GENERATING AFFINE POLYNOMIALS NONNEGATIVE ON REGIONS OF THE FORM $a_1 \le a_2 \le \cdots \le a_n \le 1$.

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ABSTRACT : Given a multilinear polynomial $q = \sum_{I\subseteq[n]} c_I \prod_{i\in I} a_i \in \mathbb{R}[a_1, ..., a_n]$, let $T(q)$ be the family of its terms and let $\deg t$ be the degree of a term t. The polynomial $\tilde{q} = \sum_{t\in T(q)}((1-a_{n+1})(b+\deg t)+a-b)t \in \mathbb{R}[a_1, ..., a_n, a_{n+1}]$, is then affine as well. It is shown that under broad conditions for reals a and b, if $q \ge 0$ whenever $a_1 \le a_2 \le \cdots \le a_n \le 1$, then $\tilde{q} \ge 0$ whenever $a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le 1$. This result implies potentially a step in a proof that the coefficient polynomials of positive degree of the power series in t of pobabilistically weighted harmonic means of the quantities $(1 - x_1t), ..., (1 - x_kt)$ are nonpositive whenever $x_1, ..., x_k$ are nonnegative.

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1. INTRODUCTION AND MOTIVATION

The author' s preprints [K1] and [K2] dealt among other things with the problem of strengthening results of F. Holland [H] concerning coefficient inequalities for the power series in t of the weighted harmonic mean of quantities $(1-x_1t), ..., (1-x_kt)$; namely $(\sum_{i=1}^{n} p_i(1-x_it)^{-1})^{-1}$, where $p_1, ..., p_k$ are nonnegative reals of sum 1. The question has connections with the nonnegative inverse eigenvalue problem: to characterize the possible spectra of nonnegative real matrices. Holland proved that if the p_i are all equal to $1/k$, then for $l \ge 1$ the coefficient of t^l is a polynomial $q_l(x_1, ..., x_k)$ which is nonpositive whenever $x_i \ge 0$ for $i = 1, ..., k$. In [K1], [K2] we provided evidence for two stronger facts: firstly it does not seem to be necessary to assume all the p_i are equal; and secondly if say (we assume without loss of generality) $0 \le x_1 \le x_2 \le \cdots \le x_n$ and we write $x_i = h_1 + \cdots + h_i$, for $i = 1, ..., n$ then the polynomial q_l transformed by this substitution into a polynomial in the hs will have only non-positive coefficients.

In the course of our work we came up with a sequence of polynomials in various variables the first few of which we indicated - up to unessential modifications - as being the following ones.

$$\begin{aligned}
q_0 &= 1\\
q_1 &= 2 - 2a_1\\
q_2 &= 6 - 8a_1 - 4a_2 + 6a_1a_2\\
q_3 &= 24 - 40a_1 - 20a_2 + 36a_1a_2 - 12a_3 + 24a_1a_3 + 12a_2a_3 - 24a_1a_2a_3\\
&\vdots
\end{aligned}$$

These polynomials occurred as a consequence of a certain complicated reduction process. We conjectured at the time that there should be an easier way to produce these polynomials, but we were unable to concretize it. We also conjectured that the polynomial q_n is nonnegative on the region $\Delta_n = \{(a_1, a_2, ..., a_n) : a_1 \le a_2 \le \cdots \le a_n \le 1\}$, that is, that $q_n|\Delta_n \ge 0$. It was shown that this would yield a strengthening of Holland's main result. We were able to prove nonnegativity for the first few inequalities but for lack of an easy definition we could not hope to prove $q_n|\Delta_n \ge 0$ for all n. But some time ago we finally found a precise conjecture to easily produce these polynomials. Given any polynomial q in various variables written in standard form as a linear combination of monomials, define $T(q) = \{\text{terms of polynomial } q\}$. For example $T(q_2) = \{6, -8a_1, -4a_2, 6a_1a_2\}$. For $t \in T(q)$, let $\deg t :=$ degree of t; for example $\deg(6a_1a_2) = 2$. The mentioned reduction process seems to deliver the following inductively defined sequence of polynomials:

$$\begin{aligned}
q_0 &= 1.\\
q_{n+1} &= \sum_{t\in T(q_n)} (2 + n + \deg t)\cdot t - \sum_{t\in T(q_n)} (2 + \deg t)\cdot t \cdot a_{n+1} = \sum_{t\in T(q_n)} ((1 - a_{n+1})(2 + \deg t) + n)t
\end{aligned}$$

We do not prove *this* conjecture here but rather we prove a result from which the second conjecture, namely that indeed we have for all $n \in \mathbb{Z}_{\ge 1}$ that $q_n|\Delta_n \ge 0$ is a very special case. We managed to prove our special polynomial inequalities only after we arrived after a number of failed tentatives at the general viewpoint which we present here. While it seems easy to show, by a method we explain in the next paragraph, that for each individual q_n there holds $q_n|\Delta_n \geq 0$, it seems to be a good deal harder to show that this method works for the totality of all the q_n. This is done here.

If $q = q(a_1, a_2, ..., a_n)$ is any real polynomial in n variables $a_1, ..., a_n$ which we wish to prove is nonnegative on Δ_n, then there is a good chance to show this by expressing it in certain other variables; namely we introduce $h_1 = 1 - a_n, h_2 = a_n - a_{n-1}, \cdots, h_n = a_2 - a_1$ from which it follows that $a_j = 1 - h_1 - h_2 - \cdots - h_{n-j+1}$. Then any monomial (i.e. any product of variables) of q is a certain product of some of the factors in $\prod_{i=0}^{n}(1-h_1-h_2-\cdots-h_i)$. By substituting such products for the monomials and expanding we get q as a polynomial in the h_i. We shall call this polynomial the *h-form* of q while the original form in which q is written is its *a-form.* It is evident that the coefficient of a monomial $h^{\mathbf{j}} = h^{(j_1,...,j_n)} := h_1^{j_1} h_2^{j_2} \cdots h_n^{j_n}$ in the h-form of q is a linear combination of the coefficients of q in the a-form. Evidently $(a_1, a_2, ..., a_n) \in \Delta_n$ if and only if $h_1, h_2, ..., h_n$ are all nonnegative. It follows that the nonnegativity of the referred linear combinations of the coefficients of q that occur writing q in h-form is a sufficient condition for having $q|\Delta_n \geq 0$.

In this vein, we present here a perhaps significant first result for the special class of affine - also called multilinear - polynomials which are those for which for each $i \in [n] = \{1, 2, ..., n\}$ the function $a \mapsto q(a_1, ..., a_{i-1}, a, a_{i+1}, ..., a_n)$ is affine; i.e. constant plus linear. Equivalently q is a polynomial in which the exponents of the a_i are all only 0 or 1.

Theorem. *Let $n \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{R}$, and consider the affine polynomials*

$$q = \sum_{I \subseteq [n]} c_I \prod_{i \in I} a_i \quad \text{and} \quad \tilde{q} = \sum_{t \in T(q)} (a + \deg t) \cdot t - \sum_{t \in T(q)} (b + \deg t) \cdot t \cdot a_{n+1} = \sum_{t \in T(q)} ((1 - a_{n+1})(b + \deg t) + a - b)t.$$

Provided $a - b - n \geq 0$ and $b \geq 0$, then the coefficients of the h-form of $\tilde{q}$ are nonnegative linear combinations of at most two of the coefficients of the h-form of q. Consequently, if the h-form of q has only nonnegative coefficients, then the h-form of $\tilde{q}$ has only nonnegative coefficients, and hence $\tilde{q}|_{\Delta_{n+1}} \geq 0$.

To see that this theorem indeed implies for the above sequence q_n that $q_n|\Delta_n \geq 0$, choose $b = 2$ and make a dependent on n, putting $a = 2 + n$. Then beginning with $q = q_0 = 1 = c_\emptyset$, the theorem applied with $n = 0$ yields as $\tilde{q}_0$ the polynomial q_1; now applying it with $q = q_1$ and $n = 1$, we get $\tilde{q}_1 = q_2$, etc. The claim follows as $q_0 \geq 0$.

We convene to see sets $I \subseteq [n]$ as lists of increasing integers; and if n is small simply as strings. So e.g. $c_{\{2,4,1,5\}}$ will usually be written c_{1245}.

To make other aspects of the theorem more palpable we present some more examples, still remaining near the original sequence $(q_i)_{i \geq 0}$. If a generic n-variable affine q is given as in the left hand side of the theorem we let $\tilde{q}$ be produced by the choice $b = 2$ and $a = 2 + n$.

Examples. In the case $n = 0, q = c_\emptyset$ and $\tilde{q} = 2c_\emptyset - 2c_\emptyset a_1 = 2c_\emptyset h_1$ and the claim of the theorem is obvious. If $n = 1$ then the a-form and the h-form of q and $\tilde{q}$ are given by

$$q = c_\emptyset + c_1 a_1 \;=\; (c_\emptyset + c_1) - c_1 h_1, \quad \text{and}$$
$$\tilde{q} = 3c_\emptyset - 2c_\emptyset a_2 + 4c_1 a_1 - 3c_1 a_1 a_2 \;=\; (c_\emptyset + c_1) + 2(c_\emptyset + c_1)h_1 - 3c_1 h_1^2 - c_1 h_2 - 3c_1 h_1 h_2,$$

respectively. So also in this case we see that if the h-form of q has only nonnegative coefficients (i.e. if $c_\emptyset + c_1 \geq 0$ and $-c_1 \geq 0$) , then the h-form of $\tilde{q}$ will have only nonnegative coefficients. If $n = 2$ then the a-form and the h-form of q are

$$q = c_\emptyset + c_1 a_1 + c_2 a_2 + c_{12} a_1 a_2 = (c_\emptyset + c_1 + c_2 + c_{12}) + (-c_1 - c_2 - 2c_{12})h_1 + c_{12} h_1^2 + (-c_1 - c_{12})h_2 + c_{12} h_1 h_2.$$

The a-form and the h-form of $\tilde{q}$ are

$$\begin{aligned}
\tilde{q} &= 4c_\emptyset + 5c_1 a_1 + 5c_2 a_2 + 6c_{12} a_1 a_2 - 2c_\emptyset a_3 - 3c_1 a_1 a_3 - 3c_2 a_2 a_3 - 4c_{12} a_1 a_2 a_3 \\
&= (2c_\emptyset + 2c_1 + 2c_{12} + 2c_2) + (2c_\emptyset + c_1 + c_2)h_1 + (-3c_1 - 6c_{12} - 3c_2)h_1^2 + 4c_{12} h_1^3 + (-2c_1 - 4c_{12} - 2c_2)h_2 \\
&\quad + (-3c_1 - 4c_{12} - 3c_2)h_1 h_2 + (8c_{12})h_1^2 h_2 + (2c_{12})h_2^2 + (4c_{12})h_1 h_2^2 + (-2c_1 - 2c_{12})h_3 \\
&\quad + (-3c_1 - 2c_{12})h_1 h_3 + 4c_{12} h_1^2 h_3 + 2c_{12} h_2 h_3 + 4c_{12} h_1 h_2 h_3.
\end{aligned}$$

Again the coefficients of the $h^{\mathbf{j}}$ of $\tilde{q}$ are nonnegative linear combinations of two of the coefficients of the h-form of q. For example,

$$(\text{coefficient of } h_1 h_2 \text{ in } \tilde{q}) = -3c_1 - 3c_2 - 4c_{12} = 3\cdot(\text{ coefficient of } h_1 \text{ in } q) + 2\cdot(\text{coefficient of } h_1^2 \text{ in } q)$$

We shall sometimes call a polynomial $\tilde{q}$ defined on the base of q (and a selection of a, b) a *successor* of q. It is quite likely that analogues of the theorems hold for further modifications or generalizations of the successor rule.

Striving for greater naturality and simplicity it is natural to ask: Given an affine polynomial q which satisfies $q|\Delta_n \geq 0$, will then a sucessor $\tilde{q}$ of typically satisfy $\tilde{q}|\Delta_{n+1} \geq 0$? We will show in Section 6 that the response to this question is in general 'no' even in the particular case that q is a polynomial of two variables and $\tilde{q}$ the polynomial just shown. We will characterize completely when a polynomial $q = c_\emptyset + c_1 x + c_2 y + c_{12} xy$ satisfies $q|_{\Delta_2} \geq 0$ and from there produce a counterexample.

In sections 2,3,4,5,6 we do the following. In Section 2 we recall notions like colex total order, prove a particular property of it, and also need one particular combinatorial interpretation of the Catalan numbers C_n. We introduce in Section 3 for an affine polynomial q a matrix, we call hc-table, that gives us an attractive form to see the coefficients of q in h-form as a linear combination of its coefficients c_I in a-form. These matrices are of size $C_{n+1} \times 2^n$. We will see that the tables of generic affine polynomials in n and $n+1$ variables are closely related. In Section 4 we express the coefficients of the a-form of $\tilde{q}$ as defined in the theorem by means of the coefficients of the a-form of q. The main result, enounced in above theorem, is proved in Section 5. Section 6 considers a 2-variable example q. We look into the question as to how far our linear inequality conditions on the coefficients c_I $I \subseteq [2]$ for q approach the complete set of necessary and sufficient conditions in order that $q|\Delta_2 \geq 0$. We also present the mentioned counterexample. and the mentioned counterexample, as said, in Section 6. What concerns references to lemmas etc., we simply write Lemma i for the lemma in Section i; and similarly of course procede with propositions, theorems, and corollaries.

2. SOME NOTATION AND PREPARATORY LEMMAS

For a subset I of $[n] = \{1, 2, ..., n\}$, and sets I in general, $|I|$ will denote the cardinality of I; while for $\mathbf{j} = (j_1, ..., j_n) \in \mathbb{Z}^n$, we define its 1-norm $|\mathbf{j}| = |j_1| + \cdots + |j_n|$. e_1 will be the first standard vector $(1, 0, 0, ..., 0)$ of size implied by the context.

We present next some concepts that skimmed over will further the smooth reading of what follows.

a. We first recall the *colex* total orders on the familiy of subsets of integers in $[n]$ and on the family of integer n-uples. Given $I, J \subseteq [n]$, assume them written as lists in which the integers in I, J increase. For $I \neq J$, we write $I < J$ if, for some l, the rightmost l integers of I and J are equal ($l = 0$ admitted) but the $l+1$-st integer counted from the right in I is either nonexistent or is smaller than the $l+1$-st integer of J. Thus we have e.g. for the case $n = 4$, that $\emptyset < 1 < 2 < 12 < 3 < 13 < 23 < 123 < 4 < 14 < 24 < 124 < 34 < 134 < 234 < 1234$. For n-uples $\mathbf{i} = (i_1, i_2, ..., i_n)$ and $\mathbf{j} = (j_1, j_2, ..., j_n)$ we similarly define $\mathbf{i} < \mathbf{j}$ if, for some l, the rightmost l integers of $\mathbf{i}$ and $\mathbf{j}$ are equal ($l = 0$ admitted) but the $l+1$-st integer counted from the right in $\mathbf{i}$ is smaller than the $l+1$-st integer of $\mathbf{j}$. Here is a system of inequalities for the set of 3-uples in $\mathcal{J}_3$ (defined below) that helps interiorizing the idea; we will suppress here and in other places parentheses and commas for n-uples with small integer entries: $000 < 100 < 200 < 300 < 010 < 110 < 210 < 020 < 120 < 001 < 101 < 201 < 011 < 111$.

Lemma. Consider

$$\mathcal{J}_n = \{(j_1, j_2, ..., j_n) \in \mathbb{Z}^n_{\geq 0} : t + j_t + j_{t+1} + \cdots j_n \leq n+1, t = 1, 2, ..., n+1\}$$

and assume $\mathcal{J}_n$ equipped with the colex order. Then the colex predecessor $\mathbf{j}' \in \mathcal{J}_n$ of a nonzero element $\mathbf{j} \in \mathcal{J}_n$ of form $\mathbf{j} = (0, j_2, j_3, ..., j_n)$ is an element of 1-norm n: $|\mathbf{j}'| = n$.

Proof. Let $\mathbf{j} \neq 0$ be in $\mathcal{J}_n$. There exists then an $i \in \{2, 3, ..., n\}$ such that $j_i \geq 1$ and the predecessor of $\mathbf{j}$ is of the form $\mathbf{j}' = (j'_1, ..., j'_{i-1}, j_i - 1, j_{i+1}, ..., j_n)$. Assume $|\mathbf{j}'| \leq n-1$. Then $l = n - |\mathbf{j}'| \geq 1$ and for $\mathbf{j}'' = (j'_1 + l, j'_2, ..., j'_{i-1}, j_i - 1, j_{i+1}, ..., j_n)$ we have evidently $|\mathbf{j}''| = |\mathbf{j}'| + l = n$, $\mathbf{j}'' \in \mathcal{J}_n$, and from the definition of the colex order it is direct that $\mathbf{j}' < \mathbf{j}'' < \mathbf{j}$. This means $\mathbf{j}'$ is not the predecessor of $\mathbf{j}$, contradicting the hypothesis. $\square$

Note the imbedding $\mathcal{J}_n \ni (j_1, ..., j_n) \mapsto (j_1, ..., j_n, 0) \in \mathcal{J}_{n+1}$. We can thus usually see $\mathcal{J}_n$ as a subset of $\mathcal{J}_{n+1}$.

b. The following fact is known.

Proposition. A monomial $x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$ occurs in the expansion of the product $x_1(x_1+x_2)(x_1+x_2+x_3)\cdots(x_1+x_2+\cdots+x_n)$ if and only if $l_1, l_2, ..., l_n$ are nonnegative integers satisfying $l_1 + l_2 + \cdots + l_t \geq t$ for $t = 1, ..., n$, with equality if $t = n$. Furthermore the number of distinct such monomials is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

These facts can be found as being two of the more than 210 different characterizations of Catalan numbers in [S1]; see there problems 6.19 (y^5) and (s^5); they can certainly also be found in [S2]. $\square$

Corollary. $h_1^{l_1} h_2^{l_2} \cdots h_{n-1}^{l_{n-1}}$ occurs in the expansion of the product $(1-h_1)(1-h_1-h_2)\cdots(1-h_1-h_2-\cdots-h_{n-1})$ if and only if $(l_1, l_2, ..., l_{n-1}) \in \mathcal{J}_{n-1}$. There exist C_n distinct monomials in the expansion of the product.

Proof. The product arises from $x_1(x_1+x_2)(x_1+x_2+x_3)\cdots(x_1+x_2+\cdots+x_n)$ by means of the substitutions $\begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ 1 & -h_1 & -h_2 & \dots & -h_{n-1}\end{pmatrix}$. So if $x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n}$ occurs in the expansion of the product $x_1(x_1+x_2)(x_1+x_2+x_3)\cdots(x_1+x_2+\cdots+x_n)$, then $h_1^{l_2}\cdots h_{n-1}^{l_n}$ will occur in the expansion of $\prod_{j=1}^{n-1}(1-\sum_{i=1}^{j}h_j)$. If conversely $h_1^{l_2}\cdots h_{n-1}^{l_n}$ occurs in the expansion of the latter product then it is clear that $l_2+\cdots+l_n\le n$; then homogenization via multiplication with $x_1^{l_1}$ with $l_1=n-l_2-\cdots-l_n$ yielding $x_1^{l_1}h_1^{l_2}\cdots h_{n-1}^{l_n}$ and back substitution of the $h_1,...,h_{n-1}$ to $x_2,...,x_n$ will yield a monomial occurring in $\prod\limits_{i=1}^{n}\sum\limits_{j=1}^{i}x_j$. Now we know by the proposition before that occurrence of $x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n}$ is the case if $l_1+l_2+\cdots+l_t\ge t$ for $t=1,...,n-1$ and $l_1+l_2+\cdots+l_n=n$. Substituting l_1 via this equality in the inequalities we get the inequalities $n-l_{t+1}-\cdots-l_n\ge t$, or $t+l_{t+1}+\cdots+l_n\le n$ for $t=1,...,n$ as necessary and sufficient conditions for occurrence of $h_1^{l_2}\cdots h_{n-1}^{l_n}$. The inequalities claimed follow by replacing variables named $l_2,...,l_n$ by $l_1,...,l_{n-1}$, respectively. The claimed number of monomials follows from the present proof and the previous proposition. $\square$

3. TRANSITION FROM a-FORM TO h-FORM

In this section, given a generic affine polynomial in a-form $q=\sum_{I\subseteq[n]}c_I\prod_{i\in I}a_i$, we shall define a matrix Q so that after arranging in increasing colex order the monomials that occur in its h-form into a column vector $\underline{h}$ and similarly arranging the coefficients c_I we get that $\underline{h}^TQ\underline{c}$ is the h-form of q. Also, if $\dot q=\sum_{I\subseteq[n+1]}c_I\prod_{i\in I}a_i$, and $\dot Q$ the matrix associated to $\dot q$, we give a 3-step rule for constructing $\dot Q$ from Q. It is useful to note for clarity that this section is completely independent from the definitions of $a,b,\tilde q$ of Section 1; that n is considered fixed; and to keep in mind that q is tied to n and $\dot q$ to $n+1$. This notation is chosen to emphasize that only two successive polynomials are envolved here.

Since the h-form of q occurs by substituting a_j by $a_j=1-h_1-\cdots-h_{n+1-j}$, for $j=1,...,n$, and expanding, the monomials occurring in the h-form of q are of the form $h^{\mathbf{j}}$ with $\mathbf{j}\in\mathcal{J}_n$, where as we know

$$\begin{aligned}\mathcal{J}_n &= \{\text{ n-uples } (j_1,j_2,...,j_n) \text{ of exponents occuring in the expansion of } \textstyle\prod_{l=0}^{n}(1-h_1-h_2-\cdots-h_l)\ \}\\ &= \{(j_1,j_2,...,j_n)\in\mathbb{Z}^n_{\ge 0}: t+j_t+j_{t+1}+\cdots j_n\le n+1, t=1,2,...,n+1\},\end{aligned}$$

and $|\mathcal{J}_n|=C_{n+1}$.

	c_0	c_1	c_2	c_{12}
1	1	1	1	1
h_1		-1	-1	-2
h_1^2				1
h_2		-1		-1
h_1h_2				1

The h-form of a polynomial q gives rise to its *hc-table*. This is a table whose columns are indexed by the c_I with the $I\subseteq[n]$ in colex order, increasing from left to right, and whose rows are indexed with the monomials $h^{\mathbf{j}}$ occurring in the h-form of the polynomial also in colex order so that the exponents $\mathbf{j}$ increase towards the bottom. The entry at the intersection of row indexed by $h^{\mathbf{j}}$ with column indexed by c_I is the numerical coefficient with which the product $c_Ih^{\mathbf{j}}$ occurs in the h-form. The inner, numerical part of the hc-table of q is the $C_{n+1}\times 2^n$ matrix Q referred before. Shown is the hc-table of q for the case $n=2$; look at the respective example in Section 1.

How to construct the hc-table of $\dot q$ from the hc-table of q ? We have

$$\begin{aligned}\dot q &= \sum_{I\in[n+1]}c_I\prod_{i\in I}a_i\\ &= \sum_{I\subseteq[n]}c_I\prod_{i\in I}a_i+\sum_{I\subseteq[n]}c_{I\uplus\{n+1\}}\prod_{i\in I}a_ia_{n+1}\\ &= q+(1-h_1)\sum_{I\subseteq[n]}c_{I\uplus\{n+1\}}\prod_{i\in I}a_i.\end{aligned}$$

The last step follows because in the h-form of $\dot q$, we have $h_1=1-a_{n+1}$. To get to the hc-table of $\dot q$ we first have to obtain the h-form of q as a subpolynomial of $\dot q$. Because of $a_1\le a_2\le\cdots\le a_n\le a_{n+1}\le 1$

we have to replace a_n by $1-h_1-h_2$, a_{n-1} by $1-h_1-h_2-h_3$, . . . , a_1 by $1-h_1-h_2-\cdots-h_n-h_{n+1}$. This corresponds to the substitution $\begin{pmatrix}1 & h_1 & h_2 & \cdots & h_{n-1} & h_n\\ 1 & h_1+h_2 & h_3 & \cdots & h_n & h_{n+1}\end{pmatrix}$ in the original h-form of q. If $q=\sum_{\mathbf{j}\in\mathcal{J}_n}\kappa_{j_1,j_2,...,j_n}h_1^{j_1}h_2^{j_2}\cdots h_{n-1}^{j_{n-1}}h_n^{j_n}$, then, what we shall call the *extended* q , is $\sum_{\mathbf{j}\in\mathcal{J}_n}\kappa_{j_1,j_2,...,j_n}(h_1+h_2)^{j_1}h_3^{j_2}\cdots h_n^{j_{n-1}}h_{n+1}^{j_n}=\sum_{\mathbf{j}\in\mathcal{J}_n}\sum_{\nu\ge0}\kappa_{j_1,j_2,...,j_n}\binom{j_1}{\nu}h_1^{\nu}h_2^{j_1-\nu}h_3^{j_2}\cdots h_n^{j_{n-1}}h_{n+1}^{j_n}$. As distinct pairs (j_1,ν) here cause the j_1+1 distinct $n+1$-uples $(\nu,j_1-\nu,j_2,...,j_n)$, $\nu=0,1,...,j_1$, the coefficient of $h_1^{\nu}h_2^{j_1-\nu}h_3^{j_2}\cdots h_n^{j_{n-1}}h_{n+1}^{j_n}$ is $\binom{j_1}{\nu}\kappa_{j_1j_2\cdots j_n}$. The coefficient $\kappa_{j_1j_2\cdots j_n}$ of $h_1^{j_1}h_2^{j_2}\cdots h_{n-1}^{j_{n-1}}h_n^{j_n}$ is of course a linear combination of certain $c_I, I\subseteq[n]$, as indicated by the hc-table of q. The full procedure to come from the hc-table of q to that of $\dot{q}$ is thus as follows.

a. Prepare an empty table by writing the possible monomials of $\dot{q}$ in h-form in right order in a column to be the leftmost and the c_I in right order in a row to be the uppermost. **b.** Note that the hc-table for $\dot{q}$ will have twice the breadth of the hc-table of q. So we can speak of a left and a right part of equal breadths. Write the coefficients of $c_I h^J$ of extended q into the left part of the the hc-table-to-be for $\dot{q}$ putting blanks or zeros for monomials that do not occur in extended q in h-form. **c.** The second part of the formula for $\dot{q}$ is $(1-h_1)\sum\limits_{I\subseteq[n]} c_{I\uplus\{n+1\}}\prod\limits_{i\in I} a_i$. The sum alone would have an easily recognized hc-table: It would be simply the left part of the current table shifted to the right. It is now not hard to see that the multiplication with $1-h_1$ leads to the following rule: The row indexed $h_1^{j_1}h_2^{j_2}\cdots h_{n+1}^{j_{n+1}}$ of the table of $\dot{q}$ is obtained from the now filled-in left part as follows: If $j_1=0$ copy the left half-row of $\dot{q}$ into the right halfrow. If $j_1\geq 1$, subtract from the left half-row with index $h_1^{j_1}h_2^{j_2}\cdots h_{n+1}^{j_{n+1}}$ the left half-row with index $h_1^{j_1-1}h_2^{j_2}\cdots h_{n+1}^{j_{n+1}}$ and write the result into the right half-row. The row indexed $h_1^{j_1-1}h_2^{j_2}\cdots h_{n+1}^{j_{n+1}}$ is according to our ordering of rows precisely the row preceding the one indexed $h_1^{j_1}h_2^{j_2}\cdots h_{n+1}^{j_{n+1}}$. In other words we have the 'formula'

$$\text{(right half-row)} = \text{(left half-row)} - \text{(left half-row of predecessor)},$$

where, if $j_1=0$, then the 'left half-row of predecessor' has to be interpreted as half-row of 0s (blanks). By Lemma 2 in fact this happens automatically for if $j_1=0$ then the predecessor of row $h^{\mathbf{j}}$ in the hc-table for $\dot{q}$ is indexed by a monomial whose exponent has norm $n+1$ and therefore $h^{\mathbf{j}}$ does not occur in the h-form of q since this polynomial has degree n.

Example.

	c_0	c_1	c_2	c_{12}
1	1	1	1	1
h_1		-1	-1	-2
h_1^2				1
h_2		-1		-1
h_1h_2				1

hc-table of q

	c_0	c_1	c_2	c_{12}	c_3	c_{13}	c_{23}	c_{123}
1	1	1	1	1	1	1	1	1
h_1		-1	-1	-2	-1	-2	-2	-3
h_1^2				1		1	1	3
h_1^3								-1
h_2		-1	-1	-2		-1	-1	-2
h_1h_2				2		1	1	4
$h_1^2h_2$								-2
h_2^2				1				1
$h_1h_2^2$								-1
h_3		-1		-1		-1		-1
h_1h_3				1		1		2
$h_1^2h_3$								-1
h_2h_3				1				1
$h_1h_2h_3$								-1

hc-table of $\dot{q}$

Consider the case $n=2$ and the example of passing from the hc-table of q (at the left) to the hc-table of $\dot{q}$ (at the right). In the sense of above explanation, the monomial $h_1^2h_2^0$ 'creates' the terms $\binom{2}{0}h_1^2$, $\binom{2}{1}h_1h_2$, $\binom{2}{2}h_2^2$; the line for h_3 is created from $h_2=h_1^0h_2^1$, and $h_1h_2=h_1^1h_2^1$ originates $\binom{1}{0}h_1^0h_2h_3=h_2h_3$ and $\binom{1}{1}h_1^1h_2^0h_3=h_1h_3$. From this finally the right hand side of that table is produced by the above formula for computing right half rows. As an example the left and the right half-rows indexed by $h_2=h_1^0h_2^1h_3^0$ in the table for $\dot{q}$ are equal (to $(0,-1,-1,-2)$), while the right half-row of h_1h_2 is the left half-row of h_1h_2 minus the left half-row of h_2: $(0,1,1,4)=(0,0,0,2)-(0,-1,-1,-2)$.

We will need in Section 5 the following lemma.

Lemma. Let $n\geq 1$, $I\subseteq[n]$, and $\mathbf{j}=(j_1,j_2,...,j_n)\in\mathcal{J}_n$. Also let $r_I(\mathbf{j})$ be the coefficient with which $c_Ih^{\mathbf{j}}$ occurs in the h-form of q or, equivalently, the entry in adress $(h^{\mathbf{j}},c_I)$ of the hc-table of q. Then
a. $j_1r_I(\mathbf{j})+(|I|+1-|\mathbf{j}|)r_I(\mathbf{j}-e_1)=0$.
b. $|I|r_I(\mathbf{j}-e_1)=(|\mathbf{j}|-1)r_I(\mathbf{j}-e_1)-j_1r_I(\mathbf{j})$.

Proof. a. $c_Ih^{\mathbf{j}}$ is born from expanding $c_I\prod_{i\in I}(1-h_1-\cdots-h_{n-i+j})$ which is a polynomial of degree $|I|$ and so we see that $|I|<|\mathbf{j}|$ implies $r_I(\mathbf{j})=0$. So for such I and $\mathbf{j}$ the claim is trivial and we assume henceforth that $|\mathbf{j}|\leq|I|$. The proof is now by induction on the smallest k for which $I\subseteq[k]$. If $k=0$, then $I=\emptyset$ and only $\mathbf{j}=(0,\ldots 0)$ is possible. Then $j_1 = 0$ and so $r_\emptyset(\mathbf{j} - e_1) = 0$. (If an n-uple $\mathbf{j}$ has negative entries we always have $r_I(\mathbf{j}) = 0$, simply because then $c_I h^{\mathbf{j}}$ cannot occur in the h-form of q.) Again the claim follows. Assume now $k < n$ and that for this k the claim is proved. A subset of $[n]$ which does not fit into $[k]$ but fits into $[k+1]$ is of the form $I \uplus \{k+1\}$ with $I \subseteq [k]$. We have $r_{I\uplus\{k+1\}}(\mathbf{j}) = r_I(\mathbf{j}) - r_I(\mathbf{j} - e_1)$. Since $|I \uplus \{k+1\}| = |I| + 1$ and, when $j_1 \geq 1$, $|\mathbf{j} - e_1| = |\mathbf{j}| - 1$, we get that the left hand side of the zero relation to prove equals

$$\begin{aligned} & j_1(r_I(\mathbf{j}) - r_I(\mathbf{j} - e_1)) + (|I| + 2 - |\mathbf{j}|)(r_I(\mathbf{j} - e_1) - r_I(\mathbf{j} - 2e_1)) \\ = \; & j_1 r_I(\mathbf{j}) + (|I| + 1 - |\mathbf{j}|) r_I(\mathbf{j} - e_1) + 1\, r_I(\mathbf{j} - e_1) - j_1 r_I(\mathbf{j} - e_1) - (|I| + 2 - |\mathbf{j}|) r_I(\mathbf{j} - 2e_1) \\ = \; & j_1 r_I(\mathbf{j}) + (|I| + 1 - |\mathbf{j}|) r_I(\mathbf{j} - e_1) - ((j_1 - 1) r_I(\mathbf{j} - e_1) + (|I| + 2 - |\mathbf{j}|) r_I(\mathbf{j} - 2e_1)) \\ = \; & 0 - 0 = 0, \end{aligned}$$

where in the last steps we used the induction hypothesis.

b. Follows by rewriting the zero relation just proved. □

4. COEFFICIENTS

Again we fix an n and write q for q_n and $\tilde{q}$ for its successor as defined in the Theorem 1. It is easy to see that $\tilde{q}$ is again an affine polynomial having variable a_{n+1} in addition to those of q. It is, so to say, a $\dot{q}$ in which the c_I are specialized in a certain manner. More precisely we have

$$q = \sum_{I \subseteq [n]} c_I \prod_{i \in I} a_i \quad \text{and} \quad \tilde{q} = \sum_{I \subseteq [n+1]} \tilde{c}_I \prod_{i \in I} a_i,$$

where the $\tilde{c}_I$ are dependent on the c_I as follows.

Lemma. Let $I \subseteq [n+1]$. Then

$$\tilde{c}_I = \begin{cases} (a + |I|) c_I & \text{if } n+1 \notin I \\ -(b - 1 + |I|) c_{I \setminus \{n+1\}} & \text{if } n+1 \in I. \end{cases}$$

Proof. We look at the formula defining the successor $\tilde{q}$ of q, namely

$$\tilde{q} = \sum_{t \in T(q)} (a + \deg t) \cdot t - \sum_{t \in T(q)} (b + \deg t) \cdot t \cdot a_{n+1},$$

and make a case distinction for I.

Case: $n+1 \notin I$. Then the term $\tilde{c}_I \prod_{i \in I} a_i = (a + \deg t) \cdot t$ for some term $t \in T(q)$. Now this $t = c_{I'} \prod_{i \in I'} a_i$ for some I' and comparison implies $I' = I$. Clearly $\deg t = |I|$ and the claim follows for this case.

Case: $n+1 \in I$. Then $\tilde{c}_I \prod_{i \in I} a_i = -(b + \deg t) \cdot t \cdot a_{n+1}$ again for some $t \in T(q)$. Writing $I = J \uplus \{n+1\}$ and $t = c_{I'} \prod_{i \in I'} a_i$ the equation transforms into $\tilde{c}_I \prod_{i \in J} a_i\, a_{n+1} = -(b + \deg t) c_{I'} \prod_{i \in I'} a_i\, a_{n+1}$ from where we extract $I' = J$ and then $\tilde{c}_I = -(b + |J|) c_J = -(b - 1 + |I|) c_{I \setminus \{n+1\}}$. □

5. PROOF OF THE THEOREM

We can now give the proof of the Theorem announced in Section 1. So choose any $\mathbf{j} \in \mathcal{J}_{n+1}$, that is, choose any monomial $h^{\mathbf{j}}$ that occurs in the h-form of $\dot{q}$ and for that matter in the h-form of $\tilde{q}$. By the $r_I(\mathbf{j})$ we shall mean here the entries in the hc-table of $\tilde{q}$. Note that $\tilde{c}_{I \uplus \{n+1\}} = -(b - 1 + |I \uplus \{n+1\}|) c_I = -(b + |I|) c_I$, and that $r_{I \uplus \{n+1\}}(\mathbf{j}) = r_I(\mathbf{j}) - r_I(\mathbf{j} - e_1)$; also if $\mathbf{j} \notin \mathbb{Z}^n_{\geq 0}$, convene to put $r_I(\mathbf{j}) = 0$. Therefore the coefficient of $h^{\mathbf{j}}$ in $\tilde{q}$ is given by

$$\begin{aligned} (**) \quad \sum_{I \subseteq [n+1]} \tilde{c}_I r_I(\mathbf{j}) \;&=\; \sum_{I \subseteq [n]} \tilde{c}_I r_I(\mathbf{j}) + \sum_{I \subseteq [n]} \tilde{c}_{I \uplus \{n+1\}} r_{I \uplus \{n+1\}}(\mathbf{j}) \\ &=\; \sum_{I \subseteq [n]} (a + |I|) c_I r_I(\mathbf{j}) - \sum_{I \subseteq [n]} (b + |I|) c_I (r_I(\mathbf{j}) - r_I(\mathbf{j} - e_1)) \\ &=\; \sum_{I \subseteq [n]} (a - b) c_I r_I(\mathbf{j}) + \sum_{I \subseteq [n]} (b + |I|) c_I r_I(\mathbf{j} - e_1) \\ &=\; \sum_{I \subseteq [n]} ((a - b) r_I(\mathbf{j}) + (b + |I|) r_I(\mathbf{j} - e_1)) c_I. \end{aligned}$$

Using the relation $|I| r_I(\mathbf{j} - e_1) = (|\mathbf{j}| - 1) r_I(\mathbf{j} - e_1) - j_1 r_I(\mathbf{j})$ found in Lemma 3b, (here for $n+1$ instead of n) we find $(a - b) r_I(\mathbf{j}) + (b + |I|) r_I(\mathbf{j} - e_1) = (a - b - j_1) r_I(\mathbf{j}) + (b + |\mathbf{j}| - 1) r_I(\mathbf{j} - e_1)$. Thus from the above,

$$\sum_{I \subseteq [n+1]} \tilde{c}_I r_I(\mathbf{j}) = (a - b - j_1) \sum_{I \subseteq [n]} c_I r_I(\mathbf{j}) + (b + |\mathbf{j}| - 1) \sum_{I \subseteq [n]} r_I(\mathbf{j} - e_1) c_I.$$

Now $\sum_{I\subseteq[n]} c_I r_I(\mathbf{j})$ and $\sum_{I\subseteq[n]} r_I(\mathbf{j}-e_1)c_I$ are the coefficients of $h^{\mathbf{j}}$ and $h^{\mathbf{j}-e_1}$ of polynomial q as defined in Theorem 1. Also, as $\mathbf{j}\in\mathcal{J}_{n+1}$, clearly $j_1\le n+1$. If $j_1\le n$, then from the hypotheses on a and b the we have $(a-b-j_1)\ge 0$. Otherwise $|\mathbf{j}|>n$ and so $r_I(\mathbf{j})=0$ for the $I\subseteq[n]$ occurring at the right hand side. Then the first term vanishes. The only case that $|\mathbf{j}|=0$ is that $\mathbf{j}=0$. In that case by our definitions, $r_I(\mathbf{j}-e_1)=0$. Hence the conditions $b\ge 0$ and $a-b-n\ge 0$ guarantee that the expression found for the coefficient of $h^{\mathbf{j}}$ of $\tilde{q}$ is a nonnegative linear combination of two of the coefficients of the h-form of q. Theorem 1 is proved. □

Examples. We use the set-up of Section 1, second example. We there have $n=2, a=4, b=2$. and pass from $q=c_\emptyset+c_1a_1+c_2a_2+c_{12}a_1a_2$ to a more complicated $\tilde{q}$. With the notation of the proof above, we have the formula

$$\sum_{I\subseteq[3]}\tilde{c}_I r_I(\mathbf{j})=(2-j_1)\sum_{I\subseteq[2]}c_I r_I(\mathbf{j})+(1+|\mathbf{j}|)\sum_{I\subseteq[2]}r_I(\mathbf{j}-e_1)c_I.$$

Accordingly, we find e.g. that the coefficient of $h_1h_2=h^{110}$ in $\tilde{q}$ is found from observing $\mathbf{j}=110$; $\mathbf{j}-e_1=010$; $|\mathbf{j}|=2$. So the coefficient is $1\sum_{I\subseteq[2]}c_I r_I(110)+3\sum_{I\subseteq[2]}r_I(010)c_I=1\cdot 2c_{12}+3(-c_1-c_2-2c_{12})=-3c_1-3c_2-4c_{12}$. Similarly we find (coefficient of $h_1^3=h^{300}$) $=-1\cdot 0+4\cdot c_{12}$ and (coefficient of $h_1^2=h^{200}$) $=0\cdot c_{12}+3\cdot(-c_1-c_2-2c_{12})=-3c_1-3c_2-6c_{12}$.

6. CONCLUDING REMARKS

Our linear inequalities give sufficient conditions on the coefficients c_I of an affine polynomial $q=q(a_1,a_2,...,a_n)$ in a-form in order that $q|\Delta_n\ge 0$. It is perhaps of interest to know how far away these conditions are from necessary and sufficient conditions. Such questions can be answered in principle by quantifier elimination. The responses given by computers, if obtainabble at all, cost even in relatively simple cases as much time as they are complicated. But the case $n=2$ can still be done and the response can be even explained.

The precise conditions in which $q(x,y)=q_2(x,y)=c_0+c_1x+c_2y+c_{12}xy$ is nonnegative on $\Delta_2=\{(x,y): x\le y\le 1\}$. were found via quantifier elimination with Mathematica© [W] by using successively the commands

```
xpr := ForAll[{x, y}, x <= y <= 1, c0 + c1 x + c2 y + c12 x y >= 0];
FullSimplify[Resolve[xpr, Reals]]
```

Proposition. A necessary and sufficient condition for the inequality $q_2|\Delta_2\ge 0$ to hold is that the following six conditions hold true:
i. $c_{12}\ge 0$, ii. $c_1+c_{12}\le 0$, iii. $c_0+c_1+c_2+c_{12}\ge 0$, iv. $c_0\ge 0$ or $c_1<0$; v. $c_{12}\ne 0$ or $c_1+c_2\le 0$.
vi. $c_{12}\le 0$ or $c_1+c_2+2c_{12}<0$ or $4c_0c_{12}\ge(c_1+c_2)^2$.

Proof. *Necessity* of i to vi for $q_2|\Delta_2\ge 0$. Assume $q_2|\Delta_2\ge 0$. Then $q(x,x)=x^2(c_{12}+(c_1+c_2)/x+c_\emptyset/x^2)\ge 0$ makes it clear that i holds, while $q_2(1,1)\ge 0$ implies iii. Suppose next ¬ii (i.e. not ii). Then $q(x,1)=c_\emptyset+c_2+x(c_1+c_{12})$ would become negative for negative x of large modulus. So ii holds. Suppose ¬iv. Then we would have $c_\emptyset<0$ and hence $q(0,0)=c_\emptyset<0$. So iv holds. Suppose ¬v. Then $c_{12}=0$ and $c_1+c_2>0$. Then $q(x,x)=c_\emptyset+x(c_1+c_2)$. This would be negative for negative x of large modulus. So v holds. Finally assume ¬vi. Then $c_{12}>0\,\&\,c_1+c_2+2c_{12}\ge 0\,\&\,4c_\emptyset c_{12}<(c_1+c_2)^2$. The quadratic $\mathbb{R}\ni x\mapsto q(x,x)$ has discriminant $\Delta=(c_1+c_2)^2-4c_\emptyset c_{12}>0$ and assumes its extreme value $\frac{-\Delta}{4c_{12}}<0$ in the point $-\frac{c_1+c_2}{2c_{12}}\le 1$ but this contradicts $q|\Delta_2\ge 0$. Thus vi is also necessary. *Sufficiency* of i to vi for $q_2|\Delta_2\ge 0$. We begin with the special case $c_{12}=0$. In this case $q(x,y)=c_\emptyset+c_1x+c_2y$. Then ii says $c_1\le 0$ and hence $\partial_x q(x,y)\le 0$, and hence $q(x,y)\ge q(y,y)=c_\emptyset+(c_1+c_2)y$. By v, now $c_1+c_2\le 0$ so $\partial_y q\le 0$ and so $q(y,y)\ge q(1,1)=c_\emptyset+c_1+c_2\ge 0$, where the latter inequality follows from iii. So the case $c_{12}=0$ is done. Assume now $c_{12}\ne 0$. Then condition i becomes $c_{12}>0$, ii,iii,iv, are as before, v is automatically satisfied and vi simplifies to $c_1+c_2+2c_{12}<0$ or $4c_\emptyset c_{12}\ge(c_1+c_2)^2$. By i,ii then $\partial_x q=c_1+c_{12}y\le 0$; so $q(x,y)\ge q(y,y)=c_\emptyset+(c_1+c_2)y+c_{12}y^2$. Now $\partial_y q(y,y)=(c_1+c_2)+2c_{12}y$. If the first inequality of modified vi holds this is by i negative and we get $q(y,y)\ge q(1,1)\ge 0$ by iii. If the first inequality in modified vi does not hold then vi implies that $c_1+c_2+2c_{12}\ge 0\,\&\,4c_\emptyset c_{12}\ge(c_1+c_2)^2$ holds. Then the discriminant for $q(y,y)=c_\emptyset+(c_1+c_2)y+c_{12}y^2$ is $\Delta=(c_1+c_2)^2-4c_\emptyset c_{12}\le 0$ and the smallest value assumed by $q(y,y)$ on $\mathbb{R}$ is $\frac{-\Delta}{4c_{12}}\ge 0$ and it is assumed in $-\frac{c_1+c_2}{2c_{12}}\le 1$. Hence again $q(x,y)\ge 0$. □

$$\begin{array}{lrcl}
l_1: & c_0+c_1+c_2+c_{12} & \ge & 0\\
l_2: & -c_1-c_2-2c_{12} & \ge & 0\\
l_3: & c_{12} & \ge & 0\\
l_4: & -c_1-c_{12} & \ge & 0
\end{array}$$

The linear inequalities l_1, l_2, l_3, l_4

vi' : $c_{12}\le 0$ or $c_1+c_2+2c_{12}<0$.
vi" : $c_{12}\le 0$ or $c_1+c_2+2c_{12}\le 0$.

The conditions vi' and vi"

Consider the hc-table of q. It encodes linear inequalities shown above at the left called l_1, l_2, l_3, l_4 guaranteeing that the coefficients of q in h-form are nonnegative and hence that $q|\Delta_2 \geq 0$. Also define the modifications vi' and vi" of item vi of the proposition, as shown at the right.

Lemma. A point $(c_0, c_1, c_2, c_{12}) \in \mathbb{R}^4$ satisfies conditions i,ii,iii,iv,v, and vi" iff it satisfies the inequalities l_1, l_2, l_3, l_4.

Proof. The inequalities i,ii,iii are respectively equivalent to the inequalities l_3, l_4, l_1.

$\Rightarrow$: Suppose we have $\neg l_2$ for a certain point $(c_0, c_1, c_2, c_{12}) \in \mathbb{R}^4$ which satisfies all the hypotheses. Then there holds $c_1 + c_2 + 2c_{12} > 0$. Now the hypothesis vi" forces $c_{12} \leq 0$. Together with i we thus have $c_{12} = 0$ which by v means $c_1 + c_2 \leq 0$. But then $c_1 + c_2 + c_{12} = c_1 + c_2 \leq 0$, a contradiction. So indeed l_2 also holds.

$\Leftarrow$: We have to show that iv,v,vi" are satisfied. In fact doing the addition $l_1 + l_2 + l_3$ we see $c_0 \geq 0$ and thus iv. Doing $l_2 + 2l_3$ yields $c_1 + c_2 \leq 0$, and thus v. Finally l_2 yields $c_1 + c_2 + 2c_{12} \leq 0$, hence vi" . □

We see that the linear conditions given by quantifier elimination are almost precisely identically to our linear conditions. But note that even though vi" is an only minimally weaker condition than vi', we cannot substitute in the implication '$\Leftarrow$' the condition vi" by vi'. To see this consider the point $(c_0, c_1, c_2, c_{12}) = (2, -2, 0, 1)$. It satisfies the hypotheses l_1, l_2, l_3, l_4 but does not satisfy vi'.

With Mathematica' s `FindInstance` command one can now easily find an example of an affine polynomial $q(x, y)$ with $q|\Delta_2 \geq 0$ and a successor $\tilde{q}$ which does not satisfy $\tilde{q}|\Delta_3 \geq 0$. Put $c_\emptyset = 3533/2048, c_1 = -1, c_2 = 29/8, c_{12} = 1$. These values of $c_\emptyset, c_1, c_2, c_{12}$ satisfy all the six conditions of the proposition and hence give rise to a polynomial q_2 which satisfies $q_2|\Delta_2 \geq 0$. In h-form $q_2 = 10957/2048 - (37h_1)/8 + h_1^2 + h_1h_2$. Using the example of Section 1 $\tilde{q} = 3533/512 - 5a_1 + (145a_2)/8 + 6a_1a_2 - (3533a_3)/1024 + 3a_1a_3 - 87/8a_2a_3 - 4a_1a_2a_3$. Its value in the point $(a_1, a_2, a_3) = (-(261/256), -1, 0) \in \Delta_3$ is $\frac{-5}{512} < 0$. This shows that in general we have $q_2|\Delta_2 \geq 0 \not\Rightarrow \tilde{q}|\Delta_3 \geq 0$, proving the claim concerning the question in Section 1. □

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