On super-solutions to fully nonlinear equations with measurable ingredients

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Abstract

We consider super-solutions to fully nonlinear elliptic equations in the presence of measurable ingredients. Our analysis explores the consequences of the weak Harnack inequality, combined with one-sided geometric control, typically related to a generalised maximum principle. We prove regularity estimates for semi-convex L^p-viscosity super-solutions in the Escauriaza regime, both in Hölder and Lipschitz-continuous spaces. We also consider super-solutions satisfying a Hölder-type modulus of continuity from below and establish local Hölder-regularity, both at the level of the function and at the level of the gradient. As a consequence, we study the quality of the diffusion associated with viscosity super-solutions; it stems from the connection of such functions with the fractional Laplacian. We close the paper with a consequence of the weak Harnack inequality concerning the class of viscosity solutions.

Keywords: Weak Harnack inequality; L^p-viscosity solutions; regularity estimates; solvability; Lévy processes.

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1 Introduction

We examine L^p-viscosity super-solutions to

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma(x)\,|Du| = f \quad \text{in} \quad \Omega, \tag{1}$$

where $\mathcal{M}^-_{\lambda,\Lambda} : S(d) \to \mathbb{R}$ is a Pucci operator with ellipticity constants $0 < \lambda \leq \Lambda$, $\gamma \in L^q(\Omega)$, and $f \in L^p(\Omega)$, for some $q > d$ and $p > p_0 > d/2$. Here, $S(d) \sim \mathbb{R}^{d(d+1)/2}$ stands for the space of symmetric matrices of order d, whereas $p_0 = p_0(d, \lambda, \Lambda)$ is the Escauriaza exponent. For an account of the theory of L^d-viscosity solutions, we refer the reader to [8].

The findings in the present paper include three classes of results. As concerns regularity theory, we start by examining semi-convex super-solutions to (1) in the Escauriaza regime. That is, when $f \in L^p(\Omega)$, for $d/2 < p_0 < p < d$. In this setting, we first prove Hölder-regularity estimates. Then we impose a condition on the norm of the source term and obtain Lipschitz continuity. Secondly, we consider L^p-viscosity super-solutions to (1) satisfying a Hölder-type modulus of continuity from below. If such a modulus is of class C^α, we establish Hölder-continuity for the super-solution. In case the modulus of continuity from below is of class $C^{1,\alpha}$, we prove the super-solution is $C^{1,\beta}$-regular, for some $\beta \in (0,1)$ depending on the data of the problem.

Our second contribution relates L^p-viscosity super-solutions to the fractional Laplacian. If a super-solution to (1) has a $C^{1,\alpha}$-modulus of continuity from below, we show it solves an equation driven by a fractional Laplacian of order s. Of particular interest is the dependence of s on the regularity regime of the super-solution. A natural consequence of this result concerns the *quality of the diffusion* underlying super-solutions to (1). Once we prove they solve a fractional Poisson equation, we associate the underlying diffusion with a $2s$-stable Lévy process. Finally, our third set of results is inspired by the work of Luis Caffarelli [5]. We define a class of viscosity solutions for (1) in terms of norms. Then we prove a function satisfying natural geometric properties belongs to such a class.

At the core of our arguments is the combination of two facts: a weak Harnack inequality and a generalised maximum principle (GMP, for short). In general, the GMP is available for sub-solutions. In the presence of those ingredients, one-sided geometric control yields two-sided information. More precisely, let u be a function satisfying a weak Harnack inequality and the GMP; if u has a modulus of continuity from below, one can produce a related modulus of continuity from above.

Working in the context of super-solutions, we resort to versions of the weak Harnack inequality available for (1) in the integrability regime of our interest; see [21]. Meanwhile, in the context of super-solutions, the GMP is not readily available. We prove that different versions of the GMP follow from the moduli of continuity satisfied by the function from below.

The idea of relating the Harnack inequality with a one-side geometric control appears firstly in the work of Luis Caffarelli [5]; see also [6]. In [5], the author establishes an equivalence between the class of viscosity solutions and the set of functions satisfying a Harnack inequality [5, Theorems 1 and 2]. As a corollary to his argument, the author also proves that quadratic control from below yields quadratic control from above for functions satisfying a Harnack inequality.

When it comes to the analysis of super-solutions, and the weak Harnack inequality is available instead, one needs to equip the argument with further geometric control. In our case, it follows from the assumption that a modulus of continuity from below is available for the function. Our rationale is to verify that some classes of moduli of continuity yield a generalised maximum principle. Hence, we prove that semi-convex functions, as well as functions with a Hölder-type modulus of continuity from below, satisfy a GMP. It builds upon the weak Harnack inequality to unlock the consequences of the Harnack inequality available for viscosity solutions.

We notice a priori geometric conditions play an important role in regularity theory. The Monge-Ampère equation is a fundamental instance where this is the case; see [4, 14, 13, 17], to name just a (very) few. Convexity is also an ingredient in the study of the special Lagrangian equation; see for instance [24, 10, 9]. In the context of fully nonlinear uniformly elliptic operators, convexity of the solutions unlocks improved regularity. This is the case in [19, Theorems 3 and 4], where the author establishes $C^{1,1}$-regularity estimates for the (unique) convex viscosity solutions to

$$F(D^2u, Du, u, x) = 0 \qquad \text{in} \qquad \mathbb{R}^d.$$

Here, F is a fully nonlinear elliptic operator satisfying natural requirements. When it comes to quasi-linear problems, one-sided geometric control also unlocks sharp regularity estimates. In [1], the authors show that a function whose p-Laplace is bounded is of class $C^{1,1/p-1)}$ provided it has a $C^{1,1/p-1}$-modulus of continuity from below [1, Theorem 5]. For the interplay of the weak Harnack inequality and the generalised maximum principle in the context of variational problems, we refer the reader to [15].

The regularity of semi-convex super-solutions to (1) has also been pursued in the literature. In the case $p > d$, a sharp regularity estimate appeared in [2]. In that article, the authors prove that semi-convex solutions to (1) are of class $C^{1,\eta}$, where η is a modulus of continuity depending on d, p and q.

Our first contribution concerns the regularity of semi-convex super-solutions in the Escauriaza range; that is, for $f \in L^p(\Omega)$ with $d/2 < p_0 < p < d$. We prove that solutions are locally of class $C^{2-d/p}$, with estimates. Moreover, we consider the additional assumption that there exist $\alpha \in (0,1)$ and $C > 0$ such that

$$\|f\|_{L^p(B_r(x_0))} \leq Cr^{\alpha-1+d/p}, \tag{2}$$

for every $x_0 \in B_1$ and $0 < r < 1$ such that $B_r(x_0) \subset B_1$. Under such a condition, we prove that semi-convex super-solutions are indeed Lipschitz continuous. This result extends to the context of super-solutions some of the findings reported in [12]. In that paper, the authors examine L^p viscosity solutions of fully nonlinear elliptic equations in Escauriaza's regime. Resorting to a truncated Riesz potential, they prove that solutions are Lipschitz continuous and continuously differentiable. Their contribution launches the fully nonlinear counterpart of a theorem due to Elias Stein on the differentiability of functions [23]. The condition in (2) resembles a Morrey-type of inequality and is strictly less smooth than Hölder continuity; see [7, Chapter 8].

In addition, we examine super-solutions satisfying a Hölder-type modulus of continuity from below. In that case, we first prove a generalised maximum principle for super-solutions. By combining this fact with the weak Harnack inequality, we produce a Hölder-type modulus of continuity *from above*. Ultimately, it leads to local regularity estimates in Hölder-spaces.

More concretely, if a super-solution to (1) satisfies a $C^{0,\alpha}$-modulus of continuity from below, it becomes locally of class $C^{0,\beta}$, for some $\beta \in (0,1)$. As an alternative, suppose a super-solution has a $C^{1,\alpha}$-modulus of continuity from below. Then it becomes locally of class $C^{1,\beta}$, for some $\beta \in (0,1)$. As a consequence, we prove that such super-solutions are weak distributional solutions to a fractional-Poisson equation.

The relevance of this consequence is in the understanding of the diffusion encoded by super-solutions to (1). We know the diffusion processes associated with the Laplace operator is governed by a standard Brownian motion. This information follows from the fact that the Laplace operator is the infinitesimal generator of the Brownian motion. In the nonlinear setting, the connection with the underlying stochastic process is more subtle. Mainly because the infinitesimal generator is rather a linear operator. By relating super-solutions to (1) with a fractional Laplacian, we find a lower bound for the quality of the diffusion described by a fully nonlinear model.

We close the paper with a discussion on the class of viscosity solutions. In [5], Luis Caffarelli proved tha a continuous function satisfying a Harnack inequality belongs to a class of viscosity solutions. To some extent, it suggests that being a viscosity solution and satisfying a geometric balance condition - as the one prescribed by the Harnack inequality - are equivalent. We examine this equivalence in the context of the weak Harnack inequality. We define a class of viscosity solutions allowing for explicit dependence on the gradient. Then we prove that a function satisfying the weak Harnack inequality and a generalised maximum principle belongs to this class. As a consequence, we verify semi-convex functions satisfying a weak Harnack inequality are viscosity solutions to some elliptic PDE.

It is worth noting the explicit dependence of the operator on the gradient entails two genuine difficulties. The first one concerns the effects of the super-solution's sub-differential in the argument. When considering gradient-related information, one subtracts an affine function ℓ from the super-solution u. The equation satisfied by $u - \ell$ relates to (1) but involves a source term of the form $f + |D\ell|$. Controlling how $|D\ell|$ conditions the argument becomes paramount.

The second main difficulty stemming from the explicit dependence on the gradient involves the integrability of γ. An interesting aspect of our results regards the effect of $\gamma \in L^q(\Omega)$ on the regularity of super-solutions. If $q > d$, the geometry of the modulus of continuity for the super-solutions does not depend on q. For instance, when proving that super-solutions with a Hölder-continuous modulus from below are C^α-regular, we notice $\alpha \in (0,1)$ does not depend on $q > d$. However, the estimates for the semi-norm of u in C^α depend on $\|\gamma\|_{L^q(\Omega)}$.

The remainder of this paper is organised as follows. Section 2.1 introduces some notation and definition, whereas Section 2.2 gathers auxiliary results. In Section 3, we prove a theorem ensuring the Harnack inequality and the GMP turn one-sided geometric control into two-sided control. Regularity for semi-convex super-solutions in the Escauriaza regime is the subject of Section 4. In Section 5 we examine the regularity of super-solutions satisfying a Hölder-type modulus of continuity. We study diffusion properties of (1) in Section 6 and close the paper with a section relating geometric properties of functions and solvability of fully nonlinear equations.

2 Preliminaries

This section gathers elementary notions and previous results used in the paper. We start by introducing some notation and terminology and recalling a few definitions.

2.1 Notation and basic definitions

We call $\Omega \subset \mathbb{R}$ a domain if it is open, connected and bounded. We denote with $S(d) \sim \mathbb{R}^{d(d+1)/2}$ the space of symmetric matrices of order d. For constants $0 < \lambda \leq \Lambda$, we define the subset $\mathcal{A}_{\lambda,\Lambda} \subset S(d)$ as

$$\mathcal{A}_{\lambda,\Lambda} := \left\{ A \in S(d) \mid \lambda|\xi|^2 \leq A\xi \cdot \xi \leq \Lambda|\xi|^2 \quad \text{for every} \quad \xi \in \mathbb{R}^d \right\}.$$

We start with the extremal Pucci operators.

Definition 1 (Extremal Pucci operators)**.** *Fix constants $0 < \lambda \leq \Lambda$. The extremal Pucci operator $\mathcal{M}^-_{\lambda,\Lambda} : S(d) \to \mathbb{R}$ is given by*

$$\mathcal{M}^-_{\lambda,\Lambda}(M) := \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{Tr}(AM).$$

We also define $\mathcal{M}^+_{\lambda,\Lambda}(M) := -\mathcal{M}^-_{\lambda,\Lambda}(-M)$.

For a list of properties of the operators $\mathcal{M}^\pm_{\lambda,\Lambda}$, see [7, Lemma 2.10]. Typically, one is interested in fully nonlinear elliptic equations of the form

$$F(D^2u, Du, u, x) = f \quad \text{in} \quad \Omega, \tag{3}$$

where $F : S(d) \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a (λ, Λ)-elliptic operator and $f \in L^p(\Omega)$, for $d/2 < p$. We suppose F satisfies the structural condition

$$\begin{aligned} \mathcal{M}^-_{\lambda,\Lambda}(M-N) &- \gamma(x)|\xi-\zeta| - \omega(|r-s|) \\ &\leq F(M,\xi,r,z) - F(N,\zeta,s,x) \\ &\qquad \leq \mathcal{M}^+_{\lambda,\Lambda}(M-N) + \gamma(x)|\xi-\zeta| + \omega(|r-s|), \end{aligned} \tag{4}$$

for every M, $N \in S(d)$, $\xi, \zeta \in \mathbb{R}^d$ and r, $s \in \mathbb{R}$, where $\gamma \in L^q(\Omega)$, for $q > d$, and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity. Next, we introduce the definition of L^p-viscosity solution.

Definition 2 (L^p-viscosity solution)**.** *Let $p > d/2$. We say that $u \in L^p(\Omega)$ is an L^p-viscosity sub-solution to* (3) *if, whenever $\phi \in W^{2,p}_{loc}(\Omega)$ is such that $u - \phi$ has a local minimum at $x_0 \in \Omega$, we have*

$$\operatorname{ess}\limsup_{x \to x_0} \big(F(D^2\phi(x), D\phi(x), u(x), x) - f(x)\big) \geq 0.$$

We say that $u \in C(\Omega)$ is an L^p-viscosity super-solution to (3) *if, whenever $\phi \in W^{2,p}_{loc}(\Omega)$ is such that $u - \phi$ has a local maximum at $x_0 \in \Omega$, we have*

$$\operatorname{ess}\liminf_{x \to x_0} \big(F(D^2\phi(x), D\phi(x), u(x), x) - f(x)\big) \leq 0.$$

If $u \in C(\Omega)$ is an L^p-viscosity sub-solution and an L^p-viscosity super-solution to (3)*, we say it is an L^p-viscosity solution to* (3)*.*

Remark 1 (C-viscosity solutions)**.** If $F = F(\,\cdot\,,\,\cdot\,,\,\cdot\,,x)$ is continuous with respect to $x \in \Omega$, test functions $\phi \in C^2(\Omega)$ become available in Definition 2. In this case, one recovers the usual definition of C-viscosity solution; see [11].

The connection between (3) and the extremal operators is clear, provided F satisfies (4). To profit from this link, we define the *class of viscosity sub/super-solutions.*

Definition 3 (Class of viscosity solutions)**.** *Let $0 < \lambda \leq \Lambda$, $q > d$, and $p > d/2$. Let $\gamma \in L^q(\Omega)$ and $f \in L^p(\Omega)$. We define the class of L^p-viscosity super-solutions $\overline{S}(\lambda, \Lambda, f)$ as the set of all L^p-viscosity super-solutions to*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma(x)|Du| = f \quad \text{in} \quad \Omega.$$

Analogously, we define the class of L^p-viscosity sub-solutions $\underline{S}(\lambda, \Lambda, f)$ as the set of all L^p-viscosity sub-solutions to

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \gamma(x)|Du| = f \quad \text{in} \quad \Omega.$$

The class of L^p-viscosity solutions is $S(\lambda, \Lambda, f) := \overline{S}(\lambda, \Lambda, f) \cap \underline{S}(\lambda, \Lambda, f)$.

Although the definition of L^p-viscosity solution requires $p > d/2$, we work under the stricter condition $p > p_0 \geq d/2$. Here, $p_0 = p_0(\lambda, \Lambda, d)$ is such that, for $p > p_0$, the Alexandrxoff-Bakelman-Pucci estimates are available for L^p-viscosity solutions. See, for instance, [8, 20]; see also [16]. We proceed by stating the definition of a semi-convex function.

Definition 4 (Semi-convex functions)**.** *Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a modulus of continuity. We say the $u : \Omega \to \mathbb{R}$ is ω-semi-convex if*

$$u(tx + (1-t)y) \leq tu(x) + (1-t)u(y) + t(1-t)|x-y|\omega(|x-y|),$$

for any x, $y \in \Omega$ and every $0 \leq t \leq 1$.

We close this section with the definition of sub-differentials for ω-semi-convex functions.

Definition 5 (Sub-differential for semi-convex functions)**.** *Let $u : \Omega \longrightarrow \mathbb{R}$ be an ω-semi-convex function. We denote with $\partial_\omega u(x)$ its sub-differential at $x \in \Omega$ and define it as*

$$\partial_\omega u(x) := \{P \in \mathbb{R} \mid u(y) \geq u(x) + \langle P, y-x\rangle - |x-y|\omega(|x-y|),\ \forall y \in B_1\}.$$

In the sequel, we collect preliminary results used in the paper.

2.2 Auxiliary results

This section gathers former results on L^p-viscosity solutions and simpler facts we use further. For simplicity, we set $\Omega \equiv B_1$. We start by recalling the weak Harnack inequality.

Proposition 1 (Weak Harnack inequality)**.** *Let $u \in C(B_1)$ be a non-negative L^p-viscosity super-solution to*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma(x)|Du| = f \quad \text{in} \quad B_1, \tag{5}$$

where $\gamma \in L^q(B_1)$ and $f \in L^p(B_1)$. The following statements hold.

1. *Suppose $q > d$ and $q \geq p \geq d$. For every $x_0 \in B_1$ and $0 < \rho < 1$ such that $B_\rho(x_0) \subset B_1$, there exist positive constants $\varepsilon = \varepsilon(d, \lambda, \Lambda)$ and $C = C\left(d, \lambda, \Lambda, q, \rho^{1-d/q}\left\|\gamma\right\|_{L^q(B_1)}\right)$ such that*

$$\left(⨍_{B_\rho(x_0)} u^\varepsilon \mathrm{d}x\right)^{\frac{1}{\varepsilon}} \leq C\left(\inf_{x \in B_\rho(x_0)} u(x) + \rho\left\|f\right\|_{L^d(B_{3\rho/2}(x_0))}\right).$$

2. *Suppose $q > d > p > p_0$. Let $x_0 \in B_1$ and $0 < \rho < 1$ be such that $B_\rho(x_0) \subset B_1$. There exist positive constants $\varepsilon = \varepsilon(d, \lambda, \Lambda)$, $C = C\left(d, \lambda, \Lambda, q, \rho^{1-d/q} \left\|\gamma\right\|_{L^q(B_1)}\right)$ and $\overline{C} = \overline{C}\left(d, \lambda, \Lambda, \left\|\gamma\right\|_{L^q(B_1)}\right)$ such that*

$$\left(\fint_{B_\rho(x_0)} u^\varepsilon \mathrm{d}x\right)^{\frac{1}{\varepsilon}} \leq C\left(\inf_{x\in B_\rho(x_0)} u(x) + \overline{C}\rho^{2-d/p}\left\|f\right\|_{L^p(B_{3\rho/2}(x_0))}\right).$$

For a proof of Proposition 1 we refer the reader to [21, Corollary 4.8]; see also Theorem 4.5 and Theorem 4.7 in [21]. In addition to the weak Harnack inequality, the generalised maximum principle plays an important role in our arguments. In the sequel, we recall three variants of this result in the context of arbitrary functions.

Proposition 2 (Maximum principle for semi-convex functions)**.** *Let $u \in L^1(B_1)$ be a ω-semi-convex function. Let $x_0 \in B_1$ and $0 < \rho < 1$ be such that $B_\rho(x_0) \subset B_1$. For every $p > 0$ there exists a positive constant $C = C(d, p, \omega)$ such that*

$$\sup_{x\in B_{\rho/2}(x_0)} |u(x)| \leq C\left[\left(\fint_{B_\rho(x_0)} |u|^p \mathrm{d}x\right)^{\frac{1}{p}} + \rho\omega(\rho)\right].$$

For the proof of Proposition 2 we refer the reader to [2, Proposition 8.2]. We continue with versions of the maximum principle for functions. They follow from *uniform* moduli of continuity touching the function from below. Before proceeding, we state an auxiliary result; see [18, Lemma 6.1].

Lemma 1 (An auxiliary inequality)**.** *Fix $\rho > 0$ and let $\Phi : [\rho, \rho/2] \to \mathbb{R}$ be a non-negative bounded function. Suppose there exist $0 < \rho/2 < t < s < \rho$, $\theta \in (0,1)$, $C > 0$ and $a > 0$ satisfying*

$$\Phi(t) \leq \theta\Phi(s) + \frac{C}{(s-t)^a} + C.$$

Then there exists $\overline{C} > 0$ such that

$$\Phi(\rho/2) \leq \overline{C}\left(\frac{2C}{\rho^a} + C\right).$$

The next two propositions provide a generalised maximum principle for functions satisfying a modulus of continuity from below. We notice that a version of these results appeared in the preprint [22]. We include them here because our statements in the present paper are simpler, starker, and their proofs are more clear-cut.

Proposition 3 (Maximum principle for one-sided regular functions I)**.** *Let $u \in L^\infty(B_1)$ and consider a modulus of continuity $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$. Suppose that*

$$u(x) - u(y) \geq -\sigma(|x-y|)$$

for every x, $y \in B_1$. Let $x_0 \in B_1$ and $0 < \rho < 1$ be such that $B_\rho(x_0) \subset B_1$. For every $p > 0$ there exists a positive constant $C = C(d,p)$ such that

$$\sup_{x\in B_{\rho/2}(x_0)} |u(x)| \leq C \left[\left(⨍_{B_\rho(x_0)} |u(x)|^p \mathrm{dx} \right)^{\frac{1}{p}} + \sigma(\rho) \right].$$

Proof. For clarity, we split the proof into 2 steps. The first one accounts for the case $p \geq 1$.

Step 1 - Let x, $y \in B_\rho(x_0)$. By assumption, we get

$$u(y) \leq u(x) + \sigma(|x-y|) \leq u(x) + \sigma(2\rho),$$

where the second inequality stems from the monotonicity of σ. Integrating the previous inequality over $B_\rho(x_0)$ for the Lebesgue measure $\mathrm{d}x$ one gets

$$u(y) \leq ⨍_{B_\rho(x_0)} |u(x)| \mathrm{d}x + \sigma(2\rho). \tag{6}$$

Now we reverse the inequality stemming from the assumption. That is, we use

$$u(y) \geq u(x) - \sigma(|x-y|).$$

Integrating both sides of the former inequality, one obtains

$$u(y) \geq ⨍_{B_\rho(x_0)} u(x) \mathrm{d}x - \sigma(2\rho) \geq -\left(⨍_{B_\rho(x_0)} |u(x)| \mathrm{d}x - \sigma(2\rho) \right). \tag{7}$$

By combining (6) e (7), and noticing that $x \in B_\rho(x_0)$ is arbitrary, we have

$$\sup_{x\in B_\rho(x_0)} |u(x)| \leq ⨍_{B_\rho(x_0)} |u(x)| \mathrm{d}x + \sigma(2\rho) \leq C \left[⨍_{B_\rho(x_0)} |u(x)| \mathrm{d}x + \sigma(\rho) \right].$$

Therefore, the result follows for $p = 1$. In case $p > 1$, a straightforward computation yields

$$⨍_{B_\rho(x_0)} |u(x)| \mathrm{d}x \leq \left(⨍_{B_\rho(x_0)} |u(x)|^p \mathrm{d}x \right)^{\frac{1}{p}};$$

hence,

$$\sup_{x\in B_\rho(x_0)} |u(x)| \leq C\left[\left(\fint_{B_\rho(x_0)} |u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} + \sigma(\rho)\right].$$

The next step addresses the case $0 < p < 1$.

Step 2 - We start by fixing $0 < \rho/2 < t < s < \rho$, where $s - t \leq t$. The previous step ensures that

$$\begin{aligned}\sup_{x\in B_t(x_0)} |u(x)| &\leq C\left(\frac{1}{t^d}\int_{B_t(x_0)} |u(y)|\mathrm{d}y + \sigma(\rho)\right)\\ &\leq C\left(\sup_{y\in B_s(x_0)} |u(y)|^{1-p}\frac{1}{(s-t)^d}\int_{B_s(x_0)} |u(y)|^p\mathrm{d}y + \sigma(\rho)\right).\end{aligned}$$

Now, a simple application of Young's inequality with ε implies

$$\sup_{B_t(x_0)} |u| \leq \frac{1}{\pi}\sup_{B_s(x_0)} |u| + C\left(\frac{1}{(s-t)^{d/p}}\left(\int_{B_s(x_0)} |u(y)|^p\mathrm{d}y\right)^{\frac{1}{p}} + \sigma(\rho)\right).$$

An application of Lemma 1 yields

$$\begin{aligned}\sup_{x\in B_{\rho/2}(x_0)} |u(x)| &\leq C\left[\left(\frac{1}{(\rho-\rho)2^{d/p}}\int_{B_\rho(x_0)} |u(y)|^p\mathrm{d}y\right)^{\frac{1}{p}} + \sigma(\rho)\right]\\ &\leq C\left[\left(\fint_{B_\rho(x_0)} |u(y)|^p\mathrm{d}y\right)^{\frac{1}{p}} + \sigma(\rho)\right],\end{aligned}$$

which completes the proof for $p \in (0, 1)$. By combining the conclusions in Step 1 and Step 2, one completes the proof. □

In addition, we provide a generalised maximum principle for functions satisfying a uniform modulus of continuity from below *at the level of sub-differentials.*

Proposition 4 (Maximum principle for one-sided regular functions II)**.** *Let $u \in C(B_1) \cap L^\infty(B_1)$ and consider a modulus of continuity $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$. Suppose that*

$$u(x) - u(y) - P_y \cdot (x - y) \geq -|x - y|\sigma(|x, y|)$$

for every x, $y \in B_1$ and some $P_y \in \mathbb{R}^d$. Let $x_0 \in B_1$ and $0 < \rho < 1$ be such that $B_\rho(x_0) \subset B_1$. For every $p > 0$ there exists a positive constant $C = C(d, p)$ such *that*

$$\sup_{x\in B_{\rho/2}(x_0)} |u(x)| \leq C\left[\left(⨍_{B_\rho(x_0)} |u(x)|^p \mathrm{dx}\right)^{\frac{1}{p}} + \rho\sigma(\rho)\right].$$

Proof. We start by supposing that $u \in C^1(B_1)$, which allows us to identify P_y with $Du(y)$. In this context, we prove the proposition by producing upper and lower bounds for u. Then we resort to a convolution argument to cover the general case. For ease of presentation, we split the proof into three steps.

Step 1 - Suppose $u \in C^1(B_1)$; hence $P_y = Du(y)$ and we infer

$$u(x) - u(y) - Du(y)\cdot(x-y) \geq -|x-y|\sigma(|x,y|), \tag{8}$$

for every x, $y \in B_\rho(x_0)$. Replace y in (8) with $z \in B_{\rho/2}(x_0)$. Integrate both sides of (8) over $B_{\rho/2}(z)$ for $\mathrm{d}x$ and multiply the resulting inequality by $|B_{\rho/2}(z)|^{-1}$. Then

$$\begin{aligned} u(z) &\leq ⨍_{B_{\rho/2}(z)} |u(x)|\mathrm{d}x + 2\rho\sigma(2\rho) - ⨍_{B_{\rho/2}(z)} Du(z)\cdot(x-z)\mathrm{d}x \\ &\leq ⨍_{B_{\rho/2}(z)} |u(x)|\mathrm{d}x + 2\rho\sigma(2\rho), \end{aligned}$$

where the second inequality follows from the mean-value formula for harmonic functions. Now,

$$⨍_{B_{\rho/2}(z)} |u(x)|\mathrm{d}x = \frac{C}{\rho^d}\int_{B_{\rho/2}(z)} |u(x)|\mathrm{d}x \leq \frac{\overline{C}}{\rho^d}\int_{B_\rho(x_0)} |u(x)|\mathrm{d}x.$$

Because $z \in B_{\rho/2}(x_0)$ was taken arbitrarily, one concludes

$$u(x) \leq C\left(⨍_{B_\rho(x_0)} |u(x)|\mathrm{d}x + \rho\sigma(\rho)\right), \tag{9}$$

for every $x \in B_{\rho/2}(x_0)$. The next step produces a lower bound for $u(x)$.

Step 2 - Let x, $y \in B_\rho(x_0)$; hence

$$u(x) \geq u(y) + Du(y)\cdot(x-y) - 2\rho\sigma(\rho). \tag{10}$$

Now, consider $\xi \in C_0^\infty(B_\rho(x_0))$ defined such that $0 \leq \xi \leq 1$ in $B_\rho(x_0)$, with $|D\xi| \leq C\rho^{-1}$, for some $C > 0$. Multiply (10) by $\xi(y)$ and integrate over $B_\rho(x_0)$

with respect to $\mathrm{d}y$ to obtain

$$\begin{aligned}\int_{B_\rho(x_0)} u(x)\xi(y)\mathrm{d}y \geq \int_{B_\rho(x_0)} u(y)\xi(y)\mathrm{d}y + \int_{B_\rho(x_0)} Du(y)\cdot(x-y)\xi(y)\mathrm{d}y \\ - 2\rho\sigma(2\rho)\int_{B_\rho(x_0)}\xi(y)\mathrm{d}y.\end{aligned} \tag{11}$$

Clearly,

$$0 \leq \int_{B_\rho(x_0)} \xi(y)\mathrm{d}y \leq |B_\rho(x_0)|.$$

Moreover, integration by parts yields

$$\begin{aligned}\int_{B_\rho(x_0)} Du(y)\cdot(x-y)\xi(y)\mathrm{d}y &= -\int_{B_\rho(x_0)} u(y)\mathrm{div}_y\left((x-y)\xi(y)\right)\mathrm{d}y \\ &\leq \int_{B_\rho(x_0)} |u(y)|\mathrm{d}y,\end{aligned}$$

where the inequality follows from the condition on ξ. Gathering the previous inequalities with (11), we obtain

$$u(x) \geq -C\left(\fint_{B_\rho(x_0)} |u(y)|\mathrm{d}y + \rho\sigma(\rho)\right) \tag{12}$$

By combining (9) and (12) one proves the proposition in the case $p=1$ and $u \in C^1(B_1)$. The next step accounts for the general situation.

Step 3 - Once the result has been obtained for $p=1$, we argue as in the proof of Proposition 3 to obtain the case $p \in (0,\infty)$. A standard regularisation argument, through the convolution of $u \in C(B_1)\cap L^\infty(B_1)$ with a standard mollifying kernel, completes the proof. $\square$

We continue by noting the sum of a semi-convex function and an affine one is a semi-convex function. This is the content of the following lemma.

Lemma 2. *Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a modulus of continuity and suppose that $u : \Omega \to \mathbb{R}$ is a ω-semi-convex function. Let $\ell : \mathbb{R}^d \to \mathbb{R}$ be an affine map. Then $u \pm \ell$ is an ω-semi-convex function.*

Proof. Because $\ell = \ell(x)$ is an affine function, we can write it as

$$\ell(x) := a + \mathrm{b}\cdot x,$$

where $a \in \mathbb{R}$ and $\mathrm{b} \in \mathbb{R}^d$. Let $t \in [0,1]$ be fixed, but arbitrary. Take x, $y \in \Omega$

and notice that

$$\begin{aligned}(u \pm \ell)(tx + (1-t)y) &\le tu(x) + (1-t)u(y) + t(1-t)|x-y|\omega(|x-y|) \\ &\quad \pm [a + \mathrm{b} \cdot (tx + (1-t)y)] \\ &= t(u \pm \ell)(x) + (1-t)(u \pm \ell)(y) \\ &\quad + t(1-t)|x-y|\omega(|x-y|),\end{aligned}$$

which completes the proof. □

The next two lemmas provide estimates for *gradient-like* quantities appearing in our analysis. First, we recall bounds for the sub-differentials of an ω-semi-convex function. Then we estimate the slope of hyperplanes touching arbitrary functions in $L^\infty(B_1)$.

Lemma 3 (Estimates for subdifferentials)**.** *Let $u \in L^\infty(B_1)$ be ω-semi-convex. Then $\partial_\omega u(x)$ is non-empty, compact, and convex for every $x \in B_1$. In addition, for any compact set $K \subset B_1$, we have*

$$\sup_{x \in B_1} \sup_{P \in \partial_\omega u(x)} |P| \le 2\left(\frac{\|u\|_{L^\infty(B_1)}}{dist(K, \partial B_1)} + \omega(dist(K, \partial B_1))\right).$$

A proof of this lemma can be found in [3, Proposition 5.1]. We close this section with a lemma comparing the opening of distinct paraboloids touching at a given point $x_0 \in B_1$. For its proof, see [22, Lemma 2].

Lemma 4 (Comparable openings)**.** *Let $A > 0$ and $x_0 \in B_1$. Let $P(x) := A|x - x_0|^2$ and suppose Q is a paraboloid touching P from below, at x_0. Then*

$$A \ge \left\| \left(D^2 Q(x_0)\right)^+ \right\|.$$

We proceed with an abstract result.

3 An abstract result

In this section, we consider functions satisfying a weak Harnack inequality and a generalised maximum principle. We suppose such functions satisfy a modulus of continuity from below. Then the weak Harnack inequality and the maximum principle combine to flip the modulus of continuity, yielding geometric control from above.

To make matters precise, we introduce two assumptions. The first one is a weak Harnack inequality.

Assumption 3.1 (Weak Harnack inequality)**.** *Let $u \in L^1(B_1)$ be a non-negative function. Let $\rho \in (0,1)$ and $x_0 \in B_1$ be such that $B_\rho(x_0) \subset B_1$. We suppose u satisfies*

$$\left(\fint_{B_\rho(x_0)} u^\varepsilon \mathrm{d}x \right)^{\frac{1}{\varepsilon}} \leq C \left(\inf_{x \in B_\rho(x_0)} u(x) + k(\rho) \right),$$

for some $C > 0$ and $\varepsilon > 0$, where $k = k(\cdot)$ is a modulus of continuity.

Our second assumption is a generalised maximum principle with a modulus of continuity.

Assumption 3.2 (Generalised maximum principle)**.** *Let $u \in C(B_1)$, and $p > 0$. Let $\rho \in (0,1)$ and $x_0 \in B_1$ be such that $B_\rho(x_0) \subset B_1$. There exist $C > 0$ such that*

$$\sup_{x \in B_{\rho/2}(x_0)} u(x) \leq C \left(\left(\fint_{B_\rho(x_0)} u^p \mathrm{d}x \right)^{\frac{1}{p}} + \sigma(\rho) \right),$$

where $\sigma = \sigma(\cdot)$ is a modulus of continuity.

Theorem 1. *Let $u \in L^\infty_{\text{loc}}(B_1)$ and $x_0 \in B_{1/2}$. Let $\ell_{x_0}(\cdot)$ be an affine function such that $\ell_{x_0}(x_0) = u(x_0)$. Suppose that $(u - \ell_{x_0}(x))$ satisfies Assumptions 3.1-3.2. Suppose further there exists a modulus of continuity $\psi(\cdot)$ for which*

$$\inf_{B_\rho(x_0)} (u - \ell_{x_0}(x)) \geq -\psi(\rho) \tag{13}$$

for every $\rho \in (0, 1/2)$. Then there exists $C > 0$ such that

$$\sup_{B_{\rho/4}(x_0)} |u - \ell_{x_0}| \leq C \left(\sigma(\rho) + k(\rho) + \psi(\rho) \right).$$

Proof. Start by fixing $x_0 \in B_{1/2}$ and taking $\rho > 0$ such that $B_\rho(x_0) \subset B_1$. For such a choice, we define $v : B_1 \to \mathbb{R}$ as

$$v(x) := u(x) - \ell_{x_0}(x) + \psi(\rho).$$

Because of (13), we notice v is non-negative; in addition, since $u - \ell_{x_0}$ satisfies Assumption 3.1 at $x_0 \in B_1$ we obtain

$$\begin{aligned} \left(\fint_{B_{\rho/2}(x_0)} v^\epsilon \mathrm{d}x \right)^{\frac{1}{\epsilon}} \leq & C \left(\inf_{B_{\rho/2}(x_0)} v + k(\rho) \right) \\ \leq & C \left(v(x_0) + k(\rho) \right) \\ \leq & C \left(\psi(\rho) + k(\rho) \right). \end{aligned} \tag{14}$$

In addition, Assumption 3.2 yields

$$
\begin{aligned}
\|v\|_{L^\infty(B_{\rho/4}(x_0))} &\le C\left[\left(\fint_{B_{\rho/2}(x_0)} |v|^p \mathrm{dx}\right)^{\frac{1}{p}} + \sigma(\rho)\right] \\
&\le C\left[\left(\fint_{B_{\rho/2}(x_0)} |v|^\varepsilon \mathrm{dx}\right)^{\frac{1}{\varepsilon}} + \sigma(\rho)\right],
\end{aligned}
\tag{15}
$$

Now, combining (14) and (15) we get

$$
\sup_{B_{\rho/4}(x_0)} v \le C\left(\psi(\rho) + k(\rho) + \sigma(\rho)\right). \tag{16}
$$

The definition of v implies

$$
\sup_{x\in B_{\rho/4}(x_0)} u(x) - \ell(x) \le \sup_{x\in B_{\rho/4}(x_0)} v(x),
$$

which builds upon (16) and completes the proof. □

Remark 2 (Regularity results via flipping)**.** Notice that, once we flip the modulus touching the function from below, we get a modulus from above that has a contribution from three isolated moduli. The one touching from below, the one from Assumption 3.1, and another coming from Assumption 3.2. It means the resulting modulus, and the regularity of the function, stems from the less regular of them. This information depends on the conditions of the problem (see Figure 1).

Remark 3 (One-sided regularity implies two-sided regularity)**.** In Theorem 1, we suppose that $u - \ell$ satisfies Assumptions 3.1-3.2, and has a modulus of continuity from below. We conclude that such a modulus of continuity flips, yielding a geometric control from above. However, had we supposed $-u + \ell$ to satisfy Assumptions 3.1-3.2, a modulus of continuity from above would generate geometric control from below. Suppose $-u + \ell$ satisfies Assumptions 3.1 and 3.2, and suppose further

$$
\sup_{B_\rho(x_0)} (u(x) - \ell_{x_0}(x)) \le \psi(\rho)
$$

where $\psi(\cdot)$ is a modulus of continuity, with $0 < \rho << 1$ and $x_0 \in B_1$ such that $B_\rho(x_0) \subset B_1$. Define the auxiliary function

$$
w(x) := \psi(\rho) - u(x) + \ell_{x_0}(x),
$$

and notice $w \geq 0$. Arguing as before we show that

$$\sup_{B_{\rho/4}(x_0)} (-u(x) + \ell_{x_0}(x)) \leq C(\sigma(\rho) + k(\rho) + \psi(\rho)).$$

That is

$$\inf_{B_{\rho/4}(x_0)} (u(x) - \ell_{x_0}(x)) \geq -C(\sigma(\rho) + k(\rho) + \psi(\rho)).$$

In conclusion, if both $u - \ell$ and $\ell - u$ satisfy Assumptions 3.1 and 3.2, then one-sided geometric control yields two-sided geometric control.

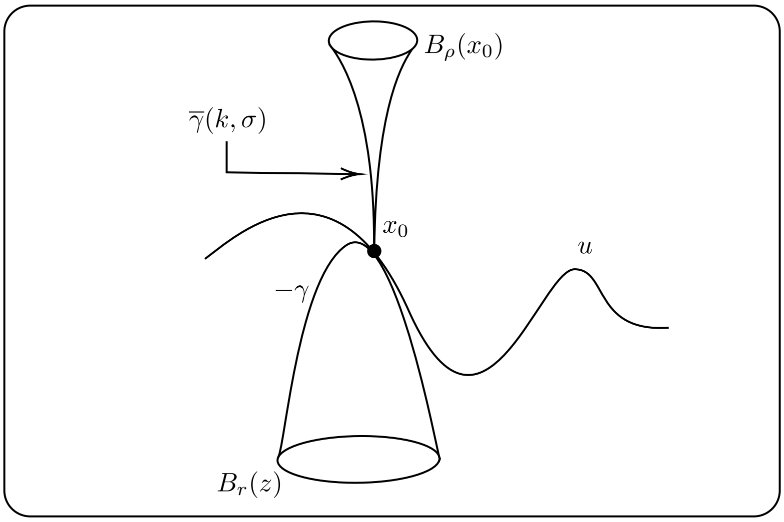

Fig. 1: Given a function $u \in L^1(B_1)$, we consider the modulus of continuity $\gamma(\cdot)$, providing geometric information from the function from below at $x_0 \in B_1$. If u satisfies Assumptions 3.1-3.2, Theorem 1 ensures the existence of a modulus of continuity $\overline{\gamma}(\cdot)$. However, $\overline{\gamma}(\cdot)$ depends also on $k(\cdot)$ and $\sigma(\cdot)$. Consequently, the resulting modulus $\overline{\gamma}$ is as smooth as the less regular among γ, σ and k.

4 Regularity for semi-convex super-solutions in the Escauriaza regime

In the sequel, we examine the regularity of semi-convex super-solutions to (1) in the case $f \in L^p(B_1)$, with $d/2 < p_0 < p < d$.

Theorem 2 (Hölder regularity)**.** *Let $\gamma \in L^q(B_1)$ and $f \in L^p(B_1)$, for some $q > d > p > p_0 \geq d/2$. Let $u \in \overline{S}(\lambda, \Lambda, \gamma, f)$, where $0 < \lambda \leq \Lambda$ are fixed, though arbitrary. Suppose u is ω-semi-convex for some modulus of continuity ω. Then*

$u \in C^{2-\frac{d}{p}}_{\text{loc}}(B_1)$. *In addition, for every* $B \Subset B_1$, *there exists a positive constant* $C = C\left(d, \lambda, \Lambda, \|\gamma\|_{L^q(B_1)}, \text{dist}(B, \partial B_1)\right)$ *such that*

$$\|u\|_{C^{2-\frac{d}{p}}(B)} \leq C\left(1 + \omega(1) + \|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right).$$

Proof. Fix $x_0 \in B_{1/2}$, arbitrary. We first notice that $u - u(x_0) \in \overline{S}(\lambda, \Lambda, \gamma, f)$. For the sake of completeness, fix $\rho \in (0, 1/2)$ such that $B_\rho(x_0) \subset B_1$. Hence, the weak Harnack inequality in Proposition 1 is available for $u - u(x_0)$ in $B_\rho(x_0)$. Moreover, because $u - u(x_0)$ is ω-semi-convex, Proposition 2 yields a generalised maximum principle for $u - u(x_0)$, with the appropriate exponent. We continue by producing a modulus of continuity from below for $u - u(x_0)$ and applying Theorem 1. For ease of presentation, we split the remainder of the proof into 2 steps.

Step 1 - Because $u - u(x_0)$ is semi-convex, Lemma 3 ensures the existence of $P \in \partial_\omega[u - u(x_0)](x_0)$. Therefore

$$u(x) - u(x_0) \geq P \cdot (x - x_0) - |x - x_0|\omega(|x - x_0|), \tag{17}$$

for every $x \in B_1$. For every $\rho \in (0, 1/16)$ and $x \in B_\rho(x_0)$, we have

$$\begin{aligned} P \cdot (x - x_0) - |x - x_0|\omega(|x - x_0|) &\leq (|P| + \omega(|x - x_0|))\,|x - x_0| \leq (|P| + \omega(1))\rho \\ &\leq (|P| + \omega(1))\rho \\ &\leq C\left(\|u\|_{L^\infty(B_1)} + \omega(1)\right)\rho, \end{aligned}$$

where the second inequality stems from the estimate in Lemma 3. Combining the former inequality with (17) and taking the infimum over $B_\rho(x_0)$, one obtains

$$\inf_{x \in B_\rho(x_0)} (u(x) - u(x_0)) \geq -C\left(\|u\|_{L^\infty(B_1)} + \omega(1)\right)\rho, \tag{18}$$

producing a modulus of continuity for $u - u(x_0)$ at $x_0 \in B_{1/2}$.

Step 2 - Propositions 1 and 2 build upon (18) in the context of Theorem 1 to yield

$$\begin{aligned} \sup_{x \in B_{\rho/2}(x_0)} u(x) - u(x_0) &\leq C\left[\rho\left(\omega(1) + \|u\|_{L^\infty(B_1)}\right) + \rho^{2-\frac{d}{p}}\|f\|_{L^p(B_1)}\right] \\ &\leq C\left(\omega(1) + \|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right)\rho^{2-\frac{d}{p}}. \end{aligned}$$

Let x, $y \in B_{1/2}$. If $|x - y| < 1/16$, set $\rho := |x - y|$ and notice there exists

$x_0 \in B_{1/2}$ such that $|x - x_0| < \rho$ and $|y - x_0| < \rho$. Hence,

$$|u(x) - u(y)| \leq C|x - y|^{2-\frac{d}{p}}. \tag{19}$$

If $|x - y| > 1/16$, notice that

$$|u(x) - u(y)| \leq 2\,\|u\|_{L^\infty(B_1)}\,|x - y|^{2-\frac{d}{p}}\,|x - y|^{\frac{d}{p}-2} \leq C\,|x - y|^{2-\frac{d}{p}}. \tag{20}$$

By combining (19) and (20), one completes the argument and finishes the proof.
□

We conclude this section with a corollary to the Theorem 2. It works under an additional assumption on the source term $f \in L^p(B_1)$ and establishes local Lipschitz continuity for semi-convex super-solutions. Indeed, we require the norm of f in L^p to have a *Hölder-like* behaviour. To make matters precise we introduce a further assumption.

Assumption 4.1 (Modulus of continuity for the L^p-norm of f)**.** *Let $d/2 \leq p_0 < p < d$, and $f \in L^p(B_1)$. There exist $C_f > 0$ and $\alpha \in (0,1)$ such that*

$$\left(⨍_{B_\rho(x_0)} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq C_f \rho^{\alpha - 1},$$

for every $x_0 \in B_1$ and $\rho \in (0,1)$ such that $B_\rho(x_0) \subset B_1$.

Under Assumption 4.1, one can improve the findings in Theorem 2. This is the content of the next corollary.

Corollary 1 (Lipschitz continuity)**.** *Let $\gamma \in L^q(B_1)$ and $f \in L^p(B_1)$, for some $q > d > p > p_0 \geq d/2$. Let $u \in \overline{S}(\lambda, \Lambda, \gamma, f)$, where $0 < \lambda \leq \Lambda$ are fixed, though arbitrary. Suppose u is ω-semi-convex for some modulus of continuity ω. Suppose further Assumption 4.1 is in force. Then u is locally Lipschitz continuous in B_q. In addition, for every $B \Subset B_1$, there exists a positive constant $C = C\left(d, \lambda, \Lambda, \|\gamma\|_{L^q(B_1)}, \mathrm{dist}(B, \partial B_1)\right)$ such that*

$$|u(x) - u(y)| \leq C\left(1 + \omega(1) + \|u\|_{L^\infty(B_1)} + C_f\right)|x - y|,$$

for every x, $y \in B$.

Proof. The argument follows along the same lines as the proof of Theorem 2. The only change appears at the beginning of Step 2. Indeed, Assumption 4.1

implies

$$\begin{aligned}\sup_{x\in B_{\rho/2}(x_0)} u(x)-u(x_0) &\leq C\left[\rho\left(\omega(1)+\|u\|_{L^\infty(B_1)}\right)+\rho^{2-\frac{d}{p}}\|f\|_{L^p(B_{3\rho/2}(x_0))}\right]\\ &\leq C\left[\rho\left(\omega(1)+\|u\|_{L^\infty(B_1)}\right)+C_f\rho^{1+\alpha}\right]\\ &\leq C\left(\omega(1)+\|u\|_{L^\infty(B_1)}+C_f\right)\rho.\end{aligned}$$

Once this inequality is available, one follows the proof of Theorem 2 to complete the argument. □

The next section reports regularity estimates for super-solutions with a Hölder-type modulus of continuity from below.

5 Flipping Hölder continuity for supersolutions

In what follows, we consider super-solutions satisfying a (local) modulus of continuity from below. Our analysis concerns Hölder-types of moduli of continuity (both at the level of the function and at the level of the sub-differentials).

We prove that one-sided control implies two-sided control, ultimately yielding a regularity result for super-solutions that are Hölder-regular from below. Our first theorem covers the Escauriaza range, yielding a Hölder-continuity result.

Theorem 3 (Hölder-continuity in the Escauriaza regime)**.** *Let $\gamma\in L^q(B_1)$ and $f\in L^p(B_1)$, for some $q>d>p>p_0\geq d/2$. Let $u\in\overline{S}(\lambda,\Lambda,\gamma,f)$, where $0<\lambda\leq\Lambda$ are fixed, though arbitrary. Fix $\delta\in(0,1]$. For $x_0\in B_{1/2}$, suppose there exists a constant $C_{x_0}>0$ such that*

$$\inf_{x\in B_\rho(x_0)}(u(x)-u(x_0))\geq -C_{x_0}\rho^\delta, \tag{21}$$

for every $\rho\in(0,1/2)$. Then there exists $C>0$ such that

$$\sup_{x\in B_{\rho/2}(x_0)}|u(x)-u(x_0)|\leq C\left(C_{x_0}+\|f\|_{L^p(B_1)}\right)\rho^\beta,$$

with

$$\beta:=\min\left\{\delta,2-\frac{d}{p}\right\}.$$

In addition, if $C_{x_0}\leq\overline{C}$, for every $x_0\in B_{1/2}$ and some $\overline{C}>0$, we conclude

$u \in C^{\beta}_{\mathrm{loc}}(B_{1/2})$. *In this case, there exists* $C > 0$ *such that*

$$\|u\|_{C^\beta(B_{1/4})} \leq \overline{C}\left(1 + \|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right).$$

Proof. We start by noticing that $u - u(x_0) \in \overline{S}(\lambda, \Lambda, \gamma, f)$. Hence, the weak Harnack inequality in Proposition 1 is available for $u - u(x_0)$.

In addition, Proposition 3 yields a generalised maximum principle for $u - u(x_0)$. By combining this information with (21), Theorem 1 ensures that

$$\begin{aligned}\sup_{B_{\rho/2}(x_0)} |u - u(x_0)| &\leq C\left(C_{x_0}\rho^\delta + \|f\|_{L^p(B_1)}\rho^{2-\frac{d}{p}}\right)\\ &\leq C\left(C_{x_0} + \|f\|_{L^p(B_1)}\right)\rho^\beta,\end{aligned} \tag{22}$$

which establishes the first claim in the theorem.

Suppose now C_{x_0} is bounded uniformly in $x_0 \in B_{1/2}$ by a constant $\overline{C} > 0$. Then (22) becomes

$$\sup_{B_{\rho/2}(x_0)} |u - u(x_0)| \leq \overline{C}\left(1 + \|f\|_{L^p(B_1)}\right)\rho^\beta.$$

for every $x_0 \in B_{1/2}$ and every $0 < \rho < 1/16$. Arguing as in the proof of Theorem 2, one concludes the proof. □

In Theorem 3, we verified a super-solution that is Hölder continuous from below is indeed Hölder continuous. However, if we replace (21) with

$$\inf_{x \in B_\rho(x_0)} (u(x) - u(x_0)) \geq -C_{x_0}\rho, \tag{23}$$

and require Assumption 4.1 to hold, one can improve Theorem 3. Indeed, under these additional conditions, one can prove that a super-solution that is Lipschitz continuous from below is indeed locally Lipschitz continuous. We state this fact in the form of a theorem.

Theorem 4 (Lipschitz continuity in the Escauriaza regime)**.** *Let* $\gamma \in L^q(B_1)$ *and* $f \in L^p(B_1)$, *for some* $q > d > p > p_0 \geq d/2$. *Let* $u \in \overline{S}(\lambda, \Lambda, \gamma, f)$, *where* $0 < \lambda \leq \Lambda$ *are fixed, though arbitrary. For* $x_0 \in B_{1/2}$, *suppose* (23) *holds with a constant* $C_{x_0} > 0$. *Suppose further Assumption 4.1 is in force. Then there exists* $C > 0$ *such that*

$$\sup_{x \in B_{\rho/2}(x_0)} |u(x) - u(x_0)| \leq C\left(C_{x_0} + 1\right)\rho.$$

In addition, if $C_{x_0} \leq \overline{C}$, *for every* $x_0 \in B_{1/2}$ *and some* $\overline{C} > 0$, *we conclude* u *is* *locally Lipschitz continuous in $B_{1/2}$. In this case, there exists $C > 0$ such that*

$$|u(x) - u(y)| \leq C|x - y|,$$

for every x, $y \in B_{1/4}$.

Proof. By arguing as in the proof of Theorem 3, and using (23) and Assumption 4.1, one obtains

$$\begin{aligned} \sup_{B_{\rho/2}(x_0)} |u - u(x_0)| &\leq C \left(C_{x_0}\rho + \|f\|_{L^p(B_{3\rho/2}(x_0))} \rho^{2-\frac{d}{p}} \right) \\ &\leq C \left(C_{x_0} + C_f \right) \rho. \end{aligned} \tag{24}$$

Once (24) is available, the argument follows along the same general lines as in the proof of Theorem 3. □

Theorem 5. *Let $\gamma \in L^q(B_1)$ and $f \in L^p(B_1)$, for some $q \geq p > d$. For $x_0 \in B_{1/2}$, suppose there exists an affine function ℓ_{x_0} and constants $C_{x_0} > 0$ and $\alpha \in (0,1)$ such that*

$$\inf_{x \in B_{\rho_{x_0}}(x_0)} (u - \ell_{x_0})(x) \geq -C_{x_0}\rho_{x_0}^{1+\alpha},$$

for some $\rho_{x_0} \in (0, 1/4)$. Let

$$\beta := \min\left\{\alpha, 1 - \frac{d}{p}\right\}.$$

Then

$$\sup_{B_{\rho_{x_0}/2}(x_0)} |u - \ell| \leq C \left(1 + \|\gamma\|_{L^d(B_1)} \left(\|u\|_{L^\infty(B_1)} + 1 \right) + \|f\|_{L^d(B_1)} \right) \rho_{x_0}^{1+\beta},$$

where $C > 0$ depends on C_{x_0}. In addition, if C_{x_0} and ρ_{x_0} are uniform in $x_0 \in B_{1/2}$, then $u \in C^{1,\beta}_{\rm loc} B_{1/2}$ and there exists $C > 0$ such that

$$\|u\|_{C^{1+\beta}(B_{1/4})} \leq C \left(1 + \|\gamma\|_{L^d(B_1)} \left(\|u\|_{L^\infty(B_1)} + 1 \right) + \|f\|_{L^d(B_1)} \right)$$

for every x, $y \in B_{1/4}$.

Proof. We split the proof into 3 steps for clarity. In what follows we write $\rho \equiv \rho_{x_0}$ to ease notation.

Step 1 - If $u \in \overline{S}(\lambda, \Lambda, \gamma, f)$, we conclude

$$u - \ell_{x_0} \in \overline{S}\left(\lambda, \Lambda, |\gamma|, f + |\gamma|\,|D\ell_{x_0}|\right).$$

As a consequence, the Harnack inequality in Proposition 3 becomes

$$\left(\fint_{B_\rho(x_0)}(u-\ell_{x_0})^\varepsilon\right)^{\frac{1}{\varepsilon}} \leq C\rho\left(\|f\|_{L^d(B_{3\rho/2}(x_0))}+\|\gamma\|_{L^d(B_{3\rho/2}(x_0))}|D\ell_{x_0}|\right) + C\inf_{x\in B_\rho(x_0)}(u-\ell_{x_0})(x)$$

Moreover, because $u-\ell_{x_0}$ has a $C^{1,\alpha}$-modulus of continuity from below, Proposition 4 yields a generalised maximum principle for $u-\ell_{x_0}$. Therefore, Theorem 1 ensures that

$$\sup_{B_{\rho/2}(x_0)}|u-\ell_{x_0}| \leq C\left(\|f\|_{L^d(B_{3\rho/2}(x_0))}+\|\gamma\|_{L^d(B_{3\rho/2}(x_0))}|D\ell_{x_0}|\right)\rho + C(C_{x_0})\rho^{1+\alpha}, \tag{25}$$

where $C(C_{x_0})>0$ is a constant depending on C_{x_0}. Next, we produce an upper bound for $|D\ell_{x_0}|$.

Step 2 - By assumption, we have

$$u(x)\geq u(x_0)+D\ell_{x_0}\cdot(x-x_0)-C_{x_0}\rho^{1+\alpha},$$

for every $x\in B_\rho(x_0)$. Set

$$x_\tau := x_0+\tau\frac{D\ell_{x_0}}{|D\ell_{x_0}|},$$

for $\tau\in(0,\rho)$. Hence, we get

$$|D\ell_{x_0}|\leq\frac{1}{\tau}\left(\|u\|_{L^\infty(B_1)}+C_{x_0}\rho^{1+\alpha}\right)\leq C\left(\|u\|_{L^\infty(B_1)}+C_{x_0}\right),$$

where the second inequality follows by taking the limit $\tau\to\rho$. The former estimate builds upon (25) to produce

$$\sup_{B_{\rho/2}(x_0)}|u-\ell_{x_0}| \leq C\|\gamma\|_{L^d(B_{3\rho/2}(x_0))}\left(\|u\|_{L^\infty(B_1)}+C_{x_0}\right)\rho + C\|f\|_{L^d(B_{3\rho/2}(x_0))}\rho+C(C_{x_0})\rho^{1+\alpha}. \tag{26}$$

Meanwhile, we notice that

$$\|f\|_{L^d(B_{3\rho/2}(x_0))}\leq C\rho^{1-d/p}\|f\|_{L^p(B_1)}$$

and

$$\|\gamma\|_{L^d(B_{3\rho/2}(x_0))}\leq C\rho^{1-d/q}\|\gamma\|_{L^q(B_1)}.$$

Hence, (25) becomes

$$\begin{aligned}\sup_{B_{\rho/2}(x_0)} |u - \ell_{x_0}| \leq C \|\gamma\|_{L^d(B_1)} \left(\|u\|_{L^\infty(B_1)} + C_{x_0}\right) \rho^{2-d/q} \\ + C \|f\|_{L^d(B_1)} \rho^{2-d/p} + C(C_{x_0})\rho^{1+\alpha},\end{aligned} \tag{27}$$

which verifies the first claim in the theorem.

Step 3 - Now, suppose there exists $\overline{C} > 0$ such that $C_{x_0} \leq \overline{C}$ for every $x_0 \in B_{1/2}$. Suppose further that ρ_{x_0} is uniform in $x_0 \in B_{1/2}$. In this case, we use (27); then

$$-C_1\rho^{1+\beta} \leq \inf_{B_{\rho/2}(x_0)} (u - \ell_{x_0}) \leq \sup_{B_{\rho/2}(x_0)} (u - \ell_{x_0}) \leq C_1\rho^{1+\beta},$$

where

$$C_1 := C \left(\|\gamma\|_{L^d(B_1)} \left(\|u\|_{L^\infty(B_1)} + C_{x_0}\right) + \|f\|_{L^d(B_1)} + C(C_{x_0})\right),$$

and the proof is complete. □

The next section builds upon $C^{1,\alpha}$-regularity to examine diffusion properties associated with the super-solutions to (1).

6 Diffusion properties of super-solutions

In Theorem 5, we find conditions for a super-solution $u \in \overline{S}(\lambda, \Lambda, \gamma, f)$ to be locally of class $C^{1,\beta}$. In the sequel, we show that under such conditions u solves an equation driven by a fractional Laplacian. As a consequence, we infer that u relates to a Lévy process whose intensity depends on β. We detail our findings in the next theorem.

Theorem 6. *Let $\gamma \in L^q(B_1)$ and $f \in L^p(B_1)$, for some $q \geq p > d$. For $x_0 \in B_{1/2}$, suppose there exists an affine function ℓ_{x_0} and constants $C > 0$, $\rho \in (0, 1/4)$, and $\alpha \in (0, 1)$, such that*

$$\inf_{x \in B_{\rho_{x_0}}(x_0)} (u - \ell_{x_0})(x) \geq -C_{x_0}\rho_{x_0}^{1+\alpha}.$$

Let $\beta \in (0, 1)$ be defined as in Theorem 5. Then, for every $r \in (1, \infty)$ and $s \in (0, (1 + \beta)/2)$, there exists $g \in L^r(B_{1/2})$ such that $(-\Delta)^s u = g$ in $B_{1/2}$, in the weak sense.

Proof. We split the proof into 3 steps.

Step 1 - Under the assumptions of the theorem, we learn that $u \in C^{1,\beta}_{\text{loc}}(B_{1/2})$, with estimates; see Theorem 5. Extend u outside of $B_{10/21}$ by $\tilde{u}$ in such a way that $\tilde{u} \equiv 0$ in $\mathbb{R}^d \setminus B_{1/2}$ and

$$\|\tilde{u}\|_{C^{1,\beta}(\mathbb{R}^d)} \leq C \|u\|_{C^{1,\beta}(B_{10/21})},$$

for some $C > 0$. Let $s \in (0, (1+\beta)/2)$. A Taylor expansion of order $(1+\beta)$ implies

$$\int_{\mathbb{R}^d} \frac{\tilde{u}(x+y) + \tilde{u}(x-y) - 2\tilde{u}(x)}{|y|^{d+2s}} \mathrm{d}y \leq C \|u\|_{C^{1,\beta}(B_{10/21})} \int_{\mathbb{R}^d} \frac{1}{|y|^{d+2s-1-\beta}} \mathrm{d}y,$$

which is finite. As a consequence, we can write

$$(-\Delta)^s \tilde{u}(x) = \int_{\mathbb{R}^d} \frac{\tilde{u}(x+y) + \tilde{u}(x-y) - 2\tilde{u}(x)}{|y|^{d+2s}} \mathrm{d}y.$$

Step 2 - Now we verify that $(-\Delta)^s \tilde{u} \in L^r(B_{1/2})$, for any $r \in (1, \infty)$. Indeed, fiz $z \in B_{1/2}$; we have

$$\begin{aligned} |(-\Delta)^s \tilde{u}(z)| &\leq \int_{B_{1/21}} \frac{|\tilde{u}(z+y) + \tilde{u}(z-y) - 2\tilde{u}(z)|}{|y|^{d+2s}} \mathrm{d}y \\ &\quad + \int_{\mathbb{R}^d \setminus B_{1/21}} \frac{|\tilde{u}(z+y) + \tilde{u}(z-y) - 2\tilde{u}(z)|}{|y|^{d+2s}} \mathrm{d}y \\ &\leq \|\tilde{u}\|_{C^{1,\beta}(\mathbb{R}^d)} \int_{B_{1/21}} \frac{1}{|y|^{d+\sigma}} \mathrm{d}y + C \|u\|_{L^\infty(B_1)}, \end{aligned}$$

where $\sigma := 2s - 1 - \beta < 0$. Hence,

$$\|(-\Delta)^s \tilde{u}\|_{L^r(B_{1/2})} \leq C \|u\|_{C^{1,\beta}(B_{1/2})}.$$

As a consequence, we can define $g \in L^r(\mathbb{R}^d)$ as

$$\tilde{g}(x) := \begin{cases} (-\Delta)^s \tilde{u}(x) & \text{if} \quad x \in B_{1/2} \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus B_{1/2}. \end{cases}$$

Clearly, $(-\Delta)^s \tilde{u}(x) = \tilde{g}(x)$ almost everywhere in $B_{1/2}$. We want to show that $\tilde{u}$ solves a fractional Poisson equation in the weak sense.

Step 3 - We know that $\tilde{u} \in C^{1,\beta}(B_{1/2})$. As a consequence, we have $\tilde{u} \in W^{s,2}(B_{1/2})$. Let $v \in W^{s,2}_0(B_{1/2})$. We have

$$\int_{\mathbb{R}^d} \tilde{g}(x) v(x) \mathrm{d}x = \int_{\mathbb{R}^d} (-\Delta)^s \tilde{u}(x) v(x) \mathrm{d}x = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v(x) \frac{\tilde{u}(x) - \tilde{u}(y)}{|x-y|^{d+2s}} \mathrm{d}x \mathrm{d}y.$$

Reversing the roles of x and y and applying Fubini's Theorem, we get

$$\int_{\mathbb{R}^d} \tilde{g}(x)v(x)\mathrm{d}x = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(y)\frac{\tilde{u}(x)-\tilde{u}(y)}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y.$$

Therefore,

$$2\int_{\mathbb{R}^d} \tilde{g}(x)v(x)\mathrm{d}x = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(v(x)-v(y))(\tilde{u}(x)-\tilde{u}(y))}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y.$$

We conclude that $\tilde{u}$ is a weak solution to

$$\begin{cases} (-\Delta)^s\tilde{u} = 2\tilde{g} & \text{in} \quad B_{1/2} \\ \tilde{u} = 0 & \text{in} \quad \mathbb{R}^d \setminus B_{1/2}. \end{cases}$$

By setting $g := 2\tilde{g}$ and restricting $\tilde{u}$ to $B_{1/2}$ one completes the proof. □

In the sequel, we examine conditions ensuring that an arbitrary function solves *some* fully nonlinear elliptic equation in the viscosity sense.

7 Geometric properties and solvability: towards an equivalence

In this section, we prove that a function satisfying a weak Harnack inequality and a generalised maximum principle is a viscosity solution to a fully nonlinear elliptic equation.

It is well-understood and documented that a viscosity super-solution to a uniformly elliptic fully nonlinear PDE satisfies a weak Harnack inequality; see, for instance, [7, 20, 21]. On the other hand, viscosity sub-solutions satisfy a generalised maximum principle [7, 8].

Our argument aims to reverse this implication. We suppose that a function satisfies a weak Harnack inequality and a generalised maximum principle. By combining both inequalities, we prove that such a function belongs to a class of viscosity solutions.

Let $u \in C(B_1)$ be a C-viscosity sub-solution to

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \gamma|Du| = f \quad \text{in} \quad B_1, \tag{28}$$

where $f \in L^\infty(B_1) \cap C(B_1)$ $\gamma \in L^\infty(B_1) \cap C(B_1)$. Here, we require f and γ to be continuous functions to frame the problem in the context of C-viscosity solutions.

Hence, if $\phi \in C^2(B_1)$ is such that $u - \phi$ attains its minimum at $x_0 \in B_1$, we have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\phi(x_0)) + \gamma(x_0)|D\phi(x_0)| \geq f(x_0).$$

Taking into account the definition of the Pucci operator, the former inequality becomes

$$\left\| \left(D^2\phi(x_0)\right)^- \right\| \leq \frac{d\Lambda}{\lambda} \left\| \left(D^2\phi(x_0)\right)^+ \right\| + \|\gamma\|_{L^\infty(B_1)} |D\phi(x_0)| + \|f\|_{L^\infty(B_1)}.$$

Similarly, if $u \in C(B_1)$ is a C-viscosity super-solution to

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma|Du| \leq f \quad \text{in} \quad B_1, \tag{29}$$

and $\phi \in C^2(B_1)$ is such that $u - \phi$ has a local maximum at $x_0 \in B_1$, then

$$\left\| \left(D^2\phi(x_0)\right)^+ \right\| \leq \frac{d\Lambda}{\lambda} \left\| \left(D^2\phi(x_0)\right)^- \right\| + \|\gamma\|_{L^\infty(B_1)} |D\phi(x_0)| + \|f\|_{L^\infty(B_1)}.$$

The former computation motivates the definition of a related class of solutions.

Definition 6 (Solution classes)**.** *Let $\gamma, f \in L^\infty(B_1) \cap C(B_1)$ and $C > 0$ be a constant, fixed though arbitrary. We say $u \in C(B_1)$ is in the class $\overline{\mathcal{S}}(C, \gamma, f)$ if, whenever a paraboloid P touches u from below at $x_0 \in B_1$, we have*

$$\left\|D^2(P(x_0))^+\right\| \leq C\left[1 + \left\|D^2(P(x_0))^-\right\| + \|\gamma\|_{L^\infty(B_1)} |D(P(x_0)| + \|f\|_{L^\infty(B_1)}\right].$$

Similarly, we say $u \in C(B_1)$ is in the class $\underline{\mathcal{S}}(C, \gamma, f)$ if, whenever a paraboloid P touches u from above at $x_0 \in B_1$, we have

$$\left\|D^2(P(x_0))^-\right\| \leq C\left[1 + \left\|D^2(P(x_0))^+\right\| + \|\gamma\|_{L^\infty(B_1)} |D(P(x_0)| + \|f\|_{L^\infty(B_1)}\right].$$

We examine the inclusion of a given function in the classes defined above. More concretely, if affine translations of $u \in C(B_1)$ satisfy a weak Harnack inequality and a generalised maximum principle, then u belongs to $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$, for parameters depending on those inequalities. The main result of this section is the following.

Theorem 7. *Let $u \in C(B_1)$, $\gamma \in L^\infty(B_1) \cap C(B_1)$, and $f \in L^\infty(B_1) \cap C(B_1)$. For any affine function $\ell(\cdot)$, suppose $u - \ell$ satisfies Assumptions 3.1 and 3.2, with*

$$k(\rho) := \|\gamma\|_{L^\infty(B_1)} |D\ell| + \|f\|_{L^\infty(B_1)} \qquad \text{and} \qquad \sigma(\rho) := \rho^2.$$

Then there exists $C > 0$ such that $u \in \overline{\mathcal{S}}(C, \gamma, f)$.

Proof. For simplicity, we suppose $x_0 \equiv 0$. The proof is split into 3 steps. The first one examines the opening of a paraboloid touching the graph of u from below at $x_0 \equiv 0$.

Step 1 - Let $P = P(x)$ be a paraboloid touching the graph of u from below at $x_0 \equiv 0$ and write

$$P(x) := P(0) + DP(0) \cdot x + D^2P(0)x \cdot x.$$

We claim that

$$-\left\|D^2P(0)^-\right\| |x|^2 \leq D^2P(0)x \cdot x, \tag{30}$$

for every $x \in \mathbb{R}^d$. To verify the claim, notice one can always write

$$D^2P(0)x \cdot x = \sum_{e_i > 0} e_i x_i^2 + \sum_{e_i < 0} e_i x_i^2,$$

for some $(e_1, \ldots, e_d) \in \mathbb{R}^d$. Hence,

$$\begin{aligned} -\left\|D^2P(0)^-\right\| |x|^2 &\leq -\left\|D^2P(0)^-\right\| \sum_{\substack{i \in \{1,\ldots,d\} \\ e_i < 0}} x_i^2 \\ &\leq -\sum_{e_i < 0} |e_i| x_i^2 \\ &\leq D^2P(0)x \cdot x, \end{aligned}$$

which establishes (30).

Step 2 - Because $P(x)$ touches u from below at $x_0 \equiv 0$, there exists $\delta > 0$ such that

$$u(x) - P(0) - DP(0) \cdot x \geq -\left\|D^2P(0)^-\right\| |x|^2,$$

for every $x \in B_\delta$. By setting $\ell(x) := P(0) - DP(0) \cdot x$, the former inequality becomes

$$\inf_{x \in B_\delta} (u - \ell)(x) \geq -\left\|D^2P(0)^-\right\| |x|^2.$$

I.e., $u - \ell$ has a modulus of continuity from below at $x_0 \equiv 0$; by assumption, we are under the scope of Theorem 1. Hence, Theorem 1 yields:

$$\sup_{B_{\delta/2}} |(u - \ell)| \leq C\left(1 + \|\gamma\|_{L^\infty(B_1)} |D\ell| + \|f\|_{L^\infty(B_1)} + \left\|D^2P(0)^-\right\|\right)\delta^2, \tag{31}$$

for some universal constant $C > 0$. We set

$$\Xi(x) := C\left(1 + \|\gamma\|_{L^\infty(B_1)} |D\ell| + \|f\|_{L^\infty(B_1)} + \left\|D^2P(0)^-\right\|\right)|x|^2.$$

Step 3 - It is clear that $(u - \ell)(0) = \Xi(0)$; we conclude from (31) that $u - \ell$

touches Ξ from below in $B_{\delta/2}$. The condition on P implies

$$D^2P(0)x \cdot x \leq (u - \ell)(x) \leq \Xi(x).$$

An application of Lemma 4 yields

$$\left\| D^2P(0)^+ \right\| \leq C \left(1 + \left\| D^2P(0)^- \right\| + \|\gamma\|_{L^\infty(B_1)} |D\ell| + \|f\|_{L^\infty(B_1)} \right)$$

and completes the proof. □

Remark 4 (Sufficient conditions to be a solution)**.** In Theorem 7, we proved that if $u - \ell$ satisfies Assumptions 3.1-3.2, for particular choices of k and σ, and every affine ℓ, u is in a class of super-solutions. Furthermore, if we suppose $-u + \ell$ satisfies Assumptions 3.1-3.2, one can prove u belongs to a class of sub-solutions. Hence, if $u \in C(B_1)$ is such that $\pm(u - \ell)$ satisfy the assumptions in Theorem 7, we can prove that u is in the class $\mathcal{S} = \overline{\mathcal{S}} \cap \underline{\mathcal{S}}$.

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