Interpolation on the space of orthonormal frames via recursive endpoint quasi-geodesics

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Abstract

The de Casteljau algorithm on Riemannian manifolds is adjusted in order to solve two-point boundary value problems that give rise to generalized cubic polynomials on Stiefel manifolds given in closed form. This adjusted approach is based on the recursive use of quasi-geodesics, which are special curves with constant geodesic curvature. Two types of interpolation problems are formulated, related, and solved explicitly.

Keywords: Interpolation, Stiefel manifolds, quasi-geodesics, adjusted de Casteljau algorithm, adjusted geometric cubic polynomials.

1 Introduction

Geodesics are the simplest curves on a Riemannian manifold, generalizing straight lines in Euclidean spaces, and being the shortest paths between two points. However, there are cases where it might be difficult to find geodesics precisely, specially those that solve two point boundary problems.

A manifold that illustrates these difficulties is the Stiefel manifold consisting of all orthonormal k-frames in n-dimensional Euclidean space, where $k \leq n$.

As far as we know, no explicit solutions for a geodesic that joins two points on the Stiefel manifold is known, except for some particular cases. This gap is a drawback when one wants to solve smooth interpolation problems on this manifold using a geometric method that generalizes the de Casteljau algorithm for Euclidean spaces, see [5] and [8] for the classical description and [13], [4] and [14] for its generalization to curved spaces. This method, based on recursive geodesic interpolation, is computationally very efficient in cases when endpoint geodesic formulas are known explicitly.

Surprisingly enough, there are certain curves that may not be true geodesics but can more easily solve a two point boundary problem, making them particularly useful in various contexts where exact geodesics might be hard to define or compute. Such curves have been named quasi-geodesics and used successfully in [12] to generate quadratic splines on the Stiefel manifold. In [12], the Riemannian metric used was the so called canonical metric. The nice properties of those curves explained the success of the results, but also raised the following natural question: are quasi-geodesics with respect to the canonical metric true geodesics with respect to another metric on Stiefel manifolds? Partial answers to this question were given in [11] and [9]. In [11] quasi-geodesics are proved to coincide with projections of sub-Riemannian geodesics on a certain Lie group that acts transitively on the Stiefel manifold. The Lie group was considered equipped with the trace metric. In general such curves are not geodesics with respect to the submersion metric induced by the action. In [9] a one-parameter family of metrics (the α metrics) on Stiefel manifolds was studied in detail and the quasi-geodesics also appeared as solutions of a variational problem associated to the limit case when $\alpha \to 0$.

Stiefel manifolds are important in theoretical and applied mathematics, as well as in interdisciplinary fields such as pattern recognition and quantum information theory. Solving interpolation problems on that manifold is crucial in all areas of application since it allows to construct new data from a discrete set of known data points.

The organization of this paper is the following. After the Introduction we review in Section 2 the main concepts related to Stiefel manifolds, in particular we present the quasi-geodesics that will be used in the main part of the paper. The generalized de Casteljau algorithm on manifolds, as well as the adjusted version of this algorithm, are described in Section 3. The formulation of the interpolation problems appear in Subsection 3.4, where also the main results, concentrated in Theorem 7 and Theorem 14, are stated and proved using several auxiliary lemmas. A list of references is also included.

2 The Stiefel manifold

For the sake of completeness, we start this section with basic definitions about Stiefel manifolds and then recall important properties of quasi-geodesics that have been introduced in [12] and will play a major role in the section dealing with interpolation.

2.1 Background & notations

The Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$ has the following matrix representation:

$$\mathbf{St}_{n,k} = \{S \in \mathbb{R}^{n\times k} \mid S^\top S = I_k\}. \tag{1}$$

There is a strong relationship between the Stiefel manifold $\mathbf{St}_{n,k}$ and the Grassmann manifold $\mathbf{Gr}_{n,k}$ consisting of all k-dimensional subspaces of $\mathbb{R}^n$. If one considers the matrix representation in terms of projection matrices, that is, $\mathbf{Gr}_{n,k} = \{P \in \mathbb{R}^{n\times n} \mid P = P^\top,\ P^2 = P,\ rank(P) = k\}$, if $S \in \mathbf{St}_{n,k}$, then $P = SS^\top \in \mathbf{Gr}_{n,k}$. This relationship will be explored later on.

In what follows, $\mathfrak{s}(n)$ denotes the set of $n \times n$ symmetric matrices, $\mathfrak{so}(n)$ denotes the set of $n \times n$ skew-symmetric matrices, and for $P \in \mathbf{Gr}_{n,k}$, $\mathfrak{so}_P(n)$ is used to denote the vector subspace of $\mathfrak{so}(n)$ defined by

$$\mathfrak{so}_P(n) = \{X \in \mathfrak{so}(n) \mid XP + PX = X\}.$$

This vector space is related to the tangent space of $\mathbf{Gr}_{n,k}$ at the point P. Indeed,

$$T_P\mathbf{Gr}_{n,k} = \{[X, P] \mid X \in \mathfrak{so}_P(n)\} \subset \mathfrak{s}(n),$$

where $[.,.]$ denotes the commutator of matrices.

The tangent space to the Stiefel manifold at a point $S \in \mathbf{St}_{n,k}$ is given by

$$T_S\mathbf{St}_{n,k} = \{V \in \mathbb{R}^{n\times k} \mid V^\top S + S^\top V = 0\}, \tag{2}$$

but another useful representation of the tangent space is the following, which already appeared in [12, Proposition 5].

Proposition 1. *Let $S \in \mathbf{St}_{n,k}$ and $P := SS^\top \in \mathbf{Gr}_{n,k}$. Then,*

$$T_S\mathbf{St}_{n,k} = \{XS + S\Omega \mid X \in \mathfrak{so}_P(n),\ \ \Omega \in \mathfrak{so}(k)\}. \tag{3}$$

Moreover, if $V = XS + S\Omega \in T_S\mathbf{St}_{n,k}$, then

$$X = VS^\top - SV^\top + 2SV^\top SS^\top, \quad \Omega = S^\top V. \tag{4}$$

We consider the Stiefel manifold equipped with the canonical metric defined in [6] by

$$\langle V_1, V_2\rangle = \mathrm{tr}\big(V_1^\top (I_n - \tfrac{1}{2}SS^\top)V_2\big), \quad V_1, V_2 \in T_S\mathbf{St}_{n,k}. \tag{5}$$

Also in [6] one can find the following second order differential equation, which is the geodesic equation with respect to the above metric.

$$\ddot\gamma + \dot\gamma\dot\gamma^\top\gamma + \gamma\big((\gamma^\top\dot\gamma)^2 + \dot\gamma^\top\dot\gamma\big) = 0. \tag{6}$$

The geometric de Casteljau algorithm to generate polynomials on Riemannian manifolds is based on successive geodesic interpolation and requires that explicit formulas for the geodesic that joins two points is available. However, as far as we know, such formulas do not exist. As an attempt to overcome this problem, the authors of [12] used quasi-geodesics, instead of geodesics, to successfully modify the de Casteljau algorithm to generate quadratic polynomials and splines on Stiefel, equipped with the metric (5). The success of that alternative approach resulted from the fact that it was possible to define quasi-geodesics that join two given points, in terms of those points only. These curves are associated to a particular retraction. Retractions on a Riemannian manifold are generalizations of the exponencial map.

2.2 Retractions and quasi-geodesics on Stiefel manifolds

Definition 2. *A retraction R on the Stiefel manifold $\mathbf{St}_{n,k}$ is a smooth mapping from the tangent bundle $T\mathbf{St}_{n,k}$ to $\mathbf{St}_{n,k}$ that, when restricted to each tangent space at a point $S \in \mathbf{St}_{n,k}$ (restriction denoted by R_S), satisfies the following properties:*

(i) $R_S(0) = S$;

(ii) $dR_S(0) = \mathrm{id}$,

where $dR_S(0)$ stands for the tangent map of R_S at $0 \in T_S\mathbf{St}_{n,k}$.

If $V \in T_S\mathbf{St}_{n,k}$, one can define a smooth curve $\beta_V : t \mapsto R_S(tV)$ associated to the retraction R. The curve β_V which satisfies $\beta_V(0) = S$ and $\dot{\beta}_V(0) = V$ is called a *quasi-geodesic*. Next, we define a particular retraction and corresponding quasi-geodesics on the Stiefel manifold, and list some of their interesting properties. Proofs and more details can be found in [12]. In what follows, e^A denotes the exponential of a matrix A and *log* is used for the principal matrix logarithm.

Proposition 3. *Let S, X and Ω be as in the Proposition 1. Then, the mapping $R\colon T\mathbf{St}_{n,k} \to \mathbf{St}_{n,k}$ whose restriction to $T_S\mathbf{St}_{n,k}$ is defined by $R_S(V) = e^X S e^\Omega$ is a retraction on the Stiefel manifold. Moreover, the curve $\beta : [0,1] \to \mathbf{St}_{n,k}$, $t \mapsto e^{tX} S e^{t\Omega}$ is a quasi-geodesic in $\mathbf{St}_{n,k}$ that satisfies*

1. $\beta(0) = S$;
2. $\dot{\beta}(t) = e^{tX}(XS + S\Omega)e^{t\Omega}$;
3. $\ddot{\beta}(t) = e^{tX}(X^2S + 2XS\Omega + S\Omega^2)e^{t\Omega}$.

In the next proposition the initial velocity of a quasi-geodesic is explicitly written in terms of the given endpoints S_0 and S_1. We use the notation $D_t\dot{\beta}$ for the covariant acceleration along the curve β and κ for the geodesic curvature.

Proposition 4. *Let S_0 and S_1 be two distinct points in $\mathbf{St}_{n,k}$ so that, for $i = 0, 1$, $P_i = S_i S_i^\top \in \mathbf{Gr}_{n,k}$. Then, if*

$$X = \frac{1}{2}\log\big((I - 2S_1S_1^\top)(I - 2S_0S_0^\top)\big) \quad \textit{and} \quad \Omega = \log\big(S_0^\top e^{-X} S_1\big), \tag{7}$$

the quasi-geodesic $\beta : [0,1] \mapsto \mathbf{St}_{n,k}$ *defined by*

$$\beta(t) = e^{tX} S_0 e^{t\Omega}, \tag{8}$$

has the following properties:

1. $\beta(0) = S_0$;
2. $\beta(1) = S_1$;
3. $\|\dot\beta(t)\|^2 = -\mathrm{tr}(S_0^\top X^2 S_0 + \frac{1}{2}\Omega^2)$ *(constant speed);*
4. $D_t \dot\beta(t) = X\beta(t)\Omega$;
5. $\|D_t\dot\beta(t)\|^2 = \mathrm{tr}(\Omega^2 S_0^\top X^2 S_0)$ *(constant norm of covariant acceleration);*
6. $\kappa = -\dfrac{\sqrt{\mathrm{tr}(\Omega^2 S_0^\top X^2 S_0)}}{\mathrm{tr}(S_0^\top X^2 S_0 + \Omega^2/2)}$ *(constant geodesic curvature).*

Remark 5. *Note that the matrices* X *and* Ω *in* (7) *are only well defined if the logarithm exists. This is always guaranteed if the points* S_0 *and* S_1 *are sufficiently close.*

The quasi-geodesic defined above is a true geodesic w.r.t. the metric (5) *only if* $X = 0$ *or* $\Omega = 0$*. In particular, these 2 situations occur when* $k = 1$ *and when* $k = n$*. Since* $\mathbf{St}_{n,1} = S^n$ *and* $\mathbf{St}_{n,n} = \mathbf{O}_n$*, for the sphere and for the orthogonal group these quasi-geodesics are geodesics.*

3 The de Casteljau Algorithm on Riemannian Manifolds

A well-known recursive procedure to generate polynomial curves in Euclidean spaces is the classical de Casteljau algorithm which was introduced, independently, by de Casteljau [5] and Bézier [2]. The algorithm is a simple and powerful tool widely used in the field of Computer Aided Geometric Design (CAGD), and is based on successive linear interpolations, cf. [7] for a modern treatise.

A generalization of that algorithm to Riemannian manifolds appeared first in [13], and the basic idea was replacing linear interpolation by geodesic interpolation. The resulting curves are also called polynomial curves as they are natural extensions to Riemannian manifolds of Euclidean polynomials. In Euclidean spaces, the most important are the cubic polynomials, due to their optimal properties, as they minimize acceleration. Generating polynomial curves and polynomial splines on manifolds was motivated by problems related to path planning of certain mechanical systems, such as spacecraft and underwater vehicles, whose configuration spaces are non-Euclidean manifolds. The rotation group, which plays an important role in this context, inspired further developments such as the work in [4] that will be used here. But first we briefly describe the de Casteljau Algorithm to generate cubic polynomials on Riemannian manifolds, assuming that they are geodesically complete.

3.1 Generating cubic polynomials

A cubic polynomial is a smooth curve that satisfies a two-point boundary value problem (initial and final points and velocities are prescribed), but may be generated from four distinct points x_0, x_1, x_2, x_3 in M, the first and last being respectively the initial and final point of the curve and the other two are auxiliary points for the geometric algorithm, but are related to the prescribed velocities. Without loss of generality, we parameterize the curves over the interval $[0, 1]$.

The next algorithm describes all steps of this construction, which is briefly described for instance in [10] and illustrated in Figure 1.

3.2 Generalized de Casteljau Algorithm

Given four distinct points x_0, x_1, x_2 and x_3 in M:

Step 1. Construct three geodesic arcs, $\beta_1(t, x_i, x_{i+1})$, $i = 0, 1, 2$, joining x_i to x_{i+1}. In the illustration below, these arcs are represented by the black dotted lines.

Step 2. For every $t \in [0, 1]$, construct two geodesic arcs

$$\beta_2(s, x_i, x_{i+1}, x_{i+2}) = \beta_1(s, \beta_1(t, x_i, x_{i+1}), \beta_1(t, x_{i+1}, x_{i+2}))$$

for $i = 0, 1$, joining $\beta_1(t, x_i, x_{i+1})$ to $\beta_1(t, x_{i+1}, x_{i+2})$. In the illustration below, these arcs are represented by the blue dotted lines.

Step 3. For every $t \in [0, 1]$, construct the geodesic arc

$$\beta_3(s, x_0, x_1, x_2, x_3) = \beta_1(s, \beta_2(t, x_0, x_1, x_2), \beta_2(t, x_1, x_2, x_3)),$$

joining $\beta_2(t, x_0, x_1, x_2)$ to $\beta_2(t, x_1, x_2, x_3)$. In the illustration below, this arc is represented by the green dotted line. The dark red dot represents the point in $\beta_3(s, x_0, x_1, x_2, x_3)$ corresponding to $s = t$.

The curve $[0, 1] \ni t \mapsto \beta_3(t) := \beta_3(t, x_0, x_1, x_2, x_3)$ obtained with this algorithm is called *geometric cubic polynomial* in M, and in Figure 1 it is represented by the red curve. Observe that this curve joins the points x_0 (at $t = 0$) and x_3 (at $t = 1$), but does not pass through the other two points x_1 and x_2. The latter are called *control points*, since they influence the shape of the curve. We also note that the whole geometric construction lives in M.

Remark 6. *There are some relationships between the velocity boundary conditions of the final curve β_3 and velocities of the curves obtained in step 1.:*

$$\begin{aligned} \dot\beta_3(0) &= 3\dot\beta_1(t, x_0, x_1)|_{t=0} \\ \dot\beta_3(1) &= 3\dot\beta_1(t, x_2, x_3)|_{t=1}. \end{aligned} \tag{9}$$

These relationships are particularly important to generate a cubic polynomial that satisfies certain Hermite conditions (initial and final points and velocities are prescribed), since the control points x_1, x_2 can be obtained from that data.

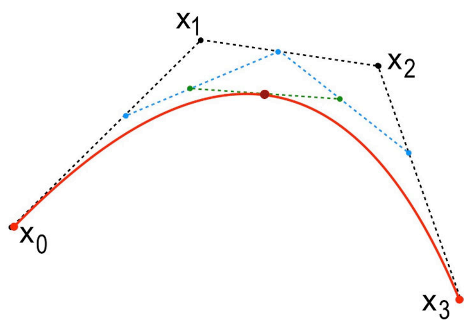

Figure 1: Illustration of the algorithm. Geometric cubic polynomial in red.

3.3 Adjusted de Casteljau Algorithm

The de Casteljau algorithm described in Subsection 3.2 can be modified in several ways. Here, for the obvious reasons, we simply replace geodesics by quasi-geodesics and name the corresponding procedure *adjusted de Casteljau algorithm*. In spite of this modification, we still call the resulting curves *geometric cubic polynomials*.

3.4 Solving a 2-point boundary problem on the Stiefel manifold, using the adjusted de Casteljau algorithm

Since no explicit formulas for the endpoint geodesics on Stiefel manifolds are known, the implementation of the de Casteljau algorithm is not possible. However, we will show next that the adjusted algorithm produces a curve that satisfies a 2-point boundary problem. We note that quasi-geodesics have already been used to generate quadratic spline curves in Stiefel manifolds in [12], however the generation of cubics requires further developments that will be done here in detail.

To be consistent with the first part of these notes, points in the Stiefel manifold will be denoted by S_i.

Problem 1. *Find a smooth curve* $\gamma : [0,1] \to \mathbf{St}_{n,k}$ *satisfying the following boundary conditions:*

$$\gamma(0) = S_0, \quad \gamma(1) = S_3, \quad \dot{\gamma}(0) = V_0, \quad \dot{\gamma}(1) = V_3, \tag{10}$$

where S_0, S_3 *are given points in* $\mathbf{St}_{n,k}$*, and* $V_0 \in T_{S_0}\mathbf{St}_{n,k}$ *and* $V_3 \in T_{S_3}\mathbf{St}_{n,k}$ *are given tangent vectors.*

The adjusted de Casteljau algorithm will be used to generate a curve that solves this problem. So, we first need to find the control points S_1, S_2 from the given data. S_1 is the end point of the quasi-geodesic that starts at the point S_0 with initial velocity equal to $\frac{1}{3}V_0$. This quasi-geodesic is given by

$$\beta_1(t, S_0, S_1) = e^{tX_0} S_0 e^{t\Omega_0}, \tag{11}$$

where, according to (4) in Proposition 1 and Remark 6,

$$X_0 = \frac{1}{3}\big(V_0 S_0^\top - S_0 V_0^\top + 2 S_0 V_0^\top S_0 S_0^\top\big), \qquad \Omega_0 = \frac{1}{3} S_0^\top V_0. \tag{12}$$

So, $S_1 = e^{X_0} S_0 e^{\Omega_0}$ defines the *first control point*.

The second control point S_2 is the end point of the quasi-geodesic that starts at the point S_3 with initial velocity equal to $-\frac{1}{3}V_3$. This quasi-geodesic is given by

$$\beta_1(t, S_3, S_2) = e^{tX_2} S_3 e^{t\Omega_2}, \tag{13}$$

where, according to (4) in Proposition 1 and Remark 6,

$$X_2 = -\frac{1}{3}\big(V_3 S_3^\top - S_3 V_3^\top + 2 S_3 V_3^\top S_3 S_3^\top\big), \qquad \Omega_2 = -\frac{1}{3} S_3^\top V_3. \tag{14}$$

So, $S_2 = e^{X_2} S_3 e^{\Omega_2}$ defines the *second control point*.

We can now proceed with the three steps of the de Casteljau algorithm.

Step 1. Construct three quasi-geodesics, defined by

$$\begin{array}{l} \beta_1(t, S_0, S_1) = e^{tX_0} S_0 e^{t\Omega_0} \\ \beta_1(t, S_1, S_2) = e^{tX_1} S_1 e^{t\Omega_1} \\ \beta_1(t, S_2, S_3) = e^{-tX_2} S_2 e^{-t\Omega_2}, \end{array} \tag{15}$$

where $X_0, \Omega_0, X_2, \Omega_2$ are given in (12) and (14), and X_1, Ω_1 are, according to Proposition 4, defined by

$$X_1 = \frac{1}{2}\log\big((I - 2S_2 S_2^\top)(I - 2S_1 S_1^\top)\big), \qquad \Omega_1 = \log\big(S_1^\top e^{-X_1} S_2\big). \tag{16}$$

Step 2. Construct two quasi-geodesic arcs

$$\beta_2(s, S_0, S_1, S_2) = e^{sX_3(t)} \beta_1(t, S_0, S_1) e^{s\Omega_3(t)} = e^{sX_3(t)} e^{tX_0} S_0 e^{t\Omega_0} e^{s\Omega_3(t)},$$

$$\beta_2(s, S_1, S_2, S_3) = e^{sX_4(t)} \beta_1(t, S_1, S_2) e^{s\Omega_4(t)} = e^{sX_4(t)} e^{tX_1} S_1 e^{t\Omega_1} e^{s\Omega_4(t)},$$

where, according to Proposition 4 and formulas (15),

$$X_3(t) = \tfrac{1}{2}\log\big((I - 2e^{tX_1} S_1 S_1^\top e^{-tX_1})(I - 2e^{tX_0} S_0 S_0^\top e^{-tX_0})\big),$$

$$\Omega_3(t) = \log\big(e^{-t\Omega_0} S_0^\top e^{-tX_0} e^{-X_3(t)} e^{tX_1} S_1 e^{t\Omega_1}\big),$$

$$X_4(t) = \tfrac{1}{2}\log\big((I - 2e^{-tX_2} S_2 S_2^\top e^{tX_2})(I - 2e^{tX_1} S_1 S_1^\top e^{-tX_1})\big),$$

$$\Omega_4(t) = \log\big(e^{-t\Omega_1} S_1^\top e^{-tX_1} e^{-X_4(t)} e^{-tX_2} S_2 e^{-t\Omega_2}\big).$$

Step 3. Construct the quasi-geodesic arc

$$\begin{aligned}\beta_3(s, S_0, S_1, S_2, S_3) &= e^{sX_5(t)}\beta_2(t, S_0, S_1, S_2)e^{s\Omega_5(t)} \\ &= e^{sX_5(t)}e^{tX_3(t)}e^{tX_0}S_0e^{t\Omega_0}e^{t\Omega_3(t)}e^{s\Omega_5(t)},\end{aligned}$$

where, according to Proposition 4 and formulas above

$$\begin{aligned}X_5(t) = \frac{1}{2}\log\big((I - 2e^{tX_4(t)}e^{tX_1}S_1S_1^\top e^{-tX_1}e^{-tX_4(t)}) \\ (I - 2e^{tX_3(t)}e^{tX_0}S_0S_0^\top e^{-tX_0}e^{-tX_3(t)})\big),\end{aligned}$$

$$\Omega_5(t) = \log\big(e^{-t\Omega_3(t)}e^{-t\Omega_0}S_0^\top e^{-tX_0}e^{-tX_3(t)}e^{-X_5(t)}e^{tX_4(t)}e^{tX_1}S_1e^{t\Omega_1}e^{t\Omega_4(t)}\big).$$

Theorem 7. *Let X_5 and Ω_5 be given by the last two formulas. Then, the geometric cubic polynomial $\gamma : [0, 1] \to \mathbf{St}_{n,k}$ defined by*

$$\begin{aligned}\gamma(t) &= e^{tX_5(t)}\beta_2(t, S_0, S_1, S_2)e^{t\Omega_5(t)} \\ &= e^{tX_5(t)}e^{tX_3(t)}e^{tX_0}S_0e^{t\Omega_0}e^{t\Omega_3(t)}e^{t\Omega_5(t)},\end{aligned} \tag{17}$$

solves **Problem 1.**

In order to prove Theorem 7, it is required some additional results that are established in the following lemmas.

Lemma 8. *The generalized cubic polynomial in $\mathbf{St}_{n,k}$ defined in* (17) *by*

$$\gamma(t) = e^{tX_5(t)}e^{tX_3(t)}e^{tX_0}S_0e^{t\Omega_0}e^{t\Omega_3(t)}e^{t\Omega_5(t)},$$

satisfies the following boundary conditions

$$\gamma(0) = S_0, \quad \gamma(1) = S_3.$$

Proof. It is immediate to see that $\gamma(0) = S_0$. In order to show that $\gamma(1) = S_3$, one needs to compute $X_3(1)$, $\Omega_3(1)$, $X_4(1)$ and $\Omega_4(1)$.

$$\begin{aligned}X_3(1) &= \frac{1}{2}\log\big((I - 2e^{X_1}S_1S_1^\top e^{-X_1})(I - 2e^{X_0}S_0S_0^\top e^{-X_0})\big) \\ &= \frac{1}{2}\log\big(\big(I - 2(e^{X_1}S_1e^{\Omega_1})(e^{X_1}S_1e^{\Omega_1})^\top\big)\big(I - 2(e^{X_0}S_0e^{\Omega_0})(e^{X_0}S_0e^{\Omega_0})^\top\big)\big) \\ &= \frac{1}{2}\log\big((I - 2S_2S_2^\top)(I - 2S_1S_1^\top)\big)\end{aligned}$$

$$\begin{aligned}\Omega_3(1) &= \log\big(e^{-\Omega_0}S_0^\top e^{-X_0}e^{-X_3(1)}e^{X_1}S_1e^{\Omega_1}\big) \\ &= \log\big((e^{X_0}S_0e^{\Omega_0})^\top e^{-X_3(1)}S_2\big) \\ &= \log\big(S_1^\top e^{-X_3(1)}S_2\big).\end{aligned} \tag{18}$$

Proceeding in an analogous way, by noticing that $S_3 = e^{-X_2}S_2e^{-\Omega_2}$, one may compute

$$\begin{aligned}
X_4(1) &= \frac{1}{2}\log\big((I - 2e^{-X_2}S_2S_2^\top e^{X_2})(I - 2e^{X_1}S_1S_1^\top e^{-X_1})\big) \\
&= \frac{1}{2}\log\Big(\big(I - 2e^{-X_2}S_2e^{-\Omega_2}\big(e^{-X_2}S_2e^{-\Omega_2}\big)^\top\big) \\
&\qquad\quad \big(I - 2\big(e^{X_1}S_1e^{\Omega_1}\big)\big(e^{X_1}S_1e^{\Omega_1}\big)^\top\big)\Big) \\
&= \frac{1}{2}\log\big((I - 2S_3S_3^\top)(I - 2S_2S_2^\top)\big)
\end{aligned} \tag{19}$$

$$\begin{aligned}
\Omega_4(1) &= \log\big(e^{-\Omega_1}S_1^\top e^{-X_1}e^{-X_4(1)}e^{-X_2}S_2e^{-\Omega_2}\big) \\
&= \log\big(S_2^\top e^{-X_4(1)}S_3\big).
\end{aligned}$$

According to the expressions derived for $X_3(1)$ and $\Omega_3(1)$, notice that the quasi-geodesic parametrized in the interval $[0, 1]$ by $\alpha(s) = e^{sX_3(1)}S_1e^{s\Omega_3(1)}$ satisfies $\alpha(0) = S_1$ and $\alpha(1) = S_2$. Therefore,

$$\begin{aligned}
\gamma(1) &= e^{X_5(1)}e^{X_3(1)}e^{X_0}S_0e^{\Omega_0}e^{\Omega_3(1)}e^{\Omega_5(1)} \\
&= e^{X_5(1)}e^{X_3(1)}S_1e^{\Omega_3(1)}e^{\Omega_5(1)} \\
&= e^{X_5(1)}S_2e^{\Omega_5(1)}.
\end{aligned}$$

It remains to prove that $e^{X_5(1)}S_2e^{\Omega_5(1)} = S_3$. But,

$$\begin{aligned}
X_5(1) &= \frac{1}{2}\log\Big(\big(I - 2e^{X_4(1)}e^{X_1}S_1S_1^\top e^{-X_1}e^{-X_4(1)}\big) \\
&\qquad\quad \big(I - 2e^{X_3(1)}e^{X_0}S_0S_0^\top e^{-X_0}e^{-X_3(1)}\big)\Big) \\
&= \frac{1}{2}\log\Big(\big(I - 2e^{X_4(1)}S_2S_2^\top e^{-X_4(1)}\big)\big(I - 2e^{X_3(1)}S_1S_1^\top e^{-X_3(1)}\big)\Big) \\
&= \frac{1}{2}\log\Big(\big(I - 2S_3S_3^\top\big)\big(I - 2S_2S_2^\top\big)\Big)
\end{aligned} \tag{20}$$

$$\begin{aligned}
\Omega_5(1) &= \log\big(e^{-\Omega_3(1)}e^{-\Omega_0}S_0^\top e^{-X_0}e^{-X_3(1)}e^{-X_5(1)}e^{X_4(1)}e^{X_1}S_1e^{\Omega_1}e^{\Omega_4(1)}\big) \\
&= \log\Big(\big(e^{X_3(1)}S_1e^{\Omega_3(1)}\big)^\top e^{-X_5(1)}S_3\Big) \\
&= \log\big(S_2^\top e^{-X_5(1)}S_3\big)
\end{aligned}$$

It is immediate that $e^{X_5(1)}S_2e^{\Omega_5(1)} = S_3$, since $e^{X_5(1)}S_2e^{\Omega_5(1)}$ represents the endpoint of the quasi-geodesic joining S_2 to S_3. So, $\gamma(1) = S_3$. □

The next result is essential to compute the derivative of γ at $t = 1$.

Lemma 9. *For the generalized cubic polynomial defined in* (17)*, one can write*

$$\gamma(t) = e^{(t-1)X_5(t)}e^{(t-1)X_4(t)}e^{(1-t)X_2}S_3e^{(1-t)\Omega_2}e^{(t-1)\Omega_4(t)}e^{(t-1)\Omega_5(t)}. \tag{21}$$

Proof. To show that the two expressions for the generalized cubic polynomial γ coincide, it suffices to rewrite each one of the curves obtained in the de Casteljau algorithm in an equivalent way. The three quasi-geodesics obtained in Step 1 can be rewritten as

$$\begin{aligned}
\beta_1(t, S_0, S_1) &= e^{tX_0} S_0 e^{t\Omega_0} \\
&= e^{tX_0} e^{-X_0} e^{X_0} S_0 e^{\Omega_0} e^{-\Omega_0} e^{t\Omega_0} \\
&= e^{(t-1)X_0} S_1 e^{(t-1)\Omega_0}
\end{aligned}$$

$$\begin{aligned}
\beta_1(t, S_1, S_2) &= e^{tX_1} S_1 e^{t\Omega_1} \\
&= e^{tX_1} e^{-X_1} e^{X_1} S_1 e^{\Omega_1} e^{-\Omega_1} e^{t\Omega_1} \\
&= e^{(t-1)X_1} S_2 e^{(t-1)\Omega_1}
\end{aligned}$$

$$\begin{aligned}
\beta_1(t, S_2, S_3) &= e^{-tX_2} S_2 e^{-t\Omega_2} \\
&= e^{-tX_2} e^{X_2} e^{-X_2} S_2 e^{-\Omega_2} e^{\Omega_2} e^{-t\Omega_2} \\
&= e^{(1-t)X_2} S_3 e^{(1-t)\Omega_2}
\end{aligned}$$

The two quasi-geodesic arcs obtained in Step 2 can be rewritten as

$$\begin{aligned}
\beta_2(s, S_0, S_1, S_2) &= e^{sX_3(t)} \beta_1(t, S_0, S_1) e^{s\Omega_3(t)} \\
&= e^{sX_3(t)} e^{-X_3(t)} \beta_1(t, S_1, S_2) e^{-\Omega_3(t)} e^{s\Omega_3(t)} \\
&= e^{(s-1)X_3(t)} \beta_1(t, S_1, S_2) e^{(s-1)\Omega_3(t)}
\end{aligned}$$

$$\begin{aligned}
\beta_2(s, S_1, S_2, S_3) &= e^{sX_4(t)} \beta_1(t, S_1, S_2) e^{s\Omega_4(t)} \\
&= e^{sX_4(t)} e^{-X_4(t)} \beta_1(t, S_3, S_2) e^{-\Omega_4(t)} e^{s\Omega_4(t)} \\
&= e^{(s-1)X_4(t)} \beta_1(t, S_2, S_3) e^{(s-1)\Omega_4(t)}
\end{aligned}$$

Finally, the curve β_3 can be reparametrized by

$$\begin{aligned}
\beta_3(s, S_0, S_1, S_2, S_3) &= e^{sX_5(t)} \beta_2(t, S_0, S_1, S_2) e^{s\Omega_5(t)} \\
&= e^{sX_5(t)} e^{-X_5(t)} \beta_2(t, S_1, S_2, S_3) e^{-\Omega_5(t)} e^{s\Omega_5(t)} \\
&= e^{(s-1)X_5(t)} \beta_2(t, S_1, S_2, S_3) e^{(s-1)\Omega_5(t)}.
\end{aligned}$$

The result follows by using the equivalent reparametrizations for the quasi-geodesic arcs derived below. □

The next goal is to show that $\dot{\gamma}(0) = V_0$ and $\dot{\gamma}(1) = V_1$. For this task, it is necessary to use the derivative of a matrix exponential (Sattinger and Weaver [15]). Notice that if $t \mapsto A(t)$ is a differentiable matrix function, then

$$\frac{de^{A(t)}}{dt} = \frac{e^u - 1}{u}\Big|_{u=ad_{A(t)}} (\dot{A}(t)) e^{A(t)}, \tag{22}$$

where ad is the adjoint operator defined by $ad_X(Y) = XY - YX$ and $\dfrac{e^u - 1}{u}$ denotes the sum of the power series $\sum_{m=0}^{+\infty} \frac{u^m}{(m+1)!}$. Moreover, if $t \mapsto Y(t)$ is a differentiable matrix function such that $\log Y(t)$ is defined for all t, then

$$\frac{d(\log Y(t))}{dt} = \frac{u}{e^u - 1}\Big|_{u=ad_{\log Y(t)}} \big(\dot{Y}(t) Y^{-1}(t)\big), \tag{23}$$

where $\dfrac{u}{e^u - 1} = 1 - \frac{u}{2} + \sum_{m=1}^{+\infty} \frac{\beta_{2m}}{(2m)!} u^{2m}$ and β_{2m} are the Bernoulli numbers.

Lemma 10. *The derivative of the generalized cubic polynomial in* $\mathbf{St}_{n,k}$ *defined by* (17) *satisfies the following boundary conditions*

$$\dot{\gamma}(0) = V_0, \quad \dot{\gamma}(1) = V_1.$$

Proof. To compute $\dot{\gamma}(0)$, we use the expression for γ given by (17). Notice first that, according to (22),

$$\frac{d(e^{tX_5(t)})}{dt} = \frac{e^u - 1}{u}\Big|_{u=\mathrm{ad}_{tX_5(t)}} \big(X_5(t) + t\dot{X}_5(t)\big) e^{tX_5(t)}.$$

Evaluating the above in $t = 0$, one simply gets

$$\frac{d(e^{tX_5(t)})}{dt}\Big|_{t=0} = X_5(0).$$

So,

$$\begin{aligned}
\dot{\gamma}(0) &= \frac{d}{dt}\big(e^{tX_5(t)} e^{tX_3(t)} e^{tX_0} S_0 e^{t\Omega_0} e^{t\Omega_3(t)} e^{t\Omega_5(t)}\big)\Big|_{t=0} \\
&= X_5(0)S_0 + X_3(0)S_0 + X_0 S_0 + S_0\Omega_0 + S_0\Omega_3(0) + S_0\Omega_5(0) \\
&= X_5(0)S_0 + S_0\Omega_5(0) + X_3(0)S_0 + S_0\Omega_3(0) + X_0 S_0 + S_0\Omega_0.
\end{aligned}$$

Now, according to (12), it is easy to check that $X_0 S_0 + S_0 \Omega_0 = \frac{1}{3} V_0$. We claim that

$$X_5(0)S_0 + S_0\Omega_5(0) = X_3(0)S_0 + S_0\Omega_3(0) = X_0 S_0 + S_0\Omega_0.$$

Indeed,

$$X_5(0) = X_3(0) = \frac{1}{2}\log\big((I - 2S_1 S_1^\top)(I - 2S_0 S_0^\top)\big)$$

and

$$\begin{aligned}
\Omega_3(0) &= \log\big(S_0^\top \mathrm{e}^{-X_3(0)} S_1\big) = \log\big(S_0^\top \mathrm{e}^{-X_5(0)} S_1\big) \\
&= \Omega_5(0).
\end{aligned}$$

It remains to prove that $X_3(0) = X_0$ and $\Omega_3(0) = \Omega_0$. According to (11), $\beta_1(t, S_0, S_1)$ is the quasi-geodesic satisfying $\beta_1(0, S_0, S_1) = S_0$ and $\beta_1(1, S_0, S_1) = S_1$. So,

$$\begin{aligned}
X_0 &= \frac{1}{2}\log\big((I - 2S_1 S_1^\top)(I - 2S_0 S_0^\top)\big) \\
\Omega_0 &= \log\big(S_0^\top \mathrm{e}^{-X_0} S_1\big).
\end{aligned}$$

This enables to conclude that $\dot{\gamma}(0) = V_0$.

To compute $\dot{\gamma}(1)$, we use the expression for γ given by (21). According to (22),

$$\frac{d(e^{(t-1)X_5(t)})}{dt} = \frac{e^u - 1}{u}\Big|_{u=\mathrm{ad}_{(t-1)X_5(t)}} \big(X_5(t) + (t-1)\dot{X}_5(t)\big)e^{(t-1)X_5(t)}.$$

Evaluating the above in $t = 1$, one simply gets

$$\frac{d(e^{(t-1)X_5(t)})}{dt}\Big|_{t=1} = X_5(1).$$

Hence,

$$\begin{aligned}
\dot{\gamma}(1) &= \frac{d}{dt}\big(e^{(t-1)X_5(t)}e^{(t-1)X_4(t)}e^{(1-t)X_2}S_3\,e^{(1-t)\Omega_2}e^{(t-1)\Omega_4(t)}e^{(t-1)\Omega_5(t)}\big)\Big|_{t=1} \\
&= X_5(1)S_3 + X_4(1)S_3 - X_2S_3 - S_3\Omega_2 + S_3\Omega_4(1) + S_3\Omega_5(1) \\
&= X_5(1)S_3 + S_3\Omega_5(1) + X_4(1)S_3 + S_3\Omega_4(1) - (X_2S_3 + S_3\Omega_2).
\end{aligned}$$

From (14), it is immediate to conclude that $X_2S_3 + S_3\Omega_2 = -\frac{1}{3}V_3$. From (19) and (20), we also conclude that

$$X_5(1)S_3 + S_3\Omega_5(1) = X_4(1)S_3 + S_3\Omega_4(1).$$

We claim that $X_5(1) = -X_2$ and $\Omega_5(1) = -\Omega_2$. To prove this fact, observe that the curve $\beta_1(t, S_3, S_2) = e^{tX_2}S_3e^{t\Omega_2}$ is the quasi-geodesic connecting S_3 (at $t = 0$) to S_2 (at $t = 1$). Therefore,

$$X_2 = \frac{1}{2}\log\big((I - 2S_2S_2^\top)(I - 2S_3S_3^\top)\big)$$

$$\Omega_2 = \log(S_3^\top e^{-X_2}S_2).$$

Following (20),

$$\begin{aligned}
X_5(1) &= \tfrac{1}{2}\log\big((I - 2S_3S_3^\top)(I - 2S_2S_2^\top) = -\tfrac{1}{2}\log\big((I - 2S_2S_2^\top)(I - 2S_3S_3^\top)\big) \\
&= -X_2,
\end{aligned}$$

$$\Omega_5(1) = \log(S_2^\top e^{-X_5(1)}S_3) = \log(S_2^\top e^{X_2}S_3) = -\log(S_3^\top e^{-X_2}S_2) = -\Omega_2,$$

where the latter comes from the orthogonality of $S_2^\top e^{X_2}S_3$ [12, Theorem 7].

We can now conclude that $\dot{\gamma}(1) = V_3$, as required. □

The goal now is to solve the following problem.

Problem 2. *Find a smooth curve $\gamma : [0, 1] \to \mathbf{St}_{n,k}$ satisfying the following boundary conditions:*

$$\gamma(0) = S_0, \quad \gamma(1) = S_3, \quad \dot{\gamma}(0) = V_0, \quad \frac{D\dot{\gamma}}{dt}(0) = W_0, \tag{24}$$

where S_0, S_3 are given points in $\mathbf{St}_{n,k}$, $V_0 \in T_{S_0}\mathbf{St}_{n,k}$ is the velocity of γ at $t = 0$ and $W_0 \in T_{S_0}\mathbf{St}_{n,k}$ is the covariant acceleration of γ at $t = 0$.

The idea is to rewrite the control points S_1 and S_2 in terms of the given new data. But, S_1 is easily computed by using the fact that it is the endpoint of the quasi geodesic $\beta_1(t, S_0, S_1)$. So,

$$S_1 = e^{\frac{1}{3}(V_0 S_0^\top - S_0 V_0^\top + 2 S_0 V_0^\top S_0 S_0^\top)} S_0 e^{\frac{1}{3} S_0^\top V_0} \tag{25}$$

It remains to compute S_2.

The covariant acceleration $\dfrac{D\dot\gamma}{dt}$ is obtained by projecting $\ddot\gamma$ into the tangent space of $\mathbf{St}_{n,k}$ which, according to [6], is given by

$$\frac{D\dot\gamma}{dt} = (I - \gamma\gamma^\top)\ddot\gamma + \gamma\,\mathbf{Skew}(\gamma^\top\ddot\gamma), \tag{26}$$

where $\mathbf{Skew}(A) = \frac{A - A^\top}{2}$.

So, it is enough to compute $\ddot\gamma$. The next two lemmas give the values for $\dot X_3(0)$ and $\dot\Omega_3(0)$.

Lemma 11. *Let X_3 be defined by*

$$X_3(t) = \frac{1}{2}\log\big((I - 2e^{tX_1} S_1 S_1^\top e^{-tX_1})(I - 2e^{tX_0} S_0 S_0^\top e^{-tX_0})\big).$$

Then,

$$\dot X_3(0) = \left.\frac{2u}{e^{2u} - 1}\right|_{u=\mathrm{ad}_{X_0}} (X_1). \tag{27}$$

Proof. According to (23),

$$\begin{aligned}
\dot X_3(0) &= \frac{1}{2}\frac{d}{dt}\log\big((I - 2e^{tX_1} S_1 S_1^\top e^{-tX_1})(I - 2e^{tX_0} S_0 S_0^\top e^{-tX_0})\big)\Big|_{t=0} \\
&= \left.\frac{u}{e^u - 1}\right|_{u=\mathrm{ad}_{2X_0}} \Big(\Big((S_1 S_1^\top X_1 - X_1 S_1 S_1^\top)(I - 2S_0 S_0^\top) \\
&\quad + (I - 2S_1 S_1^\top)(S_0 S_0^\top X_0 - X_0 S_0 S_0^\top)\Big)(I - 2S_0 S_0^\top)(I - 2S_1 S_1^\top)\Big) \\
&= \left.\frac{u}{e^u - 1}\right|_{u=\mathrm{ad}_{2X_0}} \Big((S_1 S_1^\top X_1 - X_1 S_1 S_1^\top)(I - 2S_1 S_1^\top) \\
&\quad + (I - 2S_1 S_1^\top)(S_0 S_0^\top X_0 - X_0 S_0 S_0^\top)\big)(I - 2S_0 S_0^\top)(I - 2S_1 S_1^\top)\Big).
\end{aligned}$$

Notice that, since $X_i \in \mathfrak{so}_{S_i S_i^\top}(n)$, $i = 0, 1$, then

$$\big(S_i S_i^\top X_i - X_i S_i S_i^\top\big)\big(I - 2S_i S_i^\top\big) = X_i,$$

and also

$$(I - 2S_1 S_1^\top) X_0 (I - 2S_1 S_1^\top) = -X_0.$$

Therefore,

$$\dot X_3(0) = \left.\frac{u}{e^u - 1}\right|_{u=\mathrm{ad}_{2X_0}} \big(X_1\big),$$

and the result follows. ∎

Lemma 12. *For Ω_3 given by*

$$\Omega_3(t) = \log\left(e^{-t\Omega_0} S_0^\top e^{-tX_0} e^{-X_3(t)} e^{tX_1} S_1 e^{t\Omega_1}\right),$$

we have

$$\dot{\Omega}_3(0) = \left.\frac{u}{1-e^{-u}}\right|_{u=\mathrm{ad}_{\Omega_0}} (\Omega_1). \tag{28}$$

Proof. Let's begin by noticing that $e^{-X_0} S_1 S_1^\top e^{X_0} = S_0 S_0^\top$. Thus,

$$\begin{aligned}
\dot{\Omega}_3(0) &= \frac{d}{dt} \log\left(e^{-t\Omega_0} S_0^\top e^{-tX_0} e^{-X_3(t)} e^{tX_1} S_1 e^{t\Omega_1}\right)\Big|_{t=0} \\
&= \left.\frac{u}{e^u - 1}\right|_{u=\mathrm{ad}_{\Omega_0}} \Bigg(\Big(-\Big(\Omega_0 S_0^\top + S_0^\top X_0 + S_0^\top \left.\frac{e^u-1}{u}\right|_{u=\mathrm{ad}_{-X_0}} (\dot{X}_3(0))\Big) e^{-X_0} S_1 \\
&\quad + S_0^\top e^{-X_0} (X_1 S_1 + S_1 \Omega_1)\Big) S_1^\top e^{X_0} S_0\Bigg) \\
&= \left.\frac{u}{e^u - 1}\right|_{u=\mathrm{ad}_{\Omega_0}} \Bigg(- S_0^\top \Big(X_0 + \left.\frac{1-e^{-u}}{u}\right|_{u=\mathrm{ad}_{X_0}} (\dot{X}_3(0))\Big) S_0 \\
&\quad + S_0^\top e^{-X_0} (X_1 S_1 + S_1 \Omega_1)\Big) S_1^\top e^{X_0} S_0\Bigg).
\end{aligned}$$

From [12, Proposition 3], it follows immediately that

$$S_0^\top \Big(X_0 + \left.\frac{1-e^{-u}}{u}\right|_{u=\mathrm{ad}_{X_0}} (\dot{X}_3(0))\Big) S_0 = 0,$$

since $X_0 + \left.\frac{1-e^{-u}}{u}\right|_{u=\mathrm{ad}_{X_0}} (\dot{X}_3(0)) \in \mathfrak{so}_{S_0 S_0^\top}(n)$.

Let us now prove that $Z_0 = e^{-X_0} X_1 S_1 S_1^\top e^{X_0} \in \mathfrak{gl}_{S_0 S_0^\top}(n)$. Indeed, using $S_0 S_0^\top e^{-X_0} = e^{-X_0} S_1 S_1^\top$, one can write

$$\begin{aligned}
S_0 S_0^\top Z_0 + Z_0 S_0 S_0^\top &= S_0 S_0^\top e^{-X_0} X_1 S_1 S_1^\top e^{X_0} + e^{-X_0} X_1 S_1 S_1^\top e^{X_0} S_0 S_0^\top \\
&= e^{-X_0} S_1 \underbrace{S_1^\top X_1 S_1}_{=0} S_1^\top e^{X_0} + e^{-X_0} X_1 S_1 S_1^\top S_1 S_1^\top e^{X_0} \\
&= e^{-X_0} X_1 S_1 S_1^\top e^{X_0} = Z_0.
\end{aligned}$$

This means that $S_0^\top e^{-X_0} X_1 S_1 S_1^\top e^{X_0} S_0 = 0$ and one gets

$$\begin{aligned}
\dot{\Omega}_3(0) &= \left.\frac{u}{e^u - 1}\right|_{u=\mathrm{ad}_{\Omega_0}} \left(S_0^\top e^{-X_0} S_1 \Omega_1 S_1^\top e^{X_0} S_0\right) = \left.\frac{u}{e^u - 1}\right|_{u=\mathrm{ad}_{\Omega_0}} \left(e^{\Omega_0} \Omega_1 e^{-\Omega_0}\right) \\
&= \left.\frac{u}{e^u - 1}\right|_{u=\mathrm{ad}_{\Omega_0}} (e^{\mathrm{ad}_{\Omega_0}}(\Omega_1)) = \left.\frac{u}{1 - e^{-u}}\right|_{u=\mathrm{ad}_{\Omega_0}} (\Omega_1).
\end{aligned}$$

□

Using similar computations and arguments, we can also conclude that

$$\dot{X}_5(0) = 2\dot{X}_3(0) \quad \text{and} \quad \dot{\Omega}_5(0) = 2\dot{\Omega}_3(0).$$

In order to compute $\ddot{\gamma}$, notice that, according to (22), if $t \mapsto X(t)$ is a differentiable matrix valued function, one has

$$\begin{aligned} \frac{d(e^{tX(t)})}{dt} &= \frac{e^u - 1}{u}\Big|_{u=ad_{tX(t)}} \big(X(t) + t\dot{X}(t)\big)e^{tX(t)} \\ &= X(t)e^{tX(t)} + \frac{e^u - 1}{u}\Big|_{u=ad_{tX(t)}} \big(t\dot{X}(t)\big)e^{tX(t)}. \end{aligned}$$

Therefore,

$$\frac{d(e^{tX(t)})}{dt}\Big|_{t=0} = X(0),$$

and using $\frac{e^u-1}{u} = \sum_{m=0}^{+\infty} \frac{u^m}{(m+1)!}$, one can write

$$\frac{e^u - 1}{u}\Big|_{u=ad_{tX(t)}} \big(t\dot{X}(t)\big) = t\dot{X}(t) + \frac{t^2}{2}[X(t), \dot{X}(t)] + \frac{t^3}{6}[X(t), [X(t), \dot{X}(t)]] + \cdots.$$

It can be easily seen that

$$\frac{d^2(e^{tX(t)})}{dt^2}\Big|_{t=0} = 2\dot{X}(0) + X^2(0).$$

Proposition 13. *Given the curve γ, defined in* (17) *by*

$$\gamma(t) = e^{tX_5(t)}e^{tX_3(t)}e^{tX_0}S_0e^{t\Omega_0}e^{t\Omega_3(t)}e^{t\Omega_5(t)},$$

then

$$\ddot{\gamma}(0) = \tfrac{1}{3}X_0V_0\Omega_0 + \tfrac{8}{3}(X_0V_0 + V_0\Omega_0) + 6\big(\dot{X}_3(0)S_0 + S_0\dot{\Omega}_3(0)\big), \tag{29}$$

where $\dot{X}_3(0)$ and $\dot{\Omega}_3(0)$ are given by (27) *and* (28)*, respectively.*

Proof. Differentiating the curve γ with respect to t, one gets

$$\begin{aligned} \dot{\gamma}(t) &= e^{tX_5(t)}e^{tX_3(t)}e^{tX_0}(X_0S_0 + S_0\Omega_0)e^{t\Omega_0}e^{t\Omega_3(t)}e^{t\Omega_5(t)} \\ &+ \frac{d(e^{tX_5(t)})}{dt}e^{tX_3(t)}e^{tX_0}S_0e^{t\Omega_0}e^{t\Omega_3(t)}e^{t\Omega_5(t)} \\ &+ e^{tX_5(t)}\frac{d(e^{tX_3(t)})}{dt}e^{tX_0}S_0e^{t\Omega_0}e^{t\Omega_3(t)}e^{t\Omega_5(t)} \\ &+ e^{tX_5(t)}e^{tX_3(t)}e^{tX_0}S_0e^{t\Omega_0}\frac{d(e^{t\Omega_3(t)})}{dt}e^{t\Omega_5(t)} \\ &+ e^{tX_5(t)}e^{tX_3(t)}e^{tX_0}S_0e^{t\Omega_0}e^{t\Omega_3(t)}\frac{d(e^{t\Omega_5(t)})}{dt}. \end{aligned}$$

Knowing that $X_3(0) = X_5(0) = X_0$ and $\Omega_3(0) = \Omega_5(0) = \Omega_0$, we can write

$$\begin{aligned}\ddot{\gamma}(0) &= X_0(X_0S_0 + S_0\Omega_0)\Omega_0 + 4X_0(X_0S_0 + S_0\Omega_0) + 4(X_0S_0 + S_0\Omega_0)\Omega_0 + \\ &\quad + 2X_0^2S_0 + 8X_0S_0\Omega_0 + 2S_0\Omega_0^2 + (2\dot{X}_5(0) + X_0^2)S_0 + \\ &\quad + (2\dot{X}_3(0) + X_0^2)S_0 + S_0(2\dot{\Omega}_5(0) + \Omega_0^2) + S_0(2\dot{\Omega}_3(0) + \Omega_0^2) \\ &= X_0(X_0S_0 + S_0\Omega_0)\Omega_0 + 8X_0(X_0S_0 + S_0\Omega_0) + 8(X_0S_0 + S_0\Omega_0)\Omega_0 \\ &\quad + 2(\dot{X}_5(0) + \dot{X}_3(0))S_0 + 2S_0(\dot{\Omega}_5(0) + \dot{\Omega}_3(0)).\end{aligned}$$

Using $X_0S_0 + S_0\Omega_0 = \frac{1}{3}V_0$, $\dot{X}_5(0) = 2\dot{X}_3(0)$ and $\dot{\Omega}_5(0) = 2\dot{\Omega}_3(0)$, we get the result. □

Theorem 14. *The control points S_1 and S_2 used in the Casteljau algorithm to generate the geometric cubic polynomial $\gamma : [0,1] \to \mathbf{St}_{n,k}$ satisfying the boundary conditions* (24) *are given by*

$$\begin{aligned}S_1 &= e^{\frac{1}{3}(V_0S_0^\top - S_0V_0^\top + 2S_0V_0^\top S_0S_0^\top)}S_0e^{\frac{1}{3}S_0^\top V_0} \\ S_2 &= e^{X_1}S_1e^{\Omega_1},\end{aligned}$$

where

$$\begin{aligned}\Omega_1 &= \frac{1}{6}\left.\frac{1-e^{-u}}{u}\right|_{u=\mathrm{ad}_{\Omega_0}}\Big(S_0^\top W_0 + \tfrac{1}{27}\big((S_0^\top V_0)^3 + \mathbf{Skew}(S_0^\top V_0V_0^\top V_0)\big)\Big), \\ X_1 &= \frac{1}{3}\left.\frac{e^{2u}-1}{2u}\right|_{u=\mathrm{ad}_{X_0}}\Big(\mathbf{Skew}(W_0S_0^\top) + S_0W_0^\top S_0S_0^\top - \tfrac{1}{27}\mathbf{Skew}\big((V_0S_0^\top)^3\big) \\ &\qquad + \tfrac{1}{27}S_0(S_0^\top V_0)^3S_0^\top - \tfrac{16}{9}\mathbf{Skew}\big((V_0S_0^\top)^2\big)\Big).\end{aligned}$$

Proof. Using the projection operator given by (26), we notice that

$$W_0 = S_0\,\mathbf{Skew}(S_0^\top\ddot{\gamma}(0)) + (I - S_0S_0^\top)\ddot{\gamma}(0).$$

One can observe by doing some straightforward computations that

$$S_0^\top\ddot{\gamma}(0) = S_0^\top X_0^2S_0\Omega_0 + 8(S_0^\top X_0^2S_0 + \Omega_0^2) + 6\dot{\Omega}_3(0).$$

So,

$$\mathbf{Skew}(S_0^\top\ddot{\gamma}(0)) = \tfrac{1}{2}S_0^\top X_0^2S_0\Omega_0 + \tfrac{1}{2}\Omega_0S_0^\top X_0^2S_0 + 6\dot{\Omega}_3(0).$$

Using the condition $S_0S_0^\top X_0^2S_0 = X_0^2S_0$, one gets

$$S_0\,\mathbf{Skew}(S_0^\top\ddot{\gamma}(0)) = \tfrac{1}{2}X_0^2S_0\Omega_0 + \tfrac{1}{2}S_0\Omega_0S_0^\top X_0^2S_0 + 6S_0\dot{\Omega}_3(0).$$

Moreover,

$$(I - S_0S_0^\top)\ddot{\gamma}(0) = X_0S_0\Omega_0^2 + 16X_0S_0\Omega_0 + 6\dot{X}_3(0)S_0.$$

Hence,

$$W_0 = \tfrac{1}{2}X_0^2S_0\Omega_0 + \tfrac{1}{2}S_0\Omega_0S_0^\top X_0^2S_0 + 6S_0\dot{\Omega}_3(0) + X_0S_0\Omega_0^2 + 16X_0S_0\Omega_0 \\ + 6\dot{X}_3(0)S_0.$$

Multiplying on the left the above equation by $S_0^\top$, it follows

$$S_0^\top W_0 = \tfrac{1}{2}\big(S_0^\top X_0^2S_0\Omega_0 + \Omega_0S_0^\top X_0^2S_0\big) + 6\dot{\Omega}_3(0). \tag{30}$$

Performing some computations with the expression of X_0 and Ω_0, we find that

$$\begin{aligned}
&X_0^2 = \tfrac{1}{9}\big(V_0V_0^\top S_0S_0^\top - V_0V_0^\top - S_0V_0^\top V_0S_0^\top + S_0S_0^\top V_0V_0^\top\big)\\
&S_0^\top X_0^2S_0 = \tfrac{1}{9}\big(S_0^\top V_0V_0^\top S_0 - V_0^\top V_0\big)\\
&S_0^\top X_0^2S_0\Omega_0 = -\tfrac{1}{27}\big((S_0^\top V_0)^3 + V_0^\top V_0S_0^\top V_0\big)\\
&\Omega_0S_0^\top X_0^2S_0 = \tfrac{1}{27}\big((S_0^\top V_0)^3 + V_0^\top V_0S_0^\top V_0\big)
\end{aligned}$$

By using the above into equation (30), one gets

$$S_0^\top W_0 = \tfrac{1}{27}\big(\tfrac{1}{2}V_0^\top V_0V_0^\top S_0 - \tfrac{1}{2}S_0^\top V_0V_0^\top V_0 - (S_0^\top V_0)^3\big) + 6\dot{\Omega}_3(0).$$

Using the expression for $\dot{\Omega}_3(0)$ given by (28), we obtain

$$\Omega_1 = \frac{1}{6}\left.\frac{1-e^{-u}}{u}\right|_{u=\mathrm{ad}_{\Omega_0}}\Big(S_0^\top W_0 + \tfrac{1}{27}\big((S_0^\top V_0)^3 + \mathbf{Skew}(S_0^\top V_0V_0^\top V_0)\big)\Big).$$

In order to compute X_1, we start by computing $W_0S_0^\top - S_0W_0^\top + 2S_0W_0^\top S_0S_0^\top$. Observe that

$$W_0S_0^\top - S_0W_0^\top = 2\,\mathbf{Skew}(W_0S_0^\top),$$

and since $X_0^2S_0S_0^\top = S_0S_0^\top X_0$, one gets

$$\begin{aligned}
W_0S_0^\top &= \tfrac{1}{2}X_0^2S_0\Omega_0S_0^\top + \tfrac{1}{2}S_0\Omega_0S_0^\top X_0^2S_0S_0^\top + 6S_0\dot{\Omega}_3(0)S_0^\top + X_0S_0\Omega_0^2S_0^\top\\
&\quad + 16X_0S_0\Omega_0S_0^\top + 6\dot{X}_3(0)S_0S_0^\top\\
&= \tfrac{1}{2}X_0^2S_0\Omega_0S_0^\top + \tfrac{1}{2}S_0\Omega_0S_0^\top X_0^2 + 6S_0\dot{\Omega}_3(0)S_0^\top + X_0S_0\Omega_0^2S_0^\top\\
&\quad + 16X_0S_0\Omega_0S_0^\top + 6\dot{X}_3(0)S_0S_0^\top.
\end{aligned}$$

Therefore,

$$\begin{aligned}
2\,\mathbf{Skew}(W_0S_0^\top) &= X_0^2S_0\Omega_0S_0^\top + S_0\Omega_0S_0^\top X_0^2 + 12S_0\dot{\Omega}_3(0)S_0^\top + X_0S_0\Omega_0^2S_0^\top\\
&\quad + S_0\Omega_0^2S_0^\top X_0 + 16X_0S_0\Omega_0S_0^\top - 16S_0\Omega_0S_0^\top X_0\\
&\quad + 6\dot{X}_3(0)S_0S_0^\top + 6S_0S_0^\top\dot{X}_3(0)\\
&= X_0^2S_0\Omega_0S_0^\top + S_0\Omega_0S_0^\top X_0^2 + 12S_0\dot{\Omega}_3(0)S_0^\top + X_0S_0\Omega_0^2S_0^\top\\
&\quad + S_0\Omega_0^2S_0^\top X_0 + 16X_0S_0\Omega_0S_0^\top - 16S_0\Omega_0S_0^\top X_0 + 6\dot{X}_3(0),
\end{aligned}$$

and

$$2S_0W_0^\top S_0S_0^\top = \Big(- S_0\Omega_0S_0^\top X_0^2 - X_0^2S_0\Omega_0S_0^\top - 12S_0\dot{\Omega}_3(0)S_0^\top - 2S_0\Omega_0^2S_0^\top X_0 + 32S_0\Omega_0S_0^\top X_0 - 12S_0S_0^\top \dot{X}_3(0)\Big)S_0S_0^\top = -S_0\Omega_0S_0^\top X_0^2 - X_0^2S_0\Omega_0S_0^\top - 12S_0\dot{\Omega}_3(0)S_0^\top.$$

Consequently,

$$2\,\mathbf{Skew}(W_0S_0^\top) + 2S_0W_0^\top S_0S_0^\top = = X_0S_0\Omega_0^2S_0^\top + S_0\Omega_0^2S_0^\top X_0 + 16(X_0S_0\Omega_0S_0^\top - S_0\Omega_0S_0^\top X_0) + 6\dot{X}_3(0).$$

To proceed, notice that

$$X_0S_0\Omega_0^2S_0^\top = \tfrac{1}{27}\big((V_0S_0^\top)^3 - S_0(S_0^\top V_0)^3S_0^\top\big)$$
$$S_0\Omega_0^2S_0^\top X_0 = -\tfrac{1}{27}\big((S_0V_0^\top)^3 + S_0(S_0^\top V_0)^3S_0^\top\big).$$

so

$$X_0S_0\Omega_0^2S_0^\top + S_0\Omega_0^2S_0^\top X_0 = \tfrac{2}{27}\big(\mathbf{Skew}((V_0S_0^\top)^3) - S_0(S_0^\top V_0)^3S_0^\top\big).$$

On the other hand, since

$$X_0S_0\Omega_0S_0^\top = \tfrac{1}{9}\big(V_0S_0^\top V_0S_0^\top + S_0V_0^\top S_0S_0^\top V_0S_0^\top\big),$$

then

$$X_0S_0\Omega_0S_0^\top - S_0\Omega_0^2S_0^\top X_0 = \tfrac{2}{9}\mathbf{Skew}\big((V_0S_0^\top)^2\big).$$

Therefore,

$$2\,\mathbf{Skew}(W_0S_0^\top) + 2S_0W_0^\top S_0S_0^\top = = \tfrac{2}{27}\big(\mathbf{Skew}((V_0S_0^\top)^3) - S_0(S_0^\top V_0)^3S_0^\top\big) + \tfrac{32}{9}\mathbf{Skew}\big((V_0S_0^\top)^2\big) + 6\dot{X}_3(0).$$

Finally, using the expression for $\dot{X}_3(0)$ given by (27), we conclude that

$$X_1 = \frac{1}{3}\left.\frac{e^{2u}-1}{2u}\right|_{u=\mathrm{ad}_{X_0}} \Big(\mathbf{Skew}(W_0S_0^\top) + S_0W_0^\top S_0S_0^\top - \tfrac{1}{27}\mathbf{Skew}\big((V_0S_0^\top)^3\big) + \tfrac{1}{27}S_0(S_0^\top V_0)^3S_0^\top - \tfrac{16}{9}\mathbf{Skew}\big((V_0S_0^\top)^2\big)\Big).$$

□

Remark 15. *Using formula* (17)*, one can evaluate points on the geometric cubic polynomial at different values of the parameter t (only requires computing exponentials of skew-symmetric matrices and logarithms of orthogonal matrices), in order to compare the results with curves obtained using different approaches.*

Nowadays, there are stable algorithms to compute matrix exponentials and logarithms. For instance, MatLab already uses them.

A similar strategy might be applied when the Stiefel manifold is equipped with the Euclidean metric, taking into account the numerical methods, involving the shooting method and path-straightening, to approximate endpoint geodesics, as done in [3]. Also, it would be interesting to compare the curves obtained in Theorem 7 with the approximate Riemannian cubic polynomials that result from using geometric integrators and extended retractions in [1].

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