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GEOMETRY TEACHING FROM BABYLON TO THE COMPUTER ERA[1]

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Geometry has been taught for millennia and we have rather detailed references of the geometry taught in Scribal Schools in Babylon, 4000 years ago. Today, with the computer revolution and Artificial Intelligence available, nobody seems to be proposing we stop teaching Geometry. But which Geometry should we teach, why and how? Which are the driving forces that influence each curriculum? How the new computing tools open new pathways for Geometry teaching? We will mention in some detail the controversies around the official Syllabus in Portugal in the last decades to try to identify which are the prevalent ideas in the curriculum and which are the different cultures involved in the decision process, from a purely abstract approach to a more applied approach, that answer to different societal and cultural views and mix the proposals of mathematicians, mathematics educators and teachers in the classroom. We observe that amid societal changes and advances in knowledge, Geometry is always focused on Problem Solving, use and exploration of diagrams to conjecture and achieve some kind of proof. We need to take intelligent paths to make the most of technology and not be a slave to it.

In order to mention the main ideas in some detail we have included diagrams, tables and the longer quotations in a web page available here: [***https://www.geogebra.org/m/cfnj6cb2***](https://www.geogebra.org/m/cfnj6cb2)

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THE SYLLABUS IN PORTUGAL

In the years 2021-2023 a new syllabus has been approved by the Portuguese Ministry of Education for the Mathematics courses of Basic Education (years 1 to 9) and Secondary Education (years 10 to 12), and this was done only for Mathematics. In Portugal the compulsory education includes all 12 years of school but there were no changes in other areas, as the Ministry of Education recognized there we unacceptable failure rates in Mathematics in Portugal. The Ministry began commissioning a special report to receive proposals to change the situation. The report *Recommendations for improving student learning in Mathematics* was published in 2020 (Carvalho e Silva, 2020) and was based on all the reports produced in Mathematics Education in Portugal in the last 30 years as well as on most international reports produced in this period, including international reports from OECD

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and UNESCO, as well as international assessments. It recommended a global revision of the mathematics curriculum in Portugal, that should be based on a certain number of principles, namely:

- i) Universality, internal coherence, relevance, focus and higher cognitive level.
- ii) Fulfill all the goals that justify the universality of the access to this course, taking into account the cultural, social and political dimensions of mathematics learning;
- iii) Value understanding and look for an equilibrium of problem solving, mathematical reasoning, communication, connections, multiple representations, procedural fluency, creativity, digital literacy, reflection, resilience and individual and work group.

These are general principles but particular topics like Geometry are not discussed. New syllabuses were produced in 2021 for Basic Education (Canavarro, 2021) and in 2023 for Secondary Education (Carvalho e Silva, 2023). Other details that do not relate to Geometry will not be discussed here.

Geometry in Basic Education

Geometry occupies a substantial space in the new curriculum for Basic Schools. In the first four years the students begin their "development of spatial reasoning, with an emphasis on visualization and spatial orientation" (Canavarro, 2021); they use various types of materials and also technology (namely visual programming with Scratch and robots). In the second cycle of studies (years 5 and 6) the measure of angles and the study of triangles is introduced and dynamic geometry environments, such as GeoGebra are recommended. In the last three years of Basic Education (3rd cycle, years 7 to 9) the goal for Geometry is to "continue developing students' spatial reasoning, expanding their understanding of space", as well as "the establishment of algebraic relationships from the study of geometric objects (…) accompanied by experience (where technology plays a fundamental role)" (Canavarro, 2021). Also geometric transformations are studied in a progressively more abstract and formal way.

Geometry in Secondary Education

Secondary Education in Portugal offers several paths of study that lead to further studies in Higher Education or offer a possibility of looking for a job immediately after school with a more profession oriented study. Students encounter six possible paths and in all of them we find some Geometry. The path *Mathematics Applied to Social Sciences* includes a chapter on Graph Theory and we will not discuss the details here (Carvalho e Silva, 2018). The Science and Technology path and the Economics path share the same Mathematics course spanning 3 years and 4,5 hours of class a week (Carvalho e Silva, 2023). This Mathematics course is called Mathematics A. The geometry topics studied in Mathematics A go from the notable points on the triangle with a synthetic perspective to matrices and geometric transformations, and include analytical geometry and trigonometry.

The chapter "Synthetic geometry in the plane" is a classical one but is new as such in the Portuguese curriculum. It is intended as a crossroad between classical geometry and experimentation with dynamical geometry systems, in order to "develop in students a taste for argumentation in general and proof as a central element of mathematics, for example with regard to the inscribed circle and the circumscribed circle" (Carvalho e Silva, 2023).

Analytic geometric is classic also but has a more algebraic flavor; in the curriculum students are encouraged to make software explorations (namely with *Geogebra 3D*) and work with mathematical modeling problems.

Implementation of the curricular change

We all know that an official document (intended curriculum) does not guarantee its implementation (enacted curriculum) even with good orientations (planned curriculum). Lots of countries have devised ways of connecting the intended curriculum to the enacted curriculum. We can quote the example of Costa Rica whose Reform has been very well documented by its mentor Angel Ruiz; this was a curricular change in mathematics only, that began in 2012. It developed original and innovative ideas and instruments, including blended courses for teachers, fully virtual courses with MOOC modality for high school teachers and students, and shorter virtual courses called Mini-MOOCs (Ruiz, 2015).

In Portugal several curricular changes were accompanied by a number of measures to work towards the coherence of the curriculum. Since 2021 several actions were taken: some classes (pilots) to anticipate the implementation of the new syllabus were run, tested materials were afterwards made available to the rest of the schools including comments from pilot classes, tasks were published online, continuous professional development courses were made in two phases: the first one to prepare schools leaders that in the sequence propose themselves continuous professional development courses to other teachers in all schools.

Using the history of education tool

The options taken in the new syllabus were publicly discussed, there was a preliminary version for an open discussion and public sessions were held to discuss with teachers, associations and other partners. There was no consensus on a number of issues so the team writing the curriculum (including mathematicians, mathematics educators and experienced teachers) had to choose paths in a number of issues. Why these options were ultimately held? Are they promising? Will the global situation in mathematics education improve? The tools used to make choices in the curriculum were the experience of different countries, the most successful ones including international schools with its own curriculum, and past history of mathematics education, namely in Portugal. The educational system in Portugal was structured in 1772 and greatly revamped in 1836, so there is a long history of mathematics education in Portugal. Other countries and regions have an even longer history. What can we learn from our collective past experience teaching Geometry?

HISTORY FLASHBACK ON GEOMETRY TEACHING

It is always useful to study which were the guiding principles and the school practice in past times. The Portuguese document *Recommendations for improving student learning in Mathematics* (Carvalho e Silva, 2020), produced before the last rewriting of the curriculum, made a summary of the main documents produced in Portugal in the last 30 years and discussed what happened in France, Finland, Estonia and Singapore. In the end a lot of experience was gained from the emphasis on problem solving, applications and mathematical modeling these countries have, and also on the use of technology namely with computational thinking with a large practical work in schools particularly in France (numerous classroom materials are available on the web also through the work of the well-known IREM).

Let’s go back here some millennia.

Teaching in Babylon

Scribal Schools from the Old Babylonian period are fascinating. Some 4000 years ago, we can find that “education went hand in hand with creative activity, supported by a very active milieu. (…) The most active centers were able to influence other, less creative ones, where the educational framework was less institutional. (…) These scribes conferred together and traveled.” (Proust, 2014). A lot of researchers have studied scribal schools and it is clear that Geometry was present from the beginning. As Friberg puts it: “to divide given parcels of land into shares according to some intricate set of rules, or dividing given amounts of food stuff into rations of various sizes according to some other intricate set of rules, must have been an important part of the metro-mathematical education given to the young scribes in the scribe schools” (Friberg, 2014). It is clear that Geometry, Arithmetic and Algebra were studied together. Geometrical diagrams were frequent in clay tablets.

Euclid, the *Elements* and *Pseudaria*

As Robin Hartshorne points out “Throughout most of its history, Euclid’s *Elements* has been the principal manual of geometry and indeed the required introduction to any of the sciences.” (Hartshorne, 2000). More than one thousand editions and translations are known. In the great reformation of the studies in Portugal, around 1772, Mathematics took center place at the University and so all students at the University had to study Geometry, Euclidian style, and the textbook used was a translation of the *Elements* in Portuguese.

A lost work attributed to Euclid, with a more marked pedagogical character, *Pseudaria*, never made into mainstream teaching and the knowledge of its content is almost completely lost because, possibly, as points out Fabio Acerbi, “the tradition of its pedagogical use was soon broken (…) as a consequence of a likely increase of the dogmatic character of mathematical teaching in post-Hellenistic times” (Acerbi, 2007).

Multiple translations and uses of the *Elements*

As was already mentioned, more than one thousand editions and translations of Euclid’s *Element*s are known. It was used at almost all schools in the world at some point. The first Chinese translation was made by the Jesuit Matteo Ricci (1552–1610), in 1607, when he was assigned to work in China (Joseph, 2011).

The contents and style of Euclid’s *Element*s were imitated by numerous school textbooks for Basic and Secondary schools. The most used textbook in Geometry in Portugal for some 30 years, from 1944 to 1974, was the first in Portugal to use “the treatment of demonstrations in justified steps, as Americans and English use it, [and] it seems to us very advisable” (Paulo, 1944). The price to pay with this more formal approach, as the reviewer quoted wrote, is “that despite everything, sometimes the author makes an appeal to students' intuition, it is a shame that it is no longer done often”.

Modern Mathematics

The so called *Modern Mathematics Movement* changed completely the teaching of Mathematics in numerous countries including Portugal. The Portuguese mathematics curriculum for Basic and Secondary School saw a considerable reduction of the study of classical Geometry and some emphasis on the study of geometric transformations. Euclid’s approach completely disappeared.

The coordinator of the modernization of Mathematics teaching in Portugal, mathematician José Sebastião e Silva (1914-1972), wrote that the teaching of Geometry should find an equilibrium between abstractions and the real world. He was very much influenced by Emma Castelnuovo (1913-2014).

As the 24th ICMI Study volume puts it:

> Classical synthetic geometry was completely eliminated and the main aim was not to study geometrical figures but to construct an algebraic tool to describe first the affine, then the Euclidian plane and space. Principal notions were projections, vectors, frames, transformations, etc. (Gosztonyi et al., 2023, p. 53)

Tensions in the XXIst Century – Basic School

From 1990 to 2023, there were several changes in the mathematics curriculum for Basic School and Secondary School in Portugal. The reform of the educational system of 1990 introduced new syllabuses at all levels for all courses, but other changes were mainly concentrated on changes in Mathematics, in part because in the first TIMMS and PISA Portugal was in a very modest position.

The 1990 syllabus for mathematics in Basic School initiated an extensive study of geometric transformations in a rather intuitive and balanced way; but, with an overcrowded curriculum (reformers were very ambitious) geometric transformations tended to be omitted. The next change of the curriculum in basic school in 2007 took a different approach, more visual and connected to the real world, insisting in the development of the spatial sense of students, based on exploration, manipulation and experimentation with concrete materials. Small chains of deductions should be introduced in years 7 to 9 in topics like Parallel Lines, Similar Triangles, Pythagoras Theorem and Geometric Transformations (Breda et al, 2011).

But there were radical changes again in 2012-2013, and a highly structured approach, based on logic and set theory was introduced in Basic School with an extensive study of axiomatics in the 9th grade; numerous proofs became compulsory along the official "correct" use of the vocabulary of the axiomatic method and a rather detailed knowledge of the history of the axiomatization of Geometry. The axiomatization of Geometry with points, relation "a point is between two other points" and "pairs of points are equidistant" are mentioned as objects of study, as well as Hyperbolic Geometry. This syllabus was a huge failure, some minor modifications were introduced and in 2021 a new syllabus was approved by the Ministry of Education, somehow recovering the syllabus from 2007, that had been very well received by mathematics teachers at the time.

Tensions in the XXIst Century – Secondary School

We will consider here the official documents for mathematics teaching at secondary school in Portugal on the course called "Mathematics A". There are big contrasts between the documents of 2013 and 2023. In 2013 the main goal was "the comprehension and hierarchization of mathematical concepts, the systematic study of their properties and the clear and precise argumentation, typical of this discipline" (Bivar et al. 2013). Mathematical modeling could "transmit to students a distorted vision of how one can, in fact, correctly apply Mathematics to the real world" (Bivar et al., 2013) and the use of intuition was discouraged as "conjectures formulated but not demonstrated are of limited interest" (Bivar et al., 2013). In contrast to this vision, the new documents of 2023 present the work in mathematics at secondary school as following:

> (…) develop in students the ability to identify relevant mathematical concepts to solve real problems, apply appropriate mathematical procedures and interpret results in different contexts. Mathematical reasoning is the basis of the processes of understanding mathematical concepts and objects, which can and should be analyzed, represented and related in different ways. The formulation of hypotheses, testing of conjectures, deduction, generalization and abstraction are equally important, in the construction of logical arguments and conclusions, whose communication in an appropriate way is increasingly important in today's world. (Carvalho e Silva, 2023).

Which contents and methods are the most suitable for our century?

XXIst Century: a world full of smart technology

As some people said (no quote necessary) in a world of technology that draws nice graphs and images, why study so much Mathematics (Functions or Geometry)? In fact we are in a world that already produces algorithms that can make automatic deductions in Geometry and this has the potential of changing completely the way Geometry is taught. We face three types of essentially new challenges:

a) Dynamic Geometry Software (DGS);
b) Automated proof assistants, that can be Interactive Theorem Provers (TP) or Automated Theorem provers (ATP);
c) Large language models (LLM) like ChatGPT.

DGS has been around for some time and has always been considered "rather efficient in the learning of geometry" (Balacheff, de La Tour, 2019); it is being continuously upgraded, and a surprising challenge is *Geogebra* that can run on smartphones with Augmented Reality (AR) tools; we are in a world where all students (at least from 12 years old) have a smartphone at a rate that surpasses 90%; which are the implications?

Beyond mechanizing computation, these technologies are now mechanizing mathematical reasoning and proofs with unprecedented consequences. DGS and ATP have been combined in a software with AR and used in an educational setting outdoors (Botana, Kovács, Martínez-Sevilla & Recio, 2019), (Botana, Kovács & Recio, 2020). In a recent paper Nuno Baeta and Pedro Quaresma describe an algorithm (a "Geometry Automated-Theorem-Prover") that will "be able to efficiently prove a large set of geometric conjectures, producing readable proofs" and "will open its use by third-party programs, e.g. the dynamic geometry systems" (Baeta and Quaresma, 2023). These methods began being tested in secondary schools (Teles, Baeta and Quaresma, 2022) and, even if practical difficulties are enormous, one day they will surely be used regularly in classrooms. How?

Large language models are a recent acquisition to the world of pedagogical tools and have become very controversial because they produce mathematical errors frequently: the outputs a LLM gives are just words that have a high probability of showing up together in a sentence that was produced millions of times in the texts the LLM "read". But LLM are already being used in school environments, associated to sophisticated mathematical software like Mathematica or as just a tool to give better feedback like in Geogebra as an *AI Math Assistant* (Hohenwarter, 2023) and *Khan Academy* as a text builder assistant. Will they help us teach Geometry better?

Po-Shen Loh, the coach of the American Math Olympiad team, calls the use of ChatGPT the "invasion" and thinks "the key to survival is knowing how to solve problems - and knowing which problems to solve" (Loh, 2023). He urges school to focus on "creativity, emotion and the stuff that distinguishes man from machine" (Loh, 2023).

LESSONS FOR TODAY

In this extremely brief historical excursion, we see that in each period of time, Geometry had a different character but it always had some kind of relevance related to the reasoning that it conveyed and to the real life situations it was connected to; diagrams were always used from Babylon to Euclid and the Computer Era. Somehow, connecting diagrams to its abstract representation can lead to errors (like in *Pseudaria*) but this also stimulates a healthy or corrected reasoning, enabling students to be able to deal with new problems, understanding them, looking for strategies, producing and criticizing conjectures and trying to prove them, learning how to deal with similar problems or problems that might in part benefit from the new strategies found.

The necessity to produce diagrams is a constant and our computer era is the most fertile in the availability of tools that enable us to produce a wide range of diagrams, static or dynamic. Pedagogical tools like *Geogebra 3D*, *GeoGebra AR* or *GeoGebra Discovery* (ATP) (Kovács, Recio, Vélez, 2022) are powerful and simple to use and very promising for the classroom.

Teaching and learning Geometry did not change much for millennia. Archimedes, in his *Method*, tells Eratosthenes how to investigate Geometry problems with Mechanics and then prove the results obtained using Geometry and he said the investigative part is also useful for the proof of these results (Heath, 1912). Tools and scope and interplay with other areas have changed a lot in the present century but we must see what is now important in Geometry and find new ideas to continue teaching and learning Geometry in a meaningful way. As Gila Hanna and Xiaoheng Yan (2021) put it "there is a need for new approaches to teaching proof, ones that capitalize not only on newly-available technology but also on modern theories of teaching and learning".

In the words of late President of ICMI, Miguel de Guzmán, when speaking about the impacts of new technology on mathematics teaching: "What is truly important will be the preparation [of students] for an **intelligent dialogue with the tools that already exist**, which some already have and others will have in a future that is almost present" (Guzmán, 1992) (my emphasis).

We certainly need to reflect intensely on new and significant ways to teach Geometry in Basic and Secondary School, that use technology in an intelligent and meaningful way.

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