

Free Doubly-Infinitary Distributive Categories are Cartesian Closed

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Abstract

We delve into the concept of categories with products that distribute over coproducts, which we call *doubly-infinitary distributive categories*. We show various instances of doubly-infinitary distributive categories aiming for a comparative analysis with established notions such as extensivity, infinitary distributivity, and cartesian closedness. Our exploration reveals that this condition represents a substantial extension beyond the classical understanding of infinitary distributive categories. Our main theorem establishes that free doubly-infinitary distributive categories are cartesian closed. We end the paper with remarks on non-canonical isomorphisms, open questions and future work.

A common question in category theory is how limits and colimits interact with each other. One of the most benign kinds of interaction is that of a *(pseudo)distributive law*; for instance, finitary and infinitary *distributive categories* [10], and *completely distributive categories* [37].

Explicitly, if \mathcal{C} is a category with finite products and finite coproducts, \mathcal{C} is called finitary distributive if the canonical comparison (0.2) induced by the universal property of the coproducts is invertible for any triple (A, B, C) of objects in \mathcal{C} ; in other words, \mathcal{C} is *finitary distributive* if, for any object A , the functor

$$A \times - : \mathcal{C} \rightarrow \mathcal{C} \quad (0.1)$$

preserves finite coproducts. In the presence of arbitrary coproducts, we can require even more; namely, a category is *infinitary distributive* if, for any object A and any family $\{B_i\}_{i \in I}$ of objects, the canonical comparison (0.3) is invertible, which, again, is equivalent to saying that (0.1) is a coproduct-preserving functor for any A .

$$(A \times B) \sqcup (A \times C) \xrightarrow{\cong} A \times (B \sqcup C) \quad (0.2) \qquad \bigsqcup_{i \in I} (A \times B_i) \xrightarrow{\cong} A \times \left(\bigsqcup_{i \in I} B_i \right) \quad (0.3)$$

From the perspective of two-dimensional monad theory, both notions can be realized as pseudoalgebras of suitable composites of free completion pseudomonads, arising from canonical (pseudo)distributive laws. By making use of this approach, a much stronger distributivity condition has been considered in [37]; namely, *completely distributive categories*, that is to say, categories with the distributivity property of arbitrary (small) limits over colimits.

In the present work, we explore the realm of categories with products and coproducts, featuring a distributive law between them, which we term *doubly-infinitary distributive categories*. This notion serves as an intermediary between infinitary distributive categories and completely distributive ones.

Naturally, this conceptualization emerges from the canonical pseudodistributive law between the respective free completion pseudomonads (under products and under coproducts); namely, this pseudodistributive law extends the usual (pseudo)distributive law between their finite counterparts, and is the restriction of that considered by [37].

Since we are in the context of pseudodistributive laws involving *Kock-Zöberlein pseudomonads*, the definition of the corresponding pseudodoalgebra can be given as a category with two different properties, along with the invertibility of a canonical morphism, *e.g.* [35, 46, 30]. Specifically in our case, considering a category \mathcal{C} with coproducts and products, \mathcal{C} is doubly-infinitary distributive whenever the canonical morphism

(3.1) is invertible for any family of objects $(C_{ij})_{(j,i) \in J \times I_j}$ of \mathcal{C} . To articulate this concept more concisely, where $\mathbf{Fam}(\mathcal{C})$ denotes the free coproduct completion of \mathcal{C} (as defined in Section 1), \mathcal{C} is doubly-infinitary distributive when the coproduct functor

$$\bigsqcup: \mathbf{Fam}(\mathcal{C}) \rightarrow \mathcal{C}, \quad (0.4)$$

which realizes \mathcal{C} as coproduct-complete category (pseudoalgebra of the free coproduct completion pseudomonad), preserves products.

We show various instances of doubly-infinitary distributive categories aiming for a comparative analysis with established notions such as extensivity, infinitary distributivity, and cartesian closedness. Our exploration reveals that this condition represents a substantial extension beyond the classical understanding of infinitary distributive categories. In particular, we show that cartesian closedness does not imply doubly-infinitary distributivity. Moreover, there are non-extensive categories that are doubly-infinitary distributive. Furthermore, despite being infinitary (1)extensive, we find that the category of topological spaces fails to meet the criteria for being doubly-infinitary distributive. As explained in Section 5, this observation prompts further inquiry into the conditions under which categories of generalized multicategories exhibit doubly-infinitary distributiveness, especially in the context of enriched categorical structures and (T, \mathcal{V}) -categories.

Finally, we show that there are doubly-infinitary distributive categories that are not cartesian closed, such as the category of locally connected topological spaces. In this direction, our most surprising observation, underpinning our future work on the denotational semantics of program transformations, is that, besides having many other interesting properties like extensivity, free doubly-infinitary distributive categories are cartesian closed.

Structure, literature and background In Section 1, we revisit the definition of free (co)product completion. For the interested reader, we point out that there is an extensive literature on the free (co)product completion $\mathbf{Fam}(\mathcal{C})$ on a category \mathcal{C} , as evidenced by works such as [8, 3, 31], and [40, Section 8.5].

Theorem 2.3, established in Section 2, is the main result of the present paper. It pertains to a fundamentally elementary concept. For any category \mathcal{C} , we establish that

$$\mathbf{Dist}(\mathcal{C}) \stackrel{\text{def}}{=} \mathbf{Fam}(\mathbf{Fam}(\mathcal{C}^{\text{op}})^{\text{op}}),$$

whose explicit description we give therein, is cartesian closed, that is to say, we establish the cartesian closedness of the *free coproduct completion* of the *free product completion* of any category \mathcal{C} .

Furthermore, in Section 3, we recall that we get a composite pseudomonad structure for $\mathbf{Dist}(-)$, stemming from a pseudodistributive law between the pseudomonads of free product completion and free coproduct completion. We, then, consider the 2-category of $\mathbf{Dist}(-)$ -pseudoalgebras and pseudomorphisms, whose objects we call *doubly-infinitary distributive categories*. This is the reason why $\mathbf{Dist}(\mathcal{C}) = \mathbf{Fam}(\mathbf{Fam}(\mathcal{C}^{\text{op}})^{\text{op}})$ earns the designation of the free doubly-infinitary distributive category on \mathcal{C} .

In this discourse, a familiarity with two-dimensional monad theory is assumed. Interested readers are directed to [6, 26, 28, 30] for foundational concepts and definitions surrounding biadjunctions, pseudomonads, pseudoalgebras, as well as some of the results referenced. Lastly, for an in-depth understanding of pseudodistributive laws, we recommend consulting Marmolejo's seminal work [34, 35, 36]. Additionally, works such as [24, 33, 12, 30] delve into lax idempotent 2-monads and pseudomonads, providing valuable insights to the present work.

In Section 4, we discuss examples of *doubly-infinitary distributive categories*. We compare them with established notions, such as extensiveness, cartesian closedness and distributivity. See, for instance, [10, 8], and [40, Section 7] for these notions.

Finally, we make some final remarks in Section 5. We establish one open problem on the doubly-infinitary distributivity of categories of categorical structures, especially in the context of generalized multicategories [13, 14, 40]. We also establish the result on non-canonical isomorphisms for the case of doubly-infinitary distributive categories, which follows directly from the framework introduced in [30].

Related work We note that Von Glehn considered a similar notion of distributivity for Π - and Σ -types in fibrations over a locally cartesian closed category \mathbb{B} , *e.g.* [45]. We focus on the specific case of fibrations of the form $\mathbf{Fam}(\mathcal{C}) \rightarrow \mathbf{Set}$, in which case Π - and Σ -types reduce to products and coproducts in \mathcal{C} (see [44, Theorem 3.5.2] or [43, Theorem 12]).

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1 Free completion under coproducts

We recall the basic definition of the category $\mathbf{Fam}(\mathcal{C})$ for each category \mathcal{C} . This construction has been extensively considered in the literature. We refer, for instance, to [10, 8, 3, 31, 39, 40] for further properties.

Given a category \mathcal{C} , recall that its image by the Yoneda embedding defines a strictly indexed category $\mathfrak{Fam}(\mathcal{C}) : \mathbf{Set}^{op} \rightarrow \mathbf{Cat}$, where $\mathfrak{Fam}(\mathcal{C}) = \mathbf{Cat}(-, \mathcal{C})$ is the \mathbf{Cat} -enriched hom-functor from sets considered as discrete categories. We can equivalently consider the corresponding (split) fibred category $\mathbf{Fam}(\mathcal{C}) = \Sigma_{\mathbf{Set}} \mathfrak{Fam}(\mathcal{C})$ obtained by taking the Grothendieck construction [19]. Concretely, $\mathbf{Fam}(\mathcal{C})$ has objects that consist of a pair of a set I and an I -indexed family $[C_i \mid i \in I]$ of objects C_i of \mathcal{C} , *i.e.* $\mathbf{ob}(\mathbf{Fam}(\mathcal{C})) \stackrel{\text{def}}{=} \Sigma_{I \in \mathbf{ob}(\mathbf{Set})} \mathbf{ob}(\mathcal{C})^I$. The homset sets are

$$\mathbf{Fam}(\mathcal{C})([C_i \mid i \in I], [C'_j \mid j \in J]) \stackrel{\text{def}}{=} \prod_{i \in I} \sum_{j \in J} \mathcal{C}(C_i, C'_j),$$

with identity $\text{id}_{[C_i \mid i \in I]} \stackrel{\text{def}}{=} \lambda i : I. \langle i, \text{id}_{C_i} \rangle$ and composition

$$f \circ g \stackrel{\text{def}}{=} \lambda i : I. \mathbf{let} \langle i', g' \rangle = g(i) \mathbf{in} \mathbf{let} \langle i'', f' \rangle = f(i') \mathbf{in} \langle i'', f' \circ g' \rangle.$$

Universal property of the $\mathbf{Fam}(-)$ -construction It is well-known that $\mathbf{Fam}(\mathcal{C})$ is the free coproduct-completion of \mathcal{C} . In fact, $\mathbf{Fam}(-)$ forms a lax-idempotent pseudomonad (also known as a Kock-Z oberlein pseudomonad) on \mathbf{Cat} , where the unit

$$(C \xrightarrow{f} C') \mapsto ([C \mid * \in 1] \xrightarrow{\lambda \cdot \langle *, f \rangle} [C' \mid * \in 1])$$

takes the singleton family and the multiplication

$$\begin{aligned} & ([[C_{ij} \mid i \in I_j] \mid j \in J] \xrightarrow{f} [[C'_{i'j'} \mid i' \in I'_{j'}] \mid j' \in J']) \mapsto \\ & ([C_{ij} \mid \langle j, i \rangle \in \sum_{j \in J} I_j] \xrightarrow{\lambda \langle j, i \rangle. \mathbf{let} \langle j', g \rangle = f(j) \mathbf{in} \mathbf{let} \langle i', h \rangle = g(i) \mathbf{in} \langle \langle j', i' \rangle, h \rangle} [C'_{i'j'} \mid \langle j', i' \rangle \in \sum_{j' \in J'} I'_{j'}]) \end{aligned}$$

takes the disjoint union of a family of families. Briefly, to see that this indeed is a lax-idempotent pseudomonad, observe that the biadjunction

$$\begin{array}{ccc} & F & \\ \text{CoProdCat} & \begin{array}{c} \curvearrowright \\ \perp(\epsilon, \eta) \\ \curvearrowleft \end{array} & \text{Cat} \\ & G & \end{array}$$

satisfies the Kock-Z oberlein condition for biadjunctions, namely $G\epsilon \dashv \eta G$ is a (pseudo)laxi adjunction, meaning that the coherence isomorphism $(G\epsilon)(\eta G) \cong \text{id}$ is the counit of the adjunction $G\epsilon \dashv \eta G$ (see, for instance, [26, Definition 2.5] for the biadjunctions, and [12, Theorem 3.15] for Kock-Z oberlein conditions for 2-adjunctions).

The corresponding 2-category of pseudoalgebras and pseudomorphisms **CoProdCat** consists precisely of coproduct-complete categories, coproduct-preserving functors, and natural transformations [23, 24]. Using the isomorphism of 2-categories $op : \mathbf{Cat}^{co} \rightarrow \mathbf{Cat}$, we obtain a colax idempotent pseudomonad $op \circ \mathbf{Fam}(-) \circ op$ that has the 2-category **ProdCat** of product-complete categories as its category of pseudoalgebras. As a notational convention, we denote the objects of $\mathbf{Fam}(\mathcal{C}^{op})^{op}$ as $\langle C_i \mid i \in I \rangle$ to emphasize their interpretation as a free product of a family of objects in \mathcal{C} .

2 The category $\mathbf{Dist}(\mathcal{C})$

In this section, we define the *free doubly-infinitary distributive category* $\mathbf{Dist}(\mathcal{C})$ on a category \mathcal{C} , and prove our main result on its cartesian closedness; namely Theorem 2.3. We define

$$\mathbf{Dist}(\mathcal{C}) \stackrel{\text{def}}{=} \mathbf{Fam}(\mathbf{Fam}(\mathcal{C}^{op})^{op}),$$

i.e. $\mathbf{Dist}(\mathcal{C})$ is the free coproduct completion of the free product completion of \mathcal{C} . We formalize the fact this is indeed the *free doubly-infinitary distributive category* in Section 3.

Clearly, $\mathbf{Dist}(\mathcal{C})$ is a category with coproducts, being a free coproduct completion. Surprisingly, it also has products and exponentials. To see that, it can help to give an explicit description of $\mathbf{Dist}(\mathcal{C})$.

Explicit description of the $\mathbf{Dist}(-)$ -construction We have the following concrete description of $\mathbf{Dist}(\mathcal{C})$. Objects are families of families of objects C_{ji} of \mathcal{C} :

$$[\langle C_{ji} \mid i \in I_j \rangle \mid j \in J]$$

for some set J and sets I_j (for $j \in J$). Morphisms

$$[\langle C_{ji} \mid i \in I_j \rangle \mid j \in J] \rightarrow [\langle C'_{j'i'} \mid i' \in I'_{j'} \rangle \mid j' \in J']$$

are precisely elements of the set

$$\prod_{j \in J} \sum_{j' \in J'} \prod_{i' \in I'_{j'}} \sum_{i \in I_j} \mathcal{C}(C_{ji}, C'_{j'i'}).$$

The identity on $[\langle C_{ji} \mid i \in I_j \rangle \mid j \in J]$ is

$$\lambda_j : J.\langle j, \lambda_i : I_j.\langle i, \text{id}_{C_{ji}} \rangle \rangle,$$

where we use the identities from \mathcal{C} . Composition $h' \circ h$ is

$$\lambda_j : J.\mathbf{let} \langle j', f \rangle = h(j) \mathbf{in} \mathbf{let} \langle j'', f' \rangle = h'(j') \mathbf{in} \\ \langle j'', \lambda_{i''} : I''_{j''}.\mathbf{let} \langle i', c' \rangle = f'(i'') \mathbf{in} \mathbf{let} \langle i, c \rangle = f(i) \mathbf{in} \langle i, c' \circ c \rangle$$

where we use the composition from \mathcal{C} .

Properties of $\mathbf{Dist}(\mathcal{C})$ Then, obviously, we have coproducts.

Lemma 2.1. Coproducts exist in $\mathbf{Dist}(\mathcal{C})$ and are computed as

$$\bigsqcup_{k \in K} [\langle C_{kji} \mid i \in I_{kj} \rangle \mid j \in J_k] \stackrel{\text{def}}{=} [\langle C_{kji} \mid i \in I_{kj} \rangle \mid k \in K, j \in J_k].$$

Proof. $\mathbf{Dist}(\mathcal{C})$ has coproducts by virtue of being of the form $\mathbf{Fam}(\mathcal{D})$ of a free coproduct completion, for $\mathcal{D} = \mathbf{Fam}(\mathcal{C}^{op})^{op}$. \square

We can also form products.

Lemma 2.2. Products exist in $\mathbf{Dist}(\mathcal{C})$ and are computed as

$$\prod_{k \in K} [\langle C_{kj} \mid i \in I_{kj} \rangle \mid j \in J_k] \stackrel{\text{def}}{=} [\langle C_{kf(k)i} \mid k \in K, i \in I_{kf(k)} \rangle \mid f \in \prod_{k \in K} J_k].$$

Proof. It is well-known that $\mathbf{Fam}(\mathcal{D})$ has products if \mathcal{D} does, given by $\prod_{k \in K} [D_{kj} \mid j \in J_k] = [\prod_{k \in K} D_{kf(k)} \mid f \in \prod_{k \in K} J_k]$. See, for example, [18]. Now, in $\mathcal{D} = \mathbf{Fam}(\mathcal{C}^{op})^{op}$, products are straightforward, being a free product completion: $\prod_{k \in K} \langle C_{kf(k)i} \mid i \in I_{kf(k)} \rangle = \langle C_{kf(k)i} \mid k \in K, i \in I_{kf(k)} \rangle$. \square

Surprisingly, we even have exponentials, via a Dialectica interpretation-like formula [17].

Theorem 2.3. Exponentials exist in $\mathbf{Dist}(\mathcal{C})$ and are computed as

$$\begin{aligned} \langle C_{ji} \mid i \in I_j \rangle \mid j \in J &\Rightarrow [\langle C'_{j'i'} \mid i' \in I'_{j'} \rangle \mid j' \in J'] \stackrel{\text{def}}{=} \\ &[\langle C'_{j'i'} \mid j \in J, \langle j', g \rangle = f(j), i' \in I'_{j'}, g(i') = \langle \perp, \perp \rangle \rangle \mid \\ &f \in \prod_{j \in J} \Sigma_{j' \in J'} \Pi_{i' \in I'_{j'}} \Sigma_{i \in I_j \sqcup \{\perp\}} \mathcal{C}(C_{ji}, C'_{j'i'}) \text{ if } i \neq \perp \text{ else } \{\perp\}], \end{aligned}$$

where we slightly abuse notation and leave coproduct coprojections into $I_j \sqcup \{\perp\}$ implicit to aid legibility.

Proof. We demonstrate the natural isomorphism of homsets establishing the exponential adjunction:

$$\begin{aligned} \mathbf{Dist}(\mathcal{C})([\langle C_{j_0 i_0 0} \mid i_0 \in I_{j_0 0} \rangle \mid j_0 \in J_0] \times [\langle C_{j_1 i_1 1} \mid i_1 \in I_{j_1 1} \rangle \mid j_1 \in J_1], \\ [\langle C'_{j' i'} \mid i' \in I'_{j'} \rangle \mid j' \in J']) &= \{ \text{Lemma 2.2} \} \\ \mathbf{Dist}(\mathcal{C})([\langle C_{j_k i_k k} \mid \langle k, i_k \rangle \in \Sigma_{k \in \{0,1\}} I_{j_k k} \rangle \mid \langle j_0, j_1 \rangle \in J_0 \times J_1], \\ [\langle C'_{j' i'} \mid i' \in I'_{j'} \rangle \mid j' \in J']) &= \{ \text{def. homsets } \mathbf{Dist}(\mathcal{C}) \} \\ \prod_{\langle j_0, j_1 \rangle \in J_0 \times J_1} \Sigma_{j' \in J'} \Pi_{i' \in I'_{j'}} \Sigma_{\langle k, i_k \rangle \in (\Sigma_{k \in \{0,1\}} I_{j_k k})} \mathcal{C}(C_{j_k i_k k}, C'_{j' i'}) &\cong \{ \text{currying} \} \\ \prod_{j_0 \in J_0} \prod_{j_1 \in J_1} \Sigma_{j' \in J'} \Pi_{i' \in I'_{j'}} \Sigma_{\langle k, i_k \rangle \in (\Sigma_{k \in \{0,1\}} I_{j_k k})} \mathcal{C}(C_{j_k i_k k}, C'_{j' i'}) &\cong \{ \text{assoc. } \Sigma\text{-type} \} \\ \prod_{j_0 \in J_0} \prod_{j_1 \in J_1} \Sigma_{j' \in J'} \Pi_{i' \in I'_{j'}} (\Sigma_{i_1 \in I_{j_1 1}} \mathcal{C}(C_{j_1 i_1 1}, C'_{j' i'})) \sqcup (\Sigma_{i_0 \in I_{j_0 0}} \mathcal{C}(C_{j_0 i_0 0}, C'_{j' i'})) &\cong \{ (*) \} \\ \prod_{j_0 \in J_0} \prod_{j_1 \in J_1} \Sigma_{j' \in J'} \Sigma_{h: \Pi_{i' \in I'_{j'}} (\Sigma_{i_1 \in I_{j_1 1}} \mathcal{C}(C_{j_1 i_1 1}, C'_{j' i'})) \sqcup \{\langle \perp, \perp \rangle\}} & \\ \Pi_{i' \in \{i' \in I'_{j'} \mid h(i') = \langle \perp, \perp \rangle\}} (\Sigma_{i_0 \in I_{j_0 0}} \mathcal{C}(C_{j_0 i_0 0}, C'_{j' i'})) &\cong \{ (**) \} \\ \prod_{j_0 \in J_0} \prod_{j_1 \in J_1} \Sigma_{j' \in J'} \Sigma_{h: \Pi_{i' \in I'_{j'}} \Sigma_{i_1 \in I_{j_1 1} \sqcup \{\perp\}} \mathcal{C}(C_{j_1 i_1 1}, C'_{j' i'}) \text{ if } i_1 \neq \perp \text{ else } \{\perp\}} & \\ \Pi_{i' \in \{i' \in I'_{j'} \mid h(i') = \langle \perp, \perp \rangle\}} \Sigma_{i_0 \in I_{j_0 0}} \mathcal{C}(C_{j_0 i_0 0}, C'_{j' i'}) &\cong \{ \text{assoc. } \Sigma\text{-type} \} \\ \prod_{j_0 \in J_0} \prod_{j_1 \in J_1} \Sigma_{\langle j', h \rangle \in \Sigma_{j' \in J'} \Pi_{i' \in I'_{j'}} \Sigma_{i_1 \in I_{j_1 1} \sqcup \{\perp\}} \mathcal{C}(C_{j_1 i_1 1}, C'_{j' i'}) \text{ if } i_1 \neq \perp \text{ else } \{\perp\}} & \\ \Pi_{i' \in \{i' \in I'_{j'} \mid h(i') = \langle \perp, \perp \rangle\}} \Sigma_{i_0 \in I_{j_0 0}} \mathcal{C}(C_{j_0 i_0 0}, C'_{j' i'}) &\cong \{ (***) \} \\ \prod_{j_0 \in J_0} \Sigma_{f \in \prod_{j_1 \in J_1} \Sigma_{j' \in J'} \Pi_{i' \in I'_{j'}} \Sigma_{i_1 \in I_{j_1 1} \sqcup \{\perp\}} \mathcal{C}(C_{j_1 i_1 1}, C'_{j' i'}) \text{ if } i_1 \neq \perp \text{ else } \{\perp\}} & \\ \prod_{j_1 \in J_1, \langle j', h \rangle = f(j_1)} \Pi_{i' \in \{i' \in I'_{j'} \mid h(i') = \langle \perp, \perp \rangle\}} \Sigma_{i_0 \in I_{j_0 0}} \mathcal{C}(C_{j_0 i_0 0}, C'_{j' i'}) &\cong \{ \text{uncurrying} \} \\ \prod_{j_0 \in J_0} \Sigma_{f \in \prod_{j_1 \in J_1} \Sigma_{j' \in J'} \Pi_{i' \in I'_{j'}} \Sigma_{i_1 \in I_{j_1 1} \sqcup \{\perp\}} \mathcal{C}(C_{j_1 i_1 1}, C'_{j' i'}) \text{ if } i_1 \neq \perp \text{ else } \{\perp\}} & \\ \prod_{j_1 \in J_1, \langle j', h \rangle = f(j_1), i' \in I'_{j'}, h(i') = \langle \perp, \perp \rangle} \Sigma_{i_0 \in I_{j_0 0}} \mathcal{C}(C_{j_0 i_0 0}, C'_{j' i'}) &= \{ \text{def. homsets } \mathbf{Dist}(\mathcal{C}) \} \\ \mathbf{Dist}(\mathcal{C})([\langle C_{j_0 i_0 0} \mid i_0 \in I_{j_0 0} \rangle \mid j_0 \in J_0], \\ [\langle C'_{j' i'} \mid j_1 \in J_1, \langle j', h \rangle = f(j_1), i' \in I'_{j'}, h(i') = \langle \perp, \perp \rangle \rangle \mid \\ f \in \prod_{j_1 \in J_1} \Sigma_{j' \in J'} \Pi_{i' \in I'_{j'}} \Sigma_{i_1 \in I_{j_1 1} \sqcup \{\perp\}} \mathcal{C}(C_{j_1 i_1 1}, C'_{j' i'}) \text{ if } i_1 \neq \perp \text{ else } \{\perp\}]) & \end{aligned}$$

Here, by (un)currying we mean the natural isomorphism $\prod_{\langle a, b \rangle \in \Sigma_{a \in A} B_b} C_{ab} \cong \prod_{a \in A} \prod_{b \in B_a} C_{ab}$, which holds for Σ -types with a dependent elimination rule. $(*)$ is in many ways the crux of the proof: in \mathbf{Set} , we

can characterise maps *into* a coproduct as $\prod_{a \in A} B_a \sqcup C_a \cong \sum_{h \in \prod_{a \in A} B_a \sqcup \{\perp\}} \prod_{a \in \{a \in A \mid h(a) = \perp\}} C_a$. (**) is a straightforward identity of coproducts in **Set**: $(\sum_{a \in A} B_a) \sqcup \{\perp, \perp\} \cong \sum_{a' \in A \sqcup \{\perp\}} B_{a'}$ if $a' \neq \perp$ else $\{\perp\}$. (***) is the comprehension property of Σ -types with a dependent elimination rule: $\prod_{a \in A} \sum_{b \in B_a} C_{ab} \cong \sum_{f \in \prod_{a \in A} B_a} \prod_{a \in A} C_{af(a)}$ \square

Remark 1 (Exponentials in $\mathbf{Dist}(\mathcal{C})$, inductively). Every object of $\mathbf{Dist}(\mathcal{C})$ is a coproduct of products of objects in the image of $\mathcal{C} \rightarrow \mathbf{Dist}(\mathcal{C})$. This means that we can easily give an inductive definition of the exponential as follows.

Denoting by \sqcup the coproduct in $\mathbf{Dist}(\mathcal{C})$, if $(C_i)_{i \in J}$ is a family of objects in $\mathbf{Dist}(\mathcal{C})$, A is in the image of the inclusion $\mathbf{Fam}(\mathcal{C}^{op})^{op} \rightarrow \mathbf{Dist}(\mathcal{C})$, and B is in the image of $\mathcal{C} \rightarrow \mathbf{Dist}(\mathcal{C})$:

- $A \Rightarrow B = \left(\bigsqcup_{i \in \mathbf{Dist}(\mathcal{C})(A, B)} \mathbb{1} \right) \sqcup B$,
- $A \Rightarrow \left(\bigsqcup_{i \in J} C_i \right) = \bigsqcup_{i \in J} (A \Rightarrow C_i)$.

It should be noted that the above is enough to define the exponential given the fact that $A \Rightarrow -$ and $- \Rightarrow A$ preserve products.

This surprising result of cartesian closure might remind the reader of the well-known fact that a completely distributive lattice is a complete Heyting algebra, by defining the exponential as $a \Rightarrow b \stackrel{\text{def}}{=} \bigvee \{a' \mid a \wedge a' \leq b\}$. However, as we will see, while free doubly-infinitary distributive categories are cartesian closed, other doubly-infinitary distributive categories might not be. As only the only completely distributive lattices that are of the form $\mathbf{Dist}(\mathcal{C})$ are trivial, our result really is qualitatively different. We do recover cartesian closure (and even get a Grothendieck topos) from a distributive law for non-thin categories, however, if we have more general (finite) limits and colimits that distribute over each other and have a set of generators [37].

3 $\mathbf{Dist}(-)$ as a pseudomonad

Herein, we briefly recall the fact that $\mathbf{Dist}(-)$ can be naturally endowed with a pseudomonad structure, coming from a pseudodistributive law between the pseudomonads of free completion under products $\mathbf{Fam}(-^{op})^{op}$ and under coproducts $\mathbf{Fam}(-)$. We refer the reader to [36, 34, 35] for pseudodistributive laws and compatible liftings.

Observe that we have functors $\mathcal{C} \rightarrow \mathbf{Fam}(\mathcal{C}^{op})^{op} \rightarrow \mathbf{Fam}(\mathbf{Fam}(\mathcal{C}^{op})^{op})$ given by the singleton families (the units of the pseudomonads $\mathbf{Fam}(-^{op})^{op}$ and $\mathbf{Fam}(-)$ on \mathbf{Cat}). Now, as $\mathbf{Dist}(\mathcal{C})$ has both products and coproducts, we obtain first an essentially unique coproduct preserving extension to a functor $\mathbf{Fam}(\mathcal{C}) \rightarrow \mathbf{Fam}(\mathbf{Fam}(\mathcal{C}^{op})^{op})$ and then an essentially unique product preserving functor

$$\mathbf{Fam}(\mathbf{Fam}(\mathcal{C})^{op})^{op} \rightarrow \mathbf{Fam}(\mathbf{Fam}(\mathcal{C}^{op})^{op}),$$

by the universal properties of the free coproduct and product completions.

We can give an explicit description of the resulting functor:

$$\begin{aligned} \lambda_{\mathcal{C}} : \mathbf{Fam}(\mathbf{Fam}(\mathcal{C})^{op})^{op} &\rightarrow \mathbf{Fam}(\mathbf{Fam}(\mathcal{C}^{op})^{op}) \\ (([C_{ij} \mid i \in I_j] \mid j \in J) &\xrightarrow{g} \langle [C'_{i'j'} \mid i' \in I'_{j'}] \mid j' \in J' \rangle) \mapsto \\ &([C_{f(j)j} \mid j \in J] \mid f \in \prod_{j \in J} I_j) \xrightarrow{h} (\langle [C'_{f'(j')j'} \mid j' \in J'] \mid f' \in \prod_{j' \in J'} I'_{j'} \rangle) \end{aligned}$$

where

$$h = \lambda f : \prod_{j \in J} I_j . \langle \lambda j' : J' . \mathbf{let} \langle j, g' \rangle = g(j') \mathbf{in} \pi_1(g'(f(j))) \rangle,$$

$$\lambda_{j'} : J'.\mathbf{let} \langle j, g' \rangle = g(j') \mathbf{in} \langle j, \pi_2(g'(f(j))) \rangle.$$

To parse this definition, it might be helpful to remember that

$$g \in \prod_{j' \in J'} \Sigma_{j \in J} \prod_{i \in I_j} \Sigma_{i' \in I'_j} \mathcal{C}(C_{ij}, C'_{i'j'})$$

and

$$h \in \prod_{f \in \prod_{j \in J} I_j} \Sigma_{f' \in \prod_{j' \in J'} I'_{j'}} \prod_{j' \in J'} \Sigma_{j \in J} \mathcal{C}(C_{f(j)j}, C'_{f'(j')j'}).$$

The morphisms $\lambda_{\mathcal{C}}$ are automatically pseudonatural in \mathcal{C} , as they arise from the universal properties of the free product and coproduct completions and the units of pseudomonads on \mathbf{Cat} . In fact, by [45, Theorem 7.1] with $\mathbb{B} = \mathbf{Set}$, they define a pseudodistributive law [36, Definition 11.4]

$$\lambda : (op \circ \mathbf{Fam}(-) \circ op) \circ \mathbf{Fam}(-) \rightarrow \mathbf{Fam}(-) \circ (op \circ \mathbf{Fam}(-) \circ op),$$

which is essentially unique (see [46, Remark 34/Corollary 49]). Therefore, $\mathbf{Dist}(-) \stackrel{\text{def}}{=} \mathbf{Fam}(-) \circ (op \circ \mathbf{Fam}(-) \circ op)$ is another pseudomonad on \mathbf{Cat} and $\mathbf{Fam}(-)$ lifts to a lax idempotent pseudomonad on $\mathbf{ProdCat}$, the category of pseudoalgebras of $\mathbf{Fam}(-)^{op}$ (product-complete categories).

Dist(-)-pseudoalgebras Next, we consider the pseudoalgebras of $\mathbf{Dist}(-)$.

Definition 1 (Doubly-Infinitary Distributive Category). Let \mathcal{C} be a category with products and coproducts. Then, for each family of objects $(C_{ij})_{(j,i) \in J \times I_j}$ in \mathcal{C} , we have a canonical morphism

$$[\langle \iota_{f(j)} \circ \pi_j \mid j \in J \rangle \mid f \in \prod_{j \in J} I_j] : \left(\bigsqcup_{f: \prod_{j \in J} I_j} \prod_{j \in J} C_{f(j)j} \right) \rightarrow \left(\prod_{j \in J} \bigsqcup_{i \in I_j} C_{ij} \right). \quad (3.1)$$

We say that \mathcal{C} is *doubly-infinitary distributive* (i.e., that infinite products distribute over infinite coproducts in \mathcal{C}) if (3.1) is invertible for any family $(C_{ij})_{(j,i) \in J \times I_j}$.

Remark 2 (Concise definition). As in the case of finitary/infinitary distributive categories, our definition of doubly-infinitary distributive categories can similarly be articulated in terms of a functor that preserves (co)products. As demonstrated in Lemma 3.1, this approach is directly derived from the pseudodistributive law. Specifically, a category \mathcal{C} is doubly-infinitary distributive if and only if \mathcal{C} is product and coproduct complete, and the coproduct functor

$$\bigsqcup : \mathbf{Fam}(\mathcal{C}) \rightarrow \mathcal{C} \quad (3.2)$$

preserves products.

Lemma 3.1. The 2-category $\mathbf{DistCat}$ of $\mathbf{Dist}(-)$ -pseudoalgebras consists precisely of the doubly-infinitary distributive categories, product and coproduct preserving functors and natural transformations.

Proof. $\mathbf{Dist}(-)$ -pseudoalgebras are the same as $\mathbf{Fam}(-)$ -pseudoalgebras over $\mathbf{ProdCat}$ (for the canonical lifting of $\mathbf{Fam}(-)$ to $\mathbf{ProdCat}$ that comes from the pseudodistributive law), that is categories with coproducts and products such that the coproducts are product-preserving functors in all their arguments.

In other words, a $\mathbf{Dist}(-)$ -pseudoalgebra is a triple $(\mathcal{C}, \mathbf{a}, \mathbf{b})$ where \mathbf{a}, \mathbf{b} are, respectively, $\mathbf{Fam}(-)$ and $\mathbf{Fam}(-)^{op}$ pseudoalgebra structures plus a coherence condition; namely, \mathbf{a} is a pseudomorphism of product complete categories between $\mathbf{Fam}(\mathcal{C})$ and \mathcal{C} . That is to say, a $\mathbf{Dist}(-)$ -pseudoalgebra is a category with coproducts and products, such that the coproduct functor (3.2) preserves products. By the construction of products in $\mathbf{Fam}(\mathcal{C})$, we conclude that this condition is indeed equivalent to doubly-infinitary distributivity. \square

In particular, $\mathbf{Dist}(\mathcal{C})$, is the free doubly-infinitary distributive category on \mathcal{C} . It has the universal property that any functor $\mathcal{C} \rightarrow \mathcal{D}$ to a doubly-infinitary distributive category has an essentially unique extension to a product and coproduct preserving functor $\mathbf{Dist}(\mathcal{C}) \rightarrow \mathcal{D}$.

Remark 3. In fact, an easy way to verify the existence of the distributive law λ is to observe that we have two pseudomonadic biadjunctions

$$\begin{array}{ccccc}
 & & \text{Fam}(-) & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \mathbf{Cat} & & \mathbf{ProdCat} & & \mathbf{DistCat} \\
 & \curvearrowleft & & \curvearrowright & \\
 & \perp(\epsilon, \eta) & & \perp(\epsilon', \eta') &
 \end{array}$$

that gives a lifting (as defined in [34]) of the free completion under coproducts pseudomonad to **ProdCat**.

4 Examples

In this section, we present some examples of doubly-infinitary distributive categories, and some counterexamples, aiming for a comparison with some of the related and well-established notions. We start by recalling that every doubly-infinitary distributive category is infinitary distributive. Moreover, every completely distributive (and, hence, every totally distributive) category is doubly-infinitary distributive.

4.1 The Fam example

We recall that $\mathbf{Fam}(\mathcal{C})$ is *extensive* for any category \mathcal{C} . However, $\mathbf{Fam}(\mathcal{C})$ can lack products and, hence, it might not be doubly-infinitary distributive. We, however, have:

Example 1. $\mathbf{Fam}(\mathcal{C})$ is always doubly-infinitary distributive provided that the category \mathcal{C} has products. Indeed, $\mathbf{Fam}(-)$ lifts to a pseudomonad on **ProdCat**, whose pseudoalgebras are the doubly-infinitary distributive categories. $\mathbf{Fam}(\mathcal{C})$ is precisely the free $\mathbf{Fam}(-)$ -pseudoalgebra over the product-complete category \mathcal{C} .

4.2 The category of sets

As expected, the category **Set** is doubly-infinitary distributive. More precisely:

Example 2. The category $\mathbf{Set} = \mathbf{Fam}(\mathbb{1}) = \mathbf{Dist}(\mathbb{0})$ of sets and functions is doubly-infinitary distributive. As the free $\mathbf{Dist}(-)$ -algebra on $\mathbb{0}$, it is the initial object of the category of $\mathbf{Dist}(-)$ -pseudoalgebras. To be explicit, the inverse to the canonical morphism is given by

$$\begin{aligned}
 d : \left(\prod_{j \in J} \coprod_{i \in I_j} C_{ij} \right) &\rightarrow \left(\coprod_{f: \coprod_{j \in J} I_j} \prod_{j \in J} C_{f(j)j} \right) \\
 \langle \iota_{ij}(c_j) \mid j \in J \rangle &\mapsto \iota_{\lambda_j.i_j}(\langle c_j \mid j \in J \rangle).
 \end{aligned}$$

Example 3. Similarly, any category of presheaves $[\mathcal{D}^{op}, \mathbf{Set}]$ is doubly-infinitary distributive, seeing that both products and coproducts are computed pointwise (i.e., the evaluation functors create products and coproducts).

Example 4. The category $\mathbf{Fam}(\mathbf{Set}^{op}) = \mathbf{Dist}(\mathbb{1})$ is a free doubly-infinitary distributive category. This category is commonly studied under the name of the category of polynomials or containers [1]. In particular, Theorem 2.3 implies that the category of polynomials is cartesian closed, as was previously noted in [5].

4.3 ProdCat is bicategorically semi-additive

In order to proceed with examples, we remark that the 2-category **ProdCat** has a natural enrichment over the 2-category of symmetric monoidal categories, and has products. This shows that **ProdCat** is bicategorically semi-additive (see, for instance, [25] for the notion of semi-additive category).

Lemma 4.1. The 2-category **ProdCat** has a natural enrichment over the 2-category of symmetric monoidal categories, and has products.

Proof. The product is given by the usual product of categories, while the monoidal structure on the hom-categories is given by the pointwise product of functors. \square

As a consequence, **ProdCat** has bicategorical biproducts, meaning that bicategorical products and coproducts coincide up to equivalence (see, for instance, [41, 42] and [29, 3.8] for bilimits).

With this observation, we can proceed and construct new examples of doubly-infinitary distributive categories. In particular, we have:

Example 5. The free doubly-infinitary distributive category on two objects has the following description:

$$\begin{aligned} \mathbf{Dist}(\mathbb{1} \sqcup_{\mathbf{Cat}} \mathbb{1}) &\simeq \mathbf{Fam}(\mathbf{Fam}((\mathbb{1} \sqcup_{\mathbf{Cat}} \mathbb{1})^{op})^{op}) \simeq \mathbf{Fam}(\mathbf{Fam}(\mathbb{1}^{op})^{op} \sqcup_{\mathbf{ProdCat}} \mathbf{Fam}(\mathbb{1}^{op})^{op}) \\ &\simeq \mathbf{Fam}(\mathbf{Fam}(\mathbb{1}^{op})^{op} \times \mathbf{Fam}(\mathbb{1}^{op})^{op}) \simeq \mathbf{Fam}(\mathbf{Set}^{op} \times \mathbf{Set}^{op}), \end{aligned}$$

where we use the facts that (1) $op \circ \mathbf{Fam}(-) \circ op : \mathbf{Cat} \rightarrow \mathbf{ProdCat}$ is a left biadjoint so it preserves bidimensional coproducts and (2) the bidimensional coproducts and products coincide in **ProdCat**.

4.4 Posets

There is an extensive literature on completely distributive lattice, *e.g.* [16]. We observe that a poset \mathcal{C} is a doubly-infinitary distributive category if and only if it is a completely distributive lattice.

If a poset is extensive, it is, in particular, a distributive lattice and $a < b$ implies that $a = a \wedge b = \perp$, due to disjointness of coproducts in an extensive category. Hence only the trivial poset is extensive.

This shows that doubly-infinitary distributive categories may fail to be extensive.

4.5 Categories of categorical structures

Consider the category **Cat** of small categories and functors. Observe that $\mathbf{Cat} \simeq \mathbf{Fam}(\mathbf{ConCat})$ for the full subcategory **ConCat** of *connected* categories (i.e., categories \mathcal{C} such that the free groupoid on \mathcal{C} is connected; see, for instance, [27, 1.2]). Seeing that **ConCat** has products, **Cat** is doubly-infinitary distributive, by Example 1. The same argument works to show that the category **Pos** of posets and $\omega\mathbf{CPO}$ of ω -chain cocomplete partial orders and ω -cocontinuous functors are doubly infinitary distributive.

4.6 Category of topological spaces

It is well-known that the category **Top** of topological spaces and continuous functions is infinitary distributive in the usual sense: finite products distributive over (potentially infinite) coproducts. However, infinite products in **Top** are not quite as well-behaved, making the doubly-infinitary distributive law fail, in general.

Indeed, to study the continuity of the distributor d , observe the following. Let U be an open subset of $\bigsqcup_{f:\Pi_j \in J I_j} \prod_{j \in J} C_{f(j)j}$. Then, by definition of the coproduct topology, $U = \bigsqcup_{f:\Pi_j \in J I_j} U_f$ for open subsets $U_f \subseteq \prod_{j \in J} C_{f(j)j}$. By definition of the product topology, $U_f = \bigcup_{\alpha \in A_f} \bigcap_{\beta \in B_{f\alpha}} \pi_{j_{f\alpha\beta}}^{-1}(V_{f\alpha\beta})$, where $V_{f\alpha\beta}$ is an open subset of $C_{f(j_{f\alpha\beta})j_{f\alpha\beta}}$, A_f is some potentially infinite set and $B_{f\alpha}$ is a finite set. Now,

$$d^{-1}(U) = d^{-1}\left(\bigsqcup_{f:\Pi_j \in J I_j} U_f\right)$$

$$\begin{aligned}
&= d^{-1}\left(\bigsqcup_{f:\prod_{j\in J}I_j} \bigcup_{\alpha\in A_f} \bigcap_{\beta\in B_{f\alpha}} \pi_{j_{f\alpha\beta}}^{-1}(V_{f\alpha\beta})\right) \\
&= [\langle \iota_{f(j)} \circ \pi_j \mid j \in J \rangle \mid f \in \prod_{j\in J} I_j] \left(\bigsqcup_{f:\prod_{j\in J}I_j} \bigcup_{\alpha\in A_f} \bigcap_{\beta\in B_{f\alpha}} \pi_{j_{f\alpha\beta}}^{-1}(V_{f\alpha\beta}) \right) \\
&= \bigcup_{f:\prod_{j\in J}I_j} \langle \iota_{f(j)} \circ \pi_j \mid j \in J \rangle \left(\bigcup_{\alpha\in A_f} \bigcap_{\beta\in B_{f\alpha}} \pi_{j_{f\alpha\beta}}^{-1}(V_{f\alpha\beta}) \right) \\
&= \bigcup_{f:\prod_{j\in J}I_j} \bigcup_{\alpha\in A_f} \langle \iota_{f(j)} \circ \pi_j \mid j \in J \rangle \left(\bigcap_{\beta\in B_{f\alpha}} \pi_{j_{f\alpha\beta}}^{-1}(V_{f\alpha\beta}) \right) \\
&= \bigcup_{f:\prod_{j\in J}I_j} \bigcup_{\alpha\in A_f} \bigcap_{\beta\in B_{f\alpha}} \langle \iota_{f(j)} \circ \pi_j \mid j \in J \rangle (\pi_{j_{f\alpha\beta}}^{-1}(V_{f\alpha\beta})) \quad (\text{direct images under injections preserve } \cap) \\
&= \bigcup_{f:\prod_{j\in J}I_j} \bigcup_{\alpha\in A_f} \bigcap_{\beta\in B_{f\alpha}} \langle \iota_{f(j)} \circ \pi_j \mid j \in J \rangle \left(\prod_{j\in J} \begin{cases} V_{f\alpha\beta} & \text{if } j = j_{f\alpha\beta} \\ C_{f(j_{f\alpha\beta})j_{f\alpha\beta}} & \text{otherwise} \end{cases} \right) \\
&= \bigcup_{f:\prod_{j\in J}I_j} \bigcup_{\alpha\in A_f} \bigcap_{\beta\in B_{f\alpha}} \prod_{j\in J} \begin{cases} \iota_{f(j_{f\alpha\beta})}(V_{f\alpha\beta}) & \text{if } j = j_{f\alpha\beta} \\ \iota_{f(j)}(C_{f(j)j}) & \text{otherwise} \end{cases}
\end{aligned}$$

Meanwhile, the open sets of $\prod_{j\in J} \bigsqcup_{i\in I_j} C_{ij}$, by definition of the product and coproduct topologies, are the sets of the form

$$\bigcup_{k\in K} \bigcap_{l\in L} \pi_{j_{kl}}^{-1} \iota_{i_{kl}}(W_{kl}) = \bigcup_{k\in K} \bigcap_{l\in L} \prod_{j\in J} \begin{cases} \iota_{i_{kl}}(W_{kl}) & \text{if } j = j_{kl} \\ \bigsqcup_{i\in I_j} C_{ij} & \text{otherwise} \end{cases}$$

for some set K , finite set L and open subsets W_{kl} of $C_{i_{kl}j_{kl}}$. This shows that the map d above is continuous if J is finite. However, if J is infinite, taking $I_j = \{0, 1\}$, $C_{ij} = \mathbb{R}$, $A_{\lambda_{\cdot 0}} = \{*\}$ and $A_f = \emptyset$ for all other f , $B_{f\alpha} = \{*\}$, and $V_{f**} = \mathbb{R}$ gives a counter-example to continuity. Indeed, then $U = \iota_{\lambda_{\cdot 0}}(\prod_{j\in J} \mathbb{R})$ is an open subset of $\bigsqcup_{f\in J \rightarrow \{0,1\}} \prod_{j\in J} \mathbb{R}$ and $d^{-1}(U) = \prod_{j\in J} \iota_0(\mathbb{R}) \subseteq \prod_{j\in J} \bigsqcup_{i\in\{0,1\}} \mathbb{R}$ is not open.

We see that **Top** is infinitary distributive (actually lextensive), but not doubly-infinitary distributive.

Remark 4 (Cantor space). *The proof that the category of topological spaces is not doubly-infinitary distributive can be abridged by the fact that the Cantor space \mathbb{K} is an infinite compact topological space, and, hence, not discrete.*

Specifically, \mathbb{K} can be constructed as the infinite product $\prod_{j\in\mathbb{N}} \bigsqcup_{i\in\{0,1\}} 1$ in the category **Top** of topological spaces, where 1 represents the space with only one point. Since \mathbb{K} is compact by Tychonoff's theorem, it cannot be homeomorphic to an infinite coproduct of non-trivial spaces. More explicitly, the canonical morphism 3.1, in this case, is the morphism $\underline{\mathbb{K}} \rightarrow \mathbb{K}$ between the underlying discrete topological space $\underline{\mathbb{K}}$ and \mathbb{K} , which is evidently not invertible.

Remark 5 (Pointfree topology). Recall that a *frame* is poset (partially ordered set) that is infinitary distributive when seen as a category. The study of topological features in the opposite category of the category of frames is usually called *pointfree topology*, e.g. [38]. In this setting, the objects are called locales.

Analogously, we can conclude that, despite being (l)extensive, the category of locales $\mathbb{L}oc = \mathbb{F}rm^{op}$ is not doubly-infinitary distributive (see [38, Chapter 13] for connectedness in pointfree topology).

4.7 Cartesian closedness vs. doubly-infinitary distributivity

Herein, we establish the comparison between *doubly-infinitary distributivity* and *cartesian closedness*. We start by observing the following result.

Theorem 4.2. Let \mathcal{C} be infinitary distributive category. The category \mathcal{C} is cartesian closed if and only if **Fam**(\mathcal{C}) is cartesian closed.

Proof. It is well-known that, if \mathcal{C} is cartesian closed, so is $\mathbf{Fam}(\mathcal{C})$ (we refer the reader to [3, 2] for instance). Reciprocally, if $\mathbf{Fam}(\mathcal{C})$ is cartesian closed, we know that the canonical inclusion

$$\mathcal{C} \rightarrow \mathbf{Fam}(\mathcal{C})$$

is fully faithful, and has the left adjoint given by the coproduct functor \sqcup (see (3.2)). By the definition of infinitary distributive categories, we also know that \sqcup preserves finite products. Then, by [22, Proposition 4.3.1], \mathcal{C} forms an exponential ideal in $\mathbf{Fam}(\mathcal{C})$. In particular, \mathcal{C} is cartesian closed. \square

This shows that, whenever \mathcal{C} is an example of (doubly-)infinitary distributive category that is not cartesian closed, $\mathbf{Fam}(\mathcal{C})$ is another such an example.

Doubly-infinitary distributive categories are not necessarily cartesian closed We start by considering concrete examples of doubly-infinitary distributive category that are not cartesian closed.

Example 6. Recall that the usual category $\omega\mathbf{CPO}$ of ω -cpo is not locally cartesian closed (see, for example, [4, Proposition 4]). Therefore, there is some ω -cpo X (concretely, we can take X to be the extended natural numbers \mathbb{N}_∞ with the linear order \leq), such that the slice category $\omega\mathbf{CPO}/X$ is not cartesian closed. For example, the internal hom $((\mathbb{N}, =) \hookrightarrow (\mathbb{N}_\infty, \leq)) \Rightarrow (-)$ does not exist, for the inclusion of the natural numbers with the discrete order into the extended natural numbers with the linear order. Meanwhile, products in $\omega\mathbf{CPO}/X$ are simply given by wide pullbacks in $\omega\mathbf{CPO}$, i.e. the wide pullback in \mathbf{Set} over $|X|$ equipped with the product order. Coproducts in $\omega\mathbf{CPO}/X$ are simply given by coproducts in $\omega\mathbf{CPO}$, i.e. disjoint unions of sets and orders, with the required morphisms to X induced by the cotupling. Therefore, the isomorphism

$$\prod_{j \in J} \bigsqcup_{i \in I_j} Y_{ij} \cong \bigsqcup_{f: \Pi_{j \in J} I_j} \prod_{j \in J} Y_{f(j)j}$$

restricts, if we write \prod^X and \sqcup^X for the product and coproduct in $\omega\mathbf{CPO}/X$, to one

$$\prod_{j \in J}^X \bigsqcup_{i \in I_j}^X (y_{ij} : Y_{ij} \rightarrow X) \cong \bigsqcup_{f: \Pi_{j \in J} I_j}^X \prod_{j \in J}^X (y_{f(j)j} : Y_{f(j)j} \rightarrow X),$$

showing that $\omega\mathbf{CPO}/X$ is doubly-infinitary distributive.

Example 7. For another example of a doubly-infinitary distributive category that is not cartesian closed, observe that $\mathbf{Fam}(\mathbf{Top})$ is doubly-infinitary distributive, as a free coproduct completion of a product-complete category. At the same time, \mathbf{Top} is well-known not to be cartesian closed, as the product $\mathbb{S} \times -$ with the Sierpinski space \mathbb{S} does not preserve the coequalizer of some parallel pair $f, g : X \rightarrow Y$ [7, Proposition 7.1.2]. Since \mathbf{Top} is infinitary distributive, we conclude that $\mathbf{Fam}(\mathbf{Top})$ is not cartesian closed by Theorem 4.2

Example 8. Consider the category \mathbf{ConTop} of locally connected topological spaces, and continuous functions. Observe $\mathbf{Fam}(\mathbf{ConTop})$ is, up to equivalence, the category $\mathbf{Fam}(\mathbf{ConTop}) \simeq \mathbf{LocConTop}$ of locally connected topological spaces and continuous functions (see, for instance, [8, Proposition 6.15]). \mathbf{ConTop} has products, given by the usual product topology. Therefore, $\mathbf{LocConTop}$ is doubly-infinitary distributive, as an instance of Example 1. *Again, $\mathbf{LocConTop}$ is not cartesian closed, by the same argument as for \mathbf{Top} .*

We can generalize Example 8 and the examples of 4.5 by making use of the notion of connected objects (see, for instance, [8, Definition 6.1.3]). *An object C in a category \mathcal{C} is connected if the hom-functor $\mathcal{C}(C, -)$ preserves coproducts.* We, then, define the full subcategory $\mathbf{Con}\mathcal{C}$ of the connected objects in \mathcal{C} . We get that:

Lemma 4.3. Let \mathcal{C} be a category with coproducts such that $\mathbf{Con}\mathcal{C}$ has products. If every object of \mathcal{C} is a coproduct of connected objects, then \mathcal{C} is doubly-infinitary distributive.

Proof. By [8, Proposition 6.15], we conclude that \mathcal{C} is equivalent to $\mathbf{Fam}(\mathbf{Con}\mathcal{C})$. By Example 1, we conclude that \mathcal{C} is doubly-infinitary distributive, since $\mathbf{Con}\mathcal{C}$ has products. \square

Cartesian closed categories are not necessarily doubly-infinitary distributive Since the functor $A \times -$ is left adjoint in cartesian closed categories, it is well-known that cartesian closed categories are infinitary distributive.

While the reasoning above cannot be extended to doubly-infinitary distributive categories, one might be tempted to conjecture that cartesian closed categories are doubly-infinitary distributive. However, we demonstrate the falsity of this conjecture through the presentation of the following examples.

Counter example 1. The category **FinSet** of finite sets and functions. It does not have infinite products and coproducts, so is not doubly-infinitary distributive.

Counter example 2. More interestingly, there exist categories that have all products and coproducts, are infinitary distributive and cartesian closed, but still fail to be doubly-infinitary distributive. Consider the category **Qbs** of quasi-Borel spaces [20]. **Qbs** is the category of concrete sheaves over standard Borel spaces with countable measurable covers, hence, as a Grothendieck quasi-topos, it is complete, cocomplete and cartesian closed. It is of interest as a generalized setting for probability theory as it is a concrete category that has a full, limit and countable coproduct preserving embedding of standard Borel spaces into it.

Concretely, a quasi-Borel space X consists of a set $|X|$ and a set M_X of maps $\mathbb{R} \rightarrow |X|$ such that (1) M_X contains all constant functions; (2) M_X is closed under precomposition with measurable functions $\mathbb{R} \rightarrow \mathbb{R}$; (3) M_X is closed under gluing along countable measurable partitions of \mathbb{R} . Morphisms $X \rightarrow Y$ are functions $f : |X| \rightarrow |Y|$ such that $f \circ g \in M_Y$ if $g \in M_X$.

We have products $\prod_{j \in J} X_j$ given by $|\prod_{j \in J} X_j| = \prod_{j \in J} |X_j|$ and $M_{\prod_{j \in J} X_j} = \{ \langle f_j \mid j \in J \rangle \mid f_j \in M_{X_j} \}$ and coproducts $\bigsqcup_{j \in J} X_j$ given by $|\bigsqcup_{j \in J} X_j| = \bigsqcup_{j \in J} |X_j|$ and $M_{\bigsqcup_{j \in J} X_j} = \{ \lambda r. \iota_{j(n(r))}(f_{n(r)}(r)) \mid n : \mathbb{R} \rightarrow \mathbb{N} \text{ measurable}, j : \mathbb{N} \rightarrow J, \forall m \in \mathbb{N}. f_m \in M_{X_{j(m)}} \}$. Observe that elements of $M_{\bigsqcup_{j \in J} X_j}$ always factor over countably many components of the coproduct. Countable coproducts are well-behaved in the sense that $M_{\bigsqcup_{j \in J} X_j} = \{ \lambda r. \iota_{j(r)}(f_{j(r)}(r)) \mid j : \mathbb{R} \rightarrow J \text{ measurable}, f_j \in M_{X_j} \}$ if J is countable.

The question is whether $d : \left(\prod_{j \in J} \bigsqcup_{i \in I_j} C_{ij} \right) \rightarrow \left(\bigsqcup_{f : \prod_{j \in J} I_j} \prod_{j \in J} C_{f(j)j} \right)$ is a morphism of **Qbs**. We need to show that $d \circ g \in M_{\bigsqcup_{f : \prod_{j \in J} I_j} \prod_{j \in J} C_{f(j)j}}$ for all $g \in M_{\prod_{j \in J} \bigsqcup_{i \in I_j} C_{ij}}$. This is clearly true iff $\{j \in J \mid \#I_j < 2\}$ is finite. In particular, **Qbs** is infinitary distributive, but not doubly-infinitary distributive.

5 Final remarks and future work

We discussed the concept of *doubly-infinitary distributive categories*, which is a notion that naturally arises from a canonical pseudodistributive law between the free product completion pseudomonad and the free coproduct completion pseudomonad. The most surprising result is that free doubly-infinitary distributive category **Dist**(\mathcal{C}) on a category \mathcal{C} is cartesian closed.

We could also consider free completions under *finite* coproducts and products. This would amount to replacing **Fam**($-$) with **FinFam**($-$), where $\mathbf{FinFam}(\mathcal{C}) \stackrel{\text{def}}{=} \Sigma_{\mathbf{FinSet}} \mathfrak{Fam}(\mathcal{C})$. The distributive law

$$\mathbf{FinFam}(\mathbf{FinFam}(\mathcal{C})^{op})^{op} \rightarrow \mathbf{FinFam}(\mathbf{FinFam}(\mathcal{C}^{op})^{op})$$

then gives rise to the theory of distributive categories in the usual sense (i.e., categories with finite products and coproducts that distribute), while the distributive law

$$\mathbf{FinFam}(\mathbf{Fam}(\mathcal{C})^{op})^{op} \rightarrow \mathbf{Fam}(\mathbf{FinFam}(\mathcal{C}^{op})^{op})$$

gives rise to the theory of infinitary distributive categories in the usual sense (i.e., categories with finite products and infinite coproducts that distribute). As a small variation on our results, we get that $\mathbf{FinDist}(\mathcal{C}) = \mathbf{FinFam}((\mathbf{FinFam}(\mathcal{C}^{op}))^{op})$ is cartesian closed for any *locally finite* category \mathcal{C} .

5.1 Further properties of $\mathbf{Dist}(\mathcal{C})$

We remark that $\mathbf{Dist}(\mathcal{C})$ has many other interesting properties, including most notably *extensivity*. Many of these properties are already in the literature, since they follow from the fact that $\mathbf{Dist}(\mathcal{C}) = \mathbf{Fam}(\mathcal{D})$ is the free coproduct completion of $\mathcal{D} = \mathbf{Fam}(\mathcal{C}^{\text{op}})^{\text{op}}$. We refer to [10, 8, 3, 31] for some of these properties.

5.2 Non-canonical isomorphisms

In Definition 1, doubly-infinitary distributive categories are defined in terms of a canonical morphism being invertible. It is natural to ask, then, if finding a non-canonical isomorphism would suffice (this is Pisani's question for distributive categories, *e.g.* [25]).

The answer is yes. As this is easily framed in the setting of [30], it is easy to see that a category \mathcal{C} is doubly-infinitary distributive if it has products and coproducts and there is a(ny) natural isomorphism

$$\left(\bigsqcup_{f: \Pi_{j \in J} I_j} \prod_{j \in J} C_{f(j)j} \right) \rightarrow \left(\prod_{j \in J} \bigsqcup_{i \in I_j} C_{ij} \right).$$

for each family of objects $(C_{ij})_{(j,i) \in J \times I_j}$.

Finally, since a coproduct and product complete category is doubly-infinitary distributive if and only if the coproduct functor $\bigsqcup: \mathbf{Fam}(\mathcal{C}) \rightarrow \mathcal{C}$ is product-preserving, we can achieve the non-canonical result stated above by results on preservation of limits from naturality (non-canonical isomorphisms), see, for instance, [30, 9].

5.3 Generalized categorical structures

It was recently established that the category of (T, \mathcal{V}) -categories is extensive, see [11]. This shows that a plethora of categories of topological categories and, more generally, (enriched) categorical structures share the property of being extensive.

In exploring the realm of doubly-infinitary distributivity, we unveiled that both the category of locally connected spaces and the category of (small) categories possess this property. However, notably, the category of topological spaces does not share this trait. This observation leads us to pose the following open question:

Open question 1. It remains an open question: under which conditions on T and \mathcal{V} does the category of (T, \mathcal{V}) -categories exhibit doubly-infinitary distributivity?

Remark 6. By 4.3, the obvious road map to solve the problem above would start with a characterization of the categories $(T, \mathcal{V})\text{-Cat}$ of (T, \mathcal{V}) -categories that are free doubly-infinitary distributive on its category of connected objects. Enlightened by [32, Corollary 4.7], this happens to be the case under suitable conditions, for \mathcal{V} thin and intersection-preserving T .

5.4 Gödel's Dialectica interpretation

The exponential formula for $\mathbf{Dist}(\mathcal{C})$ presented in this paper is a variation on Gödel's Dialectica interpretation (see [17] for the original reference and [21] for a categorical formulation), exponentials in categories of polynomials/containers [5], and the Diller-Nahm formulas (see [15] for the original reference and [21] and [31, Section 6.4] for categorical formulations). We plan to explain the precise relationship to these results in a separate paper, in order not to distract from the simple nature of the results presented herein.

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