HAUSDORFF MEASURE ESTIMATES FOR THE DEGENERATE QUENCHING PROBLEM

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ABSTRACT. We study analytical and geometric properties of minimizers of non-differentiable functionals epitomizing the degenerate quenching problem. Our main finding unveils finite $(n-1)-$Hausdorff measure estimates for the pertaining free boundaries. The approach hinges upon deriving optimal gradient decay estimates, coupled with a fine analysis of an intrinsic auxiliary equation stripped of the singularity.

1. Introduction

In this work, we are concerned with minimizers of $p-$energy functionals of the type

$$v \longmapsto \int_\Omega \frac{|Dv|^p}{p} + F(x,v)\,dx,$$

for non-differentiable potentials $F(x,s) \geq 0$.

This class of problems is well-understood in the case $p=2$, particularly in the context of the obstacle problem, for which $F(x,s) = s_+$ (see [6, 8] and the book [26]). Another emblematic potential is the discontinuous $F(x,s) = \chi_{\{s>0\}}$, leading to the cavity or Bernoulli free boundary problem, investigated by Alt and Caffarelli in [1]. An intermediary scenario emerges by interpolating between the two previous cases and considering potentials exhibiting $\gamma-$growth, namely with

$$F(x,s) = s_+^\gamma, \qquad \gamma \in (0,1). \tag{1.1}$$

Often known as Alt-Phillips potentials (*cf.* [2, 27, 28]), they bring about the so-called quenching problem, which models phenomena characterized by abrupt changes in certain quantities along unknown interfaces. In all these cases, the absence of differentiability for F strongly influences the efficiency of the regularity mechanisms for minimizers compared to the classical scenarios. Still, for $p=2$, it has been shown that the optimal regularity class

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for minimizers is $C^{1,\alpha}_{\text{loc}}$, with

$$\alpha = \frac{\gamma}{2-\gamma}, \qquad \gamma \in (0,1].$$

For the cavity problem, corresponding to $\gamma = 0$, minimizers are locally Lipschitz. Concerning the free boundary, in the seminal works [7] for the obstacle problem, and [28] for the case $\gamma \in (0,1)$, analytical and geometric measure theory methods were applied to establish that interfaces have finite $(n-1)-$dimensional Hausdorff measure. For the two-phase problem ($p = 2$), the free boundaries are known to be C^1 in dimension two (see [23]).

For the nonlinear case $p \neq 2$ much less is known as the challenges posed by the non-quadratic growth in the functional muddle the understanding of the regularity and geometric properties of the minimizes and the free boundary. Obstacle problems, corresponding to $\gamma = 1$, were treated in [3, 14] for non-zero obstacles φ. In particular, in [3], the authors show that a minimizer u is of class $C^{1,p'-1}$ at the free boundary $\partial\{u > \varphi\}$, for $p' = p/(p-1)$. For the sign-changing case, the intermediate scenario $\gamma \in (0,1)$ was treated in [22]. In [5], the authors considered potentials with varying non-differentiability levels

$$F(x,s) \sim (s-\varphi(x))_+^{\gamma},$$

obtaining improved regularity estimates at contact points in $\partial\{u > \varphi\}$. For the particular case of an obstacle $\varphi \equiv 0$, it is shown therein that minimizers are of class $C^{1,\alpha}$ at the quenching interface $\partial\{u > 0\}$, for

$$\alpha = \frac{\gamma}{p-\gamma},$$

revealing the precise interplay between the singularity parameter γ and the exponent p in determining the regularity of minimizers. Related problems were studied in [20, 11], establishing the Lipschitz regularity of minimizers using a singular perturbation. These efforts notwithstanding, the study of the free boundary for problems involving the $p-$Laplace operator remains virtually virgin ground, the only significant contribution being the results in [17, 21, 14, 10] for the $p-$obstacle problem, and in [12] for the cavity problem, where it was shown that near flat points, the free boundary is of class $C^{1,\beta}$, for a $\beta \in (0,1)$.

In this paper, we advance the theory one step forward, carrying the analysis in the degenerate case $p > 2$ and for the quenching scenario (1.1), for which the corresponding Euler-Lagrange equation is singular since the right-hand side blows up at the free boundary

$$\mathcal{F}(u) = \partial\{u > 0\}.$$

We consider non-negative minimizers and chiefly obtain the finiteness of the $(n-1)-$dimensional Hausdorff measure of the free boundary, a result that plays a significant role in the context of trying to universally characterize the "size" or "dimension" of the interface. These estimates are closely related to the (lack of) regularity and the geometric properties of the free boundary. Our approach relies on a pointwise gradient control based on an intrinsic Harnack-type inequality and a suitable application of regularity estimates at points relatively close to the free boundary available in [5]. This gradient control is already crucial in [28] for treating the case $p = 2$, but its proof heavily relies on the linearity of the operator and fails to work in the degenerate case. After establishing a non-degeneracy estimate and porosity of the free boundary, we remove the singularity, thereby obtaining a problem with a positive and bounded right-hand side. This strategy, complemented by ensuring that the gradient of the right-hand side of a transformed problem belongs to a suitable Morrey space, allows us to analyze the free boundary regularity as that of a refined obstacle problem.

The paper is organized as follows. In section 2, we introduce some notation, formulate the problem rigorously and gather several preliminary results. Section 3 brings the optimal gradient decay, and, in Section 4, we establish a non-degeneracy estimate and the porosity of the free boundary. In Section 5, we remove the singularity in the problem and in Section 6, we prove the main result.

2. Problem setting and preliminary results

Let Ω be an $n-$dimensional bounded domain and $p > 2$. For a given non-negative boundary data $g \in W^{1,p}(\Omega)$, we consider minimizers of the functional

$$J(v) := \int_\Omega \left(\frac{|Dv(x)|^p}{p} + [v_+(x)]^\gamma \right) dx, \tag{2.1}$$

among competitors in the set

$$\mathbb{K} := \left\{ v \in W^{1,p}(\Omega);\ v \geq 0,\ v - g \in W_0^{1,p}(\Omega) \right\}. \tag{2.2}$$

The corresponding Euler-Lagrange equation is the singular PDE, holding in the non-coincidence set $\{u > 0\} \cap \Omega$,

$$\Delta_p u := \operatorname{div}\left(|Du|^{p-2} Du\right) = \gamma u^{\gamma-1}, \tag{2.3}$$

whose right-hand side blows up at free boundary points since γ is assumed to be in $(0, 1)$.

Most of the results in the paper are valid for *local minimizers* defined as follows. Set

$$J_r(v) := \int_{B_r} \left(\frac{|Dv(x)|^p}{p} + [v_+(x)]^\gamma \right) dx.$$

Definition 2.1. *A non-negative function $u \in W^{1,p}(\Omega)$ is called a local minimizer with respect to a ball $B_r \subset \Omega$ if*

$$J_r(u) \leq J_r(v),$$

for all $v \geq 0$ such that $u - v \in W_0^{1,p}(B_r)$.

Remark 2.1. *Observe that if u is a local minimizer with respect to a ball B_r, then*

$$u_s(x) := \frac{u(sx)}{s^{\frac{p}{p-\gamma}}}$$

is a local minimizer with respect to the ball $B_{\frac{r}{s}}$. More precisely,

$$J_r(u) = s^{\frac{p\gamma}{p-\gamma}} J_{\frac{r}{s}}(u_s).$$

The existence of a minimizer for (2.1) is obtained in [5, Theorem 2.1] (see also [22], for the particular case of a zero obstacle), together with the bound

$$\|u\|_\infty \leq \|g\|_\infty,$$

for bounded boundary data. The local $C^{1,\alpha}-$regularity is the object of [5, Theorem 3.1], where the existence of a constant $C > 0$, depending only on $\operatorname{dist}(\Omega', \partial\Omega)$, $\|u\|_{L^\infty(\Omega)}$, n, p and γ, is obtained such that

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C, \tag{2.4}$$

for

$$\alpha := \min\left\{\sigma^-, \frac{\gamma}{p-\gamma}\right\}, \tag{2.5}$$

where $\sigma > 0$ is the Hölder regularity exponent for the gradient of $p-$harmonic functions. It follows from the results in [16] (see also [4]) that, for $n = 2$, one has $\alpha = \frac{\gamma}{p-\gamma}$. Note also that when $\gamma = 0$, the regularity result above recovers the Lipschitz regularity of solutions obtained in [11, Theorem 3.3]. Moreover, in [5, Theorem 4.1 and Theorem 6.1], the growth of the minimizer away from free boundary points is revealed to be

$$u \leq C\, r^{\frac{p}{p-\gamma}}, \quad \text{in } B_r(y) \tag{2.6}$$

for any $r \in (0, r_0)$ and any $y \in \partial\{u > 0\} \cap \Omega'$, where $\Omega' \subset\subset \Omega$, for universal constants $C > 0$ and $r_0 > 0$. The approach is based on geometric tangential analysis and a fine perturbation combined with an adjusted scaling argument, the idea of which is to ensure that at the limit, one gets a linear elliptic equation without the zero-order term. The intuition behind the proof is that the problem behaves essentially as an obstacle problem for a uniformly elliptic operator.

We conclude this section by showing that any minimizer u of (2.1) is $p-$subharmonic and solves the corresponding Euler-Lagrange equation. As $u \in W^{1,p}(\Omega)$, $p-$subharmonicity is equivalent to proving that u stays below any $p-$harmonic replacement, according to [24, Chapter 5]. The proof is known and can be obtained by combining tools used in [1, 11, 22, 27]. We include it below for the reader's convenience.

Lemma 2.1. *If u is a minimizer of* (2.1) *and v is a $p-$harmonic function in $B \subset \Omega$ that agrees with u in $\Omega \setminus B$, then $u \leq v$.*

Proof. Set $w := \min(u, v)$. We aim to show that, in fact, $w = u$. As $w - u \in W^{1,p}(B)$, one has $D(w-u) = 0$ a.e. on $\{w = u\}$, [18, Lemma A.4]. Hence,

$$\int_B |Dw|^{p-2} Dw \cdot D(w-u) = \int_B |Dv|^{p-2} Dv \cdot D(w-u) = 0, \tag{2.7}$$

where the last equality follows from the fact that v is $p-$harmonic and agrees with u on ∂B. On the other hand, defining for $0 \leq s \leq 1$,

$$u_s(x) := su(x) + (1-s)w(x),$$

recalling (2.7) and noting that $u_s - w = s(u-w)$, we have

$$\begin{aligned}
&\int_B (|Du|^p - |Dw|^p) \\
&= \int_0^1 \frac{d}{ds} \left(\int_B |Du_s|^p \right) ds \\
&= p \int_0^1 ds \int_B |Du_s|^{p-2} Du_s \cdot D(u-w) \\
&= p \int_0^1 ds \int_B \big(|Du_s|^{p-2} Du_s - |Dw|^{p-2} Dw \big) \cdot D(u-w) \\
&= p \int_0^1 \frac{ds}{s} \int_B \big(|Du_s|^{p-2} Du_s - |Dw|^{p-2} Dw \big) \cdot D(u_s - w).
\end{aligned}$$

Combining this with the well-known inequality

$$\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta \big) \cdot (\xi - \eta) \geq c\, |\xi - \eta|^p,$$

where $c > 0$ is a constant depending only on n and p, we reach

$$\begin{aligned}
\int_B (|Du|^p - |Dw|^p) \quad &\geq \quad cp \int_0^1 \frac{ds}{s} \int_B |D(u_s - w)|^p \\
&= \quad cp \int_0^1 s^{p-1}\, ds \int_B |D(u-w)|^p \\
&= \quad c \int_B |D(u-w)|^p \geq 0.
\end{aligned}$$

Using this and the fact that $w \leq u$, we obtain

$$J(w) - J(u) \leq \int_B (w_+^\gamma - u_+^\gamma) \leq 0.$$

By the minimality of u, we conclude that $J(w) = J(u)$, which only holds if $w = u$. □

Proposition 2.1. *If u is a minimizer of* (2.1)*, then*

$$\Delta_p u = \gamma u^{\gamma-1} \ \text{in} \ \{u > 0\}$$

in the sense of distributions.

Proof. Let $y \in \Omega$ be such that $u(y) > 0$. Since u is continuous (particularly), 2.4, then for $r > 0$ small, one has $u \geq u(y)/2$ in $B_r(y)$. Therefore, if $\xi \in C_0^\infty(\Omega)$ and $|\varepsilon|$ is small enough, then $u + \varepsilon\xi \geq u(y)/4$, *i.e.*, $u + \varepsilon\xi \in \mathbb{K}$. As u is a minimizer of (2.1), then $\varepsilon = 0$ is a minimizer of $\varepsilon \mapsto J(u + \varepsilon\xi)$. In other words, one should have $J'(u + \varepsilon\xi)|_{\varepsilon=0} = 0$, *i.e.*,

$$\int_{B_r(y)} \left(|Du|^{p-2} Du \cdot D\xi + \gamma u^{\gamma-1}\xi \right) \, dx = 0,$$

which is the desired result. □

3. Pointwise gradient estimate

In this section, we prove a pointwise gradient estimate, which plays an essential role in the analysis of the free boundary. We first establish an intrinsic Harnack-type inequality for small radii.

Lemma 3.1. *If u is a local minimizer of* (2.1) *in B_1, then there exists a constant $C > 0$, depending only on n, p, γ and $\|u\|_\infty$, such that*

$$\sup_{B_r(x)} u \leq C\, u(x),$$

for each $x \in B_{1/2}$ such that $r := [u(x)]^{\frac{p-\gamma}{p}} \leq 1$.

Proof. Suppose the conclusion fails. Then there exist a sequence of points $x_k \in B_{1/2}$ and local minimizers u_k in B_1, such that

$$s_k := \sup_{B_{r_k}(x_k)} u_k > k u_k(x_k) = k r_k^{\frac{p}{p-\gamma}}. \tag{3.1}$$

where $r_k := [u_k(x_k)]^{\frac{p-\gamma}{p}}$. Set

$$v_k(y) := s_k^{-1} u_k\left(x_k + r_k y\right) \quad \text{in } B_1.$$

Note that $v_k(y) \in [0,1]$, and, using (3.1),

$$v_k(0) = s_k^{-1} u_k(x_k) = s_k^{-1} r_k^{\frac{p}{p-\gamma}} < \frac{1}{k}.$$

Additionally,

$$\sup_{B_1} v_k = 1.$$

On the other hand, v_k minimizes

$$\int_{B_1} \left(\frac{|Dv|^p}{p} + \frac{r_k^p}{s_k^{p-\gamma}} v^\gamma \right) dx,$$

and, recalling (3.1) once more,

$$\frac{r_k^p}{s_k^{p-\gamma}} < \frac{1}{k^{p-\gamma}}. \tag{3.2}$$

Hence, from 2.4, up to a subsequence, v_k converges locally uniformly to a function v_∞ defined in B_1. Obviously, $v_\infty(0) = 0$ and

$$\sup_{B_1} v_\infty = 1. \tag{3.3}$$

Moreover, from (3.2), it follows that v_∞ minimizes

$$\int_{B_1} \frac{|Dv|^p}{p} \, dx,$$

so v_∞ is $p-$harmonic in B_1. Hence, as $v_\infty(0) = 0$, we conclude that $v_\infty \equiv 0$, which contradicts (3.3). □

As a consequence, we obtain the following pointwise gradient estimate.

Theorem 3.1. *If u is a local minimizer of* (2.1) *in B_1, then there exists a constant $C > 0$, depending only on n, p, γ and $\|u\|_\infty$, such that*

$$|Du(x)|^p \leq C\, u^\gamma(x),$$

for each $x \in B_{1/2}$.

Proof. Let $x \in B_{1/2}$. We divide the proof into two steps.

Step 1. If $\|u\|_\infty \leq 1$, then setting

$$v(y) := r^{-\frac{p}{p-\gamma}}\, u(x + ry), \quad y \in B_1,$$

for $r := [u(x)]^{\frac{p-\gamma}{p}} \leq 1$ and applying Lemma 3.1, we conclude that there exists a universal constant $C > 0$ such that

$$\sup_{B_1} v \leq C.$$

Local $C^{1,\alpha}$ regularity of u implies, in particular,

$$|Dv(0)| \leq C' C,$$

for a universal constant $C' > 0$. Observe now that

$$|Dv(0)| = r^{-\frac{\gamma}{p-\gamma}} |Du(x)| = |Du(x)| u^{-\frac{\gamma}{p}}(x),$$

and the result follows.

Step 2. If $\|u\|_\infty > 1$, set $r := [u(x)/\|u\|_\infty]^{\frac{p-\gamma}{p}} \leq 1$. Applying Lemma 3.1 for $w(x) := [u(x)/\|u\|_\infty]$, we obtain

$$\sup_{B_r(x)} w \leq Cw(x).$$

Therefore, arguing as in Step 1, we get

$$|Dw(x)|^p \leq Cw^\gamma(x),$$

i.e.,

$$|Du(x)|^p \leq C\|u\|_\infty^{p-\gamma} u^\gamma(x).$$

□

4. Non-degeneracy and porosity

In this section, we prove non-degeneracy and positive density results for minimizers of (2.1), obtaining, as a consequence, the porosity of the free boundary.

Lemma 4.1. *If u is a minimizer of (2.1) and $x_0 \in \overline{\{u > 0\}}$, then there exists a constant $c > 0$, depending only on n, p and γ, such that*

$$\sup_{\partial B_r(x_0)} u \geq c r^{\frac{p}{p-\gamma}}, \tag{4.1}$$

for any $r < \operatorname{dist}(x_0, \Omega)$.

Proof. By continuity, it is enough to prove the result for $x_0 \in \{u > 0\}$. Set

$$\lambda := \frac{p-\gamma}{p-1}$$

and define

$$v(x) := u^\lambda(x) - c|x - x_0|^{\frac{p}{p-1}} \quad \text{in} \quad \{u > 0\} \cap B_r(x_0),$$

for a constant $c > 0$. A direct calculation shows that

$$\Delta_p \left(u^\lambda\right) = \lambda^{p-1} u^{-\gamma} \left[(1-\gamma)|Du|^p + u\Delta_p u\right],$$

which, recalling Proposition 2.1, leads to

$$\Delta_p \left(u^\lambda\right) = \lambda^{p-1} u^{-\gamma} \left[(1-\gamma)|Du|^p + \gamma u^\gamma\right] \geq \gamma\lambda^{p-1} > 0.$$

Moreover,

$$\Delta_p(c|x - x_0|^{\frac{p}{p-1}}) = c^{p-1} n \left(\frac{p}{p-1}\right)^p.$$

Since $\Delta_p v$ is continuous with respect to c, we can choose $c > 0$ so small that $\Delta_p v \geq 0$ in $\{u > 0\} \cap B_r(x_0)$. As $v(x_0) > 0$, the maximum principle implies the existence of $y \in \partial\left(\{u > 0\} \cap B_r(x_0)\right)$ such that $v(y) > 0$. But $v \leq 0$ on $\partial\{u > 0\}$, hence $y \in \partial B_r(x_0)$, and therefore $u(y) \geq c r^{\frac{p}{p-\gamma}}$. □

Remark 4.1. *Note that from 2.6, we already knew that in $B_r(x_0)$, with $x_0 \in \partial\{u > 0\}$, the supremum of a minimizer grows at most like $r^{\frac{p}{p-\gamma}}$. Lemma 4.1 reveals that it grows exactly as $r^{\frac{p}{p-\gamma}}$.*

As a consequence of the lemma, we get the following positive density result.

Corollary 4.1. *If u is a minimizer of* (2.1)*, then, for every $\Omega' \subset\subset \Omega$, there exists a constant $\delta \in (0,1)$, depending only on Ω', n, p and γ, such that, for any small ball $B_r(x_0) \subset \Omega'$, with $x_0 \in \partial\{u > 0\}$, one has*

$$\frac{|\{u > 0\} \cap B_r(x_0)|}{|B_r(x_0)|} \geq \delta.$$

Proof. Indeed, Lemma 4.1 guarantees the existence of $y \in \partial B_{r/2}(x_0)$ such that

$$u(y) \geq c r^{\frac{p}{p-\gamma}} > 0.$$

Therefore, $u > 0$ in $B_{\rho r}(y) \subset B_r(x_0)$, for $\rho > 0$ small. In other words, $B_{\rho r}(y) \subset \{u > 0\} \cap B_r(x_0)$. Thus,

$$\frac{|\{u > 0\} \cap B_r(x_0)|}{|B_r(x_0)|} \geq \frac{|B_{\rho r}(y)|}{|B_r(x_0)|} = \rho^n =: \delta.$$

□

We recall the definition of a porous set to state another consequence of the non-degeneracy estimate.

Definition 4.1. *A set $E \subset \mathbb{R}^n$ is called porous, with porosity constant $\delta > 0$, if there exists a constant $\rho > 0$ such that, for each $x \in E$ and $r \in (0, \rho)$, there exists a $y \in \mathbb{R}^n$ such that*

$$B_{\delta r}(y) \subset B_r(x) \setminus E.$$

The Hausdorff dimension of a porous set does not exceed $n - C\delta^n$, where $C > 0$ is a constant depending only on n (see, for example, [25]). Hence, the Lebesgue measure of a porous set is zero.

Corollary 4.2. *Let u be a minimizer of* (2.1)*. If $x_0 \in \Omega$ and $r > 0$ are such that $B_{2r}(x_0) \subset \Omega$, then the set*

$$E := \partial\{u > 0\} \cap \overline{B_r(x_0)}$$

is porous. Hence, the free boundary $\partial\{u > 0\}$ is a set of Lebesgue measure zero.

Proof. Let $x \in E$. We have $B_{r/2}(x) \subset B_{2r}(x_0) \subset \Omega$. From Lemma 4.1, there exists $y \in \partial B_{r/2}(x)$ such that

$$u(y) \geq c r^{\frac{p}{p-\gamma}},$$

for a constant $c > 0$ depending only on n, p and γ. Hence,

$$y \in B_{2r}(x_0) \cap \{u > 0\}.$$

Set $d(y) := \operatorname{dist}\left(y, \overline{B_{2r}(x_0)} \setminus \{u > 0\}\right)$, then 2.6 provides

$$u(y) \leq C\,[d(y)]^{\frac{p}{p-\gamma}},$$

for a constant $C > 0$ depending only on n, p and γ. Therefore, setting

$$\delta := \min\left\{\frac{1}{2}, \left[cC^{-1}\right]^{\frac{p-\gamma}{p}}\right\} < 1,$$

we have

$$d(y) \geq \delta r.$$

Hence, $B_{\delta r}(y) \subset B_{d(y)}(y) \subset \{u > 0\}$. In particular,

$$B_{\delta r}(y) \cap B_r(x) \subset \{u > 0\}.$$

On the other hand, if $z \in [x, y]$ is such that $|z - y| = \delta r/2$, then

$$B_{(\delta/2)r}(z) \subset B_{\delta r}(y) \cap B_r(x).$$

Indeed, if $z^* \in B_{(\delta/2)r}(z)$, then

$$|z^* - y| \leq |z^* - z| + |z - y| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r$$

and, since $|x - y| = |x - z| + |z - y|$,

$$|z^* - x| \leq |z^* - z| + |x - y| - |z - y| < \frac{\delta r}{2} + r - \frac{\delta r}{2} = r.$$

Thus,

$$B_{(\delta/2)r}(z) \subset B_{\delta r}(y) \cap B_r(x) \subset B_r(x) \setminus \partial\{u > 0\} \subset B_r(x) \setminus E,$$

i.e., E is porous with porosity constant $\delta/2$. □

5. Removing the singularity

One of the main difficulties in analyzing equation (2.3) is the fact that the right-hand side blows up across the free boundary $\partial\{u > 0\}$. However, Proposition 2.1 combined with Theorem 3.1 allow us to transform the problem into one with a bounded right-hand side. That is the gist of this section.

Let u be a minimizer of (2.1) and set

$$v(x) := u^{\theta}(x),\ x \in \Omega, \tag{5.1}$$

where $\theta > 1$ is defined by

$$\theta := \max\left\{\frac{p-1}{p-2}, \frac{p+1}{p-1}\right\} = \begin{cases} \frac{p-1}{p-2} & \text{if} \quad 2 < p \leq 3 \\ \\ \frac{p+1}{p-1} & \text{if} \quad p \geq 3. \end{cases} \tag{5.2}$$

Recalling Proposition 2.1, in $\{u > 0\} = \{v > 0\}$, one has

$$\Delta_p v = f,$$

where

$$f := \theta^{p-1} u^{(\theta-1)(p-1)-1}\left[(\theta-1)(p-1)|Du|^p + \gamma u^\gamma\right]. \tag{5.3}$$

Observe that $(\theta - 1)(p - 1) - 1 \geq 0$, and therefore, in $\{u > 0\}$, one has

$$0 < f \leq C, \tag{5.4}$$

where the upper bound is a consequence of Theorem 3.1, with $C > 0$ a constant depending only on n, p, γ and $\|u\|_\infty$. Thus, v is a (weak) solution of the obstacle problem

$$\begin{cases} \Delta_p v = f \ \text{ in } \ \{v > 0\}, \\ v \geq 0 \ \text{ in } \ \Omega, \\ v - g^\theta \in W_0^{1,p}(\Omega), \end{cases} \tag{5.5}$$

where f is defined by (5.3).

Remark 5.1. *Note that the free boundaries for v and u coincide. Hence, the conclusions of Corollary 4.1 and Corollary 4.2 are valid for $\partial\{v > 0\}$.*

Remark 5.2. *Although the obstacle problem* (5.5) *has a unique solution (see, for example,* [9, 10, 26]*), every minimizer u of* (2.1) *generates a right-hand side f defined by* (5.3)*. Hence, there are as many v solutions as minimizers of* (2.1)*.*

Since v is the weak solution of (5.5), then we have

$$f\chi_{\{v>0\}} \leq \Delta_p v \leq f,$$

almost everywhere in Ω (cf. [9, Proposition 1.2]). Therefore, as a direct consequence of the porosity of the free boundary, we obtain

$$\Delta_p v = f \ \text{ a.e. in } \Omega.$$

Moreover, $v \in W^{2,2}_{\text{loc}}(\Omega)$ thanks to (5.4) (see [19, 29]), and, by (5.1), one has

$$|D^2 u| \leq \theta^2 \left(\frac{|Du|^2}{u} + u^{1-\theta}|D^2 v|\right), \tag{5.6}$$

in $\{u > 0\}$.

6. Hausdorff measure estimates

From Corollary 4.2, we already know that the $n-$dimensional Lebesgue measure of the free boundary is zero. In this section, we show that the $(n-1)-$dimensional Hausdorff measure of the free boundary is finite. The case $p = 2$ is treated in [28], but the nonlinear setting is considerably more demanding.

Recall that the $s-$dimensional Hausdorff measure of a set E is defined by

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E),$$

where

$$\mathcal{H}^s_\delta(E) := \inf \left\{ \sum_{j=1}^{\infty} \mu(s) \left(\frac{\operatorname{diam} E_j}{2} \right)^s \right\}.$$

Here, the infimum is taken over all countable coverings $\{E_j\}$ of E such that $\operatorname{diam} E_j \leq \delta$, $\mu(s) := \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}$ and $\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\, dt$, for $s > 0$, is the usual Gamma function.

Since the free boundaries of u and v coincide, we study the free boundary regularity for v, as the latter solves (5.5) with a bounded right-hand side f defined by (5.3).

Remark 6.1. *A Hausdorff measure estimate for problem* (5.5) *is obtained in* [21, Theorem 3.3]*, provided $f > 0$ is Lipschitz. As observed in* [10, Section 4] *(see also* [9]*), the Lipschitz continuity assumption can be relaxed, assuming instead that Df is locally in a Morrey space. More precisely, it can be replaced in our context by the assumption that*

$$\int_{B_r} |Df|\, dx \leq C r^{n-1}, \quad r \in (0,1). \tag{6.1}$$

Proposition 6.1. *The function f defined by* (5.3) *satisfies* (6.1).

Proof. Recalling (5.2) and (5.6), a direct calculation and the use of Young's inequality reveal

$$\begin{aligned} |Df| &\leq Cu^{(\theta-1)(p-1)-2}|Du|^{p+1} \\ &\quad + C' u^{(\theta-1)(p-1)-2+\gamma}|Du| \\ &\quad + C'' u^{(\theta-1)(p-1)-\theta}\left(|Du|^{2(p-1)} + |D^2 v|^2\right) \\ &\leq C\left(1 + |D^2 v|^2\right), \end{aligned}$$

where in the last inequality we used the fact that $(\theta-1)(p-1)-2 \geq 0$, $(\theta-1)(p-1)-\theta \geq 0$ and that u and its gradient are bounded. The constant $C > 0$ depends only on n, p and γ.

Then, we obtain, for $r \in (0,1)$,

$$\begin{aligned}\int_{B_r} |Df|\,dx &\leq Cr^n + C\int_{B_r} |D^2v|^2\,dx \\ &\leq Cr^n + Cr^n\|D^2v\|^2_{L^2(B_r)} \\ &\leq Cr^n, \end{aligned}\tag{6.2}$$

since $v \in W^{2,2}_{\text{loc}}(\Omega)$. The result follows because $r^n \leq r^{n-1}$. □

We next define a class of functions on the unit ball.

Definition 6.1. *We say that $v \in \mathcal{F}$ if, in B_1, it satisfies $\Delta_p v = f\chi_{\{v>0\}}$, $0 \leq v \leq 1$ and $0 \in \partial\{v > 0\}$.*

Since f is continuous and positive, one has the following estimates, deduced in [10, Lemma 4.3], [21, Lemma 2.4] and [21, Lemma 2.5].

Lemma 6.1. *If $v \in \mathcal{F}$ and $x_0 \in \partial\{v > 0\} \cap B_{1/2}$, then there exist positive constants $M = M(p,n,\gamma)$ and $0 < \sigma = \sigma(f)$ such that*

$$⨍_{B_r(x_0)} \left[|Dv(x)|^{p-2}|D^2v(x)|\right]^2\,dx \leq M, \quad \forall r \leq 1/2$$

and

$$\frac{\sigma}{(p-1)^2} \leq \left[|Dv(x)|^{p-2}|D^2v(x)|\right]^2, \quad a.e.\ in\ \{v > 0\}.$$

We now define the sets

$$Q_\varepsilon := \left\{|Dv| \leq \varepsilon^{\frac{1}{p-1}}\right\} \quad \text{and} \quad Q^i_\varepsilon := \left\{|v_{x_i}| \leq \varepsilon^{\frac{1}{p-1}}\right\}.$$

The next result is the main step towards proving the finiteness of the $(n-1)-$dimensional Hausdorff measure of the free boundary. Its proof is essentially the same as in [21, Theorem 3.3] (see also [10, Section 4]). We sketch it here for the reader's convenience.

Lemma 6.2. *If $v \in \mathcal{F}$ and $x_0 \in \partial\{v > 0\} \cap B_{1/2}$, then*

$$|Q_\varepsilon \cap B_r(x_0) \cap \{v > 0\}| \leq C\varepsilon r^{n-1}, \quad \forall r < 1/4,$$

where $C > 0$ is a constant depending only on n, p and γ.

Proof. Note that it is enough to prove that

$$\int_0^1 |Q_\varepsilon \cap B_{rs}(x_0) \cap \{v > 0\}|\,ds \leq C\varepsilon r^n, \tag{6.3}$$

for any $B_r(x_0) \subset B_1$, $r < 1/2$. Indeed, if (6.3) holds, then the lemma follows, since otherwise there exists $B_r(x_0)$ such that

$$|Q_\varepsilon \cap B_r(x_0) \cap \{v > 0\}| \geq C_0\varepsilon r^{n-1},$$

with $C_0 > 0$ arbitrarily large, and we would have

$$\begin{aligned} C_0 \varepsilon r^{n-1} &\leq |Q_\varepsilon \cap B_r(x_0) \cap \{v > 0\}| \\ &\leq \int_0^1 |Q_\varepsilon \cap B_{rs}(x_0) \cap \{v > 0\}| \, ds \\ &\leq C\varepsilon r^n, \end{aligned}$$

which leads to a contradiction for C_0 large enough. Thus, it remains to prove (6.3). The idea is to differentiate the PDE satisfied by v on its positivity set, multiply the outcome by the truncated function

$$G(\eta) := \begin{cases} \varepsilon & \text{if} \quad \eta > \varepsilon^{\frac{1}{p-1}} \\ |\eta|^{p-1}\mathrm{sign}(\eta) & \text{if} \quad |\eta| \leq \varepsilon^{\frac{1}{p-1}} \\ -\varepsilon & \text{if} \quad \eta < -\varepsilon^{\frac{1}{p-1}} \end{cases}$$

and integrate over $B_{rs}(x_0)$.

From (5.5), in $\{v > 0\}$ we have

$$D_e \Delta_p v = D_e f,$$

where e is any unit vector and D_e is the directional derivative. Thus, putting $v_e := D_e v$,

$$\mathrm{div}\left(|Dv|^{p-2}Dv_e + (p-2)|Dv|^{p-4}DvDv \cdot Dv_e\right) = D_e f,$$

in the weak sense. Multiplication by $G(v_e)$ and integration over $B_{rs}(x_0)$, $s \in (0,1)$, leads to

$$\begin{aligned} &\int_{B_{rs}(x_0)} \left(|Dv|^{p-2}Dv_e + (p-2)|Dv|^{p-4}DvDv \cdot Dv_e\right) \cdot DG(v_e)\, dx \\ &= \int_{\partial B_{rs}(x_0)} \left(|Dv|^{p-2}D_\nu v_e + (p-2)|Dv|^{p-4}D_\nu v Dv \cdot Dv_e\right) G(v_e)\, dx \qquad (6.4) \\ &\quad - \int_{B_{rs}(x_0)} D_e f G(v_e)\, dx, \end{aligned}$$

where D_ν denotes the outward normal derivative.

Integrating the first term of the right-hand side and using Hölder's inequality combined with the first estimate in Lemma 6.1, we obtain

$$\begin{aligned} &\int_0^1 \int_{\partial B_{rs}(x_0)} \left(|Dv|^{p-2}D_\nu v_e + (p-2)|Dv|^{p-4}D_\nu v Dv \cdot Dv_e\right) G(v_e)\, dx\, ds \\ &\qquad \leq (p-1) \int_{B_r(x_0)} |Dv|^{p-2}|D^2 v||G(v_e)|\, dx \\ &\qquad \leq (p-1)\varepsilon r^{n/2} \left(\int_{B_r(x_0)} \left[|Dv|^{p-2}|D^2 v|\right]^2\right)^{\frac{1}{2}} \qquad (6.5) \\ &\qquad \leq C\varepsilon r^n. \end{aligned}$$

To estimate the second term of the right-hand side, we use (6.2), obtaining

$$\int_{B_{rs}(x_0)} D_e f G(v_e)\, dx \le \int_{B_r(x_0)} |Df||G(v_e)|\, dx \le C\varepsilon r^n. \tag{6.6}$$

The constants $C > 0$ in (6.5) and (6.6) depend only on n, p and γ.

Now, to estimate below the left-hand side of (6.4), we take $e = e_i$, where $\{e_i\}_{i=1}^n$ is the standard basis in $\mathbb{R}^n$. Note that

$$G'(\eta) = (p-1)|\eta|^{p-2}\chi_{\left\{|\eta|<\varepsilon^{\frac{1}{p-1}}\right\}}$$

and $Q_\varepsilon \subset Q_\varepsilon^i$. Recalling the second estimate in Lemma 6.1, we obtain

$$\begin{aligned}
&\sum_{i=1}^n \left[\int_{B_{rs}(x_0)} |Dv|^{p-2}Dv_{x_i} + (p-2)|Dv|^{p-4}DvDv\cdot Dv_{x_i}\right]\cdot DG(v_{x_i})\\
&\ge (p-1)\int_{Q_\varepsilon^i\cap B_{rs}(x_0)\cap\{v>0\}} |Dv|^{2(p-2)}|D^2v|^2\\
&\quad + (p-1)(p-2)\int_{Q_\varepsilon^i\cap B_{rs}(x_0)\cap\{v>0\}} |Dv|^{2(p-2)}\sum_{i=1}^n\left(\frac{Dv}{|Dv|}\cdot Dv_{x_i}\right)^2\\
&\ge (p-1)\int_{Q_\varepsilon^i\cap B_{rs}(x_0)\cap\{v>0\}} \left[|Dv|^{p-2}|D^2v|\right]^2\\
&\ge \sigma\frac{|Q_\varepsilon^i\cap B_{rs}(x_0)\cap\{v>0\}|}{p-1}\\
&\ge \sigma\frac{|Q_\varepsilon\cap B_{rs}(x_0)\cap\{v>0\}|}{p-1}.
\end{aligned}$$

Combining the last inequality with (6.4), (6.5), and (6.6), we obtain (6.3). □

We are now ready to prove the main result of this section.

Theorem 6.1. *If u is a minimizer of* (2.1)*, and $x_0 \in \partial\{u > 0\} \cap B_{1/2}$, then*

$$\mathcal{H}^{n-1}(\partial\{u>0\}\cap B_r(x_0)) < Cr^n,$$

for any $r \in (0, 1/4)$, where the constant $C > 0$ depends only on n, p and γ.

Proof. It is enough to prove the result for the solution v of the obstacle problem (5.5), since $\partial\{v > 0\} = \partial\{u > 0\}$.

Recalling Besicovitch's covering lemma, let $\{B_\varepsilon(x_i)\}_{i\in I}$ be a finite covering of $\partial\{v > 0\}\cap B_r(x_0)$, with x_i at the free boundary and at most $L = L(n)$ overlapping at each point. From the proof of Corollary 4.2, there exists a constant c such that, for any $i \in I$, there exists $y_i \in B_\varepsilon(x_i)$ such that

$$B_{c\varepsilon}(y_i) \subset Q_\varepsilon \cap B_\varepsilon(x_i) \cap \{v > 0\}.$$

Using Lemma 6.2, we deduce

$$\begin{aligned}|B_1|c^n\varepsilon^n\#I &= \sum_{i\in I}|B_{c\varepsilon}(y_i)|\\ &\le \sum_{i\in I}|Q_\varepsilon\cap B_\varepsilon(x_i)\cap\{v>0\}|\\ &\le L(n)|Q_\varepsilon\cap B_\varepsilon(x_0)\cap\{v>0\}|\\ &\le L(n)C\varepsilon r^{n-1}.\end{aligned}$$

Thus,

$$\mathcal{H}^{n-1}_\delta(\partial\{v>0\}\cap B_r(x_0))\le Cr^{n-1},$$

and letting $\delta\to 0$, we arrive at

$$\mathcal{H}^{n-1}(\partial\{u>0\}\cap B_r(x_0))\le Cr^{n-1}.$$

□

We conclude with two remarks.

Remark 6.2. *Since the free boundary has locally finite $\mathcal{H}^{n-1}-$measure, the set $\{u>0\}$ has locally finite perimeter in Ω. Thus, $D(\chi_{\{u>0\}})$ is, in the sense of distributions, a vector-valued Borel measure supported on the free boundary. Moreover, its total variation is a Radon measure (see* [13]*).*

Remark 6.3. *Up to a negligible set of null perimeter, the free boundary is a union of, at most, a countable family of C^1 hypersurfaces (see* [15]*).*

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