

## OSCILLATORY FREE BOUNDARY PROBLEMS IN STOCHASTIC MATERIALS

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ABSTRACT. We investigate a class of free boundary problems with oscillatory singularities within stochastic materials. Our main result yields sharp regularity estimates along the free boundary, provided the power of the singularity varies in a Dini-continuous fashion below a certain threshold. We also reveal an interesting repelling estimate preventing the free boundary from touching the region where the singularity power oscillates above the threshold.

### 1. INTRODUCTION

In this work, we investigate non-negative local minimizers of energy functionals with varying singularities, taking into consideration the complexities stemming from the stochastic nature of the media in question. More precisely, we are interested in regularity estimates for local minimizers of the energy-functional

$$\mathcal{F}(u) := \int_{\Omega} \langle A(x)\nabla u, \nabla u \rangle + \lambda(x)u^{\gamma(x)}\chi_{\{u>0\}} dx, \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , is an open bounded set. The matrix  $A(x)$  is only assumed to be symmetric, measurable, and uniformly elliptic, *i.e.*,

$$\mu|\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \mu^{-1}|\xi|^2, \quad (1.2)$$

for some positive constant  $0 < \mu \leq 1$ . The forcing term  $\lambda(x)$  is non-negative and bounded, *i.e.*,  $0 \leq \lambda(x) \leq \Lambda$ . The exponent function  $\gamma(x)$  is non-negative, measurable, with

$$0 \leq \gamma(x) \leq \gamma^* < 2^*, \quad (1.3)$$

where  $2^* = \frac{2n}{n-2}$  stands for the critical Sobolev exponent.

Three important features of the model considered in this paper should be noted. Firstly, it is a free boundary problem which allows for a continuum

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range of singularities, *i.e.*, the exponent  $\gamma$  varies point-by-point. The static version, when  $0 < \gamma(x) \equiv \gamma_0 < 1$ , boasts a notable historical foundation, tracing its origins to the pioneering contributions of Phillips [14, 15] and Alt–Phillips [1]. These works have inspired substantial advances in the theory of free boundary problems through the years; see, for instance, [3, 6, 7, 8, 10, 9, 17, 18, 19, 20], among others.

The analysis of free boundary problems shaped via oscillatory singularities offers a robust mathematical framework to model more realistic phenomena, and it is considerably more involved. The inaugural work developed in [2], where additional insights on the importance of such models can be found, provides a thorough analysis of the problem wherein diffusion is governed by the Laplace operator, *i.e.*, assuming the matrix coefficient to be the Identity,  $A(x) = \text{Id}$ . This sets the stage for the second and primary innovation addressed in the present paper: an extension of the modelling framework to include stochastic materials. These materials, characterized by randomness or variability in their properties, introduce an element of unpredictability. Hence, the mathematical setup leads to the analysis of such problems with no predetermined structure, and consequently, we consider the coefficient matrix  $A(x)$  to be merely measurable and elliptic. Notably, in the realm of stochastic materials, the formulation of variational free boundary problems proves to be exceptionally challenging, and the current state of our understanding regarding solutions and their corresponding free boundaries in this context remains somewhat limited.

The third new feature of the model treated in this paper concerns the fact that we broaden the range in which the singularity varies, allowing  $\gamma(x)$  to oscillate between 0 and  $2^*$ . The analysis in [2] is limited to singularities oscillating between 0 and 1. While this may initially seem a minor extension, it allowed us to discover an interesting new feature of free boundary problems shaped via oscillatory singularities: a sort of free boundary repelling estimate, as described in the last part of our main theorem. We further comment that the upper bound for the oscillation range is required for the boundedness of local minimizers. This is due to classical considerations regarding the conformal invariance of the problem, as in the classical Yamabe’s problem. All the results are local, so we consider, with no loss, the simplest case,  $\Omega = B_1$ .

The main result of this paper can be stated as follows.

**Theorem 1.** *Let  $u$  be a non-negative local minimizer of the energy functional (1.1) in  $B_1 \subset \mathbb{R}^n$ . Then*

(1)  $u \in C^{0,\epsilon}(B_{1/2})$ , for some universal  $0 < \epsilon = \epsilon(n, \mu, \gamma^*) < 1$ , and

$$\|u\|_{C^{0,\epsilon}(B_{1/2})} \leq C \|u\|_{L^2(B_1)},$$

for a constant  $C$ , depending only on  $n, \mu, \gamma^*, \|\lambda\|_\infty$ .

- (2) If  $\gamma(x)$  is Dini continuous at  $x_0 \in B_{1/2} \cap \partial\{u > 0\}$  and  $\gamma(x_0) < 2$ , then

$$\sup_{B_r(x_0)} u \leq C_0 r^{\frac{2}{2-\gamma(x_0)}}, \quad (1.4)$$

for a constant  $C_0$ , depending only on  $n, \mu, \gamma^*, \|\lambda\|_\infty$ , and the Dini-modulus of continuity of  $\gamma$  at  $x_0$ .

- (3) Finally, if  $\gamma(x)$  is continuous, with modulus of continuity  $\omega$ , and  $x_0 \in B_{1/2}$  is such that  $\gamma(x_0) > 2$ , then there exists a constant  $\varrho_0 > 0$ , depending only on  $n$ , the medium  $A(x)$ ,  $\|\lambda\|_\infty$ ,  $\|u\|_\infty$ ,  $\gamma^*$  and  $\omega$ , such that

$$\text{dist}(x_0, \partial\{u > 0\}) \geq \varrho_0. \quad (1.5)$$

In particular, if  $x_0 \in Z(u) := \{u = 0\} \cap B_{1/2}$ , then  $x_0$  must be an interior point of the coincidence set  $Z(u)$ .

Probably, the primary contribution of Theorem 1 concerns the growth estimate (1.4), which yields a considerable gain of regularity along the associated free boundaries. As noted in the first part of Theorem 1, local minimizers are only  $C^{0,\epsilon}$  within the non-coincidence set  $\{u > 0\}$ . A comparable regularity estimate was previously established in [2] for the case where  $A(x) \equiv \text{Id}$ . In that context, however, local minimizers are proven to be of class  $C^{1,\alpha}$  everywhere. The sharp regularity estimate is then derived from such local regularity properties of minimizers, which are subsequently transferred to the free boundary points through the continuity properties of  $\gamma$ . The analysis to attain the corresponding estimate (1.4) in that case was considerably softer than the one carried out in this paper.

One should interpret estimate (1.5) as a sort of *unique continuation* property tailored explicitly for this class of free boundary models. Heuristically, it asserts that if  $u$  possesses an infinite order zero, then  $u$  must be identically zero within a quantifiably defined ball. The proof unfolds through an arguably charming blow-up analysis, reminiscent of the approach used in [4], albeit with a distinctive emphasis and nature.

We further comment that the proof delivered in Section 4 has additional implications. Actually, estimate (1.5) remains valid under the condition that  $\gamma(x) \geq 2$  within a specified ball. In essence, our proof establishes that the free boundary stays away from the interior of the set  $\gamma^{-1}[2, 2^*)$ .

The paper is structured as follows. In Section 2, we outline some elementary, though fundamental, results concerning local minimizers of the energy functional investigated in this study. This section includes a proof of local boundedness and universal Hölder regularity. Moving to Section 3, we establish the regularity gain along the free boundary. Lastly, in Section 4, we

establish the third and final piece of information encoded in Theorem 1: a quantified repelling estimate that prevents free boundaries from intersecting the set  $\gamma^{-1}(2, 2^*)$ .

## 2. PRELIMINARY RESULTS

In this section, we discuss some elementary results about local minimizers of the energy functional (1.1). Additionally, we explore essential concepts related to the problem, which, while somewhat well-known, are introduced here for the convenience of the readers.

Given an energy functional  $F(\mathcal{O}, v)$  acting on the Sobolev space  $H^1(\Omega)$ , a function  $w \in H^1(B_1)$  is said to be a local minimizer of  $F$  if

$$F(\mathcal{O}, w) \leq F(\mathcal{O}, v),$$

for all  $v$  such that  $v - w \in H_0^1(\mathcal{O})$ . The regularity theorems we will prove in this paper concern local minimizers of the energy functional (1.1). This is an abundant class of functions. Indeed, we start by commenting that given a boundary datum  $\phi \in H^1(\Omega)$ , there always exists a global minimizer (and thus a local minimizer) of the energy functional (1.1) taking  $\phi$  as a boundary value in the sense of the traces. The proof follows standard procedures, and we provide a brief sketch below. For the sake of notation convenience, given  $\phi \in H^1(\Omega)$ , let's denote the affine space

$$\mathcal{K}_\phi := \{v \in H_1(\Omega) \mid v - \phi \in H_0^1(\Omega)\}.$$

**Proposition 1.** *Given  $\phi \in H^1(\Omega)$ , there exists  $u \in \mathcal{K}_\phi$  satisfying*

$$\mathcal{F}(u) = \min_{v \in \mathcal{K}_\phi} \mathcal{F}(v).$$

*Furthermore,  $u$  is non-negative provided  $\phi \geq 0$ . Also if  $\lambda \geq 0$  and  $\phi \in L^\infty$ , then  $\|u\|_\infty \leq \|\phi\|_\infty$ .*

*Proof.* Set  $m := \inf_{v \in \mathcal{K}_\phi} \mathcal{F}(v)$  and let  $(v_k)_{k \geq 1} \in \mathcal{K}_\phi$  be a minimizing sequence. By ellipticity and Poincaré's inequality, the sequence  $(v_k)_{k \geq 1}$  is bounded in  $H^1$ , thus, up to a subsequence,  $u_k \rightarrow u$  weakly in  $H^1$  and, due to Sobolev embedding, strongly in  $L^{\gamma^*}(\Omega)$ . Clearly  $u \in \mathcal{K}_\phi$ . By weak lower semi-continuity and Fatou's lemma, we conclude

$$m \leq \mathcal{F}(u) \leq \liminf_{k \rightarrow \infty} \mathcal{F}(v_k) = m,$$

and thus  $u$  is a minimizer of  $\mathcal{F}(u)$  over  $\mathcal{K}_\phi$ .

Next, if  $\phi \geq 0$ ,  $u^+$  competes with  $u$  in the minimization problem. We then deduce, by ellipticity,

$$\int_{\{u < 0\}} |\nabla u|^2 dx \leq 0.$$

Finally, if  $\phi \in L^\infty$  and  $f \geq 0$ , the function  $u_M := \min\{u, \|\phi\|_\infty\}$  competes with  $u$  and we similarly deduce

$$\int_{\{u>M\}} |\nabla u|^2 dx \leq \int_{\{u>M\}} f(x) \left[ M^{\gamma(x)} - u^\gamma(x) \right] dx \leq 0.$$

□

Next, we record the scaling feature of the energy functional  $\mathcal{F}$  in a lemma. The proof follows from direct calculation via a change of variables.

**Lemma 1** (Scaling). *Let  $u \in H^1(\Omega)$  be a local minimizer of (1.1). Given  $x_0 \in \Omega$  and  $0 < \varrho < \text{dist}(x_0, \partial\Omega)$  and  $\mu > 0$ , the function*

$$v(x) := \frac{u(x_0 + \varrho x)}{\mu}, \quad x \in B_1$$

is a local minimizer of the functional

$$\mathcal{F}_{x_0, \varrho, \mu}(v) := \int_{B_1} \left( \langle A_{x_0, \varrho}(x) \nabla v, \nabla v \rangle + \lambda_{x_0, \varrho, \mu}(x) (v \chi_{\{v>0\}})^{\gamma_{x_0, \varrho}(x)} \right) dx,$$

where

$$\begin{aligned} A_{x_0, \varrho}(x) &:= A(x_0 + \varrho x), \\ \lambda_{x_0, \varrho, \mu}(x) &:= \mu^{\gamma(x_0 + \varrho x)} \left( \frac{\varrho}{\mu} \right)^2 \lambda(x_0 + \varrho x) \\ \gamma_{x_0, \varrho}(x) &:= \gamma(x_0 + \varrho x). \end{aligned}$$

**2.1.  $L^\infty$  bounds.** In what follows, we will consider a more general family of energy functionals of the form

$$\mathcal{F}_\rho(u) := \int_\Omega \langle A(x) \nabla u, \nabla u \rangle + \rho(x, u) dx, \quad (2.1)$$

where  $\rho: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$|\rho(x, u)| \leq f(x) + c|u|^\gamma, \quad (2.2)$$

with  $f \in L^q$ , for  $q > \frac{n}{2}$  and  $2 \leq \gamma < 2^* = \frac{2n}{n-2}$ .

It is critical to remark that we do not require the differentiability of  $\rho$  with respect to its  $u$  argument. We also comment that, while condition  $2 \leq \gamma$  is not restrictive, assuming  $\gamma$  is strictly less than  $2^*$  is critical for the local boundedness due to the conformal invariance of the problem. Finally, we note that the functional (1.1) trivially satisfies (2.2), for

$$\left| \lambda(x) (u \chi_{\{u>0\}})^{\gamma(x)} \right| \leq \|\lambda\|_\infty \max\{1, |u|\}^{\gamma^*} \leq \|\lambda\|_\infty (1 + |u|^{\gamma^*}).$$

We next comment on an  $L^\infty$ -bound for local minimizers of  $\mathcal{F}_\rho$ . The rationale follows a somewhat standard path along the lines of [11]; yet, we opt to include a proof as a courtesy to the readers.

**Proposition 2.** *Let  $u$  be a local minimizer of (2.1) in  $B_1$  under condition (2.2). Then,*

$$\|u\|_{L^\infty(B_{1/2})} \leq C \|u\|_{L^2(B_1)},$$

for a constant  $C$  depending only on  $n, \mu, \|f\|_q$  and  $\gamma$ .

*Proof.* The proof is based on De Giorgi's method. For  $0 < t < 1$  and  $k \in \mathbb{R}$ , denote

$$\Omega^k(t) := \{x \in B_t \mid u - k > 0\}.$$

For  $0 < r < R \ll 1$ , to be set *a posteriori*, let  $\eta$  be a radially symmetric cut-off function  $\eta \in C_0^\infty(B_R)$ , with  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  in  $B_r$ , and  $|D\eta| \leq 2(R-r)^{-1}$ . Denote  $w_k = (u - k)^+$  and consider the competing function  $v := u - \eta w_k$ . Since  $u$  is a local minimizer of  $\mathcal{F}_\rho$ , we have

$$\mathcal{F}_\rho(\Omega^k(R), u) \leq \mathcal{F}_\rho(\Omega^k(R), v),$$

which leads to

$$\int_{\Omega^k(R)} |\nabla u|^2 dx \leq C_1 \left( \int_{\Omega^k(R)} |\nabla v|^2 dx + \int_{\Omega^k(R)} f(x) + |u|^\gamma + |v|^\gamma dx \right). \quad (2.3)$$

Note that, in  $\Omega^k(R)$ , we have

$$u = (1 - \eta)u + \eta(w_k + k) \quad \text{and} \quad v = (1 - \eta)u + k\eta.$$

Hence

$$|u|^\gamma + |v|^\gamma \leq C_2 ((\eta w_k)^\gamma + |u|^\gamma (1 - \eta)^\gamma + \eta^\gamma k^\gamma)$$

and

$$|\nabla v|^2 \leq C_2 \left( (1 - \eta)^2 |\nabla u|^2 + \frac{1}{(R - r)^2} (u - k)^2 \right). \quad (2.4)$$

By Hölder's inequality, we estimate

$$\begin{aligned} \int_{\Omega^k(R)} (\eta w_k)^\gamma dx &= \int_{\Omega^k(R)} (\eta w_k)^2 (\eta w_k)^{\gamma-2} dx \\ &\leq \left( \int_{\Omega^k(R)} (\eta w_k)^{2^*} dx \right)^{\frac{2}{2^*}} \left( \int_{B_R} (\eta w_k)^{\frac{(\gamma-2)n}{2}} dx \right)^{\frac{2}{n}}. \end{aligned}$$

Note that the quantity

$$\zeta(s) := \left( \int_{B_s} (\eta w_k)^{\frac{(\gamma-2)n}{2}} dx \right)^{2/n}$$

is a modulus of continuity, uniform in  $k$ . Indeed, by Hölder's inequality, we can estimate

$$\zeta(s) \leq \left( \int_{B_s} u^{\frac{(\gamma-2)n}{2}} dx \right)^{2/n} \leq \|u\|_{L^{2^*}}^{\gamma-2} |B_s|^{1-\frac{\gamma}{2^*}}.$$

Note that we have  $\gamma < 2^*$ . Returning to (2.3), with the aid of (2.4), and applying Sobolev inequality, we can further estimate

$$\begin{aligned} \int_{\Omega^k(R)} |\nabla u|^2 dx &\leq S\zeta(R) \left( \int_{\Omega^k(R)} |\nabla \eta w_k|^2 dx \right) \\ &\leq SC_2\zeta(R) \left( \int_{\Omega^k(R)} |\nabla u|^2 dx + \frac{1}{(R-r)^2} \int_{\Omega^k(R)} w_k^2 dx \right), \end{aligned}$$

where  $S$  is the Sobolev constant. We can add to both sides in (2.3) the term

$$\int_{\Omega^k(R)} |u|^\gamma dx.$$

Now, choose  $0 < R \ll 1$  small enough so that  $SC_2\zeta(R) = 1/2$ . Combining all estimates above, keeping in mind  $\eta \equiv 1$  in  $B_r$ , yields

$$\begin{aligned} \int_{\Omega^k(R)} |\nabla u|^2 + |u|^\gamma dx &\leq C \left( \int_{\Omega^k(R) \setminus \Omega^k(r)} |\nabla u|^2 + |u|^\gamma dx \right. \\ &\quad \left. + \frac{1}{(R-r)^2} \int_{\Omega^k(R)} w_k^2 dx + k^\gamma |\Omega^k(R)| \right). \end{aligned}$$

Applying a standard analysis lemma, similar to [11, Lemma 1.1], we reach

$$\int_{\Omega^k(r)} |\nabla u|^2 + |u|^\gamma dx \leq C \left( \frac{1}{(R-r)^2} \int_{\Omega^k(R)} w_k^2 dx + k^\gamma |\Omega^k(R)| \right),$$

which readily yields

$$\int_{\Omega^k(r)} |\nabla u|^2 dx \leq \frac{C}{(R-r)^2} \int_{\Omega^k(R)} (u-k)^2 dx + k^\gamma |\Omega^k(R)|. \quad (2.5)$$

Repeating the argument to  $-u$ , which minimizes a similar functional, and then applying a standard De Giorgi iterative argument, we reach the local boundedness of  $u$ , as stated.  $\square$

**2.2. Hölder continuity.** Owing to inequality (2.5), combined with the fact that if  $u$  is a local minimizer of  $\mathcal{F}_{\rho(x,u)}$  then  $-u$  is a local minimizer of  $\mathcal{F}_{\rho(x,-u)}$ , one can eventually proceed with the classical De Giorgi's proof, as delineated in [5] and in [11], to attain interior Hölder continuity of local minimizers. We shall adopt a different strategy inspired by the recent papers [12, 13, 16]. In addition to its arguably more appealing geometric flavour, this approach provides the advantage of introducing fundamental tools that will be utilized in the subsequent sections.

Our first result concerns a local minimizer's proximity (properly measured) to its  $A$ -harmonic replacement, *i.e.*, with the unique function  $h$  in  $B_r$

satisfying  $h = u$  on  $\partial B_r$  and

$$\int_{B_r} \langle A(x) \nabla h, \nabla w \rangle dx = 0,$$

for all  $w \in H_0^1(B_r)$ .

**Lemma 2.** *Let  $u$  be a local minimizer of (2.1) such that*

$$\int_{B_1} u(x)^2 dx \leq 1. \quad (2.6)$$

*There exists a constant  $C = C(n, \mu, q)$  such that, for every  $r \in (0, \frac{1}{2})$ , we have*

$$\int_{B_r} |u(x) - h(x)|^2 dx \leq Cr^{2-n/q}, \quad (2.7)$$

*where  $h$  is the  $A$ -harmonic replacement of  $u$  in  $B_r$ .*

*Proof.* Let  $0 < r < 1/2$  and  $h$  be the  $A$ -harmonic replacement of  $u$  in  $B_r$ . Since  $h$  competes with  $u$  in the minimization problem, we readily have

$$\int_{B_r} (\langle A(x) \nabla u, \nabla u \rangle - \langle A(x) \nabla h, \nabla h \rangle) dx \leq \int_{B_r} (\rho(x, h) - \rho(x, u)) dx.$$

Now, since  $h$  is the  $A$ -harmonic replacement of  $u$ , we have

$$\int_{B_r} \langle A(x) \nabla h, \nabla h \rangle dx = \int_{B_r} \langle A(x) \nabla h, \nabla u \rangle dx,$$

and, due to the symmetry and the ellipticity of  $A$ , this implies that

$$\begin{aligned} & \int_{B_r} (\langle A(x) \nabla u, \nabla u \rangle - \langle A(x) \nabla h, \nabla h \rangle) dx \\ &= \int_{B_r} \langle A(x) \nabla(u - h), \nabla(u - h) \rangle dx \\ &\geq \mu \int_{B_r} |\nabla(u - h)|^2 dx. \end{aligned}$$

Because of (2.6) and Proposition 2,  $u$  is bounded in  $B_r$ , and, by the maximum principle, so is  $h$ . Hence, using (2.2) along with Hölder's inequality, we obtain

$$\begin{aligned} \int_{B_r} (\rho(x, h) - \rho(x, u)) dx &\leq \int_{B_r} (|\rho(x, h)| + |\rho(x, u)|) dx \\ &\leq \int_{B_r} [2|f(x)| + C] dx \\ &\leq C|B_r|^{1/q'}. \end{aligned} \quad (2.8)$$

Combining the above estimates yields

$$\int_{B_r} |\nabla(u - h)|^2 dx \leq \frac{C}{\mu} |B_r|^{-1/q}$$



and, using the Poincaré inequality in balls, we conclude

$$\int_{B_r} |u - h|^2 dx \leq Cr^{2-n/q}.$$

□

We next explore this proximity to the  $A$ -harmonic replacement and the regularity properties of the latter to start preparing the proof of the Hölder continuity of local minimizers. Recall that, by De Giorgi–Nash–Moser regularity theory,  $h$  is locally Hölder continuous, satisfying, for an (optimal) exponent  $\alpha_0 \in (0, 1)$ , the estimate

$$\|h\|_{C^{0,\alpha_0}(B_s)} \leq C \|h\|_{L^2(B_{2s})}, \quad (2.9)$$

for  $0 < s < 1/2$ .

**Lemma 3.** *Let  $u$  be a local minimizer of (2.1) satisfying (2.6). For any*

$$0 < \alpha < \min \left\{ 1 - \frac{n}{2q}, \alpha_0 \right\}, \quad (2.10)$$

where  $\alpha_0$  is from (2.9), there exists  $\tau_0 \in (0, \frac{1}{4})$  and a constant  $b$  such that

$$\int_{B_{\tau_0}} |u(x) - b|^2 dx \leq \tau_0^{2\alpha}.$$

*Proof.* With  $\tau_0 \in (0, \frac{1}{4})$  to be fixed, apply Lemma 2 with  $r = 2\tau_0 \in (0, \frac{1}{2})$ , to get

$$\int_{B_{\tau_0}} |u(x) - h|^2 dx \leq C2^{2+n/q'}\tau_0^{2-n/q} = C'\tau_0^{2-n/q},$$

where  $h$  is the  $A$ -harmonic replacement of  $u$  in  $B_{2\tau_0}$ . By local Hölder regularity, there exists  $C_0 > 0$  such that

$$|h(x) - h(0)| \leq C_0|x|^{\alpha_0},$$

for any  $x \in B_{\tau_0}$ . Then,

$$\begin{aligned} \int_{B_{\tau_0}} |u(x) - h(0)|^2 dx &\leq 2 \int_{B_{\tau_0}} |u(x) - h(x)|^2 dx \\ &\quad + 2 \int_{B_{\tau_0}} |h(x) - h(0)|^2 dx \\ &\leq 2C'\tau_0^{2-n/q} + 2C_0^2\tau_0^{2\alpha_0}. \end{aligned}$$

We conclude the proof by choosing  $\tau_0$  so small that both

$$2C'\tau_0^{2-n/q} \leq \frac{\tau_0^{2\alpha}}{2} \quad \text{and} \quad 2C_0^2\tau_0^{2\alpha_0} \leq \frac{\tau_0^{2\alpha}}{2},$$

which is possible due to (2.10). □

**Remark 1.** We remark that  $b = h(0)$  is uniformly bounded in view of the maximum principle and Proposition 2. This means that there exists a constant  $L > 0$ , depending only on the data, such that  $|b| \leq L$ .

Next, we iterate the previous result to obtain the Hölder continuity of a local minimizer in a ball.

**Lemma 4.** For any  $\alpha$  satisfying (2.10), there exist  $\tau_0 \in (0, \frac{1}{4})$  and a constant  $C > 0$ , depending only on universal quantities and  $\alpha$ , such that if  $u$  is a local minimizer of (2.1) satisfying (2.6), then

$$|u(x) - u(0)| \leq C|x|^\alpha, \quad (2.11)$$

for every  $x \in B_{\tau_0}$ .

*Proof.* Given  $\alpha$  satisfying (2.10), we consider  $\tau_0 \in (0, \frac{1}{4})$  the corresponding radius given by Lemma 3.

For  $u$  a local minimizer of (2.1), we claim that

$$\int_{B_{\tau_0^k}} |u(x) - b_k|^2 dx \leq \tau_0^{2k\alpha}, \quad \text{for } k = 1, 2, \dots, \quad (2.12)$$

for some convergent sequence  $(b_k)$ . We will use induction over  $k$  to prove the claim, the case  $k = 1$  being Lemma 3, for some constant  $b_1$  such that  $|b_1| \leq L$ , with  $L$  depending only on the data. Now, suppose that (2.12) is true for  $k > 1$  and some constant  $b_k$ , and let's show that the same is true for  $k + 1$  and a constant  $b_{k+1}$ . We define

$$w_k(x) = \frac{u(\tau_0^k x) - b_k}{\tau_0^{k\alpha}}, \quad x \in B_1,$$

and observe that  $w_k$  is a local minimizer of the functional

$$\int_{B_1} \left( \langle \tilde{A}(x) \nabla v, \nabla v \rangle + \rho_k(x, v) \right) dx,$$

where  $\tilde{A}(x) = A(\tau_0^k x)$  and  $\rho_k(x, v) = \tau_0^{2k(1-\alpha)} \rho(\tau_0^k x, \tau_0^{k\alpha} v + b_k)$ . Note that this functional remains in the same class.

Now, due to the induction hypothesis (2.12), we have

$$\int_{B_1} w_k^2 dx = \int_{B_1} \frac{|u(\tau_0^k x) - b_k|^2}{\tau_0^{2k\alpha}} dx \leq 1.$$

Thus,  $w_k$  is entitled to Lemma 3, and there exists a constant  $b$ , with  $|b| \leq L$  such that

$$\int_{B_{\tau_0}} |w_k(x) - b|^2 dx \leq \tau_0^{2\alpha}.$$

Translating this back to  $u$ , we get

$$\int_{B_{\tau_0^{k+1}}} \left| u(x) - b_k - \tau_0^{k\alpha} b \right|^2 dx \leq \tau_0^{2(k+1)\alpha}.$$

Thus, choosing  $b_{k+1} = b_k + \tau_0^{k\alpha} b$ , we obtain (2.12) for  $k+1$  as desired.

To show the sequence  $(b_k)$  is convergent, we will show it is a Cauchy sequence. First, note that, by construction,

$$|b_{k+1} - b_k| \leq L\tau_0^{k\alpha}.$$

Then, for  $b_m$  and  $b_k$ , with  $m > k$ , we have

$$\begin{aligned} |b_m - b_k| &\leq |b_m - b_{m-1}| + \dots + |b_{k+1} - b_k| \\ &\leq L \sum_{i=k}^{m-1} \tau_0^{i\alpha} \\ &= \frac{L\tau_0^{k\alpha}}{1 - \tau_0^\alpha} \left[ 1 - \tau_0^{(m-k)\alpha} \right]. \end{aligned}$$

Since  $0 < \tau_0 < 1/4$ , taking  $k \rightarrow \infty$ , we get

$$|b_m - b_k| \rightarrow 0$$

and thus  $(b_k)$  is a Cauchy sequence, so it converges to some  $b_0$ , with

$$|b_k - b_0| \leq \frac{L\tau_0^{k\alpha}}{1 - \tau_0^\alpha}.$$

Finally, for  $0 < \tau < \tau_0$ , we choose  $k$  such that  $\tau_0^{k+1} < \tau \leq \tau_0^k$ . By (2.12), we get

$$\begin{aligned} \int_{B_\tau} |u(x) - b_0|^2 dx &\leq 2 \int_{B_\tau} |u(x) - b_k|^2 dx + 2 \int_{B_\tau} |b_k - b_0|^2 dx \\ &\leq 2\tau_0^{-n} \int_{B_{\tau_0^k}} |u(x) - b_k|^2 dx + 2\tau_0^{-n} \int_{B_{\tau_0^k}} |b_k - b_0|^2 dx \\ &\leq 2\tau_0^{-n} \left( 1 + \frac{L^2}{(1 - \tau_0^\alpha)^2} \right) \tau_0^{2k\alpha} \\ &\leq 2\tau_0^{-n-2\alpha} \left( 1 + \frac{L^2}{(1 - \tau_0^\alpha)^2} \right) \tau^{2\alpha}. \end{aligned}$$

By Campanato's characterization of Hölder continuity, we obtain (2.11).  $\square$

The Hölder regularity for local minimizers of (2.1) follows readily from the previous lemma, and, in particular, we obtain the first part of Theorem 1.

*Proof of Theorem 1–(1).* For  $x_0$  an arbitrary point in  $\Omega$ , define the scaling

$$v(x) = \kappa u(x_0 + rx), \quad x \in B_1,$$

where

$$0 < r < \frac{1}{2} \text{dist}(x_0, \partial\Omega) \quad \text{and} \quad \kappa = \left( \int_{B_r(x_0)} u^2 dx \right)^{-1/2}.$$

Note that

$$\int_{B_1} v^2 dx = 1$$

and that  $v$  is a local minimizer for the functional

$$\int_{B_1} \left( \langle \tilde{A}(x) \nabla v, \nabla v \rangle + \rho_k(x, v) \right) dx,$$

where  $\tilde{A}(x) = A(x_0 + rx)$  and  $\rho_k(x, v) = \kappa^2 r^2 \rho(x_0 + rx, \kappa^{-1}v)$ . By Lemma 4, there exist  $\tau_0 \in (0, 1/4)$  and a universal constant  $C > 0$  such that

$$|v(x) - v(0)| \leq C|x|^\alpha, \quad x \in B_{\tau_0}.$$

Translating this information to  $u$ , we obtain

$$|u(x) - u(x_0)| \leq C' \|u\|_{L^2(B_r(x_0))} |x - x_0|^\alpha, \quad x \in B_{r\tau_0}(x_0),$$

with  $C'$  depending only on universal quantities,  $\alpha$  and  $\text{dist}(x_0, \partial\Omega)$ . Using Proposition 2, the result follows.  $\square$

### 3. REGULARITY GAIN

In this section, we prove the central part of Theorem 1, which provides an accurate pointwise regularity at free boundary points for local minimizers with Dini continuous oscillatory singularities. Thus, hereafter, we will focus on non-negative local minimizers of the original energy functional (1.1). For simplicity, we restrict the analysis to the case  $\Omega = B_1$ . We start with a quantitative flatness improvement result.

**Lemma 5.** *Given any  $0 < \varrho \leq 1$ , there exists  $\lambda_\varrho > 0$ , depending only on  $n$ ,  $\mu$  and  $\varrho$ , such that if  $u$  is a local minimizer of (1.1) satisfying  $0 \leq u \leq 1$  in  $B_1$  and  $u(0) = 0$ , with*

$$0 \leq \lambda(x) \leq \lambda_\varrho,$$

then

$$\sup_{B_{1/2}} u \leq \varrho.$$

*Proof.* Let  $h$  be the  $A$ -harmonic replacement of  $u$  in  $B_{1/2}$ . Revisiting the proof of Lemma 2, with a closer look at estimate (2.8), we reach

$$\int_{B_{1/2}} |u - h|^2 dx \leq C_0 \|\lambda\|_\infty, \quad (3.1)$$

where  $C_0$  depends only on dimension and ellipticity. Next we note that, since  $u$  is a local minimizer of (1.1), it satisfies

$$\operatorname{div}(A(x)\nabla u) \geq 0, \quad (3.2)$$

in the distributional sense, *i.e.*,  $u$  is  $A$ -subharmonic. In particular,  $0 \leq u \leq h$  in  $B_{1/2}$ . Applying the Harnack inequality for  $h$ , along with standard  $L^2$ - $L^\infty$  estimates for  $h$ , we obtain

$$\sup_{B_{1/2}} u \leq \sup_{B_{1/2}} h \leq C_1 h(0) \leq C_1 C_2 r^{-d/2} \|h\|_{L^2(B_r)}, \quad (3.3)$$

where  $C_1$  and  $C_2$  are universal constants and  $0 < r < 1/2$  is arbitrary. Next, using the already proven  $\epsilon$ -Hölder continuity of  $u$  at 0, together with the fact that  $u(0) = 0$ , we estimate

$$\begin{aligned} \int_{B_r} h^2 dx &\leq 2 \left( \int_{B_r} |h - u|^2 dx + \int_{B_r} u^2 dx \right) \\ &\leq 2 \left( (2r)^{-n} \int_{B_{1/2}} |h - u|^2 dx + Cr^{2\epsilon} \right). \end{aligned} \quad (3.4)$$

Combining (3.1), (3.3), and (3.4) we reach

$$\begin{aligned} \left( \sup_{B_{1/2}} u \right)^2 &\leq (C_1 C_2)^2 \omega_n \int_{B_r} h^2 dx \\ &\leq 2(C_1 C_2)^2 \omega_n \left( (2r)^{-n} C_0 \|\lambda\|_\infty + Cr^{2\epsilon} \right), \end{aligned} \quad (3.5)$$

where  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . Now, given  $0 < \varrho \leq 1$ , we select  $0 < r_\varrho \ll 1/2$  satisfying

$$2(C_1 C_2)^2 \omega_n C r_\varrho^{2\epsilon} = \frac{1}{2} \varrho^2. \quad (3.6)$$

Finally, with  $0 < r_\varrho \ll 1/2$  chosen, we take  $\lambda_\varrho > 0$  as to verify

$$2(C_1 C_2)^2 \omega_n (2r_\varrho)^{-n} C_0 \lambda_\varrho = \frac{1}{2} \varrho^2, \quad (3.7)$$

and the proof of the lemma is complete.  $\square$

Before proving our main result, we analyze the assumptions on  $\lambda$  and  $\gamma$  that guarantee the scaling invariance for minimizers of the functional (1.1). Given parameters  $0 < r \leq 1$  and  $\beta > 1$ , let

$$v(x) := \frac{u(rx)}{r^\beta}, \quad x \in B_1.$$

Recalling Lemma 1, if  $u$  is a local minimizer of (1.1) in  $B_r$ , then  $v$  is a local minimizer of the functional

$$\int_{B_1} \left( \langle A_r(x) \nabla v, \nabla v \rangle + \lambda_r(x) (v \chi_{v>0})^{\gamma_r(x)} \right) dx,$$

with

$$A_r(x) := A(rx), \quad \gamma_r(x) := \gamma(rx) \quad (3.8)$$

and

$$\lambda_r(x) := \lambda(rx) r^{\gamma(rx)\beta - 2(\beta-1)} = \lambda(rx) r^{\beta(\gamma(rx) - 2 + \frac{2}{\beta})}. \quad (3.9)$$

Observe that  $A_r$  satisfies the same ellipticity condition as  $A$  and  $\gamma_r$  satisfies the same bounds as  $\gamma$ . To guarantee that  $\lambda_r$  is universally bounded, we assume that

$$\beta \leq \frac{2}{2 - \gamma_*(r)} \implies \gamma(rx) \geq 2 - \frac{2}{\beta}, \quad x \in B_1,$$

where

$$\gamma_*(r) := \inf_{x \in B_r} \gamma(x).$$

We now prove the main result in this paper, namely the sharp higher regularity at free boundary points.

*Proof of Theorem 1-(2).* We do the analysis at the origin  $x_0 = 0$  and divide the proof into three steps. Recall we are assuming  $\gamma(0) < 2$  and that  $\gamma$  is Dini continuous at 0.

**Step 1** (Pointwise scaling invariance). Let  $M > 1$  be such that

$$\log_2 M = \frac{2}{2 - \gamma(0)} \sum_{k=1}^{\infty} \omega(2^{-k}),$$

where  $\omega$  is the modulus of continuity of  $\gamma$ . Since  $\gamma$  is Dini-continuous in  $B_1$ , the above series converges, and  $M$  is well defined.

For  $\beta = \frac{2}{2 - \gamma(0)}$ , take  $r = 2^{-k}$  in (3.9), for  $k \in \mathbb{N}$ , obtaining

$$\lambda_k(x) := \lambda_r(x) = \lambda(2^{-k}x) 2^{2k \frac{\gamma(0) - \gamma(2^{-k}x)}{2 - \gamma(0)}}.$$

If  $\gamma(0) \leq \gamma(2^{-k}x)$ , we easily get  $\lambda_k(x) \leq \lambda(2^{-k}x)$ . Otherwise,

$$\begin{aligned} \lambda_k(x) &\leq \lambda(2^{-k}x) 2^{\frac{2}{2 - \gamma(0)} k \omega(2^{-k})} \\ &\leq \lambda(2^{-k}x) 2^{\frac{2}{2 - \gamma(0)} \sum_{i=1}^k \omega(2^{-i})} \\ &\leq \lambda(2^{-k}x) 2^{\frac{2}{2 - \gamma(0)} \sum_{k=1}^{\infty} \omega(2^{-k})} \\ &= \lambda(2^{-k}x) M, \end{aligned}$$

where the second inequality follows from the fact that  $\omega(\cdot)$  is increasing because it is a modulus of continuity. Therefore,

$$\lambda_k(x) \leq (1 + M) \lambda \left( 2^{-k} x \right). \quad (3.10)$$

**Step 2** (Pointwise oscillation estimates in dyadic balls). We now assume

$$0 \leq u \leq 1 \quad \text{in } B_1. \quad (3.11)$$

Take

$$\varrho := 2^{\frac{2}{\gamma(0)-2}} \quad (3.12)$$

and let  $\lambda_\varrho > 0$  be the corresponding number given by Lemma 5. We also assume

$$\|\lambda\|_\infty \leq \frac{1}{1 + M} \lambda_\varrho. \quad (3.13)$$

Now, for each  $k \in \mathbb{N}$ , we claim that

$$\sup_{B_{2^{-k}}} u \leq 2^{\frac{2k}{\gamma(0)-2}}. \quad (3.14)$$

We proceed by induction, the case  $k = 1$  being given by Lemma 5, with  $\varrho$  as in (3.12). Let's assume (3.14) holds for  $k$  and show the same is true for  $k + 1$ . Define

$$v_k(x) := 2^{\frac{2k}{\gamma(0)-2}} u \left( 2^{-k} x \right), \quad x \in B_1.$$

Observe that  $0 \leq v_k \leq 1$  in  $B_1$ , by the induction hypothesis, and that  $v_k(0) = 0$  because  $0 \in \partial\{u > 0\}$ . In addition,  $v_k$  is a minimizer of (1.1), for  $A = A_r$ ,  $\gamma = \gamma_r$  and  $\lambda = \lambda_r$ , as defined in (3.8) and (3.9), for  $r = 2^{-k}$  and  $\beta = \frac{2}{2-\gamma(0)}$ . Thus, from (3.10) and (3.13), we have

$$\lambda_r(x) = \lambda_k(x) \leq \lambda_\varrho,$$

and so  $v_k$  is entitled to Lemma 5 and we conclude

$$\sup_{B_{1/2}} v_k \leq \varrho = 2^{\frac{2}{\gamma(0)-2}},$$

which, translated into  $u$ , gives

$$\sup_{B_{2^{-(k+1)}}} u \leq 2^{\frac{2(k+1)}{\gamma(0)-2}}.$$

The induction is complete.

**Step 3** (Smallness regime). Since  $\gamma(x_0) < 2$ , fix  $0 < r_0 \ll 1$  such that

$$\gamma_0^* := \sup_{B_{r_0}(x_0)} \gamma < 2.$$

We now need to adjust the parameter  $\kappa$  in order to guarantee that

$$w(x) := \kappa^{-1} u(x_0 + r_0 x), \quad x \in B_1,$$

satisfies conditions (3.11) and (3.13). Choosing

$$\kappa := \max \left\{ \|u\|_{L^\infty}, \left( \frac{\|\lambda\|_\infty(1+M)}{\lambda_\varrho} \right)^{\frac{1}{2-\gamma_0^*}}, 1 \right\},$$

we observe that  $w$  is a minimizer of (1.1) with coefficient  $\lambda$  satisfying (3.13) and that  $\|w\|_\infty \leq 1$ .

We then apply Step 2 to  $w$  and get the desired estimate for  $u$  with an additional dependence on  $\|u\|_\infty$ .  $\square$

#### 4. FREE BOUNDARY REPELLING ESTIMATE

In this section, we present the third and concluding piece of information incorporated into the statement of Theorem 1. The goal is to show that if  $\gamma(x_0) > 2$ , then  $x_0$  must be within a universal distance from the free boundary.

*Proof of Theorem 1–(3).* The proof involves three steps. The key step is to show that  $x_0$  cannot be a free boundary point. This will be attained by showing that if  $\gamma(x_0) > 2$ , then  $\sup_{B_r} u$  cannot be strictly positive for all  $r > 0$ . The second step is to verify that if  $x_0 \in \{u = 0\} \cap \gamma^{-1}(2, 2^*)$ , then  $B_{\varrho_0}(x_0) \subset \{u = 0\}$ . This will be done via a compactness argument, reducing the problem to the free boundary, already discussed in Step 1. The third and final step is to show that if  $x_0 \in \{u > 0\} \cap \gamma^{-1}(2, 2^*)$ , then  $\text{dist}(x_0, \partial\{u > 0\}) \geq \varrho_0 > 0$ . This will be attained by simple continuity considerations. Throughout the proof, we will denote by  $\omega$  the (fixed) modulus of continuity of  $\gamma(x)$ .

**Step 1** (The free boundary cannot intersect  $\gamma^{-1}(2, 2^*)$ ). Let us assume  $x_0$  is a free boundary point corresponding to a local minimizer of (1.1) with  $\gamma(x_0) > 2$ . By continuity, we can assume

$$\inf_{B_{r_0}(x)} \gamma \geq 2,$$

for some  $r_0 > 0$ , depending only on  $\gamma(x_0)$  and  $\omega$ . We then claim that there exists a natural number  $j_0 > -\log_2 r_0$ , such that

$$\sup_{B_{2^{-(j_0+1)}}(x_0)} u \leq \delta \sup_{B_{2^{-j_0}}(x_0)} u. \quad (4.1)$$

for all  $0 < \delta < 1$ .

To prove the claim, suppose its thesis fails to hold. This implies the existence of a positive  $\delta_0 > 0$  and a sequence  $j_k \rightarrow \infty$  such that

$$\sup_{B_{2^{-(j_k+1)}}(x_0)} u > \delta_0 \sup_{B_{2^{-j_k}}(x_0)} u,$$



for all  $k \in \mathbb{N}$ . Hereafter, for each  $0 < r < 1$ , we will denote

$$S_r := \sup_{B_r(x_0)} u.$$

Note that, by our assumption, as  $x_0 \in \partial\{u > 0\}$ , we have  $0 < S_r \leq 1$ , for all  $0 < r < 1$ . Next, we define

$$v_k(x) := \frac{u(x_0 + 2^{j_k}x)}{S_{2^{-(j_k+1)}}}.$$

We easily check that  $v_k: B_1 \rightarrow \mathbb{R}$  satisfies the following four assertions.

- $0 \leq v_k(x) \leq \delta_0^{-1}$ , for all  $x \in B_1$ ;
- $v_k(0) = 0$ ;
- $\sup_{B_{1/2}} v_k = 1$ .
- $v_k$  is a local minimizer of

$$\mathcal{F}(v) = \int \langle A_k(x) \nabla v, \nabla v \rangle + \lambda_k(x) v^{\gamma_k(x)} \chi_{\{v > 0\}} dx,$$

where  $A_k(x) = A(x_0 + 2^{j_k}x)$ ,  $\gamma_k(x) = \gamma(x_0 + 2^{j_k}x)$  and, more importantly,

$$\lambda_k(x) = [S_{2^{-(j_k+1)}}]^{\gamma_k(x)-2} 4^{-j_k} \lambda(x).$$

The key remark here is that, since  $\gamma_k(x) - 2 \geq 0$ ,  $\|\lambda_k\|_\infty = o(1)$ , as  $k \rightarrow \infty$ .

We now revisit the analysis carried out in the previous section. For each  $j_k$ , let  $h_k$  be the  $A_k$ -harmonic replacement of  $v_k$  in  $B_1$ . As in the proof of Lemma 5, we obtain

$$\int_{B_1} |\nabla(v_k - h_k)|^2 dx \leq C \|\lambda_k\|_\infty = o(1), \quad (4.2)$$

as  $k \rightarrow \infty$ . This is because, as commented above, by assumption,  $\gamma_k(x) \geq 2$ , for all  $k$  and, as also previously noted,  $0 < S_{2^{-(j_k+1)}} \leq 1$ . Applying the maximum principle and Harnack's inequality for  $h_k$ , we reach

$$\sup_{B_{2/3}} v_k \leq \sup_{B_{2/3}} h_k \leq C_1 h_k(0), \quad (4.3)$$

for a universal constant  $C_1 > 0$ . Also,  $0 \leq h_k \leq \delta_0^{-1}$ . In view of the  $C^{0,\epsilon}$  regularity estimate proven for  $v_k$ , there exists a function  $v_\infty$ , such that, up to a subsequence,  $v_k \rightarrow v_\infty$  locally uniformly in  $B_1$  and weakly in  $H^1(B_1)$ . Also, by known estimates for  $h_k$ , we can assume  $h_k$  converges locally uniformly in  $B_1$  and weakly in  $H^1(B_1)$  to a function  $h_\infty$ . Due to (4.2) and Poincaré's inequality, we conclude

$$v_\infty \equiv h_\infty,$$

in  $B_1$ . In particular, because of uniform convergence,

$$v_k(0) = v_\infty(0) = h_\infty(0) = \lim_{k \rightarrow \infty} h_k(0) = 0.$$

Thus, letting  $k \rightarrow \infty$  in (4.3), we conclude

$$v_k \equiv 0 \quad \text{in } B_{2/3}.$$

However, this leads to a contradiction as, per our construction, we have  $\sup_{B_{1/2}} v_k = 1$ , for all  $k \geq 1$ .

To conclude the proof of Step 1, we simply let  $\delta \rightarrow 0$  in (4.1), leading to the conclusion

$$\sup_{B_{\varsigma_0}(x_0)} u = 0, \tag{4.4}$$

where  $\varsigma_0 = 2^{-(j_0+1)}$ . This ultimately gives a contradiction as  $x_0$  was assumed to be a free boundary point, *i.e.*,  $x_0 \in \partial\{u > 0\}$ , whereas equation (4.4) clearly forces  $x_0$  to be in the interior of the contact zero set  $\{u = 0\}$ .

**Step 2** (From the zero set to the free boundary). Assume  $x_0 \in \{u = 0\}$ , and seeking a contradiction, that no such  $\varrho_0 > 0$  exists. This means there exists a sequence of normalized, non-negative local minimizers  $\{u_n\}$  of an energy functional

$$\mathcal{F}_n(v) = \int \langle A(x) \nabla v, \nabla v \rangle + \lambda_n(x) v^{\gamma_n(x)} \chi_{\{v > 0\}} dx,$$

with

- $x_0 \in Z(u_n)$ ;
- $\text{dist}(x_0, \partial\{u_n > 0\}) < 1/n$ ;
- $\|\lambda_n\|_\infty \leq \Lambda$ ;
- $\gamma_n(x_0) \geq \gamma(x_0) > 2$ ;
- $\gamma_n$  has the same modulus of continuity,  $\omega$ , as  $\gamma$ ;
- $0 \leq \gamma_n \leq \gamma^* < 2^*$ .

By the Arzelà–Ascoli theorem, up to a subsequence,  $\gamma_n$  converges locally uniformly to a  $\omega$ -continuous function  $\gamma$ . By the estimates already proven, passing to further subsequences, if needed, we can assume  $u_n \rightarrow u$  locally uniformly and weakly in  $H^1$ . Also,  $\lambda_n \rightarrow \lambda$ , weakly in  $L^2$ , for some uniformly bounded  $\lambda$ . It is easy to verify that  $u$  is a local minimizer of

$$\mathcal{F}(v) = \int \langle A(x) \nabla v, \nabla v \rangle + \lambda(x) v^{\gamma(x)} \chi_{\{v > 0\}} dx, \tag{4.5}$$

and that  $x_0 \in \partial\{u > 0\} \cap B_{1/2}$ .

**Step 3** (Estimate in the positive set). Assume  $x_0 \in \{u > 0\} \cap B_{1/2}$  is such that  $\gamma(x_0) = 2 + \nu$ , for some  $0 < \nu < 2^* - 2$ . Let  $z_0 \in \{u > 0\} \cap B_{1/2}$  be a free boundary point satisfying

$$|z_0 - x_0| = \text{dist}(x_0, \partial\{u > 0\}) =: d.$$

By the conclusion of Step 1, we know  $\gamma(z_0) < 2$ , and then

$$\gamma(x_0) - \gamma(z_0) > \nu.$$

On the other hand, by continuity, we can estimate

$$\gamma(x_0) - \gamma(z_0) \leq \omega(|z_0 - x_0|) = \omega(d).$$

Hence, we finally obtain

$$d \geq \omega^{-1}(\nu),$$

and the proof is complete.  $\square$

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