Inequalities and bounds for expected order statistics from transform-ordered families

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Abstract

We introduce a comprehensive method for establishing stochastic orders among order statistics in the i.i.d. case. This approach relies on the assumption that the underlying distribution is linked to a reference distribution through a transform order. Notably, this method exhibits broad applicability, particularly since several well-known nonparametric distribution families can be defined using relevant transform orders, including the convex and the star transform orders. In the context of convex-ordered families, we demonstrate that applying Jensen's inequality enables the derivation of bounds for the probability that a random variable exceeds the expected value of its corresponding order statistic.

Keywords: stochastic orders, hazard rate, odds, convex, star shaped, exceedance probability

1. Introduction

Order statistics are fundamental tools in different areas of probability, statistics, and reliability theory. Especially in the context of reliability, a major issue consists of comparing order statistics with different ranks and sample sizes, given random samples from a common baseline distribution. To be more specific, let X be a random variable (RV) and denote with $X_{k:n}$ the k-th order statistic corresponding to an i.i.d. random sample of size n from X. If X represents the lifetime of some component, then $X_{k:n}$ is the lifetime of a k-out-of-n system, that is, a system that fails if and only if at least k components stop functioning. The ageing and reliability properties of such systems, described in terms of their stochastic behaviour, are an important aspect. Hence, the issue of comparing, in some stochastic sense, the order statistics $X_{i:n}$ and $X_{j:m}$, corresponding to systems with

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a different number of components and different functioning requirements, naturally arises. This problem can be addressed by the theory of *stochastic orders* (see Shaked and Shanthikumar (2007) for general results and relationships). In particular, several results on the stochastic comparison between order statistics have been obtained, for example, by Arnold and Villaseñor (1991), Arnold and Nagaraja (1991), Kochar (2006), Kochar and Xu (2009), Kochar (2012) or Lando et al. (2021).

This paper focuses on establishing conditions under which $X_{i:n}$ dominates $X_{j:m}$ in the sense that $\mathbb{E}u(X_{i:n}) \geq \mathbb{E}u(X_{j:m})$ for every function u in some class $\mathcal{U}$. Relationships of this kind are referred to as *integral stochastic orders* with respect to a generator class $\mathcal{U}$, as defined by Müller (1997), and include comparisons of expected order statistics whenever $\mathcal{U}$ contains the identity function. Significant examples of integral stochastic orders include the *increasing concave* (ICV), *increasing convex* (ICX), and the *star-shaped* (SS) orders (Shaked and Shanthikumar, 2007). In contrast to numerous methods found in the literature (Arnold and Nagaraja, 1991; Wilfling, 1996; Kundu and Chowdhury, 2016), our approach does not presuppose a known parametric form for the cumulative distribution function (CDF) F of the RV X. Instead, we opt for a more flexible approach, leveraging nonparametric assumptions about F. Specifically, we express it in the form $G^{-1} \circ F$, where G is a carefully chosen cumulative distribution function, and $\mathcal{H}$ represents a set of increasing functions referred to as the generator class. In other words, we assume that F is related to G via a *transform order* (Lando et al., 2023b). Interesting examples of distributions satisfying transform order assumptions are the *increasing hazard rate* (IHR), *increasing odds rate* (IOR), *decreasing density* (DD), *DD on average* (DDA), *decreasing reversed hazard rate* (DRHR) families (Shaked and Shanthikumar, 2007; Marshall and Olkin, 2007; Lando et al., 2022). In this paper we show that a key step for deriving appealing probabilistic inequalities between order statistics within transform-ordered families involves combining integral and transform orders with the same generator class. Additionally, we illustrate the application of this approach by deriving bounds for expected values of order statistics. Our method's general behaviour aligns with expectations: stronger assumptions on F lead to more applicable ordering conditions between $X_{i:n}$ and $X_{j:m}$, or more stringent bounds, and vice versa.

The paper is organized as follows. In Section 2, we present formal definitions and outline our general approach. Although our result takes on a highly general form, its application extends seamlessly to well-known classes of distributions. Section 4 delves into the derivation of conditions for the ICV and ICX order between order statistics from convex-ordered families, extending some recent results of Lando et al. (2021). Moving on to Section 5, we establish conditions for the SS order between order statistics within star-ordered families. Finally, Section 6 provides bounds for the probability that X exceeds its expected order statistic $\mathbb{E}X_{i:n}$, that is, the probability that a single component surpasses the expected lifetime of the system. As a byproduct of this general result, we provide a new characterisation of the log-logistic distribution (with shape parameter 1), that is, the only distribution for which the expected order statistic $\mathbb{E}X_{i:n}$ coincide with its $\frac{i}{n}$-quantile, for every $k = 1, \ldots, n$.

2. A general method

Throughout this paper, "increasing" and "decreasing" are taken as "non-decreasing" and "non-increasing", respectively, and the generalised inverse of an increasing function v is denoted as v^{-1}. Moreover, the beta function with parameters $a, b > 0$ is denoted with $\mathcal{B}(a, b)$. Finally, given an absolutely continuous CDF, its density function is denoted by the corresponding lowercase letter.

The following general families of stochastic orders are crucial for establishing comparisons between expectations of order statistics.

Definition 1. (Müller, 1997) Let $\mathcal{U}$ be some family of functions. We say that X dominates Y in the $\mathcal{U}$-integral stochastic order, denoted as $X \geq_{\mathcal{U}}^{I} Y$, if $\mathbb{E}u(X) \geq \mathbb{E}u(Y)$ for every $u \in \mathcal{U}$, provided that the integrals exist. $\mathcal{U}$ is referred to as the generator of the integral order.

We shall be dealing with transformations of RVs, so the next definition provides means to control the behaviour with respect to stochastic orders.

Definition 2. Let $\mathcal{H}$ be some family of functions. We say that an integral stochastic order $\geq_{\mathcal{U}}^{I}$ is preserved under $\mathcal{H}$-transformations, and write $\geq_{\mathcal{U}}^{I} \in \mathcal{P}_{\mathcal{H}}$, if $X \geq_{\mathcal{U}}^{I} Y$ implies $h(X) \geq_{\mathcal{U}}^{I} h(Y)$ whenever $h \in \mathcal{H}$.

Definition 3. (Lando et al., 2023b) Let $\mathcal{H}$ be some family of increasing functions. We say that $X \sim F$ dominates $Y \sim G$ in the $\mathcal{H}$-transform order, denoted as $X \geq_{\mathcal{H}}^{T} Y$, or, equivalently, $F \geq_{\mathcal{H}}^{T} G$, if $F^{-1} \circ G \in \mathcal{H}$. $\mathcal{H}$ is referred to as the generator of the transform order $\geq_{\mathcal{H}}^{T}$.

In this article, we show that a useful approach to obtaining interesting stochastic inequalities consists of a suitable combination of integral and transform orderings based on a common generating class.

We shall be taking F as the CDF of interest, and G some suitably chosen reference CDF. It is well known that the CDF of $X_{i:n}$ is given by $F_{B_{i:n}} \circ F$, where $F_{B_{i:n}}$ is the CDF of a beta random variable with parameters i and $n-i+1$, that is, $B_{i:n} \sim beta(i, n-i+1)$ (Jones, 2004). Equivalently, one can write $X_{i:n} \stackrel{d}{=} F^{-1} \circ B_{i:n}$. This representation renders it difficult to establish conditions for a stochastic comparison between two different order statistics, say $X_{i:n}$ and $X_{j:m}$, since the result depends on the four parameters i, j, n, m and on the analytical form of F. In a parametric framework, F is assumed to be known up to defining several real parameters, so the problem boils down to a mathematical exercise, which may still be analytically complicated. However, if F is in some nonparametric class, the problem is more complicated, and, as we show in the sequel, it can be solved just by adding some shape constraints on F. In this nonparametric framework, results may still be obtained by applying a simple decomposition trick: write $X_{i:n} \stackrel{d}{=} F^{-1} \circ G \circ G^{-1} \circ B_{i,n}$ and assume that F is related to some known G by a suitable transform order. Indeed, in this case, the analytical form of G being known, the problem reduces to a simpler comparison between known RVs, namely $G^{-1} \circ B_{i:n}$ and $G^{-1} \circ B_{j:m}$.

For the sake of convenience, we introduce the following notation.

Definition 4. Let G be some CDF and $\mathcal{H}$ some family of increasing functions. We define $\mathcal{F}_{\mathcal{H}}^{G} = \{F : F \geq_{\mathcal{H}}^{T} G\}$, that is, the family of CDFs that dominate G with respect to the $\mathcal{H}$-transform order.

We may now state our main result, which establishes sufficient conditions for comparing expected order statistics.

Theorem 5. *Let $\mathcal{H}$ be a class of increasing functions. If, for some given CDF G, $X \sim F \in \mathcal{F}_{\mathcal{H}}^{G}$, $G^{-1} \circ B_{i:n} \geq_{\mathcal{H}}^{I} G^{-1} \circ B_{j:m}$ and $\geq_{\mathcal{H}}^{I} \in \mathcal{P}_{\mathcal{H}}$, then $\mathbb{E}h(X_{i:n}) \geq \mathbb{E}h(X_{j:m})$, whenever $h \in \mathcal{H}$.*

Proof. Writing $X_{i:n} \stackrel{d}{=} F^{-1} \circ G \circ G^{-1} \circ B_{i:n}$, the result follows easily from the definitions above. In fact, the order $\geq_{\mathcal{H}}^{I}$ is preserved under $\mathcal{H}$-transformations, whereas the assumption $F \in \mathcal{F}_{\mathcal{H}}^{G}$ ensures that $F^{-1} \circ G$ is an $\mathcal{H}$-transformation. Therefore, applying the transformation $F^{-1} \circ G$ to both sides of the stochastic inequality $G^{-1} \circ B_{i:n} \geq_{\mathcal{H}}^{I} G^{-1} \circ B_{j:m}$ we obtain $X_{i:n} \geq_{\mathcal{H}}^{I} X_{j:m}$, which implies the desired result by definition of integral stochastic orders, taking into account that $h \in \mathcal{H}$. □

Despite the apparent simplicity of Theorem 5, its applications are remarkably interesting, showcasing the profound implications of the interplay between integral and transform orders.

3. Types of class generators

Definitions 1 and 3 become particularly interesting when the generator classes are chosen as well-known and popular families in reliability applications. Some of these classes have been discussed in the Introduction and will be explored in more detail below.

Let us recall the defining properties of some relevant classes of functions. A non-negative function $h(x)$, defined for $x \geq 0$ and such that $h(0) = 0$, is said to be star-shaped at the origin of every segment joining the origin with the graph of h always stays above the graph. This holds if and only if $\frac{h(x)}{x}$ is increasing. Also of interest, h is said to be anti-SS (at the origin) if every segment that joins the origin with the graph of h is always below the graph, which holds if and only if $\frac{h(x)}{x}$ is decreasing.

We should note that the standard stochastic order may be seen both as an integral and a transform order. In fact, we say that X dominates Y in the standard stochastic order, abbreviated as $X \geq_{st} Y$, if $\mathbb{E}u(X) \geq \mathbb{E}u(Y)$ for every increasing function u, or, equivalently, if the $F^{-1} \circ G(x) \leq x$, for every $x \geq 0$. Besides the standard order, we will focus on the following integral stochastic orders, obtained from Definition 1 for particular generator classes.

Definition 6. Assume that $X \geq_{\mathcal{U}}^{I} Y$. We say that X dominates Y in

1. the increasing concave (ICV) order, denoted as $X \geq_{icv} Y$, if $\mathcal{U}$ is the family of increasing concave functions;
2. the increasing convex (ICX) order, denoted as $X \geq_{icx} Y$, if $\mathcal{U}$ is the family of increasing convex functions;
3. the star-shaped (SS) order, denoted as $X \geq_{ss} Y$, if $\mathcal{U}$ is the family of star-shaped functions.

The relationships among classes of functions yield the following implications:

$$\begin{array}{ccccc} X \geq_{st} Y & \Longrightarrow & X \geq_{ss} Y & \Longrightarrow & X \geq_{icx} Y \\ \Downarrow & & & & \\ X \geq_{icv} Y & & & & \end{array}$$

All these orders imply inequality of the means, $\mathbb{E}X \geq \mathbb{E}Y$ since the identity function belongs to each of the above classes.

Similarly to the integral stochastic orders defined earlier, the following transform orders may be obtained from Definition 3 by taking $\mathcal{H}$ as the class of convex and star-shaped functions, respectively.

Definition 7. Assume that $X \geq_{\mathcal{H}}^{T} Y$. We say that X dominates Y in

1. the convex transform order, denoted as $X \geq_c Y$, if $\mathcal{H}$ is the family of (increasing) convex functions;
2. the star order, denoted as $X \geq_* Y$, if $\mathcal{H}$ is the family of star-shaped functions.

Again, the identity function belongs to each of these generators, hence inequalities between expected values of the corresponding random variables hold.

Theorem 5 is quite general and it may lead to different results, according to the choice of the class $\mathcal{H}$ and of the reference CDF G. These choices also yield different classes of the type $\mathcal{F}_{\mathcal{H}}^{G}$, which, we recall, are defined via a transform order relationship. Indeed, as shown below, when $\mathcal{H}$ is the class of convex or concave functions, $\mathcal{F}_{\mathcal{H}}^{G}$ may be characterised using the convex transform order. Hence, we will refer to these choices of $\mathcal{H}$ as *convex-ordered* families. Similarly, when $\mathcal{H}$ is the class of star-shaped or anti-star-shaped functions, $\mathcal{F}_{\mathcal{H}}^{G}$ may be characterised via the star transform order, so we will refer to these choices as *star-ordered* families. For the sake of simplicity, besides the already defined classes $\mathcal{C}$ of convex functions, and $\mathcal{S}$ of functions that are star-shaped at the origin, we shall define $\mathcal{V}$ as the class of concave functions, and $\mathcal{A}$ as the class of anti-SS functions at the origin.

The results that follow from Theorem 5 also depend on the choice of G. In particular, we will consider the uniform distribution on the unit interval, with CDF $U(x) = x$, $x \in [0,1]$, the exponential distribution, with CDF $\mathcal{E}(x) = 1 - e^{-x}$, $x \geq 0$, the standard logistic distribution with CDF $L(x) = \frac{1}{e^{-x}+1}$, $x \in \mathbb{R}$, and the log-logistic distribution with shape parameter equal to 1, hereafter LL1, with CDF $LL(x) = \frac{x}{1+x}$, $x \geq 0$. We will also consider the corresponding "*negative*" versions: in general, if $Y \sim G$ then $-Y \sim G_-$, where $G_-(x) = 1 - G(-x)$. Note that, due to symmetry, for the logistic distribution we have $L = L_-$.

Combining the classes $\mathcal{C}$, $\mathcal{V}$, $\mathcal{S}$, or $\mathcal{A}$ with the choices of G discussed above, we may generate several different families of distributions, some of them well known in the literature. An application of Theorem 5 will derive inequalities that hold for each of the constructed classes of distributions. Naturally, some of these are more interesting than others. Moreover, for a few classes, we were not able to apply our method, because of some technical difficulties, which will be discussed in the subsequent open problem section. Therefore, hereafter we will focus on the following classes.

1. The class of concave CDFs, also known as *decreasing density* (DD) class, as requires the existence of a decreasing PDF (except, possibly, at the left-endpoint of its support). This may be obtained by $\mathcal{F}^U_{\mathcal{C}} = \{F : F \geq^T_{\mathcal{C}} U\} = \{F : F \geq_c U\} = \{F^{-1} \in \mathcal{C}\} = \mathcal{V}$. This class has received much attention in the literature, for instance, it is a typical assumption for shape-constrained statistical inference (Groeneboom and Jongbloed, 2014). Among known parametric models, the gamma, the log-logistic, and the Weibull distributions, with shape parameters less than or equal to 1, belong to this class.
2. The class of convex CDFs, also known as *increasing density* (ID) class, as requires the existence of an increasing PDF (except, possibly, at the left-endpoint of its support). This may be obtained by $\mathcal{F}^U_{\mathcal{V}} = \{F : F \geq^T_{\mathcal{V}} U\} = \{F : U \geq_c F\} = \{F^{-1} \in \mathcal{V}\} = \mathcal{C}$. This class is generally less applicable than the DD one, as it requires bounded support, and contains few known parametric models.
3. The class of anti-star-shaped CDFs. In the case of absolutely continuous distributions, this is also known as the class of distribution with *decreasing density on average* (DDA), as it requires the $\frac{F(x)}{x} = \frac{1}{x}\int_0^x f(t)\,dt$ to be decreasing. This may be obtained by $\mathcal{F}^U_{\mathcal{S}} = \{F : F \geq^T_{\mathcal{S}} U\} = \{F : F \geq_* U\} = \{F^{-1} \in \mathcal{S}\} = \mathcal{A}$. This is an interesting class, as it extends the applicability of the popular DD class, allowing for non-monotonicity of the PDF and jumps in the CDF.
4. The class of distributions with a convex *hazard function*, $H = -\ln(1-F)$, that is, the well-known *increasing hazard rate* (IHR) class (Marshall and Olkin, 2007), as it requires the existence of an increasing hazard rate function $h = \frac{f}{1-F}$ (except, possibly, at the left-endpoint of the support). This may be obtained by $\mathcal{F}^{\mathcal{E}}_{\mathcal{V}} = \{F : \mathcal{E} \geq^T_{\mathcal{V}} F\} = \{F : F^{-1} \circ \mathcal{E} \in \mathcal{V}\} = \{F : \mathcal{E} \geq_c F\}$. The properties and applicability of IHR models are well known.
5. The class of distributions with a concave hazard function, that is, the *decreasing hazard rate* (DHR) class (Marshall and Olkin, 2007), as it requires the existence of a decreasing hazard rate function $h = \frac{f}{1-F}$. Analogously, to the previous example, this class may be obtained by $\mathcal{F}^{\mathcal{E}}_{\mathcal{V}} = \{F : \mathcal{E} \geq^T_{\mathcal{V}} F\} = \{F : F^{-1} \circ \mathcal{E} \in \mathcal{V}\} = \{F : \mathcal{E} \leq_c F\}$.
6. The class of distributions with an anti-star-shaped hazard function. This is denoted as the

DHR on average (DHRA) class, as it requires $\frac{H(x)}{x} = \frac{1}{x}\int_0^x h(t)\,dt$, in the absolutely continuous case, to be decreasing. This class may be obtained as $\mathcal{F}_{\mathcal{A}}^{\mathcal{E}} = \{F : \mathcal{E} \leq_* F\}$. It extends the applicability of the DHR class (in the non-negative case).

7. The class of CDFs such that $\log F$ is concave, also characterised by $\frac{f}{F}$ being decreasing, known as the *decreasing reversed hazard rate* (DRHR) class. This is a rather broad class of distributions. One may obtain this class taking $\mathcal{H} = \mathcal{C}$ and $\mathcal{F}_{\mathcal{C}}^{\mathcal{E}_-} = \{F : \mathcal{E}_- \leq_c F\}$, the class of functions that dominate $\mathcal{E}_-$ w.r.t. the convex transform order.
8. The class of distributions with a convex *odds function*, $\frac{F}{1-F}$, that is, the *increasing odds rate* (IOR) class (Lando et al., 2022), as it requires the existence of an increasing odds rate function $\frac{f}{(1-F)^2}$ (except, possibly, at the left-endpoint of the support). This may be obtained by $\mathcal{F}_{\mathcal{V}}^{LL} = \{F : LL \geq_{\mathcal{V}}^{T} F\} = \{F : F^{-1} \circ LL \in \mathcal{V}\} = \{F : LL \geq_c F\}$. The properties and applicability of IOR models are discussed by Lando et al. (2022) and Lando et al. (2023a).
9. The class with a concave odds function may be similarly defined as the *decreasing odds rate* (DOR) class, which may be obtained as $\mathcal{F}_{\mathcal{C}}^{LL}$.
10. The class of distributions with a convex *log-odds function*, $\log \frac{F}{1-F}$, that is, the *increasing log-odds rate* (ILOR) class (Zimmer et al., 1998), as it requires the existence of an increasing log-odds rate function $\frac{f}{F(1-F)}$. This may be obtained by $\mathcal{F}_{\mathcal{V}}^{L} = \{F : L \geq_c F\}$.
11. The class with a concave log-odds function may be similarly defined as the *decreasing log-odds rate* (DLOR) class, which may be obtained as $\mathcal{F}_{\mathcal{C}}^{L}$.

4. Convex-ordered families

In this section, we apply Theorem 5 to families of distributions which may be obtained through the convex transform order, extending some recent results of Lando et al. (2021). All results are summarised in the following corollaries. Although some cases are already proved in Lando et al. (2021), we report them here for the sake of completeness.

Corollary 8. *[Lando et al. (2021)] If $i \geq j$, any of the following conditions imply $X_{i:n} \geq_{icv} X_{j:m}$.*

1. *F is ID and $\frac{i}{n+1} \geq \frac{j}{m+1}$;*
2. *F is IHR and $\sum_{k=n-i+1}^{n} \frac{1}{k} \geq \sum_{k=m-j+1}^{m} \frac{1}{k}$;*
3. *F is IOR class and $\frac{i}{n} \geq \frac{j}{m}$;*

4. *F is ILOR and* $\psi(i)-\psi(n-i+1)\geq\psi(j)-\psi(m-j+1)$*, where* ψ *is the digamma function.*

Corollary 9. *If* $i\leq j$*, any of the following conditions imply* $X_{i:n}\geq_{icx}X_{j:m}$*.*

1. *F is DD class and* $\frac{i}{n+1}\geq\frac{j}{m+1}$*;*
2. *F is DHR and* $\sum_{k=n-i+1}^{n}\frac{1}{k}\geq\sum_{k=m-j+1}^{m}\frac{1}{k}$*;*
3. *F is DOR class and* $\frac{i}{n}\geq\frac{j}{m}$*;*
4. *F is DLOR and* $\psi(i)-\psi(n-i+1)\geq\psi(j)-\psi(m-j+1)$*;*
5. *If F is DRHR and* $\sum_{k=i}^{n}\frac{1}{k}\leq\sum_{k=j}^{m}\frac{1}{k}$*;*
6. *F is DROR and* $\frac{n}{n-i}\leq\frac{m}{m-j}$*.*

Proof. If $G^{-1}\circ F$ is concave, then $F^{-1}\circ G$ is ICX. The ICX order is preserved under ICX transformations, so $\geq_{icx}$, equivalently defined as $\geq_{\mathcal{C}}^{I}$, belongs to $\mathcal{P}_{\mathcal{C}}$. A sufficient condition for $G^{-1}\circ B_{i:n}\geq_{icx}G^{-1}\circ B_{j:m}$ is that $i\leq j$ and $\mathbb{E}G^{-1}\circ B_{i:n}\geq\mathbb{E}G^{-1}\circ B_{j:m}$. Then, setting G as the uniform, unit exponential, LL1, standard logistic, negative exponential, and negative LL1, we obtain conditions 1–6, respectively. We verify only case 5., the less obvious one, corresponding to $G=\mathcal{E}_{-}$, where we need to compute

$$\mathbb{E}\log B_{i:n}=\int_0^1\frac{t^{i-1}(1-t)^{n-i}\log t}{\mathcal{B}(i,n-i+1)}\,dt=\psi(i)-\psi(n+1)=-\sum_{k=i}^{n}\frac{1}{k},$$

taking into account the properties of the digamma function (use repeatedly (6.44) in Viola (2016)).

□

The results of Corollaries 8 and 9 can be used to derive bounds for expected order statistics. In particular, since the ICV and the ICX orders imply the inequality between the means, we may derive conditions for the comparison between expected order statistics and the mean of the parent distribution. Setting $j=m=1$ in Corollary 8, the following result is immediate.

Corollary 10. *Each of the following conditions implies* $\mathbb{E}X_{i:n}\geq\mu$*.*

1. *F is ID and* $\frac{i}{n+1}\geq\frac{1}{2}$*;*
2. *F is ILOR and* $\psi(i)-\psi(n-i+1)\geq 0$*;*
3. *F is IHR and* $\sum_{k=n-i+1}^{n}\frac{1}{k}\geq 1$*.*

Similarly, by fixing $i=n=1$ in Corollary 9.

Corollary 11. *Each of the following conditions implies* $\mathbb{E}X_{j:m} \leq \mu$.

1. F *is in the DD class and* $\frac{1}{2} \geq \frac{j}{m+1}$*;*
2. F *is DLOR and* $\psi(j) - \psi(m-j+1) \leq 0$*;*
3. F *is DHR and* $1 \geq \sum_{k=m-j+1}^{m} \frac{1}{k}$*;*
4. F *is DRHR and* $1 \leq \sum_{k=j}^{m} \frac{1}{k}$*.*

One could try to exploit the aforementioned inequalities to derive the conditions under which $\mathbb{E}X_{i:n} = \mu$. However, this will lead to conditions that are either impossible or trivial. For instance, let's assume that F is IHR and DD, which is often possible. If one can find some i, n such that $\sum_{k=n-i+1}^{n} \frac{1}{k} \geq 1$ and $\frac{1}{2} \geq \frac{i}{n+1}$, then $\mathbb{E}X_{i:n} = \mu$. However, these conditions cannot be verified simultaneously. The only solutions are trivial, for example, consider the scenario where F is IHR and DHR, effectively reducing to $F = \mathcal{E}$. If $\sum_{k=n-i+1}^{n} \frac{1}{k} = 1$ then $\mathbb{E}X_{i:n} = \mu$. Similar to the previous example, this is possible just for the trivial case $i = n = 1$.

5. Star-ordered families

Let us start with some preliminary discussion. As starshapedness refers only to functions with domain $[0, +\infty)$, in this section, we will consider only non-negative RVs. First, a simple preservation property.

Lemma 12. $X \geq_{ss} Y$ *if and only if* $h(X) \geq_{ss} h(Y)$*, for every star-shaped function* h*.*

Proof. Let ϕ be a star-shaped function, that is, $\frac{\phi(x)}{x}$ is increasing, and consider the composition $\phi \circ h$. Now, obviously $\frac{\phi(h(x))}{x} = \frac{\phi(h(x))}{h(x)} \frac{h(x)}{x}$, where $\frac{\phi(h(x))}{h(x)}$ is increasing since ϕ is star-shaped and h is increasing, therefore $\frac{\phi(h(x))}{x}$ is increasing, being the product of two increasing functions. Thus, for every star-shaped function h, the composition $\phi \circ h$ is star-shaped as well. Now, $X \geq_{ss} Y$ if and only if $E(\phi \circ h(X)) \geq E(\phi \circ h(Y))$, for every star-shaped ϕ, that is, $h(X) \geq_{ss} h(Y)$, by definition. □

It is also useful to remark that a function ϕ is star-shaped if and only if its generalized inverse ϕ^{-1} is increasing anti-star-shaped. We now recall the following characterization of the SS order.

Theorem 13 (Shaked and Shanthikumar (2007), Theorem 4.A.54)**.** $X \geq_{ss} Y$ *if and only if, for every* $x \geq 0$, $\int_x^\infty t\,dF(t) \geq \int_x^\infty t\,dG(t)$.

In the following subsections, we will frequently deal with transformations of beta random variables using the result stated next. The proof is omitted since it straightforwardly follows, requiring a simple observation of the shape of the graphical representation of the function considered in each case.

Lemma 14. *Let $T_{a,b}(x) = x^a(1-x)^b$, where $a, b \in \mathbb{R}$, and $c > 0$. The equation $T_{a,b}(x) = c$ has one solution in $(0,1)$ if and only if $ab \le 0$. The equation $T_{a,b}(x) = c$ may have up to two solutions in $(0,1)$ if and only if $ab > 0$.*

Hereafter, let us denote by $R(a,b,c)$ the set of roots of the equation $T_{a,b}(x) = c$. The previous lemma means that when $c > 0$, $R(a,b,c)$ has at most two elements.

Using the above lemmas, it is not difficult to apply Theorem 5 to wide families of distributions, as discussed in the next subsections.

5.1. DDA distributions

Theorem 15. *Assume that F is DDA. Denote by*

$$Z(x) = \frac{i}{n+1}\left(1 - F_{B_{i+1:n+1}}(x)\right) - \frac{j}{m+1}\left(1 - F_{B_{j+1:m+1}}(x)\right). \tag{1}$$

If for every $r \in R\left(i-j, n-i-(m-j), \frac{\mathcal{B}(i,n-i+1)}{\mathcal{B}(j,m-j+1)}\right)$, it holds that $Z(r) \ge 0$, then $X_{i:n} \ge_{ss} X_{j:m}$.

Proof. Since F^{-1} is star-shaped, the result holds by Theorem 5 and Lemma 12, provided that $B_{i:n} \ge_{ss} B_{j:m}$, which, taking into account Theorem 13 and the distribution of the beta order statistics mentioned before (Jones, 2004), is equivalent to

$$\int_x^1 \frac{t^i(1-t)^{n-i}}{\mathcal{B}(i,n-i+1)}\,dt \ge \int_x^1 \frac{t^j(1-t)^{m-j}}{\mathcal{B}(j,m-j+1)}\,dt, \quad \forall x \in [0,1]. \tag{2}$$

It is easily seen that (2) is equivalent to $Z(x) \ge 0$, for every $x \in [0,1]$. Now, the extreme points of Z are at 0, 1, or among the solutions of $T_{i-j,(n-i)-(m-j)}(x) = \frac{\mathcal{B}(i,n-i+1)}{\mathcal{B}(j,m-j+1)}$, hence the result follows immediately from the assumptions $Z(0) = \frac{i}{n+1} - \frac{j}{m+1} = \mathbb{E}B_{i:n} - \mathbb{E}B_{j:m} \ge 0$, $Z(1) = 0$, and $Z(r) \ge 0$, for every $r \in R\left(i-j, (n-i)-(m-j), \frac{\mathcal{B}(i,n-i+1)}{\mathcal{B}(j,m-j+1)}\right)$. □

The results of Theorem 15 can be compared with part 1. of Corollary 9. Assume that $\frac{i}{n+1} \ge \frac{j}{m+1}$, which is equivalent to $Z(0) \ge 0$. If F is concave (DD class), then $X_{i:n} \ge_{icx} X_{j:m}$ for $i \le j$. If F is IAS (yielding the wider DDA class), then the stronger order $X_{i:n} \ge_{ss} X_{j:m}$ holds if $Z(r) \ge 0$,

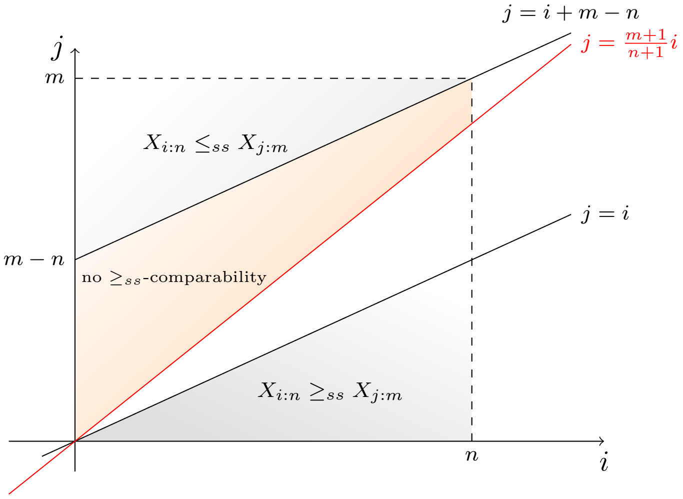

Figure 1: $\geq_{ss}$-comparability for distributions in the DDA class.

for r in the described set. Recall that the ICX order is necessary for the SS order, and $i \leq j$ is necessary for the ICX order. So, the condition $Z(r) \geq 0$ is stronger than $i \leq j$.

We may use Theorem 15 to get a complete geometric description of the $\geq_{ss}$-comparability of order statistics when F is DDA. Assume the sample sizes $n \leq m$ are given. Based on Theorem 1 in Arab et al. (2021) we know that $B_{i:n} \geq_{st} B_{j:m}$, which implies $B_{i:n} \geq_{ss} B_{j:m}$, whenever $i > j$ and $n-i < m-j$, that is, whenever $i > j$. Likewise, this result also implies that $B_{i:n} \leq_{st} B_{j:m}$, implying $B_{i:n} \leq_{ss} B_{j:m}$, whenever $i < j$ and $n-i > m-j$, which is equivalent to $n-i > m-j$ (see Figure 1). For the region $i < j < i + m - n$ we have no $\geq_{st}$-comparability. The line $j = \frac{m+1}{n+1} i$ corresponds to points such that $Z(0) = 0$, where Z is given by (1). Above this line we have $Z(0) < 0$ hence, according to Theorem 13, there is no $\geq_{ss}$-comparability. Finally, we are left with the region where $i < j$ and $\frac{i}{n+1} \geq \frac{j}{m+1}$, the region not shaded in Figure 1, where actual verification of (2) is needed. For (i, j) in the unshaded region it is easily seen that $Z'(x) = -\frac{x^i(1-x)^{n-i}}{\mathcal{B}(i+1,n-i+1)} + \frac{x^j(1-x)^{m-j}}{\mathcal{B}(j+1,m-j+1)} < 0$ whenever x is close to 0 or 1. Moreover, as Lemma 14 implies that Z has two extreme points in $(0, 1)$, the monotonicity of Z is "$\searrow\nearrow\searrow$". A numerical verification shows that the initial interval where Z is decreasing is rather small, so Z will remain nonnegative whenever $Z(0) = \frac{i}{n+1} - \frac{j}{m+1} > 0$ is large enough. Therefore, we expect that points (i, j) not satisfying the assumption in Theorem 15 will be close to the top border of the unshaded region. A few examples illustrating this behaviour are shown in Figure 2.

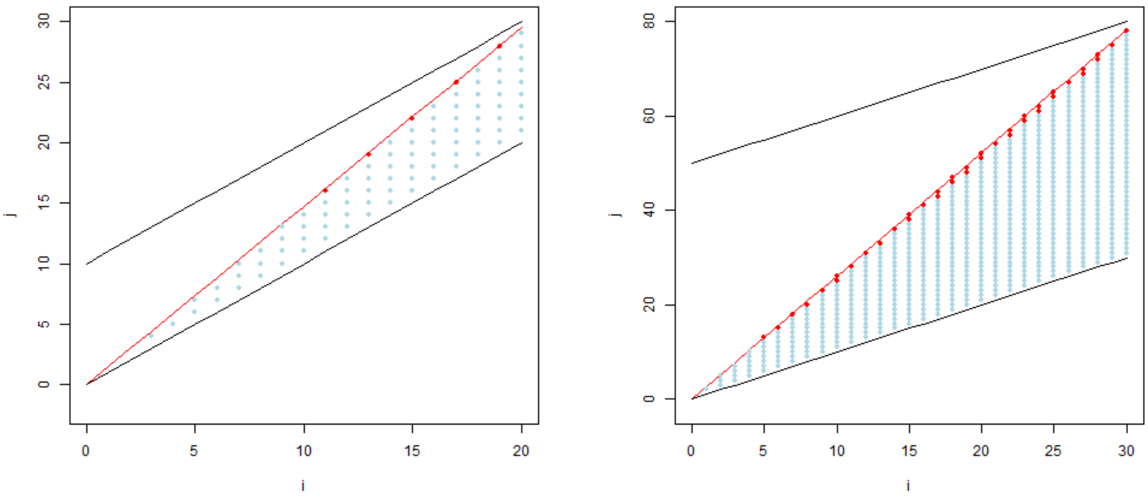

Figure 2: Blue points fulfill the assumptions of Theorem 15. Left: $n = 20$, $m = 30$. Right: $n = 30$, $m = 80$.

5.2. DHRA distributions

Theorem 16. *Assume that F is DHRA. Let*

$$Z(x) = \binom{n}{i} i \sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^{i-1-k} \frac{(e^{-x})^{n-k}(-kx + nx + 1)}{(k-n)^2}$$
$$- \binom{m}{j} j \sum_{k=0}^{j-1} \binom{j-1}{k} (-1)^{j-1-k} \frac{(e^{-x})^{m-k}(-kx + mx + 1)}{(k-m)^2}. \quad (3)$$

If $Z(-\log(1-r)) \geq 0$ for every $r \in R\left(i-j, (n-i)-(m-j), \frac{\mathcal{B}(i,n-i+1)}{\mathcal{B}(j,m-j+1)}\right)$ then $X_{i:n} \geq_{ss} X_{j:m}$.

Proof. Since $F^{-1} \circ \mathcal{E}$ is star-shaped, the result holds by Theorem 5 and Lemma 12, provided that $\mathcal{E}^{-1} \circ B_{i:n} \geq_{ss} \mathcal{E}^{-1} \circ B_{j:m}$, or, equivalently, $-\log(1-B_{i:n}) \geq_{ss} -\log(1-B_{j:m})$. This may be expressed as

$$\frac{1}{\mathcal{B}(i,n-i+1)} \int_x^\infty t\left(1-e^{-t}\right)^{i-1} e^{-(n-i+1)t}\,dt \geq$$
$$\frac{1}{\mathcal{B}(j,m-j+1)} \int_x^\infty t\left(1-e^{-t}\right)^{j-1} e^{-(m-j+1)t}\,dt, \quad \forall x \geq 0, \quad (4)$$

Using the binomial theorem, we obtain

$$\frac{1}{\mathcal{B}(i,n-i+1)}\int_x^\infty t\left(1-e^{-t}\right)^{i-1}e^{-(n-i+1)t}\,dt$$
$$=\frac{1}{\mathcal{B}(i,n-i+1)}\sum_{k=0}^{i-1}\int_x^\infty(-1)^{i-1-k}\binom{i-1}{k}te^{-(n-k)t}\,dt$$
$$=\binom{n}{i}i\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^{i-1-k}\frac{e^{(n-k)t}(-kx+nx+1)}{(k-n)^2},\quad(5)$$

and similarly for the second term, hence (4) is equivalent to $Z(x)\geq 0$, for every $x\geq 0$. The function $Z\circ\mathcal{E}^{-1}$ is continuous on $[0,1]$, so it is nonnegative if and only if its minimal value in $[0,1]$ is nonnegative. The extreme points of $Z\circ\mathcal{E}^{-1}$ are easily seen to be among the solutions of $T_{i-j,(n-i)-(m-j)}(1-e^{-x})=\frac{\mathcal{B}(i,n-i+1)}{\mathcal{B}(j,m-j+1)}$, hence the result follows immediately from the assumption $Z\circ\mathcal{E}^{-1}(r)\geq 0$ for every $r\in R\left(i-j,(n-i)-(m-j),\frac{\mathcal{B}(i,n-i+1)}{\mathcal{B}(j,m-j+1)}\right)$. □

A complete geometric picture of the $\geq_{ss}$-comparability for DHRA distributions produces a plot similar to the one in Figure 1. The shaded regions where one has comparability are the same, but the direction of the $\geq_{ss}$-comparability are reversed, taking into account that $1-B_{i:n}$ and $1-B_{j:m}$ still have *beta* distributions with the parameters swapped. Moreover, the red line in Figure 1 is now replaced by setting to 0 the two terms appearing in part 2. of Corollary 9, that is, for each $i\leq n$ going through the coordinates j and $j+1$ such that $\sum_{k=n-i+1}^{n}\frac{1}{k}-\sum_{k=m-j+1}^{m}\frac{1}{k}$ and $\sum_{k=n-i+1}^{n}\frac{1}{k}\geq\sum_{k=m-(j+1)+1}^{m}\frac{1}{k}$ have opposite signs. The region below this curve, corresponding to $Z(0)>0$, and above the diagonal is seen to be where we have no $\geq_{ss}$-comparability. The remaining region needs numerical verification. Hence, with respect to Figure 1, one reverses the direction of the comparisons, swaps the unshaded and shaded areas between the two straight lines, and, the separating red line is no longer straight.

5.3. An open problem

An interesting unexplored stochastic order can be easily defined by requiring that $\mathbb{E}g(X)\geq\mathbb{E}g(Y)$ for every increasing anti-star-shaped function. This order may be denoted as the *IAS order*. Specifically, it can be demonstrated that a function g is increasing anti-star-shaped if and only if $g=h^{-1}$, where h is star-shaped. The IAS order possesses the characteristics of an integral stochastic order. Moreover, it is weaker than the usual order and it implies the ICV order, and it is preserved under increasing anti-star-shaped transformations, making it an intriguing example for our methodology. For instance, utilizing the IAS order enables the derivation of conditions for ranking order statistics within the class of IHRA distributions, defined as the class of CDFs F such that $\mathcal{E}^{-1} \circ F$ is star-shaped. However, to the best of our knowledge, expressing the IAS order using "distributional" conditions – checking on the CDFs or suitable transformations of these – remains a challenge. More precisely, a sufficient condition for the IAS order between transformed beta random variables $G^{-1} \circ B_{i:n}$ and $G^{-1} \circ B_{j:m}$ would be enough to apply our results to classes defined as "$G^{-1} \circ F$ is star-shaped" (including the IHRA class).

6. Bounds for probabilities of exceedance

Consider a scenario where we represent the lifetime of a k-out-of-n system as $X_{k:n}$. A notable challenge in reliability analysis involves determining the probability that the individual component's lifetime falls below or exceeds the expected lifetime of the entire system, denoted as $\mathbb{E}X_{k:n}$. In a parametric setting, this probability can be precisely computed using the mathematical formula of the parent CDF F. However, when the exact form of F is unknown, we can leverage information about its overall shape to establish upper or lower bounds for this probability. This would follow from the application of Jensen's inequality, under the assumption that F belongs to a convex-ordered family $\mathcal{F}_{\mathcal{V}}^{G}$ or $\mathcal{F}_{\mathcal{C}}^{G}$. We remark that the case in which G is the uniform has been already discussed by Ali and Chan (1965).

Proposition 17. *Given a CDF G, define $p_{i:n}^{G} = G(\mathbb{E}(G^{-1} \circ B_{i:n}))$.*

1. *Let $F \in \mathcal{F}_{\mathcal{V}}^{G}$. Then, $P(X \leq \mathbb{E}X_{i:n}) \leq p_{i:n}^{G}$.*
2. *Let $F \in \mathcal{F}_{\mathcal{C}}^{G}$. Then, $P(X \leq \mathbb{E}X_{i:n}) \geq p_{i:n}^{G}$.*

In particular, given a pair of CDFs G_1 and G_2, if $G_1 \leq_{st} G_2$, then $p_{i:n}^{G_1} \geq p_{i:n}^{G_1}$.

Proof. We prove just case 1., as case 2. is proved similarly. Since, by assumption, the composition $G^{-1} \circ F$ is convex, Jensen's inequality gives

$$\mathbb{E}X_{i:n} \leq F^{-1} \circ G(\mathbb{E}(G^{-1} \circ F(X_{i:n}))).$$

Now, taking into account that $F(X_{i:n}) \sim B_{i:n}$, applying F to both sides we obtain

$$P(X \leq \mathbb{E}X_{i:n}) \leq G(\mathbb{E}(G^{-1}B_{i:n}).$$

Note that $G_1 \leq_{st} G_2$ implies that $G_1(B_{i:n}) \leq_{st} G_2(B_{i:n})$, which gives the inequality between the expectations. $\square$

Put otherwise, the above result means that, if $F \in \mathcal{F}_{\mathcal{V}}^{G}$, the expected order statistic $\mathbb{E}X_{i:n}$ is always smaller than or equal to the $p_{i:n}^{G}$-quantile of X. Similarly, if $F \in \mathcal{F}_{\mathcal{C}}^{G}$, the expected order statistic $\mathbb{E}X_{i:n}$ is always greater than or equal to the $p_{i:n}^{G}$-quantile of X, that is, $\mathbb{E}X_{i:n} \geq F^{-1}(p_{i:n}^{G})$. This result also enables a useful characterization of the LL1 distribution. Indeed, generally one may approximate $\mathbb{E}X_{i:n}$ with $F^{-1}(\frac{i}{n})$: for $n \to \infty$ and $\frac{i}{n} \to p$ (constant), $F^{-1}(\frac{i}{n}) \to \mathbb{E}X_{i:n}$. This result is exact for n finite if and only if X has an LL1 distribution.

Corollary 18 (A characterisation of the LL1 distribution)**.** $\mathbb{E}X_{i:n} = F^{-1}(\frac{i}{n})$ *if and only if* $F(x) = LL(\frac{x}{a})$*, for any scale parameter* $a > 0$.

Proof. F belongs to both $\mathcal{F}_{\mathcal{C}}^{LL}$ and $\mathcal{F}_{\mathcal{V}}^{LL}$ if and only if $F(x) = LL(\frac{x}{a})$. Without loss of generality, let $a = 1$. In this case, it is easy to verify that $p_{i:n}^{LL} = \frac{i}{n}$, so $\frac{i}{n} \leq LL(\mathbb{E}X_{i:n}) \leq \frac{i}{n}$. This means that the $\frac{i}{n}$-quantile of the LL1 is $\mathbb{E}(X_{i:n}) = \frac{i}{n-i}$. $\square$

Common choices of G yield the following explicit expressions of $p_{i:n}^{G}$:

1. If $G = U$, $p_{i:n}^{U} = \frac{i}{n+1}$.
2. If $G = \mathcal{E}$, $p_{i:n}^{\mathcal{E}} = 1 - \exp\left(-\sum_{k=n-i+1}^{n} \frac{1}{k}\right)$.
3. If $G = LL$, $p_{i:n}^{LL} = \frac{i}{n}$.
4. If $G = \mathcal{E}_-$, $p_{i:n}^{\mathcal{E}_-} = \exp\left(-\sum_{k=i}^{n} \frac{1}{k}\right)$.

Table 1 shows the $p_{i:n}^{G}$ bounds for $n = 10$ and some common choices of G. The application of our results is quite straightforward. For instance, if we know that the CDF of interest, F, is IHR and has a decreasing density, as is the case, for example, of the Pareto family, then the probability of having $X \leq \mathbb{E}X_{i:n}$ is always between $p_{i:n}^{U}$ and $p_{i:n}^{\mathcal{E}}$, that is,

$$\frac{i}{n+1} \leq P(X \leq \mathbb{E}X_{i:n}) \leq 1 - \exp\left(-\sum_{k=n-i+1}^{n} \frac{1}{k}\right).$$

If $i = 3$ and $n = 10$, this means that $P(X \leq \mathbb{E}X_{k:n}) \in [0.273, 0.285]$. Similarly, if F is IOR and DRHR, then

$$\exp\left(-\sum_{k=i}^{n} \frac{1}{k}\right) \leq P(X \leq \mathbb{E}X_{i:n}) \leq \frac{i}{n}.$$

G	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$
LL	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
$\mathcal{E}$	0.095	0.190	0.285	0.381	0.476	0.571	0.666	0.760	0.855	0.947
U	0.091	0.182	0.273	0.364	0.455	0.545	0.636	0.727	0.818	0.909
$\mathcal{E}_-$	0.053	0.145	0.240	0.334	0.429	0.524	0.619	0.715	0.810	0.905

Table 1: $p_{i:10}^G$ for different choices of G.

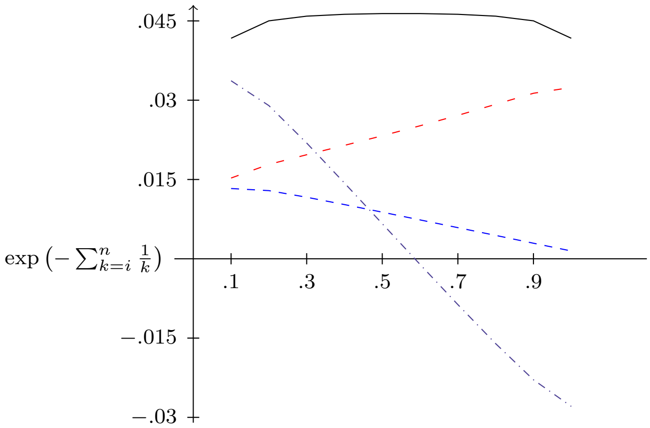

Figure 3: Upper bounds (black) and true values for $P(X \leq \mathbb{E}X_{i:n})$: Weibull(3,1) (red), $F(x) = x^3$ (blue), $F(x) = 1 - \sqrt{1-x}$ (dashdotted, dark blue) with respect to the lower bounds (the horizontal line).

These bounds are generally weaker, so, for $i = 3$ and $n = 10$, we now find $P(X \leq \mathbb{E}X_{i:n}) \in [0.240, 0.300]$. The bounds, with respect to the families of distributions, are sharp, as illustrated in Figure 3, where we plotted true probabilities for two distributions that are both IRH and DRHR (Weibull with shape parameter larger than 1, and power distribution), and the inverted power with exponent $\frac{1}{2}$, which is not IHR, hence violates the lower bound.

6.1. Application

Nichols and Padgett (2006) provides a table containing a sample of size $n = 100$ of breaking stress for carbon fibers. Applying the tests of Lando (2023) and Lando et al. (2023a), respectively, it can be tested that this dataset is not compatible with the IHR assumption, whereas it is likely to come from an IOR distribution. Moreover, the dataset also seems to satisfy the DRHR assumption. A straightforward interval for $\mathbb{E}X_{i:n}$ is then obtained by plugging in the appropriate bounds describe above to the empirical distribution function, that is $\left[\mathbb{F}_n^{-1}\left(\exp\left(-\sum_{k=i}^n \frac{1}{k}\right)\right), \mathbb{F}_n^{-1}\left(\frac{i}{n}\right)\right]$. Taking, as an example, $i = 20$, this interval reduces to a single point, as we get $1.69 \leq \widehat{\mathbb{E}}X_{20:100} \leq 1.69$. However, we may use instead estimators that take into account the available information about the shape of the distribution: we may use $\mathbb{F}_n^{IOR}$, introduced by Lando et al. (2023a), as an IOR estimator of the CDF, and $\mathbb{F}_n^{DRHR}$, proposed by Sengupta and Paul (2005), as a DRHR estimator. Hence, an interval for $\mathbb{E}X_{i:n}$ may be given by $\left[(\mathbb{F}_n^{DRHR})^{-1}\left(\exp\left(-\sum_{k=i}^{n}\frac{1}{k}\right)\right), (\mathbb{F}_n^{IOR})^{-1}(\frac{i}{n})\right]$. For this sample, thus taking into account the knowledge about the shape of the CDF, this leads to $\mathbb{E}X_{20:100} \in [1.623, 1.716]$.

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