 & \bar{3} \\ \hline \bar{3} & \bar{2} \\ \hline \bar{1} \\ \cline{1-1} \end{array} = \xi^C_{[1,4]}(T).$$

3. The symplectic Berenstein–Kirillov group

Recall the cactus group J_n is also generated by the elements $s_{[1,j]}, 1 \le j \le n$. In fact the following assignments define group epimorphisms from J_n to $\mathcal{BK}_n$: $s_{[i,j]} \mapsto q_{[i,j]}$ [9, Theorem 1.4]; $s_{[1,j]} \mapsto q_{[1,j]}$ [6, Remark 1.3], [11, Section 10.2], [12, Remark 3.9]. Thus the group $\mathcal{BK}_n$ is isomorphic to a quotient of J_n.

Definition 1. *[1] The* symplectic Berenstein–Kirillov group $\mathcal{BK}^{C_n}$*, $n \ge 1$, is the free group generated by the $2n-1$ symplectic partial Schützenberger-Lusztig involutions*

$$q^C_{[1,i]} := \xi^{C_n}_{[1,i]},\ 1 \le i < n, \quad \textit{and } q^C_{[i,n]} := \xi^{C_n}_{[i,n]},\ 1 \le i \le n,$$

on straight shaped KN tableaux on the alphabet $[\pm n]$ modulo the relations they satisfy on those tableaux.

Theorem 3. *The following is a group epimorphism from J_{C_n} to $\mathcal{BK}^{C_n}$:*

$$s^{C_n}_{[1,j]} \mapsto q^{C_n}_{[1,j]},\ 1 \le j < n, \qquad s^{C_n}_{[j,n]} \mapsto q^{C}_{[j,n]},\ 1 \le j \le n.$$

The group $\mathcal{BK}^{C_n}$ is isomorphic to a quotient of J_{C_n}.

Definition 2. *[1] For $n \ge 1$, the symplectic Bender–Knuth involutions $t^{C_n}_i$, $1 \le i \le 2n-1$, on straight shaped KN tableaux on the alphabet $[\pm n]$, are defined as*

$$\begin{aligned} t^{C_n}_i &:= q^{C_n}_{[1,i-1]} q^{C_n}_{[1,i]} q^{C_n}_{[1,i-1]} q^{C_n}_{[1,i-2]} = E^{-1} t^{A_{2n-1}}_i \tilde{t}^{A_{2n-1}}_{2n-i} E,\quad 1 \le i \le n-1, \\ \tilde{t}^{A_{2n-1}}_{2n-i} &:= q^{A_{2n-1}}_{[1,2n-1]} t^{A_{2n-1}}_i q^{A_{2n-1}}_{[1,2n-1]}\quad 1 \le i \le n-1, \\ t^{C_n}_{n-1+i} &:= q^{C_n}_{[n-i+1,n]} q^{C_n}_{[n-i+2,n]} = E^{-1} q^{A_{2n-1}}_{[n-(i-1),n+(i-1)]} q^{A_{2n-1}}_{[n-(i-2),n+(i-2)]} E,\quad 1 \le i \le n. \end{aligned} \tag{5}$$

Proposition 3. *[1] The symplectic Bender-Knuth involutions $t^{C_n}_i$, $1 \le i \le 2n-1$ also generate $\mathcal{BK}^{C_n}$. In fact, $q^{C_n}_{[1,n-1]} = t^{C_n}_1 (t^{C_n}_2 t^{C_n}_1) \cdots (t^{C_n}_{n-1} t^{C_n}_{n-2} \cdots t^{C_n}_1)$, $q^{C_n}_{[1,n]} = t^{C_n}_{2n-1} t^{C_n}_{2n-2} \cdots t^{C_n}_n$.*

Example 5. *Continue from Example 4. We have*

$$E(T) = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2&4&6\\\hline 3&6&7&8\\\hline 5&7&9\\\cline{1-3} 6&8\\\cline{1-2} 7&9\\\cline{1-2} 8\\\cline{1-1}\end{array}\ , \textit{ and}$$

$$\tilde{t}_7 E(T) = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2&4&6\\\hline 3&6&7&9\\\hline 5&7&8\\\cline{1-3} 6&8\\\cline{1-2} 7&9\\\cline{1-2} 8\\\cline{1-1}\end{array}\ ,\ t^A_3 \tilde{t}_7 E(T) = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2&3&6\\\hline 4&6&7&9\\\hline 5&7&8\\\cline{1-3} 6&8\\\cline{1-2} 7&9\\\cline{1-2} 8\\\cline{1-1}\end{array}\ \Rightarrow E t^C_3(T) = t^A_3 \tilde{t}_7 E(T).$$

Proposition 4. *[1] The symplectic Bender–Knuth involutions $t^{C_n}_i = 1$, $i = 1, \dots, 2n-1$, satisfy the following relations:*

(1) $(t^{C_n}_i)^2 = 1$, $i = 1, \dots, 2n-1$.
(2) $(t^{C_n}_{n+i-1} t^{C_n}_{n+j-1})^2 = 1$, $1 \le i, j \le n$.
(3) $(t^{C_n}_i t^{C_n}_j)^2 = 1$, $|i-j| > 1$, $1 \le i, j < n$.
(4) $(t^{C_n}_i t^{C_n}_{n+j-1})^2 = 1$, $i < n-j$.
(5) $(t^{C_n}_i q^{C_n}_{[j,k-1]})^2 = 1$, $i+1 < j < k \le n$.

(6) $(t_i^{C_n} q_{[j,n]}^{C_n})^2 = 1,\ i+1 < j \leq n.$
(7) $(t_{n+i-1}^{C_n} q_{[j,n]}^{C_n})^2 = 1,\ 1 \leq i, j \leq n.$
(8) $(t_{n+i-1}^{C_n} q_{[j,k-1]}^{C_n})^2 = 1,\ n-i+1 < j < k \leq n.$
(9) $(t_1^{C_n} t_2^{C_n})^6 = 1,\ n \geq 3.$
(10) $(t_{n-1}^{C_n} \cdots t_2^{C_n} t_1^{C_n} t_2^{C_n} \cdots t_{n-1}^{C_n} t_n^{C_n})^4 = 1.$

All relations except for 10. can be obtained using Baker's virtualization map E. Relation 10. has its origin in the braid relations of the Weyl group of type C_n (the hyperoctahedral group). The action of the Weyl group on $KN(\lambda, n)$ can be translated into an action of partial SL involutions defined on Dynkin sub-diagrams of one single node.

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