A CHARACTERIZATION OF ONE-ELEMENT COMMUTATION CLASSES

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Abstract. A reduced word for a permutation of the symmetric group is its own commutation class if it has no commutation moves available. These words have the property that every factor of length 2 is formed by consecutive integers, but in general words of this form may not be reduced. In this paper we give a necessary and sufficient condition for a word with the previous property to be reduced. In the case of involutions, we give an explicitly construction of their one-element commutation classes and relate their existence with pattern avoidance problems.

1. Introduction

Given a positive integer $n \geq 1$, let $\mathfrak{S}_{n+1}$ denote the *symmetric group* of order $n+1$ formed by all permutations of the set $[n+1] := \{1, 2, \dots n+1\}$, with composition (read from right) as group operation. We usually write a permutation σ using the *one-line notation* $\sigma = [\sigma(1), \dots, \sigma(n+1)]$. In some cases, we will also use the *cyclic notation* of a permutation, using parenthesis to represent the cycles with commas to separate the images. For example, the permutation $\sigma = [2, 5, 7, 1, 4, 3, 6]$ can be written in cyclic notation as $\sigma = (1, 2, 5, 4)(3, 7, 6)$.

The group $\mathfrak{S}_{n+1}$ is generated by the involutions $\{s_1, \dots, s_n\}$, also known as *simple refletions*, where $s_i = [1, \dots, i+1, i, \dots, n+1] = (i, i+1)$, for all $i \in [n]$. This fact can be easily understood by noticing that multiplying a permutation σ on the right by s_i interchanges the values in positions i and $i+1$ in the one-line notation of σ, that is $\sigma s_i = [\sigma(1), \dots, \sigma(i+1), \sigma(i), \dots, \sigma(n+1)]$. Since $\mathfrak{S}_{n+1}$ is generated by the involutions s_i, any permutation σ can be written as a product $s_{i_1} s_{i_2} \cdots s_{i_l}$, with $i_j \in [n]$, for all $j \in [l]$. When l is minimal, we say that the product is a *reduced decomposition* and $i_1 \cdots i_l$ a *reduced word* of σ. The integer $l(\sigma) := l$ is the *length* of σ. Let $R(\sigma)$ be the set of all reduced words of σ.

Reduced words of permutations are widely studied in combinatorics (see [1, 9, 10, 11, 13]). Perhaps, one of the most important facts concerning reduced words is a well known result of Tits [13], which says that two reduced words for the same permutation differ by a sequence of the following two types of moves:

$$ij \leftrightarrow ji, \text{ if } |i-j| > 1, \tag{1}$$

$$i(i+1)i \leftrightarrow (i+1)i(i+1), \tag{2}$$

where (1) is called a *commutation move*, and (2) is called a *braid move*.

We can define an equivalence relation on the set $R(\sigma)$ by setting $s \sim t$ if s and t differ by a sequence of commutation moves. The equivalence classes generated by this relation are called *commutation classes*. These structures were already considered by some authors (see [3, 4, 5, 6, 7]), but there is still much to understand about commutation classes. For instance, there is no known formula for the number of commutation classes of a given permutation. Recently, Tenner [12] studied the commutation classes which have only one reduced word, giving a necessary condition for a reduced word to be its own commutation class in terms of pinnacles and vales. A nice consequence of this result is that the number of one-element commutation classes of w_0, the longest element in $\mathfrak{S}_{n+1}$, is exactly 4, a result previously obtained in [5]. The goal of this paper is to extend Tenner's work on this topic. We start Section 2 by introducing the terminology used in [12] in order to define what we have called a *segment* in a word. This notion will be crucial to prove the main result of Section 3, which is a necessary and sufficient condition for a word to be a one-element commutation class of some permutation. An aplication of this caracterization will be done in Section 4, where we give an explicitly construction of the one-element commutation classes for involutions and relate their existence with pattern avoidance problems.

2. Definitions and background

Let $[n]^*$ be the set of all words with finite length over the alphabet $[n]$. A *sub-word* of a word $s = i_1 \cdots i_l$ is a word obtained from s by deleting some of its letters, and a *factor* of s is a sub-word of s of the form $s_{i_j} s_{i_{j+1}} \cdots s_{i_k}$, with $1 \leq j \leq k \leq l$. When $j = 1$, we call it a *left factor* of s. Given a permutation $\sigma \in \mathfrak{S}_{n+1}$, we denote by $R_\bullet(\sigma) \subseteq R(\sigma)$ the set of reduced words of σ that are their own commutation class. By definition, a word in $R_\bullet(\sigma)$ has no commutation moves available, which means that all of its factors of length 2 are formed by consecutive integers. These words can be descripted in terms of its "peaks", a notion introduced in [12] to study one-element commutation classes.

Definition 2.1 ([12]). *Let $s \in [n]^*$ be a word . The endpoints of s are its leftmost and rightmost letters. A* pinnacle *of s is a letter that is larger than its immediate neighbor(s), and a* vale *is a letter that is smaller than its immediate neighbor(s). We call pinnacles and vales the* peaks *of s. Write $\boldsymbol{p}(s)$ for the substring of pinnacles of s, and $\boldsymbol{v}(s)$ for the substring of vales. The substring of pinnacles and vales will be written as $\boldsymbol{pv}(s)$. If every factor of length 2 of s is formed by consecutive integers, then we say that s is a word* formed by consecutive integers.

As an example, if $s = 23454321234$ we have $\mathbf{p}(s) = 54$, $\mathbf{v}(s) = 21$ and $\mathbf{pv}(s) = 2514$. When s is a word formed by consecutive integers, each factor ij of $\mathbf{pv}(s)$ corresponds in s to the factor $i(i+1)\cdots(j-1)j$ if $i < j$, or $i(i-1)\cdots(j+1)j$ if $i > j$. Sometimes, instead of writting all of its letters, it will be more usefull to write only the endpoints of those factors, and for that we use the notation $\underline{i}$ and $\overline{j}$ to denote a vale i or to denote a pinnacle j, respectevely. When using this identification to represent the entire word, we write $s \equiv \mathbf{pv}(s)$. In the example above, we have $23454321234 \equiv \underline{2}\overline{5}\underline{1}\overline{4}$. A graphical representation for these type of words can be given using line diagrams.

Definition 2.2. *Let $s \in [n]^*$ be a word formed by consecutive integers with $\boldsymbol{pv}(s) = i_1 \cdots i_l$. The* line diagram *of s is formed by the set of points $(j, i_j) \in [l] \times [n]$, where there is a line segment connecting each pair (j, i_j) and $(j+1, i_{j+1})$, for all $j \in [l-1]$.*

The line diagram of the word in the previous example is represented in Figure 1. Each

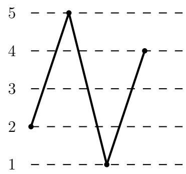

FIGURE 1. Line diagram of $\underline{2}\overline{5}\underline{1}\overline{4}$.

factor of length 2 of $\mathbf{pv}(s)$ is encoded by a line segment in the line diagram of s. Thereby, we say that $\underline{i}\,\overline{j}$ or $\overline{j}\underline{i}$ are *segments* of s if ij or ji are factors of $\mathbf{pv}(s)$, with $i < j$. For our running example, its segments are $\underline{2}\overline{5}$, $\overline{5}\underline{1}$ and $\underline{1}\overline{4}$, which we can see clearly from its diagram in Figure 1. Notice that multiplying a permutation $\sigma \in \mathfrak{S}_{n+1}$ on the right by the permutation associated to the segment $\underline{i}\,\overline{j}$ (resp. $\overline{j}\underline{i}$) has the effect of "moving" the integer $\sigma(i)$ to position $j+1$ (resp. the integer $\sigma(j+1)$ to position i) in the one-line notation of σ. In this sense, we say that the segment *moves* an integer. In the above example, the segment $\underline{1}\,\overline{4}$ acts in the permutation $[\mathbf{1}, 6, 3, 4, 5, 2]$ (the permutation associated to the left factor 2345432 of s) by moving the integer in the first position of the one-line notation to position 5, obtaining the permutation $[6, 3, 4, 5, \mathbf{1}, 2]$. Not every word formed by consecutive integers is a reduced word. For instance, the word $1232123 \equiv \underline{1}\overline{3}\underline{1}\overline{3}$ is not reduced. A set of necessary conditions for a word s to be a one-element commutation class was given in [12] using properties of the strings $\mathbf{p}(s)$, $\mathbf{v}(s)$ and $\mathbf{pv}(s)$.

Definition 2.3. *Let $s = i_1 \cdots i_l \in [n]^*$ be a word. If there exist j and k such that $1 \le j \le k \le l$ and*

$$i_1 < \cdots < i_j = i_k > \cdots > i_l,$$

then s is a wedge. *If*

$$i_1 > \cdots > i_j = i_k < \cdots < s_l,$$

then s is a vee. *If $j = k$, then that wedge or vee is strict.*

For example, the word 2 4 7 3 1 is a strict wedge and the word 2 4 5 5 2 1 is a wedge that is not strict. The words 4 2 1 5 7 and 6 3 2 2 4 5 are examples of a strict vee and a non-strict vee, respectively.

Theorem 2.1 ([12, Theorem 3.1]). *For any $\sigma \in S_{n+1}$, if $s \in R_\bullet(\sigma)$ then:*

(1) *$\boldsymbol{p}(s)$ is a wedge,*
(2) *$\boldsymbol{v}(s)$ is a vee,*
(3) *$\boldsymbol{p}(s)$ and/or $\boldsymbol{v}(s)$ is strict,*
(4) *the minimum and maximum values of $\boldsymbol{pv}(s)$ appear consecutively and,*
(5) *if $\boldsymbol{p}(s)$ (or $\boldsymbol{v}(s)$) has more than one integer i, then one of those i's is an endpoint of s.*

As a consequence, if s is a word formed by consecutive integers which does not satisfy all of the previous 5 conditions, then s cannot be a reduced word. It follows directly from condition 3 of the previous theorem that the word $s \equiv \underline{1}\,\overline{3}\,\underline{1}\,\overline{3}$ is not reduced because neither $\mathbf{p}(s) = 33$ nor $\mathbf{v}(s) = 11$ are strict.

The "converse" of this theorem is not true, *i.e.* a word s formed by consecutive integers that satisfies all the conditions of the previous theorem is not necessarily a reduced word.

Consider for instance the word $s = 2343212345654345 \equiv \underline{2}\overline{4}\underline{1}\overline{6}\underline{3}\overline{5}$, which satisfies all the conditions of Theorem 2.1 but is not a reduced word. The reason is that s contains the factor $t = 34321234565434 \equiv \underline{3}\overline{4}\underline{1}\overline{6}\underline{3}\overline{4}$ which is not reduced (the permutation associated to t is $[5, 2, 7, 3, 4, 6, 1]$ which has length 12, but t has 14 letters). Notice that t contains two occurrences of the segment $\underline{3}\overline{4}$. This is not a coincidence, as we are going to see in the next section. The line diagrams of s and t are depicted in Figure 2.

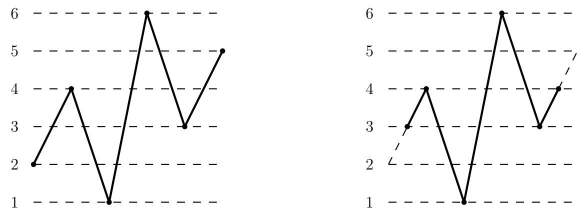

FIGURE 2. Line diagrams of $s \equiv \underline{2}\overline{4}\underline{1}\overline{6}\underline{3}\overline{5}$ and $t \equiv \underline{3}\overline{4}\underline{1}\overline{6}\underline{3}\overline{4}$.

We end this section with the following lemma which will be usefull more ahead.

Lemma 2.2. *Let $s \in [n]^*$ be a word formed by consecutive integers. Suppose that t is a factor of s. Then:*

(1) *Every peak of t that is not an endpoint of t is also a peak of s.*
(2) *If $\underline{i}\,\overline{j}$ (resp. $\overline{j}\,\underline{i}$) is a segment of t which does not contain any endpoint of t, then $\underline{i}\,\overline{j}$ (resp. $\overline{j}\,\underline{i}$) is also a segment of s.*

Proof. If i is a vale (resp. pinnacle) of t that is not one of its endpoints, then it is between two letters that are larger (resp. smaller) in the word t. Since t is a factor of s, that letter i is also between the same letters in the word s, implying that i is also a vale (resp. pinnacle) of s. To prove condition 2, if $\underline{i}\,\overline{j}$ or $\overline{j}\,\underline{i}$ is a segment of t which does not contain any endpoint, then i and j are also peaks of s, by condition 1, which appear consecutively in $\mathbf{pv}(s)$. Therefore, it is also a segment of s. ☐

In other words, if we have a word s formed by consecutive integers and t a factor of s, then the only segments of t that may not be segments of s are its leftmost and rightmost ones. The reason is that endpoints are always considered peaks, and so the endpoints of t will be always peaks of t, but not necessary peaks of s. Considering s and t as in Figure 2, the only segments of t that are segments of s are $\overline{4}\underline{1}$, $\underline{1}\overline{6}$ and $\overline{6}\underline{3}$. If we consider the word $u = 123456543 \equiv \underline{1}\overline{6}\underline{3}$ which is also a factor of s, then every segment of u is a segment of s, because the endpoints of u are also peaks in s. The word $v = 1234565434 \equiv \underline{1}\overline{6}\underline{3}\overline{4}$ is a factor of s where only one of its endpoints is a peak of s. The line diagrams of u and v are depicted in Figure 3.

3. A characterization of one-element commutation classes

As we saw in the previous section, the conditions stated in Theorem 2.1 are not enough to completely characterize one-element commutation classes, as there are words formed by consecutive integers satisfying all five conditions of the theorem which are not reduced. In this section, we give a necessary and sufficient condition for a word formed by consecutive integers to be reduced.

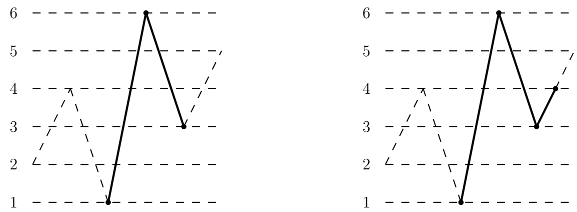

FIGURE 3. Line diagrams of $u \equiv \underline{1}\overline{6}\underline{3}$ and $v \equiv \underline{1}\overline{6}\underline{3}\overline{4}$.

Given $s = i_1 \cdots i_l \in [n]^*$, define the words

- $s^r = i_l \cdots i_1$, called the *reverse word* of s,
- $s^c = i'_1 \cdots i'_l$ with $i'_j = n + 1 - i_j$, called the *complement word* of s,
- $s^{rc} = (s^r)^c = (s^c)^r$, called the *reverse complement word* of s.

These words are called the *symmetries* of s. It is easy to check that if $s \in R_\bullet(\sigma)$, then $s^r \in R_\bullet(\sigma^{-1})$, $s^c \in R_\bullet(w_0\sigma w_0)$ and $s^{rc} \in R_\bullet(w_0\sigma^{-1}w_0)$, since all permutations σ, σ^{-1} and $w_0\sigma w_0$ have the same length. (see [1]). The following result is a well known property about reduced words of permutations, which we will use in a moment.

Lemma 3.1 ([1]). *Let $\sigma \in \mathfrak{S}_{n+1}$ and $i \in [n]$. Then*

$$l(\sigma s_i) = \begin{cases} l(\sigma) + 1, & \text{if } \sigma(i) < \sigma(i+1) \\ l(\sigma) - 1, & \text{if } \sigma(i) > \sigma(i+1) \end{cases}.$$

As a consequence we have the following.

Corollary 3.2. *Let $s = t \cdot i \in R(\sigma)$ be a reduced word for some $\sigma \in \mathfrak{S}_{n+1}$, with $t \in [n]^*$ and $i \in [n]$. Then, $\sigma(i) > \sigma(i+1)$.*

Proposition 3.3. *Let $s = \underline{i}\,\overline{j}\ t\ \underline{i}\,\overline{j} \in [n]^*$ be a word formed by consecutive integers, with $1 \leq i < j \leq n$ and $t \in [n]^*$. Then, s is not reduced.*

Proof. Suppose that s is reduced, and let σ be its associated permutation so that $s \in R_\bullet(\sigma)$. We know that t must contain a peak of s, otherwise $s \equiv \underline{i}\overline{j}\underline{i}\overline{j}$ which is not reduced by condition 3 of Theorem 2.1. Let $\mathbf{v}(s) = i\ v_1 \cdots v_l\ i$ and $\mathbf{p}(s) = j\ p_1\ \cdots p_l\ j$ be the strings of vales and pinnacles of s, respectively, for some integer l. From Theorem 2.1, since $\mathbf{v}(s)$ is a vee (resp. $\mathbf{p}(s)$ is a wedge) we have $v_k < i$ (resp. $p_k > j$), for all $k \in [l]$. This allow us to write s as

$$s = \boldsymbol{(i \cdots j \cdots i)(i-1 \cdots v_1 \cdots i-1) \cdots (i \cdots p_k \cdots i)}(i+1 \cdots j).$$

(See Figure 4) The permutation associated to the bold left factor of s is

$$\pi = (i, j+1)(i, v_1) \cdots (i, p_k + 1) = (i, p_k + 1, \cdots, v_1, j+1).$$

Notice that the permutation associated to $(i+1 \cdots j)$ moves the integer $\pi(i+1)$ to position $j+1$. Since $\pi(i+1) = i+1$ and $\pi(j+1) = i$, we have that $\sigma(j) = i$ and $\sigma(j+1) = i+1$, which contradicts Corollary 3.2. Therefore, s cannot be reduced. $\square$

It follows from the previous lemma and the definition of reverse word that a word of the form $\overline{j}\,\underline{i}\ t\ \overline{j}\,\underline{i}$, with $i < j$ that is formed by consecutive integers is also not reduced.

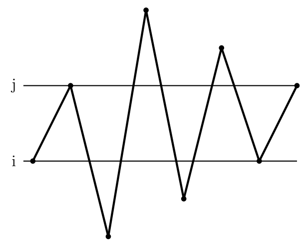

FIGURE 4. Diagram of a word of the form $\underline{i}\,\overline{j}t\underline{i}\,\overline{j}$

These words contain at least two occurences of the same segment, a fact that motivated the following definition.

Definition 3.1. *A word formed by consecutive integers is said to have a factor with repeated segments if it contains a factor* $\underline{i}\,\overline{j}\,t\,\underline{i}\,\overline{j}$ *or* $\overline{j}\,\underline{i}\,t\,\overline{j}\,\underline{i}$*, for some* $i < j$ *and* $t \in [n]^*$.

For example, the word $s \equiv \underline{2}\overline{4}\underline{1}\overline{6}\underline{3}\overline{5}$ has a factor with repeated segments since $u = \underline{3}\overline{4}t\underline{3}\overline{4}$, with $t = 3212345654$, is a factor of s. As a consequence of Proposition 3.3, we have a new necessary condition for a word to be a one-element commutation class of some permutation.

Theorem 3.4. *Let* $\sigma \in \mathfrak{S}_{n+1}$ *and* $s \in R_\bullet(\sigma)$*. Then* s *does not have a factor with repeated segments.*

Proof. If s has a factor with repeated segments, then it contain a factor $\underline{i}\,\overline{j}\,t\,\underline{i}\,\overline{j}$ or $\overline{j}\,\underline{i}\,t\,\overline{j}\,\underline{i}$ for some word $t \in [n]^*$, which by the previous proposition is not reduced. □

A natural question that one may ask is whether this new condition plus the ones stated in Theorem 2.1 are sufficient to completely characterize one-element commutation classes. As we are going to see, we just need this new one and condition 5 of Theorem 2.1 to complete this characterization. We will start by giving a criteria to identify words that contain factors with repeated segments just by looking at its line diagram.

Proposition 3.5. *Let* $s \in [n]^*$ *be a word formed by consecutive integers. The following statements are equivalent:*

1. *There is no factor of* s *with repeated segments.*
2. *The word* s *does not contain a factor* $\underline{x}\,\overline{j}\ t\ \underline{i}\,\overline{y}$ *with* $x \leq i < j \leq y$*, or a factor* $\overline{y}\,\underline{i}\ t\ \overline{j}\,\underline{x}$ *with* $y \geq j > i \geq x$*, for some word* $t \in [n]^*$*, where* x *and* y *are peaks of* s.

Proof. We prove the contra-positive assertions. A word s contains a factor with repeated segments if and only if s contains a factor $\underline{i}\,\overline{j}\ t\ \underline{i}\,\overline{j}$ or $\overline{j}\,\underline{i}\ t\ \overline{j}\,\underline{i}$. Consider the first case (the second is analogous). From Lemma 2.2, the only peaks of $\underline{i}\,\overline{j}\ t\ \underline{i}\,\overline{j}$ that may not be peaks of s are its endpoints. Therefore s contains a factor $\underline{x}\,\overline{j}\ t\ \underline{i}\,\overline{y}$ with $x \leq i < j \leq y$, with x and y peaks of s.

Reciprocally, suppose that s contains a factor $\underline{x}\,\overline{j}\ t\ \underline{i}\,\overline{y}$ with $x \leq i < j \leq y$ (the other case is analogous). Then, $\underline{i}\,\overline{j}\ t\ \underline{i}\,\overline{j}$ is a factor of s, and we have the result. □

It follows that if s is a one-element commutation class, then its line diagram must avoid the two shapes depicted in Figure 5.

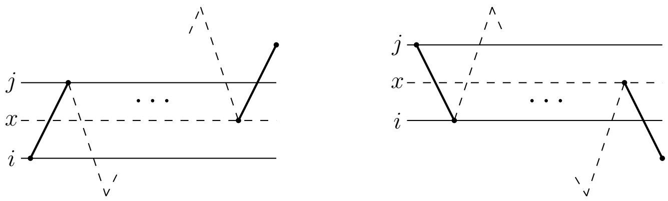

FIGURE 5. Avoiding shapes for one-element commutation classes.

The following definition will allows us to translate condition 5 of Theorem 2.1 into the language os segments.

Definition 3.2. *A word formed by consecutive integers is said to have a factor with symmetric segments if it has a factor* $\underline{i}\,\overline{j}\,t\,\overline{j}\,\underline{i}$ *or* $\overline{j}\,\underline{i}\,t\,\underline{i}\,\overline{j}$*, for some* $i < j$ *where* $t \in [n]^*$ *contains at least one peak of* s*.*

For instance, the word $s \equiv \underline{2}\,\overline{4}\,\underline{1}\,\overline{4}\,\underline{2}$, depicted in Figure 6, has symmetric segments. The word $s \equiv \overline{4}\underline{2}\overline{4}$ does not contains symmetric segments.

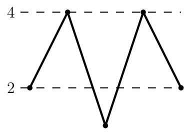

FIGURE 6. Diagram of the word $s = \underline{2}\,\overline{4}\,\underline{1}\,\overline{4}\,\underline{2}$.

Proposition 3.6. *Let* $s \in [n]^*$ *be a word formed by consecutive integers. The following statements are equivalent:*

(1) *There is no factor of* s *with symmetric segments.*
(2) *If* $\boldsymbol{p}(s)$ *(or* $\boldsymbol{v}(s)$*) has more than one integer* i*, then one of those* i*'s is an endpoint of* s*.*

Proof. We prove the contra-positive assertions. If s contains a factor with symmetric segments, then s must contain a factor $u = \underline{i}\,\overline{j}\ t\ \overline{j}\,\underline{i}$, or $u = \overline{j}\,\underline{i}\ t\ \underline{i}\,\overline{j}$, with $i, j \in [n]$ and $t \in [n]^*$ a word which contains some peak of s. Considering the first case (the other is analogous), by Lemma 2.2 we have that both $\overline{j}$ in u are pinnacles of s, and neither of them is an endpoint. Therefore, condition 2 does not hold.

Now suppose that $\mathbf{v}(s)$ contains two letters i such that neither of them is an endpoint (the $\mathbf{p}(s)$ case is analogous). Then s will contain a factor u of the form $u = \overline{j}\,\underline{i}\ t\ \underline{i}\,\overline{k}$, with $j, k \in [n]$ and t a word which contain some peak of s. If $j \leq k$, then s will contain the factor $\overline{j}\,\underline{i}\ t\ \underline{i}\,\overline{j}$. If $j > k$, then s will contain the factor $\overline{k}\,\underline{i}\ t\ \underline{i}\,\overline{k}$ (see Figure 7). In either case, we have a factor with symmetric segments. □

Given $i < j$, let $[i, j] := \{i, i + 1, \ldots, j\}$ and $[i, j]^*$ the set of words with finite length in the alphabet $[i, j]$. The following lemma will be usefull more ahead.

Lemma 3.7. *Let* $s \in R(\sigma)$ *a reduced word for some* $\sigma \in \mathfrak{S}_{n+1}$*. If* $s \in [i, j]^*$*, then* $\sigma(k) = k$ *for all* $k \in [n + 1] \setminus [i, j + 1]$*.*

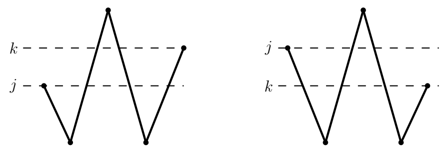

FIGURE 7. Diagram of the word $u = \overline{j}\,\underline{i}\,t\,\underline{i}\,\overline{k}$ with $j \leq k$ and $j > k$.

Proof. Suppose that there is $k \in [n+1] \setminus [i, j+1]$ such that $\sigma(k) \neq k$. Then, every reduced word for σ must contain a letter k or a letter $k-1$. But this is a contradiction because k and $k-1$ belong to $[n+1] \setminus [i, j+1]$ and $s \in [i, j]^*$. □

Before stating the main result of this section, we need a better understanding of how segments behave. Suppose that $s = \underline{i}\overline{j}t \in R_\bullet(\sigma)$, for some word $t \in [n]^*$. The segment $\underline{i}\overline{j}$ moves the integer i to position $j+1$. If s contains another segment that moves the integer i, then the following segment to move this integer is of the form $\underline{j}\overline{y}$, with $j < y$, or $\overline{j}\underline{x}$ with $j > x$. Notice that the second case cannot hold, otherwise we would have two pinnacles j in s where neither is an endpoint, contradicting condition 5 of Theorem 2.1. We can do a similar reasoning if $s = \overline{j}\underline{i}t$ and concluding the following results:

Lemma 3.8. *Let $s \in R_\bullet(\sigma)$ for some $\sigma \in \mathfrak{S}_{n+1}$. If a segment $\underline{i}\overline{j}$ (resp. $\overline{j}\underline{i}$) of s moves an integer $k \in [n+1]$, then every segment of s that moves k is of the form $\underline{x}\overline{y}$ (resp. $\overline{y}\underline{x}$).*

Lemma 3.9. *Let $s \in R_\bullet(\sigma)$ for some $\sigma \in \mathfrak{S}_{n+1}$ and $i \in [n]$.*

(1) *If $\sigma(i+1) < i$, then there is a segment $\underline{x}\overline{y}$ of s that moves the integer $\sigma(i+1)$.*
(2) *If $i+1 < \sigma(i)$, then there is a segment $\overline{y}\underline{x}$ of s that moves the integer $\sigma(i)$.*

There is also a restriction on the integers that are moved by segments.

Lemma 3.10. *Let $s \in R_\bullet(\sigma)$ for some $\sigma \in \mathfrak{S}_{n+1}$. If $\underline{i}\overline{j}$ (resp. $\overline{j}\underline{i}$) is a segment of s, then it moves an integer $k \in [i+1]$ (resp. $k \in [j, n+1]$).*

Proof. Assume that s contains a segment $\underline{i}\overline{j}$ (the other case is similiar) which moves an integer k. Then, we can write $s = t_1\underline{i}\overline{j}t_2$, for some words $t_1, t_2 \in [n]^*$. Suppose by contradiction that $i+1 < k$. Since s is a reduced word, the word t_1 is also a reduced word for some permutation $\pi \in \mathfrak{S}_{n+1}$ with $\pi(i) = k$ (because $\underline{i}\overline{j}$ moves the integer k). But $i+1 < k$, which from the previous lemma implies that t contains a segment $\overline{y}\underline{x}$ that moves k, contradicting Lemma 3.8. Therefore $k \leq i+1$. □

Proposition 3.11. *Let $s \in [n]^*$ be a word formed by consecutive integers such that $s = t \cdot j$ with $t \in R_\bullet(\sigma)$ for some $\sigma \in \mathfrak{S}_{n+1}$ and $j \in [n]$. If s is not reduced, then s contains a factor with repeated or symmetric segments.*

Proof. Since s is not reduced, from Lemma 3.1 we have $\sigma(j) > \sigma(j+1)$. Assume that j is a pinnacle of s (the case when j is a vale follows from the application of complement word). Then, we can write $s = u\underline{i}\overline{j}$ for some $i < j$, and we need to consider two cases.

Case 1: $j > i+1$

If $j > i+1$, we have $t = u\underline{i}\overline{j-1}$ and from the previous lemma, since t is a one-element commutation class, the segment $\underline{i}\overline{j-1}$ moves an integer $k \in [i+1]$ to position j. But then, $\sigma(j) = k$ and we have $j > i+1 \geq k > \sigma(j+1)$ which implies, from Lemma 3.9, that t contains a segment $\underline{x}\overline{y}$ that moves the integer $\sigma(j+1)$ (call it l), for some $x < y$. The rightmost such segment $\underline{x}\overline{y}$ must have $y = j$, so we can write

$$t = u'\underline{x}\overline{j}u''\underline{i}\overline{j-1},$$

for some words $u', u'' \in [n]^*$. Our goal is to prove that $x \leq i$. If $x > i$, the fact that $\mathbf{v}(t)$ is a vee implies that u' is a word in the alphabet $[x+1, n]$, because u' cannot contain vales of t that are smaller than or equal to x. Considering π the permutation associated to u', from Lemma 3.7 we have $\pi(x) = x$. Since the segment $\underline{x}\overline{j}$ moves the integer l, we have $x = l$. But this is a contradicion because $x \geq i+1 > l$. Therefore $x \leq i$ and we have $s = u'\underline{x}\overline{j}u''\underline{i}\overline{j}$, which, by Lemma 3.5, contain a factor with repeated segments (the factor $\underline{x}\overline{j}u''\underline{i}\overline{j}$).

Case 2: $j = i+1$

In this case we have $s = u\underline{i}\overline{i+1}$, with $t = u\ i$. If u does not contain any letter i, then u is a word in the alphabet $[i+1, n]$. But then, its associated permutation fixes i, so $\sigma(i+1) = i$. Since $\sigma(i+1) > \sigma(i+2)$, we have $i+1 > i > \sigma(i+2)$, which from Lemma 3.9 implies that t contains a segment $\underline{x}\overline{y}$ that moves the integer $\sigma(i+2)$, for some $x < y$. The rightmost such segment $\underline{x}\overline{y}$ must have $y = i+1$, and so we can write

$$s = u'\underline{x}\overline{i+1}u''\underline{i}\overline{i+1},$$

for some words $u', u'' \in [n]^*$. Since $x \leq i$, we have that s contains a factor with repeated segments (the factor $\underline{x}\overline{i+1}u''\underline{i}\overline{i+1}$).

For the case where u contains letters i, we need to consider two sub-cases:

Sub-case 1: u does not contain vales i

Our goal is to prove that $i+1 > \sigma(i+2)$ in order to use the previous argument. If the first appereance of a letter i in t is preceed by a letter $i-1$, then $\sigma(i+1) = i+1$. If it's preceed by a letter $i+1$, then $\sigma(i+1) = i$. Since $\sigma(i+1) > \sigma(i+2)$ we have the result.

Sub-case 2: u contains vales i

We can write $s = u'\ \underline{i}\overline{y}\ u''\ \underline{i}\overline{i+1}$. If u' is not empty, then s contains two vales i where neither is an endpoint, which from Proposition 3.6 implies that s contains symmetric segments. If u' is empty, then $t = \underline{i}\overline{y}\ u''\ \underline{i}$, which from the proof of Proposition 3.3 we have $\sigma(i+1) = i+1$. Using the same argument as in Sub-case 1, we have that s contains a factor with repeated segments. □

We are now in condition to state and prove the main result of this section.

Theorem 3.12. *Let $s \in [n]^*$ be a word formed by consecutive integers and let $\sigma \in \mathfrak{S}_{n+1}$ be the corresponding permutation. Then, $s \in R_\bullet(\sigma)$ if and only if there is no factor of s with repeated or symmetric segments.*

Proof. From Theorem 3.4, if $s \in R_\bullet(\sigma)$, then s cannot contain factors with repeated segments. Moreover, s must satisfy condition 5 of Theorem 2.1, which is equivalent to say that s does not have a factor with symmetric segments by Proposition 3.6. Reciprocally, suppose that $s \notin R_\bullet(\sigma)$ for all $\sigma \in \mathfrak{S}_{n+1}$ (we want to prove the contra-positive assertion). Then, s cannot be a reduced word and so it must contain a left factor $s' = t \cdot j$ where t is reduced but s' is not, for some $j \in [n]$. From the previous proposition, s' contains a factor with repeated or symmetric segments and we have the result. □

Notice that when a word formed by consecutive integers contains repeated or symmetric segments, then it repeats vales and pinnacles at the same time. Therefore, another way to interpret the previous is that a word formed by consecutive integers is reduced if and only if does not contain any factor that repeats vales and pinnacles at the same time.

4. One-element classes for involutions

In this section we give an explicit construction of one-element commutation classes for involutions and relate their existence with pattern avoidance problems. We start by recalling the following result proved in [5].

Lemma 4.1. *The word* $\underline{1}\,\overline{n}$ *(resp.* $\overline{n}\,\underline{1}$*) is the only reduced word formed by consecutive integers with length* $\geq n$ *over the alphabet* $[n]$*, having left (resp. right) endpoint the letter* 1 *and right (resp. left) endpoint the letter* n*.*

In other words, there are no peaks between letters 1 and n in a one-element commutation class. As a consequence, we have the following.

Lemma 4.2. *Let* σ *be a permutation in* $\mathfrak{S}_{n+1}$ *that fixes neither* 1 *nor* $n+1$*. If* $|R_\bullet(\sigma)| > 0$*, then* $\sigma(n+1) = 1$ *or* $\sigma(1) = n+1$*. Moreover, if* $\sigma(n+1) = 1$ *(resp.* $\sigma(1) = n+1$*) every one-element commutation class of* σ *contains a segment* $\underline{1}\overline{n}$ *(resp.* $\underline{n}\overline{1}$*).*

Proof. Since 1 and $n+1$ are not fixed points of σ, every reduced word for σ must contain at least a letter 1 and a letter n. Let $s \in R_\bullet(\sigma)$ and suppose that there is a letter 1 proceeded by a letter n in s. By Lemma 4.1, we can write s as

$$s = u\underline{1}\,\overline{n}v,$$

where u is a word that does not contain letters 1 and v is a word that does not contain letters n. But then, the permutation associated to u fixes the integer 1, and so the segment $\underline{1}\,\overline{n}$ will move the integer 1 to position $n+1$. Since the permutation associated to v fixes the integer $n+1$, we have $\sigma(n+1) = 1$. With analogous arguments one can prove that if there is a letter n proceeded by a letter 1, then $\sigma(1) = n+1$ and every one-element commutation class contains a segment $\overline{n}\underline{1}$. ☐

The previous lemma gives us a necessary condition for a permutation that does not fix 1 nor $n+1$ to contain one-element commutation classes. It is not, however, a sufficient condition; consider for instance the permutation $\sigma = [3,4,5,2,1] \in \mathfrak{S}_5$, which does not fix 1 nor 5 and $\sigma(5) = 1$. One can check that this permutation contains 4 commutation classes and neither of them is a one-element commutation class. In the case of involutions, one can get more information. Before that, let's recall that the symmetries of a word s are the words s, s^r, s^c and s^{cr}.

Lemma 4.3. *Let* σ *be an involution in* $\mathfrak{S}_{n+1}$ *that fixes neither* 1 *nor* $n+1$*. If* $|R_\bullet(\sigma)| > 0$*, then* $\sigma(n+1) = 1$ *and* $\sigma(1) = n+1$*. Moreover, if* $s \in R_\bullet(\sigma)$*, then* s *is a symmetry of* $\underline{1}\overline{n}\underline{1}t$*, for some word* $t \in [2, n-1]^*$*.*

Proof. The equalities $\sigma(1) = n+1$ and $\sigma(n+1) = 1$ follows from the previous lemma and from the fact that σ is an involution. We have also from the previous lemma that, if $s \in R_\bullet(\sigma)$, then s contains a segment $\underline{1}\overline{n}$ (because $\sigma(n+1) = 1$) and a segment $\overline{n}\underline{1}$ (because $\sigma(1) = n+1$). The only possibility for s to contain those segments at the same time is to have the factor $\underline{1}\overline{n}\underline{1}$ or $\overline{n}\underline{1}\overline{n}$, which must contain an endpoint, by Theorem 3.12. □

The previous two lemmas can be generalized for any permutation σ by replacing 1 and $n+1$ with the minimum and maximum non-fixed points of σ, respectively.

Notice that there is always an endpoint of a one-element commutation class of an involution that does not fix 1 nor $n+1$ that is the letter 1 or n. We have the following.

Proposition 4.4. *Let σ be an involution in $\mathfrak{S}_{n+1}$ that fixes neither 1 nor $n+1$. Then:*

(1) *σ has a one-element commutation class that contains a letter 1 as endpoint if and only if $\sigma = \prod_{k=1}^{l}(k, j_k+1)$, with $j_1 = n > j_2 > \cdots > j_l$ for some integer l, and $k < j_k + 1$ for all $k \in [l]$.*

(2) *σ has a one-element commutation that contains a letter n as endpoint if and only if $\sigma = \prod_{k=1}^{l}(i_k, n+2-k)$, with $i_1 = 1 < i_2 < \cdots < i_l$ for some integer l, and $i_k < n+2-k$ for all $k \in [l]$.*

Proof. We prove only statement 1 (the proof of 2 is analogous). Since σ is an involution, if there is a one-element commutation class of σ that contains a letter 1 as endpoint, there must be $s \in R_\bullet(\sigma)$ such that $s = \underline{1}\overline{n}\underline{1}t$, for some word $t \in [2, n-1]^*$. Note that the permutation associated to t (call it π) is also an involution because the left factor $\underline{1}\overline{n}\underline{1}$ is a reduced word for the cycle $(1, n+1)$ and π fixes 1 and $n+1$. Since the left-endpoint of t is a letter 2, we must have $t = \underline{2}\overline{j}\underline{2}t'$ for some $j < n$ and $t' \in [3, j-1]^*$. Continuing this procedure, we conclude that

$$s = (1 \cdots j_1 \cdots 1)(2 \cdots j_2 \cdots 2) \cdots (l \cdots j_l \cdots l), \tag{3}$$

with $n = j_1 > \cdots > j_l$. Notice that l can be equal to j_l if the right endpoint of s is a pinnacle. In that case $(l \cdots j_l \cdots l)$ would be just the letter l. Each factor $(k \cdots j_k \cdots k)$ encodes the involution (k, j_k+1), so $\sigma = \prod_{k=1}^{l}(k, j_k+1)$. Moreover, s does not contain any factor with repeated or symmetric segments. Therefore, from Theorem 3.12, s is a reduced word of σ.

For the converse, just consider s as in (3), which is a reduced word for σ and is a word formed by consecutives integers. □

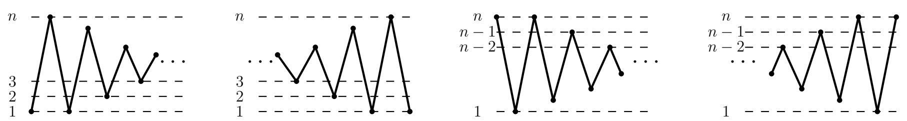

FIGURE 8. Possible diagrams for a one-element commutation class of an involution that fixes neither 1 nor $n+1$

If $s = \underline{1}\overline{n}\underline{1}t$, then its associated permutation is completly determined by its pinnacles, as we saw in the previous proof. Hence we can conclude the following.

Lemma 4.5. *Let σ be an involution in $\mathfrak{S}_{n+1}$ that fixes neither* 1 *nor* $n+1$*. Then, there is at most one word in $R_\bullet(\sigma)$ of the form $\underline{1}\overline{n}\underline{1}t$ (the same is true for its symmetries).*

Before stating one of the main results of this section, we need two auxiliar lemmas.

Lemma 4.6. *Let σ be an involution in $\mathfrak{S}_{n+1}$ that fixes neither* 1 *nor* $n+1$ *and $s \in R_\bullet(\sigma)$. If $\boldsymbol{p}(s)$ or $\boldsymbol{v}(s)$ is a symmetry of the word $1\ 2\ \cdots l$ for some l, then $s^c \in R_\bullet(\sigma)$.*

Proof. Suppose that $s = \underline{1}\overline{n}\underline{1}t$ for some $t \in [2, n-1]^*$ (the other cases are analogous). Since $\mathbf{v}(s)$ is not strict we must have that $\mathbf{p}(s)$ is the string that belong to the symmetries of $1\ 2\ \cdots l$ for some l. From the proof of the previous lemma $\sigma = \prod_{k=1}^{l}(k, n+2-k)$, and since $w_0\sigma w_0 = \sigma$, we have $s^c \in R_\bullet(\sigma)$. □

Lemma 4.7. *Let σ be an involution in $\mathfrak{S}_{n+1}$ that fixes neither* 1 *nor* $n+1$ *and $s \in R_\bullet(\sigma)$. If $s = s^r$, then $s \equiv \underline{1}\overline{n}\underline{1}$ or $s \equiv \overline{n}\underline{1}\overline{n}$.*

Proof. The assumption that $s = s^r$ implies that its endpoints are equal. Therefore, both of its endpoints are the letter 1 or n, which from Lemma 4.5 we have the result. □

Theorem 4.8. *Let σ be an involution in $\mathfrak{S}_{n+1}$ that fixes neither* 1 *nor* $n+1$*. Suppose that $s \in R_\bullet(\sigma)$.*

(1) *If $n = 1$, then $R_\bullet(\sigma) = \{1\}$.*
(2) *If $\sigma = (1, n+1)$, then $R_\bullet(\sigma) = \{\underline{1}\overline{n}\underline{1}, \overline{n}\underline{1}\overline{n}\}$.*
(3) *If $n > 1$ and neither $\boldsymbol{p}(s)$ nor $\boldsymbol{v}(s)$ is a symmetry of the word $1\ 2\ \cdots l$ for some integer l, then $R_\bullet(\sigma) = \{s, s^r\}$.*
(4) *If $n > 1$, $\sigma \neq (1, n+1)$ and one of the strings $\boldsymbol{p}(s)$ or $\boldsymbol{v}(s)$ is a symmetry of the word $1\ 2\ \cdots l$, then $R_\bullet(\sigma) = \{s, s^r, s^c, s^{cr}\}$*

Proof. Condition 1 is trivial. From Lemma 4.5, we have at most one word $s \in R_\bullet(\sigma)$ with left (resp. right) endpoint the letter 1 or with left (resp. right) endpoint the letter n. If $\sigma = (1, n+1)$, then $\underline{1}\overline{n}\underline{1}$ and $\overline{n}\underline{1}\overline{n}$ are one-element commutation classes and they must be the only ones. To prove conditions 3 and 4, we are going to assume that $s = \underline{1}\overline{n}\underline{1}t$ (the other cases are analogous). We need to consider two cases:

Case 1: $\mathbf{p}(s)$ is not a symmetry of $1\ 2\ \cdots l$

From Lemma 4.4 we have $\sigma = \prod_{k=1}^{l}(k, j_k+1)$, where j_k is the k-th pinnacle of s and $k < j_k$. Our goal is to show that σ cannot contain one-element commutation classes with endpoint the letter n. If it contains such class, from Lemma 4.4 we have $\sigma = \prod_{k=1}^{l}(i_k, n+2-k)$. Since $i_k < n+2-k$ for all $k \in [l]$, and since σ has a unique decomposition into disjoint cycles, we must have $\mathbf{p}(s) = j_1 j_2 \cdots j_l = n\ n \cdots n+1-l$, which is a symmetry of the word $1\ 2 \cdots l$, contradicting our assumption. Since $s^r \in R_\bullet(\sigma)$, we have $R_\bullet(\sigma) = \{s, s^r\}$.

Case 2: $\mathbf{p}(s)$ is a symmetry $1\ 2\ \cdots l$

From Lemma 4.6 we have $s^c \in R_\bullet(\sigma)$, and since σ is an involution, $s^{cr} \in R_\bullet(\sigma)$. The fact that $\sigma \neq (1, n+1)$ implies that $s \neq s^r$. Therefore, we have $R_\bullet(\sigma) = \{s, s^r, s^c, s^{cr}\}$. □

For an arbitrary involution $\sigma \in \mathfrak{S}_{n+1}$, the previous theorem can be generalized by defining the complementary word of $s = i_1 i_2 \cdots i_l$ as

$$(M+m-1-i_1)(M+m-1-i_2)\cdots(M+m-1-i_l),$$

where m and M are the minimum and maximum non-fixed points of σ, respectively.

We end this section with a relation between involutions that contains one-element commutation classes and pattern avoidance problems.

Definition 4.1. *Let $\sigma \in \mathfrak{S}_{n+1}$ and $p \in \mathfrak{S}_k$ with $k \leq n+1$. We say that σ* contains *the patters p if there is a substring of the one-line notation of σ order isomorphic to p. If not, we say that σ is p*-avoiding.

When writting permutation as patterns, we drop the brackets and commas. For instance, the permutation $[4, 1, 2, 5, 3]$ is 321-avoiding, but contains two patterns 123, namely the substrings 1 2 5 and 1 2 3.

Proposition 4.9. *Let σ be an involution in $\mathfrak{S}_{n+1}$ that fixes neither* 1 *nor $n+1$. If σ is* 132 *and* 3412*-avoiding, then $\sigma(1) = n+1$ and $\sigma(n+1) = 1$.*

Proof. Assume by contradiction that $\sigma(1) \neq n+1$ and $\sigma(1) \neq 1$. Since σ does not fix 1 nor $n+1$, we must have

$$\sigma = [i, \ldots, 1, \ldots, n+1, \ldots, j]$$

or

$$\sigma = [i, \ldots, n+1, \ldots, 1, \ldots, j],$$

for some integers $i, j \in [2, n]$. The first case cannot hold because we have the subword $1 \ n+1 \ j$, which is a 132-pattern. In the second case, if $i < j$, then $i \ n+1 \ j$ is a 132-pattern. If $i > j$, then σ contains the substring $i \ n+1 \ 1 \ j$, which is a 3412-pattern. Therefore, we must have $\sigma(1) = n+1$ or $\sigma(1) = n+1$. The fact that σ is an involution implies $\sigma(1) = n+1$ and $\sigma(n+1) = 1$. ☐

Before stating the main result of this section, we need to recall two known results about pattern avoidance.

Proposition 4.10 ([8])**.** *A permutation σ is* 2143 *and* 3412*-avoiding if and only if σ can be partitioned into an increasing and decreasing sequence.*

For instance, the permutation $\sigma = [3, 6, 4, 7, 5, 2, 1]$ is 2143 and 3412-avoiding because we can partioned σ into the sequences 3 4 7 and 6 5 2 1.

Proposition 4.11 ([2])**.** *Let $\sigma \in \mathfrak{S}_{n+1}$ and $p \in \mathfrak{S}_k$ with $k \leq n+1$. Then σ contains the pattern p if and only if $w_0 \sigma w_0$ contains the patterns $w_0 p w_0$.*

We have the following.

Theorem 4.12. *Let σ be an involution in $\mathfrak{S}_{n+1}$ that fixes neither* 1 *nor $n+1$. Then, $|R_\bullet(\sigma)| > 0$ if and only if σ avoids the patterns* 132 *and* 3412 *or the patterns* 213 *and* 3412.

Proof. Assume that $|R_\bullet(\sigma)| > 0$. Using Lemma 4.4, we start by consider $\sigma = \prod_{k=1}^{l}(k, j_k + 1)$, with $j_1 = n > j_2 > \cdots > j_l$ for some integer l. To prove that σ is 3412-avoiding, just notice that the non-fixed points forms a decreasing sequence and the fixed points forms an increasing sequence. Then, by Proposition 4.10 we have the result. It remains to prove that σ is 132-avoiding. By way of contradiction, assume that σ contains a pattern 132. Then σ contains a subword xzy such that $x < y$, $x < z$ and $z > y$. We then have two cases.

Case 1: x is not a fixed point.

Since the sequence of non-fixed points of σ is decreasing, we have that y and z are fixed points. But that cannot happen because $z > y$ and the sequence of fixed points is increasing.

Case 2: x is a fixed point.

If x is a fixed point of σ, then $x > l$. We have that all non-fixed points of σ that are to the right of x are smaller than x. Since $x < y$ and $x < z$, the integers y and z are also fixed points of σ. But that cannot be possible because $z > y$. Therefore, σ must be 132-avoiding.

Now considering the case where $\sigma = \prod_{k=1}^{l}(i_k, n+2-k)$, with $i_1 = 1 < i_2 < \cdots < i_l$ for some integer l, we have that $w_0\sigma w_0 = \prod_{k=1}^{l}(k, n+2-i_k)$, which we already proved is 132 and 3412-avoiding. From Proposition 4.11, σ is 213 and 3412-avoiding.

For the converse, suppose that σ avoids the patterns 132 and 3412. From Lemma 4.9, we have $\sigma(1) = n+1$ and $\sigma(n+1) = 1$, so σ contains the cycle $(1, n+1)$ in its disjoint cycle decomposition. Without loss of generality, the disjoint decomposition of σ can be written as

$$\sigma = (i_1, j_1)(i_2, j_2)\cdots(i_l, j_l),$$

with $1 = i_1 < i_2 < i_3 < \cdots < i_l$, $j_1 = n+1$ and $i_k < j_k$ for all $k \in [l]$. Notice that $i_2 = 2$, otherwise $2 < i_2 < j_2$ and 2 would be a fixed point of σ implying that $2\ j_2\ i_2$ would be a substring of σ, which is a 132-pattern. We also have that $n+1 = j_1 > j_2$. Now suppose that

$$\sigma = (1\ n+1)(2\ j_2)\cdots(m\ j_m)(i_{m+1}\ j_{m+1})\cdots(i_l, j_l),$$

with $n+1 > j_2 > j_3 > \cdots j_m$ and $i_k < j_k$ for all $k \in [m+1, l]$. Our goal is to show that $i_{m+1} = m+1$ and $j_m > j_{m+1}$. If $i_{m+1} \neq m+1$, then $m+1 < i_{m+1} < j_{m+1}$ and we have two cases. If $m+1$ is a fixed point, then σ will contain the subword $m+1\ j_{m+1}\ i_{m+1}$, which is a 132-pattern. If $m+1$ is not a fixed point, then $(m, j_m) = (m, m+1)$. But then, σ will contain the subword $m\ j_{m+1}\ i_{m+1}$, which is a 132-pattern. Therefore $i_{m+1} = m+1$. If $j_m < j_{m+1}$, then σ will contain the subword $j_m\ j_{m+1}\ m\ m+1$, which is a 3412-pattern. Using an inductive argument we show that $\sigma = (1, j_1)(2, j_2)\cdots(l, j_l)$, with $n+1 = j_1 > j_2 > \cdots > j_l$. From Lemma 4.4, we have that σ contains a one-element commutation class with some endpoint the letter 1. If σ avoids the patters 213 and 3412, then $w_0\sigma w_0$ avoids the patterns 132 and 3412 from Proposition 4.11, which we proved to have a one-element commutation class s. Hence s^c is a one-element commutation class for σ, proving that $R_\bullet(\sigma) > 0$. $\square$

Notice that this result is not true in general for any permutation that does not fix 1 nor $n+1$. For instance, the permutation $[3, 4, 5, 2, 1]$ is 132, 213 and 3412- avoiding and does not contain one-element commutation classes.

As a corollary, we have a necessary and sufficient condition for an involution to contain 4 one-element commutation classes.

Corollary 4.13. *Let σ be an involution in $\mathfrak{S}_{n+1}$ that fixes neither 1 nor $n+1$. Assume that $\sigma \neq (1, n+1)$. The following are equivalent:*

(1) $|R_\bullet(\sigma)| = 4$.

(2) *σ is* 132*,* 213 *and* 3412*-avoiding.*

Proof. From Theorem 4.8, if $|R_\bullet(\sigma)| = 4$, then $R_\bullet(\sigma) = \{s, s^r, s^c, s^{cr}\}$ for some word $s \in [n]^*$ formed by consecutive integers. We can assume that $s = \underline{1}\overline{n}\underline{1}t$ for some word $t \in [2, n-1]^*$. From the proof of the previous theorem, its associated permutation is 132 and 3412 avoiding. Since $s^c \in R_\bullet(\sigma)$, we have $\sigma = w_0\sigma w_0$ and from Proposition 4.11, σ will also avoid the pattern 213. Reciprocally, since σ avoids the patterns 132 and 3412, from the proof of the previous theorem we have that σ contains a one-element commutation class s with some endpoint the letter 1. We also have that σ is 213 and 3412-avoiding, so σ contains a one-element commutation class t with some endpoint the letter n. Since $\sigma \neq (1, n+1)$, from Lemma 4.5 we have $R_\bullet(\sigma) = \{s, s^r, t, t^r\}$. □

Corollary 4.13 allow us to recover the result from [5] and [12] which states that for $n > 1$ the longest permutation w_0 contains 4 one-element commutation classes, since w_0 is an involution that fix neither 1 nor $n+1$ and the only patterns of length 3 and 4 that w_0 contains are 321 and 4321, respectively. One can generalize the previous results to any involution.

Corollary 4.14. *Let $\sigma \in S_{n+1}$ an involution such that m and M are the minimum and maximum non-fixed points of σ. Consider the permutation $\pi = [\sigma(m), \sigma(m+1), \cdots, \sigma(M)]$. Then:*

(1) *$|R_\bullet(\sigma)| > 0$ if and only if π avoids the patterns* 132 *and* 3412 *or the patterns* 213 *and* 3412*.*
(2) *Suppose that $\sigma \neq (m, M)$. Then $|R_\bullet(\sigma)| = 4$ if and only if π avoids the patterns* 132*,* 213 *and* 3412*.*

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