EFFECTIVE DESCENT MORPHISMS OF FILTERED PREORDERS

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Abstract. We characterize effective descent morphisms of what we call filtered preorders, and apply these results to slightly improve a known result, due to the first author and F. Lucatelli Nunes, on the effective descent morphisms in lax comma categories of preorders. A filtered preorder, over a fixed preorder X, is defined as a preorder A equipped with a profunctor $X \to A$ and, equivalently, as a set A equipped with a family $(A_x)_{x\in X}$ of upclosed subsets of A with $x' \leqslant x \Rightarrow A_x \subseteq A_{x'}$.

1. Introduction

Throughout this paper, by a *preorder* we mean a set equipped with a reflexive transitive relation, and we write $X = (X, \leqslant)$ for a fixed preorder. Such preorders were simply called ordered sets, e.g. in [1], and keeping the notation of [1], we will write:

- Ord for the category of preorders;
- $\mathsf{Ord}//X$ for the lax comma category of Ord over X, and we recall that its objects are the same as in $\mathsf{Ord}/X = (\mathsf{Ord} \downarrow X)$, while a morphism

$$f : (A, \alpha) \to (B, \beta)$$

in $\mathsf{Ord}//X$ is a morphism $f : A \to B$ in Ord with $\alpha \leqslant \beta f$, that is, with

$$\alpha(a) \leqslant \beta f(a)$$

for all $a \in A$.

Considering one of the problems formulated in [1], namely the problem of characterizing effective descent morphisms in $\mathsf{Ord}//X$, we found a 'nicer' category Ord_X, of what we called 'filtered preorders', where this problem turned out to be easy enough to solve it completely. It also turned out that dealing with Ord_X helps to improve what was done in [1] with $\mathsf{Ord}//X$. Here, a filtered preorder, over a fixed preorder X, is defined as a preorder A equipped with a profunctor $X \to A$ and, equivalently, as a set A equipped with a family $(A_x)_{x\in X}$ of upclosed subsets of A with $x' \leqslant x \Rightarrow A_x \subseteq A_{x'}$.

The paper is organized as follows:

Section 2 is devoted to general categorical remarks, notably involving the notion of map of adjunctions which was briefly mentioned in [6]. These remarks should be reformulated in the style of [7] one day, but we did not go further than what we needed for our purposes. We should particularly mention the contrast between 'strict' and 'pseudo-', which appears as the contrast between Corollary 9.6 in [5] and our Corollary 2.6(b). We are satisfied with 'strict' since we are using it for functors that preserve relevant chosen limits and colimits (specifically, pullbacks and coequalizers).

Date: December 21, 2023.

2010 *Mathematics Subject Classification.* 18C15, 18A20, 18A25, 18B35, 18E50.

Key words and phrases. filtered preorder, lax comma category, effective descent morphism, map of adjunctions, monadic functor, chosen pullbacks, chosen coequalizers.

Partially supported by the Centre for Mathematics of the University of Coimbra – UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.

Section 3 introduces filtered preorders and shows how their category extends the category $\mathsf{Ord}//X$, while the purpose of Section 4 is to characterize effective descent morphisms in Ord_X, and Section 5 gives a new class of effective descent morphisms in $\mathsf{Ord}//X$ slightly improving Theorem 3.9 of [1].

2. General remarks on monadicity and descent

In this section we first deal with a fixed *map* $(P, P')\colon \Phi \to X$ *of adjunctions* in the sense of [6] (see Section 7 of Chapter IV therein) displayed as

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Phi} & \mathcal{U}' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ \mathcal{W} & \xrightarrow[X]{} & \mathcal{W}' \end{array}$$

In this diagram P and P' are functors while

$$\Phi = (\Phi_!, \Phi^*, \eta^\Phi, \varepsilon^\Phi)\colon \mathcal{U} \to \mathcal{U}' \;\text{ and }\; X = (X_!, X^*, \eta^X, \varepsilon^X)\colon \mathcal{W} \to \mathcal{W}'$$

are adjunctions with

$$X_! P = P'\Phi_!,\; P\Phi^* = X^* P',\; P\eta^\Phi = \eta^X P,\; P'\varepsilon^\Phi = \varepsilon^X P'.$$

Equivalently, we can present it as an adjunction

$$((\Phi_!, X_!), (\Phi^*, X^*), (\eta^\Phi, \eta^X), (\varepsilon^\Phi, \varepsilon^X))\colon (\mathcal{U}, P, \mathcal{W}) \to (\mathcal{U}', P', \mathcal{W}'),$$

in the 2-category ArrCAT ($= \mathsf{Cat}^\mathbf{2}$ in the notation of, e.g., [4]) of arrows of the category of categories. In order to make this clear, let us just recall that:

- the objects of ArrCAT are all functors $F\colon \mathcal{A} \to \mathcal{B}$, written as triples $(\mathcal{A}, F, \mathcal{B})$;
- a morphism $(\mathcal{A}, F, \mathcal{B}) \to (\mathcal{A}', F', \mathcal{B}')$ is a pair of functors (K, L) making the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{K} & \mathcal{A}' \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathcal{B} & \xrightarrow[L]{} & \mathcal{B}' \end{array}$$

  commute;
- for morphisms $(K, L), (M, N)\colon (\mathcal{A}, F, \mathcal{B}) \to (\mathcal{A}', F', \mathcal{B}')$, a 2-cell from (K, L) to (M, N) is a pair of natural transformations $(\sigma, \tau) =$

$$\begin{array}{ccc} \mathcal{A} & \underset{M}{\overset{K}{\rightrightarrows}}\;{\scriptstyle \downarrow\sigma} & \mathcal{A}' \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathcal{B} & \underset{N}{\overset{L}{\rightrightarrows}}\;{\scriptstyle \downarrow\tau} & \mathcal{B}' \end{array}$$

  with $F'\sigma = \tau F$.

Our first remark is: the equations $P\eta^\Phi = \eta^X P$ and $P'\varepsilon^\Phi = \varepsilon^X P'$ immediately imply:

Lemma 2.1. *For the map of adjunctions above:*

(1) *if η^Φ is an isomorphism and P is surjective on objects, then η^X is an isomorphism;*
(2) *if ε^Φ is an isomorphism and P' is surjective on objects, then ε^X is an isomorphism;*
(3) *in particular, if Φ is a category equivalence and both P and P' are surjective on objects, then X is a category equivalence.* □

Next, our map of adjunctions above determines, whenever the categories $\mathcal{U}'$ and $\mathcal{W}'$ have chosen coequalizers preserved by the functor P', its *derived map of adjunctions*

$$\begin{array}{ccc} \mathcal{U}^{T^{\phi}} & \longrightarrow & \mathcal{U}' \\ {\scriptstyle \tilde{P}}\downarrow & & \downarrow{\scriptstyle P'} \\ \mathcal{W}^{T^{X}} & \longrightarrow & \mathcal{W}' \end{array}$$

where: T^{ϕ} and T^{X} are the monads determined by the adjunctions Φ and X, respectively; $\tilde{P}$ is induced by (P, P'); and the horizontal arrows are the comparison adjunctions. If (P, P') is a split epimorphism of adjunctions, then $\tilde{P}$ also is a split epimorphism and, using Lemma 2.1, one can easily show that monadicity of Φ implies monadicity of X. The precise statement is:

Lemma 2.2. *Let $(P, P') \colon \Phi \to X$ and $(I, I') \colon X \to \Phi$ be maps of adjunctions with $PI = 1_{\mathcal{W}}$ and $P'I' = 1_{\mathcal{W}'}$. If the categories $\mathcal{U}'$ and $\mathcal{W}'$ have chosen coequalizers preserved by the functors P' and I', then premonadicity of Φ implies premonadicity of X and monadicity of Φ implies monadicity of X.* □

Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be categories with chosen pullbacks, and $S_1 \colon \mathcal{C}_1 \to \mathcal{C}_0$ a functor that preserves them. Then any morphism $p_1 \colon E_1 \to B_1$ in $\mathcal{C}_1$ determines a map

$$\begin{array}{ccc} (\mathcal{C}_1 \downarrow E_1) & \xrightarrow{(p_{1!}, p_1^*, \eta^{p_1}, \varepsilon^{p_1})} & (\mathcal{C}_1 \downarrow B_1) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ (\mathcal{C}_0 \downarrow E_0) & \xrightarrow[(p_{0!}, p_0^*, \eta^{p_0}, \varepsilon^{p_0})]{} & (\mathcal{C}_0 \downarrow B_0) \end{array}$$

of adjunctions, in which $(p_0 \colon E_0 \to B_0) = S_1(p_1 \colon E_1 \to B_1)$, P and P' are induced by S_1, and the horizontal arrows are the suitable change-of-base adjunctions. From Lemma 2.2, having in mind the monadic approach to descent theory (see e.g. [3]) we obtain:

Corollary 2.3. *Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be categories with chosen pullbacks and chosen coequalizers, and $S_1 \colon \mathcal{C}_1 \to \mathcal{C}_0$ and $J_1 \colon \mathcal{C}_0 \to \mathcal{C}_1$ functors that preserves them and have $S_1 J_1 = 1_{\mathcal{C}_0}$. Then the functor S_1 sends descent morphisms in $\mathcal{C}_1$ to descent morphisms in $\mathcal{C}_0$ and effective descent morphisms in $\mathcal{C}_1$ to effective descent morphisms in $\mathcal{C}_0$.* □

Now consider a cube diagram

$$\begin{array}{ccccccc} & & \mathcal{V} & & \xrightarrow{\quad\Psi\quad} & & \mathcal{V}' \\ & {\scriptstyle \Pi_2}\nearrow & & & & {\scriptstyle \Pi_2'}\nearrow & \\ \mathcal{U} \times_{\mathcal{W}} \mathcal{V} & & \xrightarrow{\quad\Phi \times_X \Psi\quad} & & \mathcal{U}' \times_{\mathcal{W}'} \mathcal{V}' & & \\ & & \downarrow{\scriptstyle Q} & & & & \downarrow{\scriptstyle Q'} \\ {\scriptstyle \Pi_1}\downarrow & & & & {\scriptstyle \Pi_1'}\downarrow & & \\ & & \mathcal{W} & & \xrightarrow[\quad X\quad]{} & & \mathcal{W}' \\ & {\scriptstyle P}\nearrow & & & & {\scriptstyle P'}\nearrow & \\ \mathcal{U} & & \xrightarrow[\quad\Phi\quad]{} & & \mathcal{U}' & & \end{array}$$

whose left-hand and the right-hand faces are pullbacks in the category of categories, while all other faces are maps of adjunctions. Here we assume that $\mathcal{U} \times_{\mathcal{W}} \mathcal{V}$ and $\mathcal{U}' \times_{\mathcal{W}'} \mathcal{V}'$ are constructed 'as usually', that is, as suitable categories of pairs, and the adjunction $\Phi \times_X \Psi$ is defined by

$$(\Phi \times_X \Psi)_!(U, V) = (\Phi_!(U), \Psi_!(V)),\ (\Phi \times_X \Psi)^*(U', V') = (\Phi^*(U'), \Psi^*(V')),$$

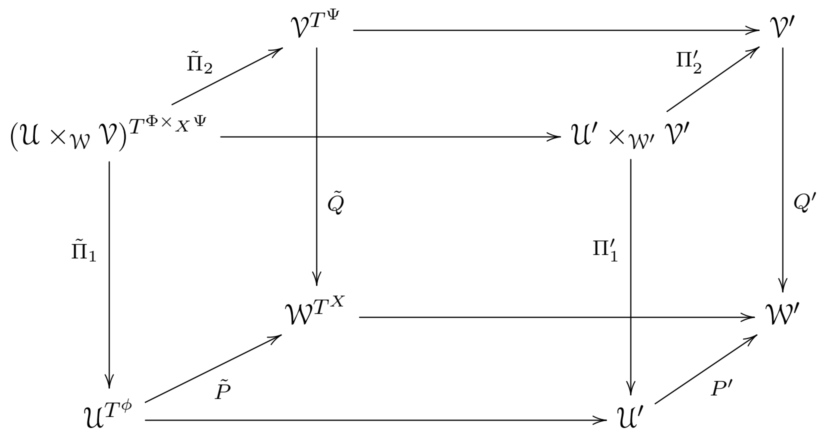
$$\eta^{\Phi\times_X\Psi}_{(U,V)} = (\eta^{\Phi}_{U}, \eta^{\Psi}_{V}),\ \varepsilon^{\Phi\times_X\Psi}_{(U',V')} = (\varepsilon^{\Phi}_{U'}, \varepsilon^{\Psi}_{V'}).$$

From the construction of $\eta^{\Phi\times_X\Psi}_{(U,V)}$ and $\varepsilon^{\Phi\times_X\Psi}_{(U',V')}$, we obtain:

Lemma 2.4. *Under the assumptions above:*

(1) *if η^{Φ} and η^{Ψ} are isomorphisms, then so is $\eta^{\Phi\times_X\Psi}$;*
(2) *if ε^{Φ} and ε^{Ψ} are isomorphisms, then so is $\varepsilon^{\Phi\times_X\Psi}$;*
(3) *if Φ and Ψ are category equivalences, then so is $\Phi\times_X\Psi$.* □

If the categories $\mathcal{U}'$, $\mathcal{V}'$, and $\mathcal{W}'$ have chosen coequalizers preserved by the functors P' and Q', then we have the *derived cube diagram*



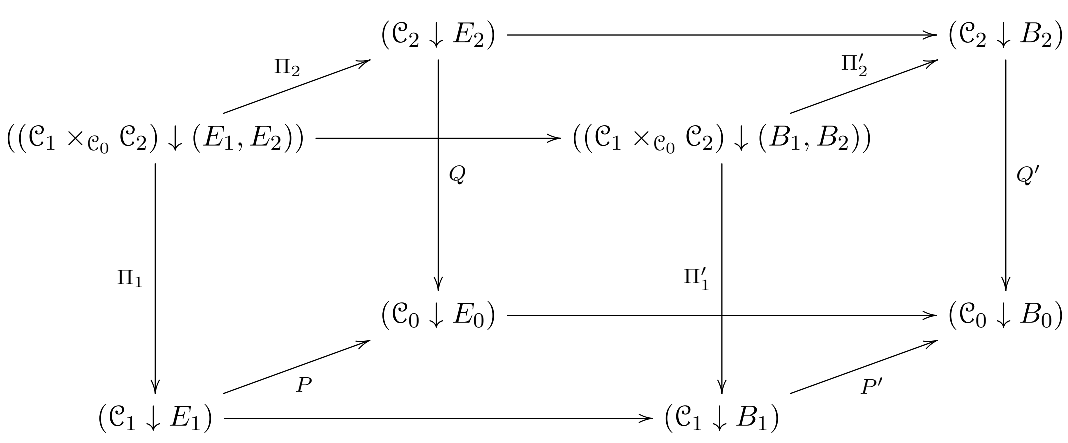
obtained in an obvious way using derived maps of adjunctions. Note, in particular, that $(\mathcal{U}\times_{\mathcal{W}}\mathcal{V})^{T^{\Phi\times_X\Psi}}$ can be identified with $\mathcal{U}^{T^{\phi}}\times_{\mathcal{W}^{T^X}}\mathcal{V}^{T^{\Psi}}$. And, from Lemma 2.4, we obtain

Lemma 2.5. *For maps $(P,P')\colon \Phi\to X$ and $(Q,Q')\colon \Psi\to X$ of adjunctions, premonadicity of Φ and Ψ implies premonadicity of $\Phi\times_X\Psi$ and monadicity of Φ and Ψ implies premonadicity of $\Phi\times_X\Psi$, provided the categories $\mathcal{U}'$, $\mathcal{V}'$, and $\mathcal{W}'$ have chosen coequalizers preserved by the functors P' and Q'.* □

Let

$$\mathcal{C}_1 \xrightarrow{S_1} \mathcal{C}_0 \xleftarrow{S_2} \mathcal{C}_2,$$

be a cospan of categories having chosen pullbacks preserved by the functors S_1 and S_2. Given a morphism $(p_1,p_2)\colon (E_1,E_2)\to(B_1,B_2)$ in $\mathcal{C}_1\times_{\mathcal{C}_0}\mathcal{C}_2$, we can take the originally considered cube diagram to be



where (P,P') is induced by S_1, (Q,Q') is induced by S_2, and horizontal arrows are the suitable change-of-base adjunctions. From Lemma 2.5, since the change-of-base adjunction along a morphism is premonadic/monadic if and only if it is a descent/effective descent morphism (see e.g. [3]), we obtain:

Corollary 2.6. *If the categories $\mathcal{C}_i$ $(i = 0, 1, 2)$ have chosen pullbacks and chosen coequalizers preserved by the functors S_1 and S_2, then:*

(1) *if p_1 and p_2 are descent morphisms, then so is (p_1, p_2);*
(2) *if p_1 and p_2 are effective descent morphisms, then so is (p_1, p_2).* □

Example 2.7. Let us choose the data above as follows:

- $\mathcal{C}_0$ to be the category of sets.
- $\mathcal{C}_1 = \mathsf{Ord}$.
- $\mathcal{C}_2$ to be the category of pairs $(A, (A_x)_{x\in X})$, where A is a set and $(A_x)_{x\in X}$ is a family of subsets of A with $x' \leqslant x \Rightarrow A_x \subseteq A_{x'}$. A morphism

$$f\colon (A, (A_x)_{x\in X}) \to (B, (B_x)_{x\in X})$$

  in $\mathcal{C}_2$ is a morphism $f\colon A \to B$ in $\mathcal{C}_0$ with $f(A_x) \subseteq B_x$ for each $x \in X$.
- S_1 and S_2 to be the underlying set functors. Note that in all our categories here the pullbacks and coequalizers are *chosen as for sets* and preserved by the functors S_1 and S_2; moreover, one could easily choose suitable right inverses of S_1 and S_2.

We can write the objects of $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ simply as $A = (A, (A_x)_{x\in X})$ assuming that A is a set equipped with a preorder relation and a family $(A_x)_{x\in X}$ of subsets satisfying the condition above. A morphism $p\colon E \to B$ in $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ is then a map $p\colon E \to B$ that preserves the preorder relation and has $p(E_x) \subseteq B_x$ for each $x \in X$. It is easy to show that p is an effective descent morphism in $\mathcal{C}_2$ if and only if p and all induced maps $p_x\colon E_x \to B_x$ are surjective. Then, putting this together with the description of effective descent morphisms in Ord in Proposition 3.4 of [2], and applying Corollaries 2.3 and 2.6, we conclude that

Theorem 2.8. *A morphism $p\colon E \to B$ in $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ is an effective descent morphism if and only if:*

(a) *for every $b_2 \leqslant b_1 \leqslant b_0$ in B, there exist $e_2 \leqslant e_1 \leqslant e_0$ in E with $p(e_i) = b_i$ for each $i = 0, 1, 2$;*
(b) *p and all induced maps $p_x\colon E_x \to B_x$ are surjective.*

3. X-filtered preorders

Definition 3.1. An *X-filtered preorder* is a pair (A, a), where $\mathsf{a}\colon X \to A$ is a $\{0, 1\}$-profunctor; that is, $\mathsf{a} \subseteq X \times A$ is a relation satisfying

$$(x' \leqslant x \ \&\ a \leqslant a') \Rightarrow ((x, a) \in \mathsf{a} \Rightarrow (x', a') \in \mathsf{a})$$

for all $x, x' \in X$ and $a, a' \in A$. A morphism $f\colon (A, \mathsf{a}) \to (B, \mathsf{b})$ in the category Ord_X of X-filtered preorders is a morphism $f\colon A \to B$ in Ord with $f\mathsf{a} \leqslant \mathsf{b}$, that is, with

$$(x, a) \in \mathsf{a} \Rightarrow (x, f(a)) \in \mathsf{b}$$

for all $x \in X$ and $a \in A$.

The reason for calling such pairs (A, a) X-filtered preorders is that a profunctor $\mathsf{a}\colon X \to A$ can be equivalently described as an *X-filtration* on A defined as a family $(A_x)_{x\in X}$ of upclosed subsets of A with $x' \leqslant x \Rightarrow A_x \subseteq A_{x'}$. The relationship between these two types of structure is straightforward and well known at various levels of generality; it is given by

$$A_x = \mathsf{a}(x, -) = \{a \in A \mid (x, a) \in \mathsf{a}\}$$

or, equivalently, by

$$\mathsf{a} = \{(x, a) \in X \times A \mid a \in A_x\}.$$

If there is no danger of confusion, we will simply write

$$(A, \mathsf{a}) = A = (A, (A_x)_{x \in X})$$

(the second equality here is what we have already used in a more general situation in Example 2.7).

We could also similarly describe these structures as families of A-indexed (for various A) families of subsets

$$\mathsf{a}(-, a) = \{x \in X \mid (x, a) \in \mathsf{a}\}$$

of X, but that would be less useful since we are considering a fixed X, not a fixed A.

We will use the fully faithful functors

$$\mathsf{Ord}//X \xrightarrow{F_1} \mathsf{Ord}_X \xrightarrow{F_2} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$$

where F_1 is defined by

$$F_1(A, \alpha) = (A, \{(x, a) \mid x \leqslant \alpha(a)\})$$

and by requiring the diagram

$$\begin{array}{ccc} \mathsf{Ord}//X & \xrightarrow{\quad F_1 \quad} & \mathsf{Ord}_X \\ & {\scriptstyle\text{forgetful}}\searrow \quad \swarrow{\scriptstyle\text{forgetful}} & \\ & \mathsf{Ord} & \end{array}$$

to commute, $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ is as in Example 2.7, and F_2 is the inclusion functor. The following proposition is also well known at various levels of generality, but instead of explaining that we give a (simple) direct proof:

Proposition 3.2. *The image of* $F_1 \colon \mathsf{Ord}//X \to \mathsf{Ord}_X$*, which is the same as its replete image, consists of those filtered preorders* A *in which, for every* $a \in A$*, the subset* $\mathsf{a}(-, a)$ *of* X *has a largest element; equivalently, there is a largest* $x \in X$ *with* $a \in A_x$*.*

Proof. For (A, α) and (B, β) in $\mathsf{Ord}//X$, a morphism $f \colon A \to B$ in Ord, and $a \in A$, we have

$$\alpha(a) \leqslant \beta f(a) \Leftrightarrow \forall_x (x \leqslant \alpha(a) \Rightarrow x \leqslant \beta f(a)),$$

which means that f is a morphism from (A, α) to (B, β) in $\mathsf{Ord}//X$ if and only if it is a morphism from $F_1(A, \alpha)$ to $F_1(B, \beta)$ in Ord_X. That is, F_1 is indeed fully faithful.

Let (A, a) be an object in Ord_X such that each $\mathsf{a}(-, a)$ has a largest element. Let $\alpha(a)$ be such a largest element (for each $a \in A$), and consider the so defined (A, α). We have

$$a \leqslant a' \Rightarrow \alpha(a) \in \mathsf{a}(-, a') \Rightarrow \alpha(a) \leqslant \alpha(a'),$$

and so (A, α) is an object in $\mathsf{Ord}//X$. We also have $x \leqslant \alpha(a) \Leftrightarrow (x, a) \in \mathsf{a}$, and so $F_1(A, \alpha) = (A, \mathsf{a})$.

Conversely, if $(A, \mathsf{a}) = F_1(A, \alpha) = (A, \{(x, a) \mid x \leqslant \alpha(a)\})$, then, obviously, for each $a \in A$, $\alpha(a)$ is the largest $x \in X$ with $x \leqslant \alpha(a)$, which is equivalent to $x \in \mathsf{a}(-, a)$. □

4. Effective descent morphisms in Ord_X

We will often use a pullback diagram

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

in a given category, and refer to it simply as the pullback of p and f. Recall, e.g. from part 2 of Corollary 2.7 in [3]:

Proposition 4.1. *Let $\mathcal{D}$ be a category with pullbacks and $\mathcal{C}$ a full subcategory in $\mathcal{D}$ closed under pullbacks. For an effective descent morphism $p\colon E \to B$ in $\mathcal{D}$, the following conditions are equivalent:*

(i) *p is an effective descent morphism in $\mathcal{C}$;*
(ii) *for every morphism $f\colon A \to B$ in $\mathcal{D}$, we have $E \times_B A \in \mathcal{C} \Rightarrow A \in \mathcal{C}$.* □

Using this proposition applied to the inclusion $F_2\colon \mathsf{Ord}_X \to \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ and the results of Section 2, we will prove

Theorem 4.2. *A morphism $p\colon E \to B$ in Ord_X is an effective descent morphism if and only if:*

(a) *for each $b_2 \leqslant b_1 \leqslant b_0$ in B, there exist $e_2 \leqslant e_1 \leqslant e_0$ in E with $p(e_i) = b_i$ for each $i = 0, 1, 2$;*
(b) *for each $x \in X$ and each $b_1 \leqslant b_0$ in B_x, there exist $e_1 \leqslant e_0$ in E_x with $p(e_i) = b_i$ for each $i = 0, 1$.*

Proof. "Only if": Suppose p is an effective descent morphism in Ord_X. In Corollary 2.3, take S_1 to be the forgetful functor $\mathsf{Ord}_X \to \mathsf{Ord}$ and $J_1\colon \mathsf{Ord} \to \mathsf{Ord}_X$ to be defined by

$$J_1(A) = (A, (A_x)_{x \in X}) \text{ with } A_x = A \text{ for all } x \in X.$$

According to the description of effective descent morphisms in Ord, it follows that condition (a) is satisfied.

To prove that condition (b) is also satisfied, take any $x \in X$ and $b_1 \leqslant b_0$ in B_x, and consider the pullback of p and f in $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ with $A = \{b_1, b_0\}$ having $b_i \leqslant b_j \Leftrightarrow j \leqslant i$, f being the inclusion map, and

$$A_{x'} = \begin{cases} \{b_1, b_0\}, & \text{if } x' < x; \\ \{b_1\}, & \text{if } x' \sim x; \\ \emptyset, & \text{if } x' \nleqslant x, \end{cases}$$

where by $x' \sim x$ we mean $x' \leqslant x$ and $x \leqslant x'$, and by $x' < x$ we mean $x' \leqslant x$ and $x' \not\sim x$. In this case

$$E \times_B A = (p^{-1}\{b_1\} \times \{b_1\}) \cup (p^{-1}\{b_0\} \times \{b_0\})$$

with

$$(E \times_B A)_{x'} = \begin{cases} ((E_{x'} \cap p^{-1}\{b_1\}) \times \{b_1\}) \cup ((E_{x'} \cap p^{-1}\{b_0\}) \times \{b_0\}), & \text{if } x' < x; \\ (E_{x'} \cap p^{-1}\{b_1\}) \times \{b_1\}, & \text{if } x' \sim x; \\ \emptyset, & \text{if } x' \nleqslant x. \end{cases}$$

Since A_x is not upclosed, A does not belong to Ord_X. Hence, since p is an effective descent morphism in Ord_X, it follows, by Proposition 4.1, that $E \times_B A$ does not belong to Ord_X. Therefore, at least one $(E \times_B A)_{x'}$ is not upclosed in $E \times_B A$. However, suppose $x' < x$. Then

$$(E \times_B A)_{x'} = (E_{x'} \cap p^{-1}\{b_1\}) \times \{b_1\}) \cup ((E_{x'} \cap p^{-1}\{b_0\}) \times \{b_0\})$$

and, since $E_{x'}$ is upclosed in E, it is upclosed in $E \times_B A$. Or, suppose $x' \nleqslant x$. Then $(E \times_B A)_{x'}$ is empty and so it is upclosed in $E \times_B A$. Hence upclosedness of $(E \times_B A)_{x'}$ must fail for some $x' \sim x$; that is, $(E \times_B A)_{x'}$, and therefore $(E \times_B A)_x$, is not upclosed in $E \times_B A$. Since E_x is upclosed in E,

$$(E \times_B A)_x = (E_x \cap p^{-1}\{b_1\}) \times \{b_1\}$$

is upclosed in $p^{-1}\{b_1\} \times \{b_1\}$, and so there exist $(e_1, b_1) \in (E_x \cap p^{-1}\{b_1\}) \times \{b_1\}$ and $(e_0, b_0) \in p^{-1}\{b_0\} \times \{b_1\}$ with $(e_1, b_1) \leqslant (e_0, b_0)$. This gives $e_1 \leqslant e_0$, both in E_x, since e_1 belongs to E_x and E_x is upclosed in E, with $p(e_1) = b_1$ and $p(e_0) = b_0$.

"If": Suppose conditions (a) and (b) hold. This makes p an effective descent morphism in $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ (see Example 2.7). Therefore, according to Proposition 4.1, it suffices to prove that, for every morphism $f\colon A \to B$ in $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ with each $(E \times_B A)_x$ upclosed, each A_x is also upclosed.

Suppose $a \leqslant a'$ in A with $a \in A_x$. As follows from condition (b), there exist $e, e' \in E$ with $e \leqslant e'$, $p(e) = a$, and $p(e') = a'$. This gives $(e, a) \leqslant (e', a')$ in $E \times_B A$ with $(e, a) \in (E \times_B A)_x$. Since $(E \times_B A)_x$ is upclosed in $E \times_B A$, it follows that (e', a') is in $(E \times_B A)_x$, and so a' is in A_x, since the projection $\pi_2\colon E \times_B A \to A$ is a morphism in $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$. □

5. Effective descent morphisms in $\mathsf{Ord}//X$

In this section we assume that X is *locally complete*, in the sense that, for each $x \in X$, the preorder $\{x' \in X \mid x' \leqslant x\}$ is equivalent to a complete lattice. We will also identify the category $\mathsf{Ord}//X$ with its F_1-image in Ord_X; note that $\mathsf{Ord}//X$ is then closed under pullbacks in Ord_X, thanks to the local completeness. While working with the morphism $p\colon E \to B$ we will write $B = (B, \beta)$, $E = (E, \varepsilon)$, and $E \times_B A = (E \times_B A, \gamma)$ in the notation we used for $\mathsf{Ord}//X$.

Lemma 5.1. *Let $p\colon E \to B$ be a morphism in $\mathsf{Ord}//X$ such that each induced map $p_x\colon E_x \to B_x$ ($x \in X$) is surjective. If $f\colon A \to B$ is a morphism in Ord_X with $E \times_B A$ in $\mathsf{Ord}//X$, then A is in $\mathsf{Ord}//X$.*

Proof. According to Proposition 3.2, we have to prove that, given $a \in A$, there is a largest $x \in X$ with $a \in A_x$. Following the proof of Theorem 3.9 in [1], we are going to put

$$\alpha(a) = \bigvee \{x \in X \mid a \in A_x\}$$

and prove that it is such an element. First we observe that, if a belongs to A_x, then $f(a)$ belongs B_x and so $x \leqslant \beta f(a)$. Therefore

$$\{x \in X \mid a \in A_x\} = \{x \leqslant \beta f(a) \mid a \in A_x\},$$

and so the join above is well defined. By definition of $\alpha(a)$, it suffices to prove that $a \in A_{\alpha(a)}$. As $f(a) \in B_{\alpha(a)}$, we have $\alpha(a) \leqslant \beta f(a)$, and so $f(a)$ belongs to $B_{\alpha(a)}$. By our hypotheses, there exist $e \in E_{\alpha(a)}$ with $p(e) = f(a)$. Hence, for every $x \leqslant \alpha(a)$, (e, a) belongs to $E_x \times_B A_x = (E \times_B A)_x$, and so $x \leqslant \gamma(e, a)$. Since $\alpha(a)$ is the join of such elements x, it follows that $\alpha(a) \leqslant \gamma(e, a)$. Then $a = \pi_2(c, a) \in A_{\gamma(c,a)} \subseteq A_{\alpha(a)}$, which completes our proof. □

From Proposition 4.1, Theorem 4.2, and Lemma 5.1, we obtain:

Theorem 5.2. *If X is a locally complete preordered set, then a morphism $p\colon E \to B$ in $\mathsf{Ord}//X$ is effective for descent in $\mathsf{Ord}//X$ provided that:*

(a) *for each $b_2 \leqslant b_1 \leqslant b_0$ in B, there exist $e_2 \leqslant e_1 \leqslant e_0$ in E with $p(e_i) = b_i$ for each $i = 0, 1, 2$;*

(b′) *for each $x \in X$ and each $b_1 \leqslant b_0$ with $x \leqslant \beta(b_1)$, there exist $e_1 \leqslant e_0$ with $x \leqslant \varepsilon(e_1)$ and $p(e_i) = b_i$ for $i = 0, 1$.* □

Our next result shows that conditions (a) and (b′) characterize effective descent morphisms in $\mathsf{Ord}//X$ among those $p\colon E \to B$ with $p_x\colon E_x \to B_x$ surjective for every $x \in X$.

Theorem 5.3. *Let X be a locally complete preordered set with bottom element. A morphism $p\colon E \to B$ in $\mathsf{Ord}//X$ such that $p_x\colon E_x \to B_x$ is surjective for every $x \in X$ is effective for descent in $\mathsf{Ord}//X$ if and only if:*

(a) *for each $b_2 \leqslant b_1 \leqslant b_0$ in B, there exist $e_2 \leqslant e_1 \leqslant e_0$ in E with $p(e_i) = b_i$ for each $i = 0, 1, 2$;*

(b′) *for each $x \in X$ and each $b_1 \leqslant b_0$ with $x \leqslant \beta(b_1)$, there exist $e_1 \leqslant e_0$ with $x \leqslant \varepsilon(e_1)$ and $p(e_i) = b_i$ for $i = 0, 1$.* □

Proof. We only need to prove the necessity of conditions (a) and (b′). Its proof follows directly the "only if" proof of Theorem 4.2. Let $\bot$ be the bottom element of X. As in the proof of Theorem 4.2 we apply Corollary 2.3 to the functors

$$\mathsf{Ord}//X \underset{J_1}{\overset{S_1}{\rightleftarrows}} \mathsf{Ord}$$

where S_1 is the forgetful functor and J_1 assigns to each preordered set A the pair $(A, \bot)$, with $\bot(a) = \bot$ for every $a \in A$, and conclude that an effective descent morphism in $\mathsf{Ord}//X$ is in particular effective for descent in Ord, that is, it satisfies condition (a).

To show the necessity of (b′), let $x \in X$ and $b_1 \leqslant b_0$ in B with $x \leqslant \beta(b_1)$. Given a pullback diagram of p and f in $\mathcal{C}_0 \times_{\mathcal{C}_1} \mathcal{C}_2$ with A and A_x as described in the proof of 4.2, both A and $E \times_B A$ do not belong to Ord_X and therefore they do not belong to $\mathsf{Ord}//X$. □

There is another convenient way to express the surjectivity of each $p_x\colon E_x \to B_x$:

Proposition 5.4. *The following conditions on a morphism $p : E \to B$ in $\mathsf{Ord}//X$ are equivalent:*

(i) *$p_x\colon E_x \to B_x$ is surjective for every $x \in X$;*

(ii) *for every $b \in B$ there exists $e \in E$ with $p(e) = b$ and $\varepsilon(e) \sim \beta(b)$.*

Proof. (ii)⇒(i) is obvious and so we only need to prove (i)⇒(ii). Suppose (i) holds. Given $b \in B$, since $b \in B_{\beta(b)}$, there exists $e \in E_{\beta(b)}$ with $p(e) = b$. Then we have $\beta(b) \leqslant \varepsilon(e)$ since e belongs to $E_{\beta(b)}$, and $\varepsilon(e) \leqslant \beta p(e) = \beta(b)$ since p is a morphism in $\mathsf{Ord}//X$. □

Lemma 5.5. *Suppose that, for every $x \in X$, every subset of $\{x' \in X \mid x' \leqslant x\}$ has a largest element. If $p\colon E \to B$ is a pullback stable extremal epimorphism in $\mathsf{Ord}//X$, then $p_x\colon E_x \to B_x$ is surjective for every $x \in X$.*

Proof. Given $b \in B$, consider the pullback diagram

$$\begin{array}{ccc} p^{-1}(b) & \longrightarrow & \{b\} \\ \downarrow & & \downarrow \\ E & \xrightarrow[p]{} & B \end{array}$$

in $\mathsf{Ord}//X$, where the vertical arrow are the inclusion maps, and $\{b\} = (\{b\}, \beta')$ with $\beta'(b) = \beta(b)$; accordingly $p^{-1}(b) = (p^{-1}(b), \varepsilon')$ with $\varepsilon'(e) = \varepsilon(e)$ for each $e \in p^{-1}(b)$. We have $\varepsilon(e) \leqslant \beta(b)$ for each $e \in p^{-1}(b)$, and we define $\beta'' : \{b\} \to X$ by taking $\beta''(b)$ to be a largest element in the set $\{\varepsilon(e) \mid e \in p^{-1}(b)\}$. Then $(p^{-1}(b), \varepsilon') \to (\{b\}, \beta')$, which is the top arrow in our pullback diagram, factors through $(\{b\}, \beta'') \to (\{b\}, \beta')$. Since p is a pullback stable extremal epimorphism, it follows that $(\{b\}, \beta'') \to (\{b\}, \beta')$ is an isomorphism. Therefore $\beta''(b) \sim \beta'(b)$, which implies the existence of $e \in p^{-1}(b)$ with $\varepsilon(e) \sim \beta(b)$ and so completes the proof. □

From Theorem 5.3 and Lemma 5.5, we immediately obtain:

Theorem 5.6. *Suppose that, for every $x \in X$, every subset of $\{x' \in X \mid x' \leqslant x\}$ has a largest element. Then a morphism $p\colon E \to B$ is effective for descent in $\mathsf{Ord}//X$ if and only if:*

(a) *for each $b_2 \leqslant b_1 \leqslant b_0$ in B, there exist $e_2 \leqslant e_1 \leqslant e_0$ in E with $p(e_i) = b_i$ for each $i = 0, 1, 2$;*

(b′) *for each $x \in X$ and each $b_1 \leqslant b_0$ with $x \leqslant \beta(b_1)$, there exist $e_1 \leqslant e_0$ with $x \leqslant \varepsilon(e_1)$ and $p(e_i) = b_i$ for $i = 0, 1$.* □

Remark 5.7. Although Theorem 5.3 is stronger than Theorem 3.9 in [1], we still do not know how far it is from a complete characterization of effective descent morphisms in $\mathsf{Ord}//X$. Theorem 5.6 answers this question, but only under a strong additional condition on X.

References

[1] M. M. Clementino and F. Lucatelli Nunes, *Lax comma categories of ordered sets*, Quaestiones Mathematicae 46 (2023) Issue sup1, 145–159

[2] G. Janelidze and M. Sobral, *Finite preorders and topological descent I*, Journal of Pure and Applied Algebra 175(1-3) (2002), 187–205

[3] G. Janelidze and W. Tholen, *Facets of Descent I*, Applied Categorical Structures 2 (1994), 245–281

[4] S. Lack and R. H. Street, *The formal theory of monads II*, Journal of Pure and Applied Algebra 175(1-3) (2002), 243–265

[5] F. Lucatelli Nunes, *Pseudo-Kan extensions and descent theory*, Theory and Applications of Categories 33 (2018), Paper No. 15, 390–444

[6] S. Mac Lane, *Categories for the Working Mathematician*, Springer 1971; 2nd Edition 1998

[7] R. H. Street, *The formal theory of monads*, Journal of Pure and Applied Algebra 2 (1972), 149–168

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