ON TOPOSES, ALGEBRAIC THEORIES, SEMI-ABELIAN CATEGORIES AND COMPACT HAUSDORFF SPACES

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ABSTRACT. In this paper we study the categories $\mathcal{C}_*^{\mathsf{op}}$ and $(\mathcal{C}_*^{\mathsf{op}})^{\mathbb{T}}$ of $\mathbb{T}$-models in $\mathcal{C}_*^{\mathsf{op}}$ for an arbitrary algebraic theory $\mathbb{T}$, when $\mathcal{C}$ is a topos or the category CHaus of compact Hausdorff spaces. It is well-known that, when $\mathcal{C}$ is a topos, $\mathcal{C}_*^{\mathsf{op}}$ is semi-abelian. We show that $\mathsf{CHaus}_*^{\mathsf{op}}$ is semi-abelian, as well as $(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}}$, and that, when $\mathcal{C}$ is a topos having locales of subobjects, $(\mathcal{C}_*^{\mathsf{op}})^{\mathbb{T}}$ is also semi-abelian. In addition, we prove the representability of actions in $\mathsf{CHaus}_*^{\mathsf{op}}$.

Introduction

It is well-known that, given a topos $\mathcal{E}$, the dual $\mathcal{E}_*^{\mathsf{op}}$ of the category of pointed objects of $\mathcal{E}$ is semi-abelian (see [5]). We first prove that an analogous result holds in the context of compact Hausdorff spaces: the dual $\mathsf{CHaus}_*^{\mathsf{op}}$ of the category of pointed compact Hausdorff spaces is semi-abelian. The Bourn-Janelidze characterization of semi-abelian algebraic theories (see [10]), and its generalization by Gran-Rosický (see [14]), indicate at once that adding arbitrarily operations (other than constants) and axioms to such a theory, one keeps a semi-abelian theory. This is what suggested us to investigate what occurs when adding arbitrary operations and axioms to $\mathcal{E}_*^{\mathsf{op}}$ or $\mathsf{CHaus}_*^{\mathsf{op}}$, that is, when considering the categories $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ and $(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}}$ of models of an arbitrary algebraic theory $\mathbb{T}$ in $\mathcal{E}_*^{\mathsf{op}}$ or $\mathsf{CHaus}_*^{\mathsf{op}}$. And the answer is: all categories $(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}}$ are semi-abelian. And, except for the existence of binary coproducts, the categories $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ satisfy all the other axioms for being semi-abelian, thus are homological (see [5]) and exact. And binary coproducts exist, thus $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ is semi-abelian, as soon as, in the topos $\mathcal{E}$, the subobjects of every object constitute a locale. This is the case when $\mathcal{E}$ is a Grothendieck topos, but also when $\mathcal{E}$ is a topos of sheaves or presheaves of finite sets on a finite site.

In a semi-abelian category, one has the notion of an object G acting on an object X, in terms of algebras for some monad. Actions on X are representable when the functor mapping G to the set of G-actions on X is representable. This recaptures a well-known property of the category of groups, where the actions on a group X are represented by

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the group $\mathsf{Aut}(X)$ of automorphisms of X. The representability of actions in a semi-abelian category is a strong property which does not hold in general. Given a topos $\mathcal{E}$, the representability of actions in $\mathcal{E}_*^{\mathsf{op}}$ has been studied in [7]; we prove that it holds in $\mathsf{CHaus}_*^{\mathsf{op}}$.

1. Semi-abelianess versus duality

Convention

Through this paper, $\mathcal{E}$ will always denote a topos and CHaus*, the category of compact Hausdorff spaces. $\mathbb{T}$ will denote a Lawvere algebraic theory. We write $\mathcal{E}_*$ and CHaus_* for the categories of pointed objects of $\mathcal{E}$ and CHaus: the categories of pairs (A, a) where A is an object and $a\colon 1 \longrightarrow A$ is a "base point", that is, a morphism from the terminal object 1 to A; the morphisms of $\mathcal{E}_*$ and CHaus_* respect the base points. We write further $\mathcal{E}_*^{\mathsf{op}}$ and $\mathsf{CHaus}_*^{\mathsf{op}}$ for the duals of the categories $\mathcal{E}_*$ and CHaus_*, and $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$, $(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}}$ for the categories of $\mathbb{T}$-models in $\mathcal{E}_*^{\mathsf{op}}$ and $\mathsf{CHaus}_*^{\mathsf{op}}$. We write $\mathbf{1}$ for the zero object of $\mathcal{E}_*$, CHaus_*, $\mathcal{E}_*^{\mathsf{op}}$, and $\mathsf{CHaus}_*^{\mathsf{op}}$, corresponding thus to the terminal object of $\mathcal{E}$ or CHaus. We denote by $+$ the coproduct in $\mathcal{E}_*$ and CHaus_*, which is thus the pushout under $\mathbf{1}$ in $\mathcal{E}$ and CHaus. We keep the notation $\amalg$ for the coproduct in $\mathcal{E}$ and CHaus.*

We shall most often develop the proofs in $\mathcal{E}_*$ and CHaus_*, instead of $\mathcal{E}_*^{\mathsf{op}}$ and $\mathsf{CHaus}_*^{\mathsf{op}}$, working thus with $\mathbb{T}$-coalgebras in $\mathcal{E}_*$ and CHaus_*: contravariant functors from $\mathbb{T}$ to $\mathcal{E}_*$ or CHaus_*, transforming finite products in finite coproducts. We write ${}^{\mathbb{T}}\mathcal{E}_*$, ${}^{\mathbb{T}}\mathsf{CHaus}_*$ for these categories of coalgebras; they are thus the duals of $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ and $(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}}$.

To avoid any confusion, let us make clear that the dual of the category of pointed objects of a category $\mathcal{C}$ with a terminal object $\mathbf{1}$ is by no means the category of pointed objects of $\mathcal{C}^{\mathsf{op}}$. For example, the category of pointed objects in the dual of Set is the terminal category, reduced to the (dual object) of the empty set.

Let us first recall some definitions (see [5]), in the special context where we shall need them.

1.1. Definition. *A finitely complete Barr-regular category with a zero object is*

- homological *when given a commutative diagram*

$$\begin{tikzcd}
K \arrow[r, tail, "k"] \arrow[d, "u"'] & A \arrow[r, shift right, "p"'] \arrow[d, "v"'] & P \arrow[l, shift right, "s"'] \arrow[d, "w"] \\
L \arrow[r, tail, "l"'] & B \arrow[r, shift right, "q"'] & Q \arrow[l, shift right, "r"']
\end{tikzcd}
\qquad
\begin{array}{ll}
ps = \mathsf{id}_P & k = \ker p \\
 & \\
qr = \mathsf{id}_Q & l = \ker q
\end{array}$$

if u and w are isomorphisms, so is v;

- semi-abelian *when it is homological, Barr-exact and admits binary coproducts;*
- *such categories are* arithmetical *(also called* congruence distributive*) when their lattices of equivalence relations are distributive.*

The diagram condition in the definition of a homological category is – in the pointed case – the so-called *protomodularity axiom*, due to D. Bourn (see [9]). D. Bourn and G. Janelidze characterized those Lawvere algebraic theories $\mathbb{T}$ whose category of models in Set is protomodular (see [10]). M. Gran and J. Rosický generalized this characterization to the case of multi-sorted infinite theories (see [14]). In particular, every theory with a single constant and containing a group operation is semi-abelian. Homological categories (see [5]) satisfy the basic lemmas of homological algebra: the five lemma, the nine lemma, the snake lemma, and so on. Let us recall that the notion of abelian category is self-dual: it turns out that a semi-abelian category is abelian as soon as its dual is semi-abelian as well (see [5]). All this somehow justifies the terminology. The *arithmetical* axiom is certainly less popular, but will play an important role in Section 4.

Let us now focus our attention on some "non-algebraic" examples of semi-abelian categories.

1.2. Theorem. *The dual $\mathcal{E}_*^{\mathsf{op}}$ of the category of pointed objects of an elementary topos $\mathcal{E}$ is semi-abelian and arithmetical*

Proof. The semi-abelianess is proved in [5], Example 5.1.8.

Lattices of equivalence relations in the exact category $\mathcal{E}_*^{\mathsf{op}}$ correspond to lattices of regular epimorphisms in $\mathcal{E}_*^{\mathsf{op}}$, thus to lattices of regular – that is all – subobjects in $\mathcal{E}_*$; these inherit distributivity from $\mathcal{E}$. □

1.3. Theorem. *The dual $\mathsf{CHaus}_*^{\mathsf{op}}$ of the category of pointed compact Hausdorff spaces is semi-abelian and arithmetical.*

Proof. First of all, let us recall that $\mathsf{CHaus}^{\mathsf{op}}$ is an exact category, because it is monadic over Set. Indeed, it is shown in [13], Example 5.15.3, and [20], Theorem 1.7, that the functor

$$\mathsf{CHaus}^{\mathsf{op}} \longrightarrow \mathsf{Set}, \quad X \mapsto \mathcal{C}\big(X,[0,1]\big)$$

is monadic.

The functor $\mathsf{CHaus}_* \longrightarrow \mathsf{CHaus}$ creates limits, coequalizers of arbitrary families of parallel morphisms, and pushouts. An arbitrary coproduct in CHaus_* is obtained by computing the corresponding coproduct in CHaus, and, next, the generalized coequalizer identifying all the base points. In particular, CHaus_* is complete and cocomplete, thus $\mathsf{CHaus}_*^{\mathsf{op}}$ as well.

Let us now prove that $\mathsf{CHaus}_*^{\mathsf{op}}$ is regular. We work in the dual category CHaus_*. In CHaus, the monomorphisms are the closed embeddings. A (regular) monomorphism $f\colon (X,x_0) \longrightarrow (Y,y_0)$ in CHaus_* is a monomorphism in CHaus, thus a closed embedding.

Conversely if f is a closed embedding, it is the kernel in CHaus_* of the quotient of (Y, y_0) by the closed equivalence relation $(X \times X) \cup \Delta_Y$. To form the (regular epi-mono) factorization of a morphism $g\colon (B, b_0) \longrightarrow (A, a_0)$ in $\mathsf{CHaus}_*^{\mathsf{op}}$, one considers the image factorization in CHaus_*

$$\begin{tikzcd}
(A, a_0) \arrow[rr, "g"] \arrow[dr, two heads, "e"'] & & (B, b_0) \\
& \big(g(A), g(a_0)\big) \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

where $g(A)$ is equipped with the final topology for e, or equivalently, since the spaces involved are compact Hausdorff, the initial topology for m. We still have to prove that the pushout of a (regular) monomorphism in CHaus_* remains a (regular) monomorphism. But, as already observed, pushouts in CHaus_* are computed as in CHaus, that is as pullbacks in $\mathsf{CHaus}^{\mathsf{op}}$. And since $\mathsf{CHaus}^{\mathsf{op}}$ is exact, the pullback of a regular epimorphism is a regular epimorphism.

We prove next that $\mathsf{CHaus}_*^{\mathsf{op}}$ is exact. Again we work in CHaus_*. Consider a co-equivalence relation $r_0, r_1\colon (X, x_0) \rightrightarrows (Y, y_0)$ in CHaus_*. We know that every equivalence relation in $\mathsf{CHaus}^{\mathsf{op}}$ is effective, because the category is exact. Thus, in CHaus (r_0, r_1) is the cokernel pair of its equalizer $k\colon K \longrightarrow X$. Since r_0, r_1 preserve the base point, $x_0 \in K$ and (r_0, r_1) is then the cokernel pair of its equalizer k in CHaus_*.

To prove that $\mathsf{CHaus}_*^{\mathsf{op}}$ is semi-abelian, it remains to check the protomodularity (homological) axiom of Definition 1. The given diagram yields a corresponding diagram in $\mathsf{Set}_*^{\mathsf{op}}$, because the forgetful functor $\mathsf{CHaus}_* \longrightarrow \mathsf{Set}_*$ preserves cokernels. But $\mathsf{Set}_*^{\mathsf{op}}$ is semi-abelian by Theorem 1.2, thus v is an isomorphism in $\mathsf{Set}_*^{\mathsf{op}}$, that is, a bijection $B \longrightarrow A$ in Set_*. So v is a continuous bijection in CHaus_*, hence it is also a closed map, thus a homeomorphism.

Let us finally check the arithmetical property. We must prove that the lattice of equivalence relations on an object $(X, x_0) \in \mathsf{CHaus}_*^{\mathsf{op}}$ is distributive. But, since $\mathsf{CHaus}_*^{\mathsf{op}}$ is exact, this reduces to proving that the lattice of regular quotients of (X, x_0) in $\mathsf{CHaus}_*^{\mathsf{op}}$ is distributive. This is further equivalent to the lattice of regular subobjects of (X, x_0) in CHaus_* being distributive. But, as already observed above, the (regular) subobjects in CHaus_* are the closed embeddings. And since set theoretical finite unions and finite intersections of closed embeddings in compact Hausdorff spaces are still closed embeddings, the lattice of subobjects of (X, x_0) in CHaus_* is isomorphic to a sublattice of the lattice of subobjects of (X, x_0) in Set_*, and is therefore distributive. $\square$

Gelfand duality expresses the duality between CHaus and the category of commutative unital $\mathbb{C}^*$-algebras (see [1]). In [14], it is proved that the category of commutative non-unital $\mathbb{C}^*$-algebras is semi-abelian. To avoid any ambiguity, let us observe that this category is not equivalent to $\mathsf{CHaus}_*^{\mathsf{op}}$. The initial commutative unital $\mathbb{C}^*$algebra is the algebra $\mathbb{C}$ of complex numbers. The semi-abelian category $\mathsf{CHaus}_*^{\mathsf{op}}$ is thus equivalent to the category of pairs (A, φ), where A is a commutative unital $\mathbb{C}^*$-algebra and $\varphi\colon A \longrightarrow \mathbb{C}$ is a character of A, or thus, equivalently, a maximal ideal of A (see [1]).

2. Adding algebraic structures

We arrive at the point at the origin of this paper: what occurs when adding operations and axioms to $\mathcal{E}_*^{\mathsf{op}}$ and $\mathsf{CHaus}_*^{\mathsf{op}}$? Let us first recall a standard result, borrowed from [3].

2.1. THEOREM. *Let $\mathcal{C}$ be an exact category and $\mathbb{T}$ an algebraic theory. The category $\mathcal{C}^{\mathbb{T}}$ of $\mathbb{T}$-models in $\mathcal{C}$ is exact. The forgetful functor $U\colon \mathcal{C}^{\mathbb{T}} \longrightarrow \mathcal{C}$ preserves limits and coequalizers of kernel pairs, thus in particular regular epimorphisms; it also reflects isomorphisms.* $\square$

2.2. COROLLARY. *Let $\mathcal{C}$ be a semi-abelian category and $\mathbb{T}$ an algebraic theory. The category $\mathcal{C}^{\mathbb{T}}$ of $\mathbb{T}$-models in $\mathcal{C}$ is homological and exact. And, when $\mathcal{C}$ is arithmetical, so is $\mathcal{C}^{\mathbb{T}}$.*

PROOF. Since $\mathcal{C}$ has a zero object, all constants of the theory $\mathbb{T}$ in a $\mathbb{T}$-algebra A are realized by the zero morphism $\mathbf{1} \longrightarrow A$ in $\mathcal{C}$. And, of course, $\mathbf{1}$ becomes the zero object of $\mathcal{C}^{\mathbb{T}}$.

To prove the protomodularity of $\mathcal{C}^{\mathbb{T}}$, consider the diagram in Definition 1.1; we must prove that v is an isomorphism. Since the forgetful functor $\mathcal{C}^{\mathbb{T}} \longrightarrow \mathcal{C}$ preserves kernels and reflects isomorphisms, this follows at once from the protomodularity of $\mathcal{C}$.

The statement concerning the arithmetical axiom follows from Example 2.9.5 in [5], because the forgetful functor $\mathcal{C}^{\mathbb{T}} \longrightarrow \mathcal{C}$ preserves pullbacks and reflects isomorphisms. $\square$

By Theorems 1.2, 1.3 and 2.1, Corollary 2.2 applies in particular to all categories $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ and $(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}}$. To have the semi-abelianess of these categories, it remains to prove the existence of binary coproducts in $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ and $(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}}$, that is, the existence of binary products in ${}^{\mathbb{T}}\mathcal{E}_*$ and ${}^{\mathbb{T}}\mathsf{CHaus}_*$. The existence of products of coalgebras is a quite involved problem (see [15]). To support the intuition, let us first give an explicit description of binary products of coalgebras in ${}^{\mathbb{T}}\mathsf{Set}_*$.

Notation. *Given an n-ary operation α in an algebraic theory $\mathbb{T}$, we shall write $\tilde{\alpha}$ for the 1-ary operation*

$$\tilde{\alpha}\colon T^1 \xrightarrow{\ \Delta\ } T^n \xrightarrow{\ \alpha\ } T^1$$

where Δ is the diagonal.

Given a $\mathbb{T}$-coalgebra A, the operation $\tilde{\alpha}_A$ on A is thus the composite

$$A \xrightarrow{\ \alpha_A\ } A + \cdots + A \xrightarrow{\ \nabla_A\ } A$$

where ∇_A is the codiagonal. Thus roughly speaking, $\tilde{\alpha}_A$ takes the same values as α_A but forgets in which component of the coproduct.

2.3. PROPOSITION. *Let $\mathbb{T}$ be an algebraic theory. The category ${}^{\mathbb{T}}\mathsf{Set}_*$ has binary products.*

Proof. Let A and B be two $\mathbb{T}$-coalgebras in Set_*. We shall write $A \otimes B$ for the product of these coalgebras in ${}^{\mathbb{T}}\mathsf{Set}_*$ and keep the notation $A \times B$ for their product as objects of Set_*.

Given an operation α of arity n, we shall say that an element of the coalgebra A admits i as an α-rank when it is mapped by α_A in the i-th copy of A in the coproduct

$$\alpha_A \colon A \longrightarrow A + \cdots + A.$$

An element can admit several ranks: this occurs precisely when it is mapped on the base point of the coproduct; in that case, it admits all possible α-ranks.

We define $A \otimes B$ to be the set of those pairs $(a, b) \in A \times B$, such that, for every operation β, the elements a and b have a common β-rank. Notice in particular that all pairs $(a, \star)$ and $(\star, b)$ belong to $A \otimes B$, since the base point $\star$ admits all ranks. We must now provide $A \otimes B$ with the structure of a $\mathbb{T}$-coalgebra.

Given an operation γ of arity n, we must thus define

$$\gamma_{A\otimes B} \colon A \otimes B \longrightarrow (A \otimes B) + \cdots + (A \otimes B).$$

Given an element $(a, b) \in A \otimes B$, we have by definition that the elements a and b have a common γ-rank i; we define

$$\gamma_{A\otimes B}(a, b) = \big(\tilde{\gamma}_A(a), \tilde{\gamma}_B(b)\big)$$

in the i-th component of the right hand coproduct. In the case where the common γ-rank is not unique, both a and b have thus a multiple γ-rank: this occurs, as we have seen, when they are mapped on the base point by γ_A and γ_B. In that case, the choice of the index i does not matter.

Of course we must verify that $\big(\gamma_A(a), \gamma_B(b)\big)$ lies in $A \otimes B$. That is, given an arbitrary operation β, we must prove that $\tilde{\gamma}_A(a)$ and $\tilde{\gamma}_B(b)$ have the same β-rank. This is indeed the case because $\tilde{\gamma} \circ \beta$ is an operation in $\mathbb{T}$ and, by definition of $A \otimes B$, a and b have the same α-rank for each operation α. (Since we are working with coalgebras, thus contravariant functors, one has indeed $(\beta \circ \tilde{\gamma})_A = \beta_A \circ \tilde{\gamma}_A$.)

It remains to observe that the $\mathbb{T}$-axioms are satisfied, that is, given a commutative diagram in $\mathbb{T}$, the corresponding composites of co-operations on $A \otimes B$ yield the expected commutativity. This is the case since by definition of the $\mathbb{T}$-algebra structure of $A \otimes B$, these axioms are satisfied in each component.

Next, by definition of $A \otimes B$, the projections

$$p_A \colon A \otimes B \longrightarrow A,\ p_A(a, b) = a, \quad p_B \colon A \otimes B \longrightarrow B,\ p_B(a, b) = b$$

are trivially morphisms of $\mathbb{T}$-coalgebras.

We must still prove the universal property of the product $A \otimes B$. Consider thus a $\mathbb{T}$-coalgebra C and two morphisms $f \colon C \longrightarrow A$, $g \colon C \longrightarrow B$ of $\mathbb{T}$-coalgebras. By definition of a morphism of $\mathbb{T}$-coalgebras, when an element $x \in C$ admits i as a β-rank (unique or not), then $f(x)$ and $g(x)$ also admit i as a β-rank and the pair $\big(f(x), g(x)\big)$ lies in $A \otimes B$. Therefore the unique factorization $f \colon C \longrightarrow A \times B$ through the product in Set_* factors further through $A \otimes B$; it follows at once that this factorization is a morphism of $\mathbb{T}$-coalgebras, since so are f and g. $\square$

Of course as a corollary we get at once:

2.4. Corollary. *Given an algebraic theory* $\mathbb{T}$*, the category* $(\mathsf{Set}_*^{\mathsf{op}})^{\mathbb{T}}$ *is semi-abelian.*

Proof. By Propositions 2.2 and 2.3. $\square$

Let us now use the construction in Proposition 2.3 to handle the case of compact Hausdorff spaces. But first, let us point out some basic facts about CHaus_*.

2.5. Lemma. *In the category* CHaus_**, finite unions of subobjects are computed as in* Set_**; they are effective and universal.*

Proof. The monomorphisms in CHaus_* are the closed embeddings. Since a finite union of closed subsets is closed, finite unions of subobjects are computed as in Set_*. In particular they are universal, i.e. stable under pullbacks, since this is the case in Set_*.

A finite union of subobjects $A_i \subseteq A$ in CHaus_* is effective when given morphisms $f_i\colon A_i \longrightarrow B$, such that f_i and f_j coincide on $A_i \cap A_j$, then there is a unique extension $f\colon \cup_{i\in I} A_i \longrightarrow B$. This property holds trivially in Set_* and it remains to check the continuity of f. If $C \subseteq B$ is closed, each $f_i^{-1}(C)$ is closed in the closed subset A_i, thus is closed in A. And $f^{-1}(C)$ is closed as the finite union of these closed subsets. $\square$

2.6. Lemma. *A finite coproduct* $A_1 + \cdots + A_n$ *in* CHaus_* *is computed as in* Set_**; in particular each* A_i *is a subobject of the coproduct which is itself the union of these* A_i*'s.*

Proof. The coproduct in CHaus_* is obtained by computing the corresponding coproduct in CHaus and identifying all the base points.

$$p : A_1 \amalg \ldots \amalg A_n \longrightarrow\!\!\!\!\!\rightarrow A_1 + \cdots + A_n.$$

A finite coproduct of compact Hausdorff spaces is just their set theoretical and topological disjoint union. The equivalence relation defining $A_1 + \cdots + A_n$ is the union of the diagonal of $A_1 \amalg \ldots \amalg A_n$ and the finitely many pairs of base points. Since the spaces are Hausdorff, this is a closed equivalence relation, as a finite union of closed subsets. Then p is a closed continuous map whose domain is normal and Hausdorff, and therefore its image is Hausdorff (see [12], Theorem 15.4). It is compact as well as image of a compact space. $\square$

2.7. Proposition. *Let* $\mathbb{T}$ *be an algebraic theory. The category* ${}^{\mathbb{T}}\mathsf{CHaus}_*$ *has binary products.*

Proof. We use the notation of the proof of Proposition 2.3. We have now $A, B \in {}^{\mathbb{T}}\mathsf{CHaus}_*$, their product $A \otimes B$ in ${}^{\mathbb{T}}\mathsf{Set}_*$ and their product $A \times B$ in CHaus_*. Let us first prove that $A \otimes B$ is closed in $A \times B$, thus is compact Hausdorff.

Given an operation β of rank n on the $\mathbb{T}$-coalgebra A, let us write $A_i^\beta \subseteq A$ for the subset of elements of β-rank i, which is thus the inverse image of the i-th component of the coproduct along β_A.

$$\begin{tikzcd}
A_i^\beta \arrow[r] \arrow[d, tail] & A \arrow[d, tail, "s_i"] \\
A \arrow[r, "\beta_A"'] & A + \cdots + A
\end{tikzcd}$$

It is therefore a subobject of A in CHaus_* (see Lemma 2.6).

The same argument can be developed on B, yielding the closed subset $B_i^\beta \subseteq B$ of elements of β-rank i. The product $A_i^\beta \times B_i^\beta$ is thus the set of those pairs (a, b) having the same fixed β-rank i. Therefore,

$$(A, B)^\beta = (A_1^\beta \times B_1^\beta) \cup \cdots \cup (A_n^\beta \times B_n^\beta)$$

is the set of those pairs (a, b) having a common β-rank. This is again a closed subset of $A \times B$, as a finite union of closed rectangles. The product $A \otimes B$ is the intersection of all these closed subsets $(A, B)^\beta$, for all operations β: it is thus a closed subset of the compact Hausdorff space $A \times B$, thus it is itself compact Hausdorff.

Let us verify that the co-operations of $A \otimes B$ are continuous. Given a $\mathbb{T}$-operation γ of arity n, by definition of $\gamma_{A\otimes B}$ we have a commutative diagram

$$\begin{tikzcd}
+^n(A \otimes B) \arrow[r, tail] & +^n(A \times B) \arrow[r, tail, "s"] & (+^n A) \times (+^n B) \\
A \otimes B \arrow[u, dotted, "\gamma_{A\otimes B}"] \arrow[rr, tail] & & A \times B \arrow[u, "\gamma_A \times \gamma_B"']
\end{tikzcd}$$

in Set_*, with s being $s_i^A \times s_i^B$ on the i-th copy of $A \times B$ in its domain. All the plain arrows are continuous and the horizontal ones are injective, thus are closed embeddings, since all spaces are compact Hausdorff. Therefore $\gamma_{A\otimes B}$ is the restriction on subspaces of the continuous mapping $\gamma_A \times \gamma_B$ and is therefore continuous.

Of course the projections of the product $A \otimes B$ in ${}^{\mathbb{T}}\mathsf{CHaus}_*$ are continuous, since they are the restrictions of the projections of the product $A \times B$ in CHaus_*

The conclusion is now easy. Given $f : C \longrightarrow A$ and $g : C \longrightarrow B$ in ${}^{\mathbb{T}}\mathsf{CHaus}_*$, the unique continuous factorization $(f, g) : C \longrightarrow A \times B$ through the product in CHaus_* takes values in $A \otimes B$, which is provided with the induced topology. Thus the factorization in ${}^{\mathbb{T}}\mathsf{CHaus}_*$ is continuous. $\square$

One should observe that Proposition 2.7 does not generalize as such to the case of an infinite product, because the rectangles $A_i^\beta \times B_i^\beta$ above would then have to be replaced by infinite products of closed subsets, which are not closed in general. Of course, in the situation of Proposition 2.3, this objection disappears and the given construction can be extended as such to the case of an arbitrary product in ${}^{\mathbb{T}}\mathsf{Set}_*$.

2.8. Theorem. *Given an algebraic theory $\mathbb{T}$, the category $(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}}$ is semi-abelian.*

Proof. By Propositions 2.2 and 2.7. □

The constructions developed in the proof of Propositions 2.3 and 2.7 apply also to the case of toposes. First an easy observation.

2.9. Lemma. *Let $\mathcal{E}$ be a topos. Given a coproduct $A \cong A_1 + \cdots + A_n$ in $\mathcal{E}_*$, A is the effective universal union of the various A_i, while $A_i \cap A_j \cong \mathbf{1}$ when $i \neq j$.*

Proof. As in every category with a zero object, the canonical inclusions of a coproduct admit a retraction, thus are monomorphisms. The coproduct in $\mathcal{E}_*$ is the pushout under $\mathbf{1}$ in $\mathcal{E}$, from which the union condition and its universality follow, because colimits are universal in $\mathcal{E}$ (see [18]). Effectiveness holds because unions are effective in a topos (see [4]). Now $A_i + A_j$ is the pushout of these objects under $\mathbf{1}$ in $\mathcal{E}$; this pushout square is also a pullback (see [18]), proving that $A_i \cap A_j = \mathbf{1}$ as subobjects of $A_i + A_j$. This is also the case as subobjects of $A_1 + \cdots + A_n$, because the canonical inclusion of $A_i + A_j$ in this coproduct is a monomorphism. □

2.10. Proposition. *Let $\mathbb{T}$ be an algebraic theory and $\mathcal{E}$ a topos admitting intersections of arbitrary families of subobjects. Then the category ${}^{\mathbb{T}}\mathcal{E}_*$ has binary products.*

Proof. Using the notation of the proof of Proposition 2.7, we consider two coalgebras A and B in $\mathcal{E}_*$. Given an operation β, we write A_i^β for the inverse image along β_A of the i-th component of $+^n A$, and analogously for B_i^β. By Lemma 2.9, A is the effective union of the various A_i^β's, while two distinct of these have as intersection the inverse image of $\mathbf{1}$ along β_A, that is, the kernel of β_A. Analogously for B.

We consider next the subobject

$$(A,B)^\beta = (A_1^\beta \times B_1^\beta) \cup \cdots \cup (A_n^\beta \times B_n^\beta) \subseteq A \times B$$

and we define $A \otimes B$ to be the intersection of all these subobjects $(A,B)^\beta$, for all operations β.

Observe now that, given an arbitrary operation γ, the morphism $\tilde{\gamma}_A \times \tilde{\gamma}_B$ factors through $A \otimes B$

$$\begin{tikzcd}
A \times B \arrow[r, "\tilde{\gamma}_A \times \tilde{\gamma}_B"] & A \times B \\
A \otimes B \arrow[u, tail] \arrow[r, dotted] & A \otimes B \arrow[u, tail]
\end{tikzcd}$$

By definition of $A \otimes B$ as an intersection, we must prove that, for every n-ary operation β, $\tilde{\gamma}_A \times \tilde{\gamma}_B$ maps $A \otimes B$ in $(A, B)^\beta$. Considering the composite

$$A \xrightarrow{\tilde{\gamma}_A} A \xrightarrow{\beta_A} A + \cdots + A$$

and pulling back the i-th component of the coproduct along it, we conclude that $\tilde{\gamma}_A$ maps $A_i^{\tilde{\gamma}\beta}$ in A_i^β. An analogous result holds for B and thus, $\tilde{\gamma}_A \times \tilde{\gamma}_B$ maps $(A, B)^{\tilde{\gamma}\beta}$ in $(A, B)^\beta$. We obtain so the expected morphism

$$A \otimes B \rightarrowtail (A, B)^{\tilde{\gamma}\beta} \xrightarrow{\tilde{\gamma}_A \times \tilde{\gamma}_B} (A, B)^\beta.$$

Let us now define the $\mathbb{T}$-coalgebra structure on $A \otimes B$. With the notation above, given the n-ary operation γ

$$A \otimes B \subseteq (A_1^\gamma \times B_1^\gamma) \cup \cdots \cup (A_n^\gamma \times B_n^\gamma)$$

thus

$$A \otimes B = \big((A \otimes B) \cap (A_1^\gamma \times B_1^\gamma)\big) \cup \cdots \cup \big((A \otimes B) \cap (A_n^\gamma \times B_n^\gamma)\big)$$

This union is effective as every finite union in a topos. Therefore to define

$$\gamma_{A \otimes B} \colon A \otimes B \longrightarrow (A \otimes B) + \cdots + (A \otimes B)$$

it suffices to define it coherently on each piece $(A \otimes B) \cap (A_i^\gamma \times B_i^\gamma)$ of the union. This is the composite

$$(A \otimes B) \cap (A_i^\gamma \times B_i^\gamma) \rightarrowtail A \otimes B \xrightarrow{\tilde{\gamma}_A \times \tilde{\gamma}_B} A \otimes B \xrightarrow{s_i} (A \otimes B) + \cdots + (A \otimes B)$$

since we know already that $\tilde{\gamma}_A \times \tilde{\gamma}_B$ restricts on $A \otimes B$. It remains to observe that these definitions coincide on the intersection of two pieces. But such an intersection has the form

$$(A \otimes B) \cap (\mathsf{Ker}\gamma_A \times \mathsf{Ker}\gamma_B)$$

and, on this intersection, we have in both cases the zero morphism. This concludes the definition of $\gamma_{A \otimes B}$. All axioms are satisfied since they are on both components.

It remains to prove the universal property. Just by definition, given an operation β and still with the notations as above, a morphism of coalgebras $f \colon C \longrightarrow A$ maps the corresponding subobject C_i^β in A_i^β. Therefore, given another morphism $g \colon C \longrightarrow B$, the factorization $(f, g) \colon C \longrightarrow A \times B$ through the product maps C_i^β in $A_i^\beta \times B_i^\beta$. Since C is the union of the subobjects C_i^β, (f, g) maps C in

$$(A, B)^\beta = (A_1^\beta \times B_1^\beta) \cup \cdots \cup (A_n^\beta \times B_n^\beta).$$

This shows that (f, g) factors through $A \otimes B$, the intersection of all these subobjects $(A, B)^\beta$. Thus $A \otimes B$ is the product of A and B in ${}^{\mathbb{T}}\mathcal{E}_*$. $\square$

In a topos, the lattices of subobjects are Heyting algebras. To be locales, they have to be complete, that is, to admit arbitrary unions or, equivalently, arbitrary intersections (see [17]). The assumption on $\mathcal{E}$ in Proposition 2.10 is thus requiring that the lattices of subobjects are locales.

2.11. Theorem. *Let $\mathbb{T}$ be an algebraic theory and $\mathcal{E}$ a topos having locales of subobjects. Then the category $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ is semi-abelian.*

Proof. By Propositions 2.2 and 2.10. □

2.12. Corollary. *Let $\mathcal{E}$ be a Grothendieck topos and $\mathbb{T}$ an algebraic theory. The category $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ is semi-abelian.*

Proof. By Theorem 2.11.

In the case of a Grothendieck topos, an alternative existence proof could have been given. A Grothendieck topos is locally presentable and, as a consequence, $\mathcal{E}_*$ is locally presentable as well (see [2]). It follows that the category ${}^{\mathbb{T}}\mathcal{E}_*$ of $\mathbb{T}$-coalgebras in $\mathcal{E}_*$ is itself locally presentable (see Theorem 17 in [21]), thus complete. □

2.13. Corollary. *Let $\mathbb{T}$ be an algebraic theory. When $\mathcal{E}$ is the topos of sheaves or presheaves of finite sets on a finite site $(\mathcal{E}, \mathcal{T})$, the category $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ is semi-abelian.*

Proof. Every object of $\mathcal{E}$ has only finitely many subobjects, thus every intersection reduces to a finite one. □

Notice the difference with the case of $\mathbb{T}$-algebras in $\mathcal{E}$. The category of finite groups is not semi-abelian, because it does not have binary coproducts: the coproduct of two finite groups is generally not finite.

Let us conclude this section with a comment. When $\mathbb{T}$ is an algebraic theory, the category of $\mathbb{T}$-models in Set is monadic with finite rank over Set. When $\mathcal{C}$ is monadic over Set, and the corresponding monad has a rank, the category $\mathcal{C}^{\mathbb{T}}$ of $\mathbb{T}$-models in $\mathcal{C}$ is monadic over Set (see [11]) for the tensor product of the two monads; thus, in particular, it is complete and cocomplete. But a topos, even a Grothendieck one, is generally not monadic over Set; and the monad corresponding to $\mathsf{CHaus}_*^{\mathsf{op}}$ does not have a rank.

3. More on limits and colimits

In Section 2, we focused our attention on the existence of binary coproducts in $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ and $(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}}$. Let us investigate further limits and colimits in these categories and observe that additional properties hold, compared with the classical case of $\mathsf{Set}^{\mathbb{T}}$. But first of all, in the most general case:

3.1. Proposition. *Let $\mathcal{C}$ be a (finitely) complete category and $\mathbb{T}$ an algebraic theory. The forgetful functor $U\colon {}^{\mathbb{T}}\mathcal{C}_* \longrightarrow \mathcal{C}_*$ creates (finite) limits.*

Proof. This is essentially contained in Theorem 2.1, but, in view of future use, let us be explicit. Limits in $(\mathcal{C}_*^{\mathsf{op}})^{\mathbb{T}}$ are computed as in $\mathcal{C}_*^{\mathsf{op}}$, that is, colimits in ${}^{\mathbb{T}}\mathcal{C}_*$ are computed as in $\mathcal{C}_*$. A $\mathbb{T}$-coalgebra in $\mathcal{C}_*$ is given by an object A of $\mathcal{C}$ and, for each n-ary operation α in $\mathbb{T}$, a morphism

$$\alpha_A\colon A \longrightarrow +^n A;$$

these data must make commutative the diagrams expressing the axioms of $\mathbb{T}$. When having a diagram $(A_i)_{i\in I}$ of $\mathbb{T}$-coalgebras and a colimit $A = \mathsf{colim}_{i\in I} A_i$ in $\mathcal{C}_*$, A becomes itself a $\mathbb{T}$-coalgebra when considering, for each n-ary operation α

$$\alpha_A \colon A \cong \mathsf{colim}\, A_i \xrightarrow{\mathsf{colim}\,\alpha_{A_i}} \mathsf{colim}\,(+^n A_i) \cong +^n(\mathsf{colim}\, A_i),$$

just by commutativity of colimits between themselves. □

Let us now handle the case of colimits in $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}}$ and $(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}}$, that is, limits of $\mathbb{T}$-coalgebras. The case of coproducts (products of coalgebras) has already been treated in Section 2. As far as coequalizers are concerned, Theorem 2.1 tells us only that, in the conditions of Proposition 3.1, the forgetful functor $U\colon (\mathcal{C}_*^{\mathsf{op}})^{\mathbb{T}} \longrightarrow \mathcal{C}_*^{\mathsf{op}}$ preserves coequalizers of kernel pairs. In our cases of interest, let us extend this result to arbitrary coequalizers.

3.2. Proposition. *Let $\mathcal{E}$ be a topos and $\mathbb{T}$ an algebraic theory. The forgetful functor $(\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}} \longrightarrow \mathcal{E}_*^{\mathsf{op}}$ creates coequalizers.*

Proof. We work in the dual categories. We consider two morphisms $f, g\colon A \rightrightarrows B$ of $\mathbb{T}$-coalgebras in $\mathcal{E}_*$ and their equalizer $k\colon K \rightarrowtail A$ as morphisms of $\mathcal{E}_*$. We shall prove that the $\mathbb{T}$-coalgebra structure of A restricts uniquely on K and yields the expected equalizer in ${}^{\mathbb{T}}\mathcal{E}_*$. Given an n-ary operation β of the theory $\mathbb{T}$, we have thus the situation

$$\begin{tikzcd}
K \arrow[r, "k"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
+^n K \arrow[u, "\nabla_K"] \arrow[r, tail, "+^n k"] & +^n A \arrow[u, "\nabla_A"'] \arrow[r, shift left, "+^n f"] \arrow[r, shift right, "+^n g"'] & +^n B \arrow[u, "\nabla_B"'] \\
K \arrow[u, dotted, "\beta_K"] \arrow[r, tail, "k"] & A \arrow[u, "\beta_A"'] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[u, "\beta_B"']
\end{tikzcd}$$

By Lemma 2.9, pulling back the coproduct $+^n A$ along β_A allows writing A as an effective union $A = A_1^\beta \cup \ldots \cup A_n^\beta$ of subobjects, with the intersection of two distinct pieces being the inverse image of $\mathbf{1}$, that is, the kernel of β_A. This allows writing

$$K = (K \cap A_1^\beta) \cup \ldots \cup (K \cap A_n^\beta).$$

By commutativity of the diagram,

$$K \cap A_i^\beta \rightarrowtail A_i^\beta \rightarrowtail A \xrightarrow{\beta_A} +^n A \xrightarrow{\nabla_A} A$$

is equalized by f and g, proving that β_A maps $K \cap A_i^\beta$ in K. Since A_i^β is mapped by β_A in the i-th component A of $+^n A$, we conclude that β_A maps $K \cap A_i^\beta$ in the subobject K of the i-th component of $+^n A$. And since K is the union of the various $K \cap K_i$ and the intersection of two of these is the kernel of β_A, this yields, by effectiveness of the union, the expected factorization β_K. □

3.3. Corollary. *Let $\mathcal{E}$ be a topos and $\mathbb{T}$ an algebraic theory. The forgetful functor* $U\colon (\mathcal{E}_*^{\mathsf{op}})^{\mathbb{T}} \longrightarrow \mathcal{E}_*^{\mathsf{op}}$ *preserves and reflects short exact sequences.*

Proof. By Propositions 3.1 and 3.2, we know that U creates kernels and cokernels, thus preserves and reflects short exact sequences. □

Let us now handle the case of compact Hausdorff algebras.

3.4. Proposition. *Let $\mathbb{T}$ be an algebraic theory. The forgetful functor*

$$(\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}} \longrightarrow \mathsf{CHaus}_*^{\mathsf{op}}$$

creates coequalizers.

Proof. In view of Lemmas 2.5 and 2.6, the proof of Proposition 3.2 carries at once over to the case of compact Hausdorff spaces. □

3.5. Corollary. *Let $\mathbb{T}$ be an algebraic theory. The forgetful functor*

$$U\colon (\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}} \longrightarrow \mathsf{CHaus}_*^{\mathsf{op}}$$

preserves and reflects short exact sequences.

Proof. The proof of Corollary 3.3 applies as such. □

3.6. Corollary. *Given an algebraic theory $\mathbb{T}$, the forgetful functor*

$$U\colon (\mathsf{CHaus}_*^{\mathsf{op}})^{\mathbb{T}} \longrightarrow (\mathsf{Set}_*^{\mathsf{op}})^{\mathbb{T}}$$

creates finite colimits.

Proof. This follows at once from Proposition 3.4 and the construction of binary products of coalgebras in the proof of Proposition 2.7. □

4. The representability of actions in $\mathsf{CHaus}_*^{\mathsf{op}}$

In a semi-abelian category one can define, in terms of algebras for a monad, the actions of an object G on an object X: a notion which recaptures a very classical one in the case of groups. In this paper, we shall use the following alternative characterization of actions:

4.1. Proposition. *In a semi-abelian category $\mathcal{C}$, the actions of an object G on an object X are in natural bijection with the isomorphism classes of split short exact sequences with kernel X*

$$\begin{tikzcd} \mathbf{1} \arrow[r] & X \arrow[r, tail, "k"] & A \arrow[r, shift right, "q"'] & G \arrow[l, shift right, "r"'] \arrow[r] & \mathbf{1} \end{tikzcd}$$

where thus $qr = \mathsf{id}_G$.

Proof. See [8]. □

Fixing X, we obtain a contravariant functor $\mathsf{Split}(-,X)\colon \mathcal{C}\longrightarrow\mathsf{Set}$, mapping an object G on the set of isomorphism classes of split exact sequences as in Proposition 4.1; this functor acts by pullback on the split epimorphism part, and next by computing the corresponding kernel.

$$\begin{tikzcd}
\mathbf{1} \arrow[r] & X' \arrow[r, "k'"] \arrow[d, dotted] & A' \arrow[r, shift right, "q'"'] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & Q' \arrow[l, shift right, "r'"'] \arrow[r] \arrow[d] & \mathbf{1} \\
\mathbf{1} \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift right, "q"'] & Q \arrow[l, shift right, "r"'] \arrow[r] & \mathbf{1}
\end{tikzcd}$$

Let us recall that, in a semi-abelian category, every regular epimorphism is normal, thus normal epimorphisms are stable under pullbacks (see [5]). And an obvious diagram chasing shows that, indeed, $X' \cong X$. One says that actions on X are representable when the functor $\mathsf{Split}(-,X)$ is representable. Like every contravariant functor to Set, $\mathsf{Split}(-,X)$ is representable precisely when its category of elements – the category of split short exact sequences with kernel X – has a terminal object.

In the case of groups, it is well-known that the representing object is the group of automorphisms of X. But the representability of actions in a semi-abelian category is a strong property which does not hold in general.

Let us consider a clearly related problem: the representability of the contravariant functor $\mathsf{Ext}(-,X)\colon \mathcal{C}\longrightarrow\mathsf{Set}$, associating with an object G the set of equivalence classes of short exact sequences

$$\mathbf{1}\longrightarrow X \xrightarrow{\ k\ } A \xrightarrow{\ q\ } G \longrightarrow \mathbf{1}$$

with kernel X, and acting again by pullback and kernel. This turns out to be a different problem from that involving $\mathsf{Split}(-,X)$: the representing objects – when both exist – are generally different; see the proof of Theorem 4.4 for an example. But nevertheless:

4.2. Proposition. *Let $\mathcal{C}$ be an arithmetical semi-abelian category and X an object of $\mathcal{C}$. The following conditions are equivalent:*

- *the functor $\mathsf{Split}(-,X)$ is representable;*
- *the functor $\mathsf{Ext}(-,X)$ is representable.*

In these conditions, actions on X are thus representable.

Proof. See [6], Theorem 8.1. □

When working with $\mathsf{Ext}(-,X)$, we get at once the following criterion:

4.3. Proposition. *Let $\mathcal{C}$ be an arithmetical semi-abelian category and $X \in \mathcal{C}$. The following conditions are equivalent:*

1. *actions on X are representable;*
2. *in the category $X/\mathcal{C}$, the full subcategory of normal monomorphisms with domain X admits a terminal object.*

Proof. Via the consideration of their cokernels, the category of normal monomorphisms with domain X is equivalent to the category of short exact sequences with kernel X, that is, to the category of elements of the contravariant functor $\mathsf{Ext}(-, X)$. And, as every contravariant functor to Set, $\mathsf{Ext}(-, X)$ is representable if and only if its category of elements admits a terminal object (see [19]). □

In the semi-abelian category $\mathcal{E}_*^{\mathsf{op}}$, with $\mathcal{E}$ a topos, the representability of actions has been studied in [7], from which it follows that the result holds in particular for two important classes of toposes: the Boolean toposes and the toposes of presheaves. Let us prove that actions are representable in the semi-abelian category $\mathsf{CHaus}_*^{\mathsf{op}}$.

First a warning. By Proposition 3.3, a short exact sequence

$$\mathbf{1} \longrightarrow G \longrightarrow A \longrightarrow X \longrightarrow \mathbf{1}$$

in CHaus_* is a short exact sequence in Set_*. In particular, as a pointed set, $A \cong G + X$. The example

$$\mathbf{1} \longrightarrow \big([0,1], 0\big) \longrightarrow \big([0,2], 0\big) \longrightarrow \big([1,2], 1\big) \longrightarrow \mathbf{1}$$

underlines a major difference when considering that situation in CHaus_* or Set_*: $\big([0,2], 0\big)$ is isomorphic to $\big([0,1], 0\big) + \big([1,2], 1\big)$ in Set_*, but not in CHaus_*. Thus in general, we do not have $A \cong G + X$ in CHaus_*. Nevertheless, this occurs in some particular cases.

4.4. Proposition. *Let X be a pointed compact Hausdorff space. When the base point of X is open, given a short exact sequence in CHaus_**

$$\mathbf{1} \longrightarrow G \xrightarrow{\;k\;} A \xrightarrow{\;p\;} X \longrightarrow \mathbf{1}$$

one has $A \cong G + X$ in CHaus_* and actions on X in $\mathsf{CHaus}_*^{\mathsf{op}}$ are represented by X itself.

Proof. Since the base point $\star$ is open in X, $p^{-1}(\star) = G$ is open in A, thus its complement is closed. The union X' of this complement and the base point of A is thus closed in A and so, is compact Hausdorff. But the continuous mapping p restricts as a bijection between X' and X; the spaces being compact Hausdorff, this restriction is an homeomorphism. This yields an inverse homeomorphism $h\colon X \longrightarrow X' \subseteq A$ in CHaus_* and thus a continuous bijection – that is, again an homeomorphism – $(k, h)\colon G + X \longrightarrow A$ in CHaus_*.

This proves that every short exact sequence in CHaus_*, with cokernel X, has the form

$$\mathbf{1} \longrightarrow G \xrightarrow{\;k\;} G + X \xrightarrow{(0, \mathsf{id}_X)} X \longrightarrow \mathbf{1}.$$

The functor $\mathsf{Ext}(-, X)$ is thus the constant functor on the singleton, that is, is represented by $\mathbf{1}$. Giving a short exact sequence with cokernel X in CHaus_* reduces to giving the object G.

As a consequence, giving a split short exact sequence with cokernel X in CHaus_*

$$\begin{tikzcd} \mathbf{1} \arrow[r] & G \arrow[r, shift right, "k"'] & G + X \arrow[l, shift right, "r"'] \arrow[r, "{(0, \mathsf{id}_X)}"] & X \arrow[r] & \mathbf{1}. \end{tikzcd}$$

reduces to giving the retraction r. But, as a retraction, r has the form (id_G, f) for some arbitrary morphism $f\colon X \longrightarrow G$; giving the split short exact sequence reduces thus to giving f. Therefore $\mathsf{Split}(G, X)$ is isomorphic to $\mathsf{CHaus}_*(X, G)$, that is, to $\mathsf{CHaus}_*^{\mathsf{op}}(G, X)$, proving that $\mathsf{Split}(-, X)$ is represented by X. □

In the general case, we have the following existence theorem:

4.5. Theorem. *Actions are representable in the semi-abelian category* $\mathsf{CHaus}_*^{\mathsf{op}}$.

Proof. We shall apply Proposition 4.2 and prove that the category $\mathsf{Norm}(X)$ of normal epimorphisms with codomain X in CHaus_* has an initial object. To achieve this, we shall prove that $\mathsf{Norm}(X)$ is stable in CHaus_*/X under limits, while the solution set condition holds. The inclusion will then have a left adjoint, mapping the initial object $\mathbf{1} \longrightarrow X$ of CHaus_*/X on an initial object of $\mathsf{Norm}(X)$.

Consider a family of normal epimorphisms $(p_i\colon A_i \longrightarrow X)_{i \in I}$ in CHaus_*. Their product $p\colon A \longrightarrow X$ in CHaus_*/X is their generalized pullback over X in CHaus; it is computed as in Set and provided with the topology induced by the product topology on $\prod_{i \in I} A_i$. Let us write G_i for the kernel of p_i. Set theoretically, we have thus $A_i \cong G_i + X$, with G_i mapped by p_i on the base point of X, while p_i restricts as the identity on the component X. This proves that the set A is simply $\left(\prod_{i \in I} G_i\right) + X$, with p restricting as the constant map on the base point on the term $\prod_{i \in I} G_i$, and the identity on X on the second term. This is thus a normal epimorphism with kernel $\prod_{i \in I} G_i$ in Set_*.

It remains to prove that the topology of X is the quotient one for p. But $p\colon A \twoheadrightarrow X$ is a surjective continuous mapping and since the spaces are compact Hausdorff, this forces X to have the quotient topology for p (see [12]).

Consider now p, q in $\mathsf{Norm}(X)$, two morphisms f, g from p to q and their equalizer k in CHaus_*/X.

$$\begin{tikzcd} K \arrow[r, tail, "k"] \arrow[dr, "pk"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "p"] & B \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

We must prove that pk is a normal epimorphism. Set theoretically, $A \cong G + X$ and $B \cong H + X$, while f, g, p, q, k restrict as the identity on the X components of these sums. By commutativity of the diagram, f and g factor through the kernels of p and q, that is, map G in H. Thus $K \cong L + X$, where L is the equalizer of f and g restricted to G and H. Therefore pk maps L on the base point and restricts as the identity on X. It is thus a normal epimorphism in Set_*. The same argument as in the case of products allows to conclude that the quotient topology for pk is precisely the topology of X.

It remains to check the solution set condition. Thus, given a morphism $p\colon A \longrightarrow X$ in CHaus_*, we must find a family $(q_i\colon B_i \longrightarrow X)_{i\in I}$ of normal epimorphisms in CHaus_*, together with morphisms $f_i\colon p \longrightarrow q_i$ in CHaus_*/X such that,

$$\begin{tikzcd}
A \arrow[rr, "f_i"] \arrow[dr, "f"] \arrow[ddr, "p"'] & & B_i \arrow[dl, "h_i"'] \arrow[ddl, two heads, "q_i"] \\
& B \arrow[d, two heads, "q"] & \\
& X &
\end{tikzcd}$$

given a normal epimorphism $q\colon B \longrightarrow X$ and a morphism $f\colon p \longrightarrow q$ in CHaus_*/X, f factors through one of the f_i's, making the whole diagram commutative.

Set theoretically, $B \cong G + X'$, with G the kernel of q, while q maps X' bijectively on X. Of course f factors through $f(A)$, whose cardinality is less than the cardinality of A. And $f(A)$ is compact, thus closed, as image of a compact. But $q\colon f(A) \longrightarrow X$ has no reason to be surjective, thus to be a normal epimorphism. So one would like to consider instead $f(A) \cup X' \ldots$ but this subset has no reason to be closed in B, thus to be compact Hausdorff. Therefore we consider instead the closure B' of $f(A) \cup X'$ in B, through which f factors. This time, the restriction $q\colon B' \longrightarrow X$ is surjective and therefore, since the spaces are compact Hausdorff, keeps yielding a quotient topology, thus is a normal epimorphism. Moreover the cardinality of B' is bounded by the cardinality of all possible (Cauchy) sequences of elements in $f(A) \cup X'$, that is, also by the cardinality of $(A \amalg X)^{\mathbb{N}}$. The solution set is thus given by all the normal epimorphisms $q_i\colon B_i \longrightarrow X$ in CHaus_*, where the cardinality of B_i is less than $\#(A \amalg X)^{\mathbb{N}}$. There is only (up to isomorphism) a set of such sets B_i and, on each of them, there is only a set of possible (compact Hausdorff) topologies. □

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