HESSIAN REGULARITY IN HÖLDER SPACES FOR A SEMI-LINEAR BI-LAPLACIAN EQUATION

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ABSTRACT. We examine a semi-linear variant of the bi-Laplacian equation in the superlinear, subquadratic setting and obtain $C^{2,\sigma}$-regularity estimates, depending on the growth regime of the nonlinearity. Our strategy is to render this fourth-order problem as a system of two Poisson equations and explore the interplay between the integrability and smoothness available for each equation taken isolated.

Keywords: bi-Laplacian operator; semi-linear equations; Hessian regularity; Hölder spaces.

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1. Introduction

We examine the regularity of weak solutions to the semi-linear bi-Laplacian equation

$$\Delta^2 u = f(x, u, Du) \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded smooth domain, $\Delta^2 u := \Delta(\Delta u)$ denotes the bi-Laplacian operator, and the nonlinearity $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies a polynomial growth condition. Our findings report on $C^{2,\sigma}$-regularity estimates for the solutions to (1), depending on the growth regime of the nonlinearity.

Elliptic equations driven by operators of higher order play an important role across disciplines in pure mathematics and find relevant applications in various realms of life and social sciences. We mention differential geometry, calculus of variations, free boundary problems, the mechanics of deformable media (mainly in the mathematical theory of elasticity), and the dynamics of slow viscous fluids; see the monograph [17] and the references therein.

From the perspective of partial differential equations (PDE), the study of bi-Laplacian equations has covered a plethora of topics. These include the

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existence of solutions, fundamental properties (such as the validity of the maximum principle and the positivity of the Green's function), and regularity estimates. We notice the existence of relevant literature examining those properties in connection with the geometry of the domain. Of particular interest is the fact that merely Lipschitz-regular domains entail further difficulties for the analysis. In this connection, we refer the reader to [16] and the extensive list of references therein.

Regarding regularity estimates, the study of the bi-Laplacian operator has been pursued in several contexts. In the realm of obstacle problems, it appears as the operator governing a variational inequality. More precisely, given a function $\varphi : \Omega \to \mathbb{R}$, let $u \in W^{2,2}(\Omega)$ be such that

$$\Delta^2 u \geq 0 \quad \text{in } \Omega, \tag{2}$$

with $u \geq \varphi$, and

$$\Delta^2 u \cdot (u - \varphi) = 0, \quad \text{in } \Omega. \tag{3}$$

In [9], the author proves that weak solutions to this problem are in $W^{2,\infty}(\Omega)$. The analysis of the free boundary associated with (2)-(3) is the subject of [2]. In that paper, the authors establish the local boundedness of the Hessian and verify that $\Delta^2 u$ is a non-negative measure with finite mass. In addition, they examine the planar case, showing that solutions are C^2-regular and that the free boundary is contained in a continuously differentiable curve.

The bi-Laplacian operator has also been studied in [3] in the context of the two-phase free boundary obstacle problem

$$\begin{cases} \Delta^2 u = 0 & \text{in} \quad B_1^+ \\ u = g & \text{on} \quad (\partial B_1)^+ \\ \partial_{x_{d+1}} u = 0 & \text{on} \quad B_1' \\ \partial_{x_{d+1}} \Delta u = \lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1} & \text{on} \quad B_1', \end{cases} \tag{4}$$

where $p > 1$, λ_- and λ_+ are positive constants, $g \in W^{2,q}(B_1^+)$ for $q := \max(2, p)$, and the sets B_1^+ and B_1' are defined as

$$B_1^+ := \left\{ (x, x_{d+1}) \in B_1 \subset \mathbb{R}^d \times \mathbb{R} \,|\, x_{d+1} > 0 \right\}$$

and

$$B_1' := B_1 \cap \{x_{d+1} = 0\}.$$

The authors prove that both u and Δu are locally bounded. They also establish regularity estimates in Hölder spaces of the type $C^{p+1,\gamma}$, for every $\gamma \in (0, 1)$, and show their findings are optimal in the case of $p \in \mathbb{N}$. They also consider an Almgren's type frequency formula and a Monneau monotonicity formula and perform a thorough analysis of the singular set associated with (4). If B_1 is replaced with $\mathbb{R}^d$, it is worth noticing the formulation in (4) can be regarded as a Dirichlet-to-Neumann extension, in the spirit of Caffarelli and Silvestre, for the operator

$$(-\Delta)^{3/2} u = \lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1} \quad \text{in} \quad \mathbb{R}^d,$$

with $u \to 0$ as $|x| \to \infty$.

Interior regularity estimates for the pure equation $\Delta^2 u = f$ prescribed in a domain Ω have also been pursued in the literature. Of particular interest is the analysis of polyharmonic equations of the form

$$(-\Delta)^m u = f \quad \text{in} \quad \Omega,$$

where $2 \leq d \leq 2m+1$ and $f \in C_c^\infty(\Omega)$. In [16], the authors prove that solutions to this problem satisfy

$$D^{m-\frac{d}{2}+\frac{1}{2}} u \in L^\infty(\Omega) \quad \text{if} \quad d = 2k+1,$$

and

$$D^{m-\frac{d}{2}} u \in L^\infty(\Omega) \quad \text{if} \quad d = 2k,$$

with $k \in \mathbb{N}$. In the concrete case of (1), were $f = f(x)$ a smooth function, the analysis would lead to $u \in L^\infty(\Omega)$ for dimensions $d = 4, 5$, and $Du \in L^\infty(\Omega)$ for dimensions $d = 2, 3$.

Concerning the semi-linear formulation of the bi-Laplacian equation, we mention the developments reported in [4]. In that paper, the authors produce regularity estimates for the weak solutions to

$$\Delta^2 u + a(x)u = g(x,u) \quad \text{in} \quad \mathbb{R}^d. \tag{5}$$

They work under natural assumptions on the functions $a = a(x)$ and $g = g(x,u)$, including polynomial growth conditions on g, and prove that solutions to (5) are in $W^{4,2}(\mathbb{R}^d) \cap W^{2,s}(\mathbb{R}^d)$ for every $1 \leq s \leq \infty$. Their arguments rely on asymptotic properties of the fundamental solution associated with the operator $\Delta^2 + k^2$, for $k \in \mathbb{N}$. Apparently, these methods fall short in addressing the dependence on the gradient Du, not covering the case of (1). To the best of our knowledge, the analysis of semi-linear bi-Laplacian equations with explicit dependence on Du has hitherto not been addressed in the literature.

Our reasoning in this note relies on a reduction argument. Namely, we write the fourth-order PDE as a system of two second-order equations. This strategy is inspired by ideas introduced in the works [6, 7] of Evans. The gist of those papers is to design new PDE methods to address the Mather minimization principle in dynamics, as well as the weak KAM theory. The analysis starts with an approximating functional of the form

$$I_k[v_k] := \int_{\mathbb{T}^d} e^{kH(P+Dv_k,x)}\mathrm{d}x,$$

where $P \in \mathbb{R}^d$ is given, $v_k \in C^1(\mathbb{T}^d)$, and $k \in \mathbb{N}$. The Euler-Lagrange equation associated with I_k is

$$\operatorname{div}\left(e^{kH(P+Dv_k,x)}D_pH(P+Dv_k,x)\right) = 0 \quad \text{in} \quad \mathbb{T}^d. \tag{6}$$

Now, consider the Hamiltonian $\overline{H}^k = \overline{H}^k(P)$ given by

$$\overline{H}^k(P) := \frac{1}{k}\ln\left(\int_{\mathbb{T}^d} e^{kH(P+Dv_k,x)}\mathrm{d}x\right).$$

By defining $u_k := P \cdot x + v_k$ and

$$\sigma_k(x) := \frac{e^{kH(P+Dv_k,x)}}{\displaystyle\int_{\mathbb{T}^d} e^{kH(P+Dv_k,x)}\mathrm{d}x}$$

the Euler-Lagrange equation in (6) becomes the system

$$\begin{cases} e^{k\left(H(Du_k,x)-\overline{H}^k(P)\right)} = \sigma_k & \text{in} \quad \mathbb{T}^d \\ \operatorname{div}\left(\sigma_k D_pH(Du,x)\right) = 0 & \text{in} \quad \mathbb{T}^d. \end{cases} \tag{7}$$

That is, the Euler-Lagrange equation associated with I_k turns into the *coupling* of a generalized eikonal equation, whose unknown is u_k, and a transport equation for σ_k. The rationale then is to take the limit $k \to \infty$ and recover information on the effective Hamiltonian $\overline{H}$, as well as on the trajectories of the Hamiltonian system

$$\begin{cases} \dot{\mathrm{x}} = D_pH(\mathrm{p},\mathrm{x}) \\ \dot{\mathrm{p}} = -D_xH(\mathrm{p},\mathrm{x}). \end{cases}$$

We refer the reader to [8, 14, 15] for a discussion on the Mather problem and weak KAM theory. A similar approach involving second-order equations appears in [11], where the authors examine a stochastic variant of the Evans-Aronsson problem, unveiling new properties of the model. For completeness, we mention the theory of mean-field games as an instance where the coupling of two distinct equations has been examined from the viewpoint of regularity theory; see [13, 12]. For a related strategy in the context of semi-linear elliptic equations, see [1].

Our approach stems from this class of ideas as we write the single equation in (1) as a coupling. Indeed, the integrability available for D^2u in the class of weak solutions allows us to define a function $m \in L^q(\Omega)$ as $m := \Delta u$. The definition of weak solution to (1) then implies that m is a *very* weak solution to the Poisson equation $\Delta m = f$. As a consequence, we render the bi-Laplacian equation as the system

$$\begin{cases} \Delta u = m & \text{in } \Omega, \\ \Delta m = f(x,u,Du) & \text{in } \Omega. \end{cases} \tag{8}$$

The unknown for (8) is a pair (u,m) in a suitable functional space, where u is a strong solution for the first equation in the system, while m satisfies the second one in the very weak sense (we make these matters precise further on in the note).

In this context, the regularity of the solutions to (1) benefits from the interplay between the regularity estimates available at the level of the equations taken on their own. An L^q-regularity theory for very weak solutions builds upon standard Sobolev embeddings to produce improved integrability for m in Lebesgue spaces. In turn, the integrability of m affects the regularity of u. Indeed, we prove that very weak solutions to

$$\Delta m = f(x,u,Du) \quad \text{in} \quad \Omega$$

are in $W^{2,s}_{\text{loc}}(\Omega)$, for some $s \in (d/2, d]$. Hence, u solves a Poisson equation with right-hand side in $C^{0,2-d/s}_{\text{loc}}(\Omega)$. This fact unlocks a Schauder regularity theory for the solutions to (1).

The remainder of this note is split into two sections, one detailing preliminary definitions and useful facts and the other encompassing our main result.

2. Notions of solution and auxiliary results

Because our strategy is to render the bi-Laplacian equation as a system, we rely on three different notions of solutions. Namely, weak distributional solutions, very weak solutions, and L^d-strong solutions. Requiring u to be in $W^{2,q}(\Omega)$, for values of $q \in (d/2, d]$ depending on the growth regime satisfied by f, allows us, in particular, to switch between these different notions of solution seamlessly. We start by defining a weak solution for the bi-Laplacian equation in a distributional sense.

Definition 1 (Local weak solution of the bi-Laplacian equation). *A function $u \in W^{2,2}_{loc}(\Omega)$ is a local weak solution to (1) if*

$$\int_\Omega \Delta u \, \Delta \varphi \, dx = \int_\Omega f(x, u, Du) \, \varphi \, dx,$$

for every $\varphi \in C_c^\infty(\Omega)$.

Concerning Poisson's equation

$$\Delta w = g \quad \text{in } \Omega, \tag{9}$$

we will explore two notions of solution, those of very weak and L^q-strong solution. For completeness, we recall these notions in what follows.

Definition 2 (Very weak solution of the Poisson equation). *Let $g \in L^1_{loc}(\Omega)$. A function $w \in L^1_{loc}(\Omega)$ is a very weak solution to (9) if*

$$\int_\Omega w \, \Delta \varphi \, dx = \int_\Omega g \, \varphi \, dx,$$

for every $\varphi \in C_c^\infty(\Omega)$.

Definition 3 (L^q-strong solution of the Poisson equation). *Let $g \in L^q(\Omega)$ for $q > 1$. We say that $w \in W^{2,q}(\Omega)$ is an L^q-strong solution to (9) if*

$$\Delta w(x) = g(x), \quad \text{a.e. } x \in \Omega.$$

We proceed by defining a notion of solution to the system in (8), relating (1) with the latter.

Definition 4. *The pair $(u, m) \in W^{2,q}(\Omega) \times L^q(\Omega)$ is a solution to (8) if u is an L^q-strong solution to the first equation in (8) whereas m is a very weak solution to the second equation in (8).*

Next, we recall a result on the regularity of very weak solutions to the Poisson equation.

Proposition 1 (Sobolev regularity for very weak solutions). *Fix $1 < s < \infty$. Let $w \in L^1_{loc}(\Omega)$ be a very weak solution to (9), with $g \in L^s_{loc}(\Omega)$. Then $Dw \in W^{1,s}_{loc}(\Omega)$. If $w \in L^s_{loc}(\Omega)$, then $w \in W^{2,s}_{loc}(\Omega)$. Moreover, for $\Omega'' \Subset \Omega' \Subset \Omega$, there exists $C > 0$ such that*

$$\|w\|_{W^{2,s}(\Omega'')} \leq C \left(\|w\|_{L^s(\Omega)} + \|g\|_{L^s(\Omega)} \right). \tag{10}$$

Proof. To verify that $w \in W^{2,s}_{loc}(\Omega)$, we resort to [5, Theorem 3]. Once we have $w \in W^{2,s}_{\text{loc}}(\Omega)$, w becomes a strong solution to $\Delta w = g$ in Ω'. Standard results in elliptic regularity theory (see, for instance, [10, Theorem 9.11]) yield (10). $\square$

We conclude this section by verifying that a weak solution to (1) yields a pair (u, m) in a suitable functional space, solving (8).

Lemma 1. *Let $u \in W^{2,q}(\Omega)$ be a local weak solution to (1), with $q \geq 2$. Then, there exists $m \in L^q(\Omega)$ such that (u, m) is a solution to (8) according to Definition 4.*

Proof. It is clear that if $u \in W^{2,q}(\Omega)$, $\Delta u \in L^q(\Omega)$. Set $m := \Delta u$ and notice that $m \in L^q(\Omega)$ is defined almost everywhere in Ω. In addition, the weak formulation of (1) implies

$$\int_\Omega m\, \Delta\varphi\, dx = \int_\Omega \Delta u\, \Delta\varphi\, dx = \int_\Omega f(x, u, Du)\, \varphi\, dx,$$

for every $\varphi \in C_c^\infty(\Omega)$. Hence, m is a very weak solution to

$$\Delta m = f(x, u, Du) \quad \text{in } \Omega,$$

and the proof is complete. $\square$

3. Improved regularity in Hölder spaces

In this section, we state and prove the main result in this note. For ease of presentation, we set $\Omega \equiv B_1$, where B_1 stands for the unit ball in $\mathbb{R}^d$; standard covering arguments ensure this reduction entails no further restrictions on the problem.

Theorem 1. *Let $2 \leq q \leq d$ and $u \in W^{2,q}(B_1)$ be a local weak solution to*

$$\Delta^2 u = f(x, u, Du) \quad \textit{in } B_1.$$

Assume the nonlinearity satisfies the growth condition

$$|f(x, r, p)| \leq h(x) + C\left(|r|^\alpha + |p|^\beta\right), \tag{11}$$

for $h \in L^d(B_1)$ and fixed constants $C > 0$ and

$$\alpha, \beta \in [1, 2). \tag{12}$$

Suppose further that

$$\max(\alpha, \beta)\, \frac{d}{2} < q \leq d. \tag{13}$$

Then $u \in C^{2,\sigma}_{\mathrm{loc}}(B_1)$ *for*

$$\sigma := 2 - \frac{d \max(\alpha,\beta)}{q} \in (0,1).$$

Moreover, there exists $C > 0$ *such that*

$$\|u\|_{C^{2,\sigma}(B_{7/8})} \leq C \left(\|h\|_{L^d(B_1)} + \|u\|^{\max(\alpha,\beta)}_{W^{2,q}(B_1)} \right).$$

Proof. We start by choosing

$$s := \frac{q}{\max(\alpha,\beta)},$$

noting that, due to (12) and (13), we have $s \in (d/2, q]$. Moreover, for $B_{9/10} \Subset B_1$ it follows from (11) that

$$\begin{aligned}\|f(\cdot,u,Du)\|_{L^s(B_{9/10})} &\leq C \left(\|h\|_{L^s(B_{9/10})} + \|u\|^{\alpha}_{L^{\alpha s}(B_{9/10})} + \|Du\|^{\beta}_{L^{\beta s}(B_{9/10})} \right) \\ &\leq C \left(\|h\|_{L^d(B_1)} + \|u\|^{\max(\alpha,\beta)}_{W^{2,q}(B_1)} \right),\end{aligned}$$

which is finite since $u \in W^{2,q}(B_1)$. The constant C here depends on q and the ingredients in (11). By Lemma 1, there exists $m \in L^q(B_1)$ such that m is a very weak solution to

$$\Delta m = f(x,u,Du) \quad \text{in } B_1.$$

Due to Proposition 1, we conclude $Dm \in W^{1,s}_{\mathrm{loc}}(B_{99/100})$. But, since $d/2 < s \leq q$, we also have $m \in L^s(B_1)$ and therefore $m \in W^{2,s}_{\mathrm{loc}}(B_{99/100})$. In addition, there exists $C > 0$ such that

$$\begin{aligned}\|m\|_{W^{2,s}(B_{9/10})} &\leq C \left(\|h\|_{L^d(B_1)} + \|m\|_{L^s(B_1)} + \|u\|^{\max(\alpha,\beta)}_{W^{2,q}(B_1)} \right) \\ &\leq C \left(\|h\|_{L^d(B_1)} + \|u\|_{W^{2,q}(B_1)} + \|u\|^{\max(\alpha,\beta)}_{W^{2,q}(B_1)} \right),\end{aligned}$$

where the second inequality follows from the fact that

$$\|m\|_{L^s(B_{9/10})} = \|\Delta u\|_{L^s(B_{9/10})} \leq \|u\|_{W^{2,q}(B_1)},$$

Because of Gagliardo-Nirenberg-Sobolev's embedding theorem, we obtain $m \in C^{0,\sigma}(\overline{B_{8/9}})$, with

$$\sigma := 2 - \frac{d \max(\alpha,\beta)}{q}.$$

Since u is an L^q−strong solution to $\Delta u = m$, we have $u \in C^{2,\sigma}(\overline{B_{8/9}})$ (see [10, Thm. 9.19]). Also, by Schauder's theory, there exists a positive constant

$C = C(d, \alpha, \beta, q)$, such that

$$\|u\|_{C^{2,\sigma}(B_{7/8})} \leq C\left(\|u\|_{L^\infty(B_{8/9})} + \|m\|_{C^{0,\sigma}(B_{8/9})}\right).$$

To complete the proof, we notice that

$$\begin{aligned} \|m\|_{C^{0,\sigma}(B_{8/9})} &\leq C\,\|m\|_{W^{2,s}(B_{8/9})} \\ &\leq C\,\|f(\cdot, u, Du)\|_{L^s(B_{9/10})} \\ &\leq C\left(\|h\|_{L^d(B_1)} + \|u\|^{\max(\alpha,\beta)}_{W^{2,q}(B_1)}\right), \end{aligned}$$

and the estimate in the theorem follows. □

Remark 1. A fundamental question arises in the context of assumption (13): one must ensure that

$$\max(\alpha, \beta)\,\frac{d}{2} < d,$$

so the range for q is nonempty. The above inequality is indeed satisfied due precisely to (12).

The explicit description of the modulus of continuity is appealing, as it provides asymptotic information. As $q \to d$ and the growth conditions for f approach the linear regime, the exponent $\sigma \to 1$, yielding asymptotic estimates for (1) in $C^{2,1}$. We also notice the explicit gains of regularity stemming from (1). Indeed, we start with a function $u \in W^{2,q}(\Omega)$ and the equation yields $u \in C^{2,\sigma}_{\rm loc}(\Omega)$, with estimates.

We conclude this note with a corollary to Theorem 1 yielding smoothness of the solutions to (1) in the case $\alpha = \beta = 1$, under the assumption that $h \in C^\infty(\Omega)$.

Corollary 1 (C^∞-regularity estimates)**.** *Let $u \in W^{2,q}(B_1)$ be a weak solution to* (1)*, with $q \geq 2$ satisfying* (13)*. Suppose* (11) *is in force, with*

$$f(x, r, p) := h(x) + a(x)r + c(x) \cdot p,$$

where $h, a \in C^\infty(B_1)$ and $c \in C^\infty(B_1, \mathbb{R}^d)$. Suppose further there exists $C > 0$ such that

$$\|h\|_{C^\infty(B_1)} + \|a\|_{C^\infty(B_1)} + \|c\|_{C^\infty(B_1,\mathbb{R}^d)} \leq C.$$

Then $u \in C^{\infty}_{\text{loc}}(B_1)$. Moreover, for every $k \in \mathbb{N}$ and every multi-index α with $|\alpha| = k$, we have

$$\sup_{B_{7/6}} |D^\alpha u| \leq C \left(1 + \|u\|_{W^{2,q}(B_1)}\right).$$

Proof. Fix a direction $i \in \{1, \ldots, d\}$ and define

$$v := \frac{\partial u}{\partial x_i}.$$

Clearly, v solves

$$\Delta^2 v = g(x) + a(x)v(x) + c(x) \cdot Dv,$$

with g given by

$$g(x) := \frac{\partial}{\partial x_i} h(x) + \frac{\partial}{\partial x_i} a(x)u(x) + \frac{\partial}{\partial x_i} c(x) \cdot Du(x).$$

One easily notices that

$$|g(x) + a(x)v(x) + c(x) \cdot Dv| \leq C \left(1 + \|u\|_{W^{2,q}(B_1)} + |v(x)| + |Dv(x)|\right).$$

Hence, Theorem 1 implies $v \in C^{2,\sigma}_{\text{loc}}(B_1)$. Because the direction i is arbitrary, we conclude $u \in C^{3,\sigma}_{\text{loc}}(B_1)$. An induction argument on the order of differentiation completes the proof. $\square$

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