Cauchy convergence in $\mathcal{V}$-normed categories

Maria Manuel Clementino*, Dirk Hofmann†, Walter Tholen‡

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Building on the notion of normed category as suggested by Lawvere, we introduce notions of Cauchy convergence and cocompleteness for such categories that differ from proposals in previous works. Key to our approach is to treat them consequentially as categories enriched in the monoidal-closed category of normed sets, *i.e.*, of sets which come with a norm function. Our notions largely lead to the anticipated outcomes when considering individual metric spaces or normed groups as small normed categories (in fact, groupoids), but they can be challenging when trying to establish them for large categories, such as those of semi-normed or normed vector spaces and all linear maps as morphisms, not just because norms of vectors need to be allowed to have value ∞ in order to guarantee the existence of colimits of (sufficiently many) infinite sequences. These categories, along with categories of generalized metric spaces, are the key example categories discussed in detail in this paper.

Working with a general commutative quantale $\mathcal{V}$ as a value recipient for norms, rather than only with Lawvere's quantale $\mathcal{R}_+$ of the extended real half-line, we observe that the categorically atypical, and perhaps even irritating, structure gap between objects and morphisms in the example categories is already present in the underlying normed category of the enriching category of $\mathcal{V}$-normed sets. To show that this normed category and, in fact, all presheaf categories over it, are Cauchy cocomplete, we assume the quantale $\mathcal{V}$ to satisfy a couple of light alternative extra properties which, however, are satisfied in all instances of interest to us. Of utmost importance to the general theory is the fact that our notion of Cauchy convergence is subsumed by the notion of weighted colimit of enriched category theory. With this theory and, in particular, with results of Albert, Kelly and Schmitt, we are able to prove that all $\mathcal{V}$-normed categories have correct-size Cauchy cocompletions, for $\mathcal{V}$ satisfying our light alternative assumptions.

We also show that our notions are suitable to prove a Banach Fixed Point Theorem for contractive endofunctors of Cauchy cocomplete normed categories.

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Contents

1 Introduction

A category $\mathbb{X}$ is *normed* if it comes with a function which assigns to every morphism $f : x \to y$ a value $|f| \in [0, \infty]$, such that

$$0 \geq |1_x| \qquad \text{and} \qquad |f| + |g| \geq |g \cdot f|$$

for all morphisms $g : y \to z$. Hence, this paper adopts Lawvere's [30] original notion of normed category, as a category enriched in a certain monoidal-closed category. Mentioned by him only rather covertly as a remark in [30], the notion was worked out in full generality soon afterwards by Betti and Galuzzi [7]. Some authors call such categories *weighted* (see [16, 12, 35]) rather than normed. We avoid this change of the original terminology, mostly for being able to distinguish between our notion of normed colimit and the established notion of weighted colimit of enriched category theory when we prove the non-trivial fact that the former notion is subsumed by the latter. We do not aim at special categorical environments, like that of triangulated categories (see [34] for an overview), and we refrain from imposing any further *a priori* conditions on the notion as some papers do (such as [29, 24]) but treat or discuss these as special add-on properties of Lawvere's fundamental notion.

Given Lawvere's minimalistic and logic-oriented approach, it cannot surprise that examples of normed categories abound. First of all, individual mathematical objects may often be seen as small normed categories. We mention here only the most obvious example: for every metric space (X, d) one forms the (indiscrete) category $\mathrm{i}X$ with object set X in which, for any $x, y \in X$, there is precisely one morphism $x \to y$, denoted by (x, y), and its norm is given by the metric: $|(x, y)| = d(x, y)$. In fact, the small normed categories in which all hom-sets are singletons correspond precisely to the generalized metric spaces $(X, d : X \times X \to [0, \infty])$, required only to satisfy the point and triangle inequalities

$$0 \geq d(x, x) \qquad \text{and} \qquad d(x, y) + d(y, z) \geq d(x, z)$$

for all $x, y, z \in X$, with this atypical way of stating them (following [30, 31]) to be explained shortly.

Many of the large, and sufficiently interesting, normed categories have objects with some metric structure which, however, is hardly, or not at all, respected by the morphisms. But the metric structure of the objects may then be used to specify classes of well-behaved morphisms. For instance, let $\mathbb{X}$ be the (large) category NVec_∞ whose objects are all normed real vector spaces in the usual sense, except that we allow norms to assume the value ∞ (see further below for some justification), along with an adjustment of the real arithmetic for this value, and whose morphisms are *all* linear maps (*i.e.*, the linear ∞-Lipschitz maps). From a standard categorical perspective, forming this category appears to be highly questionable since it makes two objects X and Y isomorphic as soon as they are algebraically isomorphic, regardless of their norms. (In fact, with a choice of a basis for every space granted, $\mathbb{X}$ becomes equivalent to the category of all real vector spaces and their linear maps.) Nevertheless, $\mathbb{X}$ has a *raison d'être* when regarded as a *normed* category, as it may allow us to investigate morphisms of interest within the same category, such as the (uniformly) continuous maps, or the maps with a given Lipschitz value ≥ 1. Indeed, for a linear map $f : X \to Y$, with $\|\cdot\|$ denoting the given norms of vectors in X and Y (which, as already observed in [29], may also be considered as categorical norms when X and Y are treated as one-object categories under addition), writing $\log^\circ \alpha := \max\{0, \log \alpha\}$ when $\alpha > 0$ one simply considers

$$|f| = \sup_{x \neq 0} \log^\circ\left(\frac{\|fx\|}{\|x\|}\right),$$

so that $|f|$ becomes minimal in $[0, \infty]$ with respect to the natural order and the property that

$$e^{|f|}\|x\| \geq \|fx\|$$

holds for all $x \in X$. This makes $e^{|f|}$ the Lipschitz value $\mathrm{L}(f)$ of the map f whenever $\mathrm{L}(f) \geq 1$ under a natural extension of the real arithmetic to ∞), so that the condition $|f| < \infty$ characterizes f as bounded (or, equivalently, as (uniformly) continuous), while $|f| = 0$ describes it as non-expanding, or 1-Lipschitz.

A seemingly trivial situation arises when we consider a normed category $\mathbb{X}$ in which norms are allowed to take as values only 0 or ∞. Interpreting 0 as $\top$ (true) and ∞ as $\bot$ (false), and rewriting the relation $\geq$ as the implication $\Rightarrow$ and the operation $+$ as the logical connective $\wedge$, then the norm conditions determining the class $\mathcal{S}$ of all morphisms f that satisfy $|f| = \top$ or, equivalently, $\top \Rightarrow |f|$, may be recorded equivalently as

$$1_x \in \mathcal{S} \qquad \text{and} \qquad f \in \mathcal{S} \,\wedge\, g \in \mathcal{S} \Longrightarrow g \cdot f \in \mathcal{S} \,.$$

In other words, such 2-valued normed categories $\mathbb{X}$ are just categories that come equipped with a distinguished wide subcategory $\mathcal{S}$ (*i.e.,* a subcategory with the same class of objects as $\mathbb{X}$). The wide-subcategory conditions as stated above explain why it is logically preferable to use the natural $\geq$ of the reals as the relevant order, a perspective that gets justified further by Lawvere's enriched category-theoretical view of metrics and norms which we adopt in this paper.

Just as for metric spaces, the concept of *Cauchy convergence* should be fundamental in the study of normed categories. But what is it? And once defined, what does completeness mean? Do there exist completions, and are there protagonistic normed categories in this context, like the presheaf categories in the completion theory of ordinary categories? In this paper we try to give answers to these questions and test them in examples. Taking seriously the enriched categorical perspective that is already present in [30, 7, 31], our answers differ from those presented in other papers, such as [29, 34, 24]. Also, instigated not just by the third type of examples above, we increase the potential range of applications by allowing norms to take values in an arbitrary (commutative and unital) *quantale*, *i.e.*, in a complete lattice $(\mathcal{V}, \leq)$ which, in addition, has a commutative monoid structure $(\mathcal{V}, \otimes, \mathrm{k})$, such that $\otimes$ distributes in each variable over arbitrary joins. Hence, we consider $\mathcal{V}$*-normed* categories where $(\mathcal{V}, \leq, \otimes, \mathrm{k})$ will, amongst others, take on the role of the *Lawvere quantale* $\mathcal{R}_+ = ([0, \infty], \geq, +, 0)$, or of the *Boolean quantale* $\mathsf{2} = (\{\top, \bot\}, \Rightarrow, \wedge, \top)$ used in the examples above.

Seeing the study of $\mathcal{V}$-normed categories as embedded into enriched category theory, we must take seriously the enriching monoidal-closed category $\mathsf{Set}/\!/\mathcal{V}$ whose objects are mere sets equipped with a $\mathcal{V}$-valued function; morphisms are maps which keep or increase the $\mathcal{V}$-value of elements. Hence, $\mathcal{V}$-normed categories are categories enriched in $\mathsf{Set}/\!/\mathcal{V}$. Despite its simplicity, $\mathsf{Set}/\!/\mathcal{V}$ has an unexpected feature: the internal hom of objects A and B is given by *all* mappings $A \to B$, not just by the $(\mathsf{Set}/\!/\mathcal{V})$-morphisms $A \to B$. Hence, being central to the understanding of the theory, right from start we have to consider the somewhat strange $\mathcal{V}$-normed category induced by the internal hom of $\mathsf{Set}/\!/\mathcal{V}$. This leads us to the $\mathcal{V}$-normed category

$$\mathsf{Set}|\!|\mathcal{V}$$

of $\mathcal{V}$-normed sets with arbitrary maps as their morphisms. It contains the ordinary category $\mathsf{Set}/\!/\mathcal{V}$ as a non-full subcategory. As the recipient category for the presheaves over any given $\mathcal{V}$-normed category $\mathbb{X}$, it is key in the study of any kind of completions of $\mathbb{X}$.

For this reason, in Sections 2-4 we take time to present the fundamentals of $\mathcal{V}$-normed category theory in detail. Alongside many examples of such small and large categories, we note that the category $\mathsf{Cat}/\!/\mathcal{V}$ of all small $\mathcal{V}$-normed categories and $\mathcal{V}$-normed functors is *topological* [1, 22] over

$\mathsf{Set}/\!/\mathcal{V}$, which gives the recipe for the construction of limits and colimits of $\mathcal{V}$-normed categories. We show that it is *symmetric monoidal closed* [26, 33], as well as *locally presentable* [15, 3]. Other than $\mathsf{Set}|\!|\mathcal{V}$, we also introduce the $\mathcal{V}$-normed categories $\mathcal{V}$-Lip and $\mathcal{V}$-Dist, both having as their objects small $\mathcal{V}$-categories, *i.e.*, Lawvere metric spaces when $\mathcal{V} = \mathcal{R}_+$, whilst their morphisms are respectively arbitrary maps and $\mathcal{V}$-distributors. The former category facilitates the study of norms for categories of metric spaces, and the norm of the latter category naturally leads to non-symmetrized Hausdorff metrics (as considered in [30] and studied in [4, 36]).

In Sections 5 and 6, introducing the key notions of *Cauchy sequence* and of *normed convergence* of a sequence in a $\mathcal{V}$-normed category, we tighten the corresponding definitions as proposed by Kubiś [29] in the context of $\mathcal{V} = \mathcal{R}_+$, in such a way that, unlike in Kubiś's work, normed colimits become unique and conform with the enriched setting. This allows us to prove in Sections 11-13 the central general theorems of this paper, under two very weak alternative assumptions on the ambient quantale $\mathcal{V}$, the status of which we discuss further in the appended Section 15. Hence, first we show that the normed category of $(\mathsf{Set}|\!|V)$-valued presheaves of a $\mathcal{V}$-normed category $\mathbb{X}$ is *Cauchy cocomplete*[1], *i.e.*, that all of its Cauchy sequences have a normed colimit. Then we exhibit the normed colimits as *weighted* colimits in the sense of enriched category theory, and finally we invoke the machinery developed by Albert, Kelly and Schmitt [5, 28] to show the existence of a Cauchy cocompletion of $\mathbb{X}$ belonging to the same universe as the given normed category $\mathbb{X}$. Unlike these results, it is easy to see that our notions lead to known concepts and outcomes when applied to individual (Lawvere) metric spaces seen as small normed categories; see in particular [9, 39, 23, 21]. We leave to future work the question whether the methods used in these and other papers may be generalized to produce a more direct construction of the Cauchy cocompletion of a normed category than the one presented here.

Our discussion of norms for categories of normed vector spaces and of metric spaces does not require any advanced categorical tools and can be read independently from the abstract completion theory. We therefore present it earlier (in Sections 8 and 9), even though matters are not as straightforward as one may have hoped. Consider the sequence of normed vector spaces

$$\mathbb{R} = \mathbb{R}_1 \longrightarrow \mathbb{R}_{\frac{1}{2}} \longrightarrow \mathbb{R}_{\frac{1}{3}} \longrightarrow \dots \longrightarrow \mathrm{colim}_n\, \mathbb{R}_{\frac{1}{n}} = \mathbb{R}_0\,,$$

where $\mathbb{R}_c$ is the 1-dimensional vector space of real numbers normed by $\|1\| = c$ for a constant $c > 0$. The connecting identity maps being strictly contractive, it seems reasonable to work in a categorical context which gives as the colimit the space $\mathbb{R}_0$, and thus to admit the case $c = 0$, *i.e.*, to allow norms of non-zero vectors to be 0, rather than to force the sequence to collapse in the null space in order to obtain a normed vector space. Hence, just like for Lawvere metric spaces, we do not insist on separation and consider *semi-normed* vector spaces, rather than just normed spaces. Moreover, to have enough limits and colimits, just as it is the case for Lawvere metrics, we should allow norms to have value ∞. Here is a simple example of an inverse sequence, again involving only 1-dimensional normed spaces and strict contractions given by identity maps; in the same notation as above, it reads as

$$\mathbb{R} = \mathbb{R}_1 \longleftarrow \mathbb{R}_2 \longleftarrow \mathbb{R}_3 \longleftarrow \dots \longleftarrow \lim_n \mathbb{R}_n = \mathbb{R}_\infty\,.$$

[1] The term, or its dual, is not to be confused with Cauchy completeness in the sense of idempotent completeness; see Corollary 7.3 and the footnote there.

To be able to work in a satisfactory categorical environment, we therefore consider in Section 8 the normed category SNVec_∞ whose objects are semi-normed vector spaces with norms permitted to attain the value ∞, and whose morphisms are just linear maps. Consequently, we had to extend (and restrict) not only the real addition and subtraction, but also the multiplication and division to $[0,\infty]$ and, in order to have the category normed, do the same for the exponential function and the logarithm. This works well when one extends inverse operations systematically by *adjunction*, rather than by *ad hoc* conventions. We consider the proof showing that, by traditional analytic epsilon arguments, the normed category SNVec_∞ is Cauchy cocomplete as this paper's main result in the realm of concrete normed categories. By contrast, the category NVec_∞ fails to be Cauchy cocomplete.

Expanding on previous work (see in particular [29, 35]), in Section 9 we establish the purely metric version of the vector space result, by proving that the normed category Met_∞ of all Lawvere metric spaces and arbitrary maps is Cauchy cocomplete. We do so by first showing that the previously used epsilon argumentation works well also in the more general setting of the category $\mathcal{V}$-Lip, under reasonable (but no longer mild) conditions on the quantale $\mathcal{V}$ which align with the methods used in Flagg's pioneering work [13, 14] and, more recently, in [20]. After a change of base, which trades the quantale $\mathcal{R}_+$ for its multiplicative counterpart, the Cauchy cocompleteness of Met_∞ follows. Finally, improving a result by Kubiś [29], in Section 14 we present a relatively easily established *Banach Fixed Point Theorem* for a contractive endofunctor of a Cauchy cocomplete normed category which replicates the classical theorem in the case of a complete metric space, considered as a small normed category.

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2 $\mathcal{V}$-normed sets

Throughout this paper $\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$ is a unital and commutative *quantale*, that is: $(\mathcal{V}, \leq)$ is a complete lattice and $(\mathcal{V}, \otimes, \mathrm{k})$ is a commutative monoid with neutral element k, such that, for all $v \in \mathcal{V}$, the map $-\otimes v : \mathcal{V} \to \mathcal{V}$ preserves arbitrary suprema:

$$(\bigvee_{i\in I} u_i) \otimes v = \bigvee_{i\in I} (u_i \otimes v);$$

in particular, $\perp \otimes v = \perp$ for the bottom element $\perp$ of $\mathcal{V}$. In the (small and thin) symmetric monoidal-closed category $\mathcal{V}$, for all $v, w \in \mathcal{V}$ one has the internal hom, $[v, w]$, characterized by

$$u \leq [v, w] \iff u \otimes v \leq w$$

for all $u \in \mathcal{V}$. The standard quantales considered in this paper are the *Boolean quantale*, $2 = \{\bot, \top\}$ with $\otimes = \wedge$ and $\mathrm{k} = \top$, and the *Lawvere quantale*, $\mathcal{R}_+ = ([0,\infty], \geq, +, 0)$, ordered by the natural $\geq$-order of the extended real half-line. In $\mathcal{R}_+$, the internal hom is computed as $[v,w] = \max\{0, w-v\}$, $[v,\infty] = \infty$, $[\infty, w] = [\infty,\infty] = 0$ for all $v, w < \infty$, and in 2 it is given by the implication: $[v,w] = (v \Rightarrow w)$.

Definition 2.1. A $\mathcal{V}$-*normed set* is a set A that comes with a function $|\text{-}|_A : A \to \mathcal{V}$, and a $\mathcal{V}$-*normed map* $(A, |\text{-}|_A) \to (B, |\text{-}|_B)$ is a mapping $f : A \to B$ satisfying $|a|_A \leq |fa|_B$ for all $a \in A$:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "|\text{-}|_A"'] & {} \arrow[d, phantom, "\leq"] & B \arrow[dl, "|\text{-}|_B"] \\ & \mathcal{V} & \end{tikzcd}$$

Henceforth, we usually drop the subscripts. This defines the category $\mathsf{Set}/\!/\mathcal{V}$.

This category is simply the formal coproduct completion $\mathrm{Fam}\mathcal{V}$ of the category $(\mathcal{V}, \leq)$. Applying the Fam-construction to the unique functor $\mathcal{V} \to 1$ of $\mathcal{V}$ to the terminal quantale one obtains the forgetful functor $\mathsf{Set}/\!/\mathcal{V} \to \mathsf{Set}$, which is *topological* [1]; that is: given a family of any size of mappings $f_i : A \to B_i$ $(i \in I)$ with a fixed set A and all B_i $\mathcal{V}$-normed, then there is an "initial" $\mathcal{V}$-norm on A, namely

$$|a| = \bigwedge_{i \in I} |f_i a|.$$

Equivalently: given any family of mappings $g_i : A_i \to B$ $(i \in I)$ from $\mathcal{V}$-normed sets A_i to a given set B, then there is a "final" $\mathcal{V}$-norm on B that is described by

$$|b| = \bigvee_{i \in I} \bigvee_{a \in g_i^{-1} b} |a|.$$

Consequently, $\mathsf{Set}/\!/\mathcal{V}$ is complete and cocomplete. Moreover, the forgetful functor has a left adjoint, putting on every set the *discrete* $\mathcal{V}$-*norm* with constant value $\bot$, as well as a right adjoint, putting on every set the *indiscrete* $\mathcal{V}$-*norm* with constant value $\top$. In particular, $\mathsf{Set}/\!/\mathcal{V} \to \mathsf{Set}$ is represented by the discrete singleton $\mathcal{V}$-normed set $\mathrm{E}_\bot$, *i.e.*, by $\{*\}$ with $|*| = \bot$.

More importantly, one has:

Proposition 2.2. *The category* $\mathsf{Set}/\!/\mathcal{V}$ *is symmetric monoidal-closed.*

Proof. For $\mathcal{V}$-normed sets A and B, their tensor product $A \otimes B$ is carried by the cartesian product $A \times B$, normed by $|(a,b)| = |a| \otimes |b|$ in $\mathcal{V}$, and the tensor-neutral set E_k is the set $\{*\}$ normed by $|*| = \mathrm{k}$.

To describe the internal hom-object, denoted by $[A, B]$, we first observe that its elements are necessarily described by $\mathcal{V}$-normed maps $\mathrm{E}_\bot \to [A,B]$, which must correspond to $\mathcal{V}$-normed maps $\mathrm{E}_\bot \otimes A \to B$. But these correspond precisely to arbitrary Set-maps $A \to B$, since $\mathrm{E}_\bot \otimes A$ puts just the discrete structure on the set A. Consequently, $[A, B]$ has carrier set $\mathsf{Set}(A, B)$, *i.e.*, the set of *all* mappings $\varphi : A \to B$, with their norm defined by

$$|\varphi| = \bigwedge_{a \in A} [|a|, |\varphi a|].$$

This turns out to be the largest structure (in the order induced pointwise by $\mathcal{V}$) making the evaluation map $[A, B] \otimes A \to B$ $\mathcal{V}$-normed; *i.e.*, $|\varphi|$ is maximal with the property

$$|\varphi| \otimes |a| \leq |\varphi a|$$

for all $a \in A$. □

Remarks 2.3. (1) We note that, for $\varphi \in [A, B]$, one has $\mathrm{k} \leq |\varphi|$ precisely when $|a| \leq |\varphi a|$ for all $a \in A$, that is, when $\varphi : A \to B$ is a $\mathcal{V}$-normed map. Hence, $|\varphi|$ is to be seen as the "degree" to which the arbitrary map φ is a morphism of $\mathsf{Set}/\!/\mathcal{V}$.

(2) When we consider the lattice $\mathcal{V}$ as a small thin category, the functor $\mathsf{1} \to \mathsf{Set}$ of the terminal category $\mathsf{1}$ pointing to the terminal object $\{*\}$ of Set "lifts" to the functor $\mathrm{i} : \mathcal{V} \to \mathsf{Set}/\!/\mathcal{V}$, which assigns to every $v \in \mathcal{V}$ the set $\mathrm{E}_v = \{*\}$, normed by $|*| = v$. It has a left adjoint, s, which assigns to every object A its "sum", or "supremum" $\mathrm{s}A = \bigvee_{a \in A} |a|$, also regarded as its "optimal value" [35]. In the commutative diagram

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\mathrm{i}"] \arrow[d] \arrow[r, phantom, shift right=2, "\top"] & \mathsf{Set}/\!/\mathcal{V} \arrow[l, bend left, "\mathrm{s}"] \arrow[d, "\text{forget}"] \\
\mathsf{1} \arrow[r] \arrow[r, phantom, shift right=2, "\top"] & \mathsf{Set} \arrow[l, bend left]
\end{tikzcd}$$

all arrows are monoidal homomorphisms and they, except possibly for s, preserve also the internal homs.

(3) As a left adjoint, the functor s preserves all colimits, and it also preserves products if (and only if) the lattice $\mathcal{V}$ is completely distributive.

(4) Other than the forgetful functor $\mathsf{Set}/\!/\mathcal{V} \to \mathsf{Set}$ as in (2), one may, for every $v \in \mathcal{V}$, consider more generally the functor $P_v : \mathsf{Set}/\!/\mathcal{V} \to \mathsf{Set}$ which assigns to a $\mathcal{V}$-normed set A the set $\{a \in A \mid v \leq |a|\}$. It has a left adjoint which puts the $\mathcal{V}$-norm with constant value v onto every set, and it is represented by the $\mathcal{V}$-normed set E_v as defined in (2). The set of objects

$$\{\mathrm{E}_v \mid v \in \mathcal{V}\}$$

distinguishes itself as being a *strong generator* of the category $\mathsf{Set}/\!/\mathcal{V}$. Indeed, for any $\mathcal{V}$-normed set B, the family of all morphisms $\mathrm{E}_v \to B$ with some $v \in \mathcal{V}$ is not only jointly epic, but in fact strongly so, since B carries the final structure (as described above) with respect to this family.

Before exploiting the strong-generator property of the set $\{\mathrm{E}_v \mid v \in \mathcal{V}\}$, we show that every individual member of this set has an important property (as introduced in [15], see also [3]):

Lemma 2.4. *For the least regular cardinal λ larger than the size of the set $\mathcal{V}$, and for every element $v \in \mathcal{V}$, the $\mathcal{V}$-normed set E_v is λ-presentable in $\mathsf{Set}/\!/\mathcal{V}$, that is: the representable functor $P_v : \mathsf{Set}/\!/\mathcal{V} \to \mathsf{Set}$ preserves λ-directed colimits.*

Proof. For a λ-directed ordered set I (so that any subset of size $< \lambda$ has an upper bound in I), we consider an I-indexed diagram $(f_{i,j} : A_i \to A_j)_{i \le j}$ with colimit cocone $(g_i : A_i \to B)_{i \in I}$ in $\mathsf{Set}/\!/\mathcal{V}$. We must show that every morphism $b : \mathrm{E}_v \to B$ in $\mathsf{Set}/\!/\mathcal{V}$ has an essentially unique factorization through some g_i. But such morphism is described by an element $b \in B$ with $v \le |b|$, and since B carries the final structure with respect to the colimit cocone, we have $|b| = \bigvee C_b$ with

$$C_b = \{|a| \mid \exists i \in I : a \in g_i^{-1}b\} \subseteq \mathcal{V}.$$

For every $u \in C_b$ we choose an index $i_u \in I$ and an element $a_u \in g_{i_u}^{-1}b$ with $|a_u| = u$. Since the terminal object $\{*\}$ is λ-presentable in Set and I is λ-directed, there are $j \in I$ and $a \in A_j$ so that, for all $u \in C_b$, one has $i_u \le j$ and $f_{i_u,j}(a_u) = a$. Therefore $g_j(a) = b$ and, for all $u \in C_b$, $u \le |a|$. Consequently $v \le |a|$, that is, the map $\mathrm{E}_v \to A_j$ with value a is actually a morphism in $\mathsf{Set}/\!/\mathcal{V}$, and we conclude that $b \colon \mathrm{E}_v \to B$ factors through $g_j : A_j \to B$. The essential uniqueness of this factorization follows from the fact that a singleton set is λ-presentable in Set. □

Proposition 2.5. *The category $\mathsf{Set}/\!/\mathcal{V}$ is locally presentable.*

Proof. It suffices to note that the Set-based topological category $\mathsf{Set}/\!/\mathcal{V}$ is cocomplete, and that every object of its strong generator $\{\mathrm{E}_v \mid v \in \mathcal{V}\}$ is locally λ-presentable, with λ as in Lemma 2.4. □

3 $\mathcal{V}$-normed categories

Definition 3.1. A *$\mathcal{V}$-normed category* $\mathbb{X}$ is a $(\mathsf{Set}/\!/\mathcal{V})$-enriched category. This just means that $\mathbb{X}$ is an ordinary category with (small) $\mathcal{V}$-normed hom-sets such that, for all objects x, y, z, the maps

$$\mathrm{E}_\mathrm{k} \to \mathbb{X}(x,x),\ * \mapsto 1_x, \qquad \text{and} \qquad \mathbb{X}(x,y) \otimes \mathbb{X}(y,z) \to \mathbb{X}(x,z),\ (f,g) \mapsto g \cdot f,$$

are $\mathcal{V}$-normed; equivalently, for all morphisms $f : x \to y$ and $g : y \to z$ one has

$$\mathrm{k} \le |1_x| \qquad \text{and} \qquad |f| \otimes |g| \le |g \cdot f|.$$

A functor $F : \mathbb{X} \to \mathbb{Y}$ is *$\mathcal{V}$-normed* if it makes its hom maps $\mathbb{X}(x,y) \to \mathbb{Y}(Fx,Fy)$ $\mathcal{V}$-normed; that is, if

$$|f| \le |Ff|$$

holds for all morphisms f in $\mathbb{X}$. For the emerging categories

$$(\mathsf{Set}/\!/\mathcal{V})\text{-}\mathsf{Cat} \quad \text{and} \quad (\mathsf{Set}/\!/\mathcal{V})\text{-}\mathsf{CAT}$$

of all small $\mathcal{V}$-normed categories with their $\mathcal{V}$-normed functors and its (higher-universe) counterpart of all $\mathcal{V}$-normed categories we use respectively the more familiar notation

$$\mathsf{Cat}/\!/\mathcal{V} \quad \text{and} \quad \mathsf{CAT}/\!/\mathcal{V}$$

as justified by the following observation.

Facts 3.2. (1) Considering the monoid $(\mathcal{V}, \otimes, \mathrm{k})$ as a one-object 2-category with its 2-cells given by the order of $\mathcal{V}$, we may describe a $\mathcal{V}$-normed category $\mathbb{X}$ equivalently as a 2-category with only identical 2-cells, equipped with a lax functor $|\text{-}| : \mathbb{X} \to \mathcal{V}$. A $\mathcal{V}$-normed functor $F : \mathbb{X} \to \mathbb{Y}$ is then a (lax, but necessarily strict) 2-functor producing the lax-commutative diagram

$$\begin{tikzcd}
\mathbb{X} \arrow[rr, "F"] \arrow[dr, "|\text{-}|_{\mathbb{X}}"'] & {} \arrow[d, phantom, "\leq"] & \mathbb{Y} \arrow[dl, "|\text{-}|_{\mathbb{Y}}"] \\
& \mathcal{V} &
\end{tikzcd}$$

(2) The (monoidal) functors of the diagram of Remarks 2.3(2) induce the diagram

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathsf{Cat} \arrow[r, "\mathrm{i}"] \arrow[d, "\text{forget}"'] \arrow[r, phantom, shift right=2, "\top"] & \mathsf{Cat}/\!/\mathcal{V} \arrow[l, bend left, "\mathrm{s}"] \arrow[d, "\text{forget}"] \\
\mathsf{Set} \arrow[r, "\text{indiscrete}"] \arrow[r, phantom, shift right=2, "\top"] & \mathsf{Cat} \arrow[l, bend left, "\text{ob}"]
\end{tikzcd}$$

of change-of-base functors. Here an object of $\mathcal{V}$-Cat is (as in [30] and [22]) a set X which, for all $x, y \in X$, comes with a value $X(x, y) \in \mathcal{V}$, satisfying the laws

$$\mathrm{k} \leq X(x, x) \qquad \text{and} \qquad X(x, y) \otimes X(y, z) \leq X(x, z).$$

The functor i describes the $\mathcal{V}$-category X equivalently as an indiscrete category $\mathbb{X} = \mathrm{i}X$ with $\mathrm{ob}\mathbb{X} = X$, putting the $\mathcal{V}$-norm

$$|x \to y| = X(x, y)$$

on the only morphism in $\mathbb{X}(x, y)$. The functor s takes an arbitrary small $\mathcal{V}$-normed category $\mathbb{X}$ to the $\mathcal{V}$-category $\mathrm{s}\mathbb{X} = \mathrm{ob}\mathbb{X}$ with

$$(\mathrm{s}\mathbb{X})(x, y) = \bigvee\{|f| \mid f \in \mathbb{X}(x, y)\}.$$

(3) The norm-forgetting functor $\mathsf{Cat}/\!/\mathcal{V} \to \mathsf{Cat}$ must be carefully distinguished from the functor

$$(-)_\circ : \mathsf{Cat}/\!/\mathcal{V} \to \mathsf{Cat}$$

which sends a small $\mathcal{V}$-normed category $\mathbb{X}$ to the category $\mathbb{X}_\circ$, defined (as in enriched category theory [26]) to have the same objects as $\mathbb{X}$, but the morphisms of which are only those morphisms $f : x \to y$ in $\mathbb{X}$ with $\mathrm{k} \leq |f|$ (since these are equivalently described by the $(\mathsf{Set}/\!/\mathcal{V})$-morphisms

$\mathrm{E_k} \to \mathbb{X}(x,y)$). Extending the terminology of [29] from $\mathcal{R}_+$ to arbitrary $\mathcal{V}$, we call the morphisms of $\mathbb{X}_\circ$ the k-*morphisms* of $\mathbb{X}$, and we say that the (ordinary and generally non-full) subcategory

$$\mathbb{X}_\circ$$

of $\mathbb{X}$ is the *category of* k-*morphisms in* $\mathbb{X}$. An isomorphism f in $\mathbb{X}_\circ$ is called a k-*isomorphism* of $\mathbb{X}$; *i.e.*, f is an isomorphism in $\mathbb{X}$ such that both, f and f^{-1}, are k-morphisms.

Caution: An isomorphism in the ordinary category $\mathbb{X}$ may not belong to $\mathbb{X}_\circ$, and even if it does, it may not be an isomorphism in $\mathbb{X}_\circ$: for a (non-symmetric) two-point metric space $X = \{a, b\}$ with $X(a, b) = 1$ and all other distances 0, just consider $\mathbb{X} = \mathrm{i}X$, as formed in Examples 3.5(3).

(4) Being of the form $\mathcal{W}$-Cat for some symmetric monoidal-closed category $\mathcal{W}$, all four categories of the diagram in (2) are again symmetric monoidal closed (if not cartesian closed). In particular, recall that the tensor product of X and Y in $\mathcal{V}$-Cat is carried by the cartesian product and structured by

$$(X \otimes Y)((x, y), (x', y')) = X(x, x') \otimes Y(y, y'),$$

and that their internal hom, $[X, Y]$, is carried by the hom-set $\mathcal{V}$-$\mathsf{Cat}(X, Y)$ and structured by

$$[X, Y](f, g) = \bigwedge_{x \in X} Y(fx, gx).$$

Of course, $\mathrm{E} = \{*\}$ with $\mathrm{E}(*, *) = \mathrm{k}$ is the monoidal unit in $\mathcal{V}$-Cat (see [22] for details).

We define the *tensor product* $\mathbb{X} \otimes \mathbb{Y}$ of the $\mathcal{V}$-normed categories $\mathbb{X}$ and $\mathbb{Y}$ to be carried by the ordinary category $\mathbb{X} \times \mathbb{Y}$, structured by

$$|(f, f')| = |f| \otimes |f'|.$$

One then routinely shows that their *internal hom* $[\mathbb{X}, \mathbb{Y}]$ is given by the $\mathcal{V}$-normed functors $\mathbb{X} \to \mathbb{Y}$ and all natural transformations between them, normed by

$$|\alpha| = \bigwedge_{x \in \mathrm{ob}\mathbb{X}} |\alpha_x|.$$

The terminal category $\mathbb{E}$ in Cat with $\mathrm{ob}\mathbb{E} = \mathrm{E} = \{*\}$ becomes the *monoidal unit* when one puts $|1_*| = \mathrm{k}$ (but note that it is terminal in $\mathsf{Cat}/\!/\mathcal{V}$ only if $\mathrm{k} = \top$). Clearly, the functors i and s preserve the monoidal structure, and i preserves even the closed structure.

We also observe that the forgetful functor $\mathsf{Cat}/\!/\mathcal{V} \to \mathsf{Cat}$ is, like $\mathcal{V}$-$\mathsf{Cat} \to \mathsf{Set}$, *topological*; that is: for any (arbitrarily large) family $F_i : \mathbb{X} \to \mathbb{Y}_i$ $(i \in I)$ of functors of a fixed category $\mathbb{X}$ to $\mathcal{V}$-normed categories, there is the "initial" $\mathcal{V}$-normed structure on $\mathbb{X}$, given by

$$|f| = \bigwedge_{i \in I} |F_i f|.$$

Let us summarize the main points of these observations:

Proposition 3.3. *The category* $\mathsf{Cat}/\!/\mathcal{V}$ *is symmetric monoidal closed and topological over* Cat. *In particular,* $\mathsf{Cat}/\!/\mathcal{V}$ *is complete and cocomplete, and the forgetful functor* $\mathsf{Cat}/\!/\mathcal{V} \to \mathsf{Cat}$ *has both, a right and a left adjoint, providing an ordinary small category respectively with the indiscrete and the discrete* $\mathcal{V}$*-norm. The restriction to small* $\mathcal{V}$*-normed categories whose carrier category is indiscrete reproduces the corresponding statements for the category* $\mathcal{V}$-Cat *and its forgetful functor to* Set.

We also mention that, since $\mathsf{Set}/\!/\mathcal{V}$ is locally presentable (Proposition 2.5), by the general result shown in [27] this important property gets inherited by $(\mathsf{Set}/\!/\mathcal{V})$-$\mathsf{Cat}$:

Corollary 3.4. *The category* $\mathsf{Cat}/\!/\mathcal{V}$ *is locally presentable.*

Examples 3.5. (1) A $\mathsf{1}$-normed category (for the terminal quantale $\mathsf{1}$) is just an ordinary category, and for $\mathcal{V} = \mathsf{1}$ the diagram of Facts 3.2(2) flattens to

$$\mathrm{s} \dashv \mathrm{i} : \mathsf{1}\text{-}\mathsf{Cat} = \mathsf{Set} \longrightarrow \mathsf{Cat} = \mathsf{Cat}/\!/\mathsf{1}.$$

For the Boolean quantale $\mathcal{V} = \mathsf{2}$, the diagram of Facts 3.2(2) takes the form

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "\mathrm{i}"] \arrow[d, "\text{forget}"'] & \mathsf{Cat}/\!/\mathsf{2} \arrow[l, bend left, "\mathrm{s}", "\top"'] \arrow[d, "\text{forget}"] \\
\mathsf{Set} \arrow[r, "\text{indiscrete}"] & \mathsf{Cat} \arrow[l, bend left, "\text{ob}", "\top"']
\end{tikzcd}$$

Here $\mathsf{Ord} = \mathsf{2}$-$\mathsf{Cat}$ is the category of preordered sets and monotone maps. Objects in $\mathsf{Cat}/\!/\mathsf{2}$ may be described as small categories $\mathbb{X}$ which come with a distinguished class $\mathcal{S}$ of morphisms that is closed under composition and contains all identity morphisms; necessarily then, as a category, $\mathcal{S} = \mathbb{X}_\circ$ as defined in Facts 3.2(3). Morphisms in $\mathsf{Cat}/\!/\mathsf{2}$ are functors preserving the distinguished morphisms.

(2) In an $\mathcal{R}_+$-normed category $\mathbb{X}$, henceforth often called just *normed* and written with the natural $\leq$ for the real numbers, rather than with the natural $\geq$ as in the Introduction, the norm conditions read as

$$|1_x| = 0 \qquad \text{and} \qquad |g \cdot f| \leq |f| + |g|$$

for all $f : x \to y$ and $g : y \to z$. A ($\mathcal{R}_+$-)normed functor $F : \mathbb{X} \to \mathbb{Y}$ must satisfy $|Ff| \leq |f|$ for all morphisms f in $\mathbb{X}$.

(3) Every Lawvere metric space $X = (X, d)$ is equivalently described as a small indiscrete category $\mathbb{X}$ whose objects are the points of X, such that for every pair $x, y \in X$, the only morphism $x \to y$ in $\mathbb{X}$ gets normed by $|x \to y| = X(x, y)$, where we have written $X(x, y)$ for $d(x, y)$ in accordance with Facts 3.2(2). Hence, the categorical norm conditions as shown in (2) just generalize the defining conditions of a Lawvere metric space. Non-expanding maps between Lawvere metric

spaces are equivalently presented as normed functors of indiscrete small normed categories. This describes the full reflective embedding

$$\mathrm{i} : \mathsf{Met}_1 := \mathcal{R}_+\text{-}\mathsf{Cat} \longrightarrow \mathsf{NCat}_1 := \mathsf{Cat}/\!/\mathcal{R}_+$$

of Facts 3.2(2) (with the subscript 1 indicating the Lipschitz constant defining the morphisms). Its left adjoint s provides the set X of all objects of a small normed category $\mathbb{X}$ with the (Lawvere) metric

$$X(x,y) = \inf\{|f| \mid f \in \mathbb{X}(x,y)\}.$$

(4) (Lawvere [30]) The subsets of a (Lawvere) metric space X are the objects of the small normed category $\mathbb{H}X$ whose morphisms $\varphi : A \to B$ are arbitrary Set-maps, normed (like in the internal hom $[A, B]$ of $\mathsf{Set}/\!/\mathcal{V}$) by

$$|\varphi| = \sup_{x\in A} X(x, \varphi x).$$

The reflector s provides the powerset of X with the non-symmetrized Hausdorff metric

$$d(A, B) = \inf_{\varphi:A\to B} \sup_{x\in A} X(x, \varphi x) = \sup_{x\in A} \inf_{y\in B} X(x, y),$$

where the validation of the second equality (presenting the metric in its more usual form) requires the Axiom of Choice.

(5) (See also [29, 35].) Here is an $\mathcal{R}_+$-norm that measures the degree to which an arbitrary mapping between metric spaces fails to be 1-Lipschitz (*i.e.*, fails to be non-expanding). Just form the (somewhat strange) category Met_∞ whose objects are Lawvere metric spaces, and whose morphisms $\varphi : X \to Y$ are mere Set-maps, normed by

$$|\varphi| = \sup_{x,x'\in X} \log^\circ(\frac{Y(\varphi x, \varphi x')}{X(x, x')}),$$

where we have used the abbreviation $\log^\circ \alpha = \max\{0, \log \alpha\}$ for $\alpha \in [0, \infty]$ and extended the real arithmetic to $[0, \infty]$, the details of which are given in Section 8; see also Corollary 9.4. If X is a metric space in the classical sense, then this extension may be largely avoided since $|\varphi| = \log^\circ \mathrm{L}(\varphi)$, where

$$\mathrm{L}(\varphi) = \sup\{\frac{Y(\varphi x, \varphi x')}{X(x, x')} \mid x, x' \in X,\ X(x, x') \neq 0\})$$

is the Lipschitz value of φ in $[0, \infty]$. Since the 0-morphisms in the normed category Met_∞, *i.e.*, the morphisms φ with $|\varphi| = 0$, are precisely the 1-Lipschitz maps, we have

$$(\mathsf{Met}_\infty)_\circ = \mathsf{Met}_1\ .$$

If X and Y are the underlying metric spaces of normed vector spaces and φ is linear, then (with $\|\text{-}\|$ denoting the given norms of the vector spaces), the above formula reads equivalently as

$$|\varphi| = \sup_{x\neq 0} \log^\circ(\frac{\|\varphi x\|}{\|x\|})\ ,$$

as recorded in the Introduction. Hence, for φ 1-Lipschitz, $e^{|\varphi|} = \|\varphi\|$ is the usual operator norm of φ.

(6) For every commutative monoid $(M, +, 0)$ we have the *free* quantale $(\mathcal{P}M, \subseteq, +, \{0\})$ over the monoid M, with the powerset of M structured by $A + B = \{a + b \mid a \in A, b \in B\}$ for all $A, B \subseteq M$. (Note that this quantale drastically fails to be *integral*, *i.e.*, here, for the tensor-neutral element $\mathrm{k} = \{0\}$ one has $\mathrm{k} < \top = M$, unless M is trivial.) A $\mathcal{P}M$-normed category $\mathbb{X}$ may be thought of as a category that comes equipped with a mapping that assigns to every $\alpha \in M$ a class $\mathcal{S}_\alpha$ of morphisms in $\mathbb{X}$, which we may call "of type α", or "bounded by α", subject to the rules that identity morphisms are bounded by 0, and that, for composable morphisms f and g that are respectively bounded by α and β, one has $g \cdot f$ bounded by $\alpha + \beta$. A $\mathcal{P}M$-normed functor must preserve bounds.

We note that the trivial monoid $\{0\}$ returns the case $\mathcal{V} = \mathsf{2}$ of (1).

(7) Other quantales of interest include the quantale Δ of distance distribution functions [14, 21, 22], categorically characterized as a coproduct of two copies of $\mathcal{R}_+$ in the category of commutative unital quantales and their homomorphisms [38]. The small categories enriched in Δ are the probabilistic metric spaces. Its consideration as a value quantale for norms of categories, however, we leave for future work.

4 The $\mathcal{V}$-normed categories $\mathsf{Set}||\mathcal{V}$, $\mathcal{V}$-Lip, and $\mathcal{V}$-Dist

Every symmetric monoidal-closed category $\mathcal{W}$ becomes a $\mathcal{W}$-enriched category with the same objects, *qua* its internal hom. Exploiting this fact for $\mathcal{W} = \mathsf{Set}/\!/\mathcal{V}$ we obtain a $\mathcal{V}$-normed category whose objects are $\mathcal{V}$-normed sets, but whose hom-sets of morphisms $A \to B$ are given by the internal hom $[A, B]$ of $\mathsf{Set}/\!/\mathcal{V}$, *i.e.*, by *all* Set-maps $A \to B$. The emerging normed category must be carefully distinguished from the category $\mathsf{Set}/\!/\mathcal{V}$ and, as it plays an important role in what follows, deserves a separate notation,

$$\mathsf{Set}||\mathcal{V}\ ,$$

not to be confused with its (generally non-full) subcategory $\mathsf{Set}/\!/\mathcal{V}$. For clarity, with the proof of Proposition 2.2 we summarize these points, as follows.

Proposition 4.1. *The category* $\mathsf{Set}||\mathcal{V}$ *of* $\mathcal{V}$*-normed sets with arbitrary mappings as morphisms becomes a* $\mathcal{V}$*-normed category with*

$$|A \xrightarrow{\varphi} B| = \bigwedge_{a \in A} [|a|, |\varphi a|].$$

In the notation and terminology of Facts 3.2(3)*, the ordinary category* $\mathsf{Set}/\!/\mathcal{V}$ *is precisely the category of* k*-morphisms of the* $\mathcal{V}$*-normed category* $\mathsf{Set}||\mathcal{V}$*:*

$$(\mathsf{Set}||\mathcal{V})_\circ = \mathsf{Set}/\!/\mathcal{V}.$$

Hence, at the level of sets we have

$$(\mathsf{Set}||\mathcal{V})(A,B) = [A,B] = \mathsf{Set}(A,B) \quad \text{and} \quad (\mathsf{Set}||\mathcal{V})_\circ(A,B) = (\mathsf{Set}//\mathcal{V})(A,B)$$

for all $\mathcal{V}$-normed sets A and B.

Remarks 4.2. (1) As an ordinary category, $\mathsf{Set}||\mathcal{V}$ is equivalent to the category Set. The introduction of a separate notation is justifiable only when $\mathsf{Set}||\mathcal{V}$ is regarded as a $\mathcal{V}$-normed category.

(2) The monoid $(\mathcal{V}, \otimes, \mathrm{k})$ may be regarded as a one-object $\mathcal{V}$-normed category, with an identical norm function. As the monoid acts on itself, we obtain a functor

$$\lambda : \mathcal{V} \longrightarrow \mathsf{Set}||\mathcal{V}, \quad u \longmapsto (\lambda_u : \mathcal{V} \to \mathcal{V},\, v \mapsto u \otimes v),$$

where $\mathcal{V}$, as a domain and codomain of the (left-)translation λ_u, is regarded just as an identically $\mathcal{V}$-normed set. The functor λ is $\mathcal{V}$-normed, actually norm-preserving, since

$$|\lambda_u| = \bigwedge_{v \in \mathcal{V}} [v, u \otimes v] = u\,.$$

In generalization of Example 3.5(5), next we consider another category in which morphisms are not required to respect the structure of the objects: the objects of the category $\mathcal{V}$-Lip are small $\mathcal{V}$-categories, with arbitrary maps as morphisms (so that, as an ordinary category, $\mathcal{V}$-Lip is actually equivalent to Set again, as in Remarks 4.2(1)). Their $\mathcal{V}$-norm measures to which extent they may fail to be $\mathcal{V}$-functors, as follows.

Proposition 4.3. *Defining the* Lipschitz $\mathcal{V}$-norm *of a mapping $\varphi : X \to Y$ between small $\mathcal{V}$-categories by*

$$|\varphi| = \bigwedge_{x,x' \in X} [X(x,x'), Y(\varphi x, \varphi x')]$$

makes $\mathcal{V}$-Lip a $\mathcal{V}$-normed category, such that $\mathrm{k} \leq |\varphi|$ holds precisely when φ is a $\mathcal{V}$-functor:

$$(\mathcal{V}\text{-}\mathsf{Lip})_\circ = \mathcal{V}\text{-}\mathsf{Cat}\,.$$

Furthermore, the forgetful functor

$$\mathcal{V}\text{-}\mathsf{Lip} \longrightarrow \mathsf{Set}||\mathcal{V}, \; X \longmapsto X \times X,$$

defined to remember just that every $\mathcal{V}$-category X comes with a function $X \times X \to \mathcal{V}$, is not only $\mathcal{V}$-normed but actually norm-preserving. Restricting it to k*-morphisms gives a faithful functor $\mathcal{V}$-$\mathsf{Cat} \longrightarrow \mathsf{Set}//\mathcal{V}$.*

Proof. For arbitrary maps $\varphi : X \to Y$ and $\psi : Y \to Z$ of $\mathcal{V}$-categories X, Y and Z, utilizing the fact that $\mathcal{V}$ with its internal hom [-,-] is a $\mathcal{V}$-category, we obtain

$$\begin{aligned}
|\varphi| \otimes |\psi| &= \Big(\bigwedge_{x,x' \in X} [X(x,x'), Y(\varphi x, \varphi x')]\Big) \otimes \Big(\bigwedge_{y,y' \in Y} [Y(y,y'), Z(\psi y, \psi y')]\Big) \\
&\leq \bigwedge_{x,x' \in X} [X(x,x'), Y(\varphi x, \varphi x')] \otimes [Y(\varphi x, \varphi x'), Z(\psi\varphi x, \psi\varphi x')] \\
&\leq \bigwedge_{x,x' \in X} [X(x,x'), Z(\psi\varphi x, \psi\varphi x')] = |\psi \cdot \varphi|\,.
\end{aligned}$$

Since trivially $\mathrm{k} \le |\mathrm{id}_X|$, this makes $\mathcal{V}$-Lip $\mathcal{V}$-normed. The other statements are even easier to verify. $\square$

We will apply the Proposition in Section 9 in order to obtain results for categories of generalized metric spaces.

There is another well-known way of weakening the notion of $\mathcal{V}$-functor. Recall that a *$\mathcal{V}$-distributor* $\rho : X \looparrowright Y$ (also *$\mathcal{V}$-(bi)module* or *-profunctor*) of $\mathcal{V}$-categories X and Y is given by a $\mathcal{V}$-functor $\rho : X^{\mathrm{op}} \otimes Y \to \mathcal{V}$, *i.e.*, by a function ρ satisfying

$$X(x', x) \otimes \rho(x, y) \otimes Y(y, y') \le \rho(x', y')$$

for all $x, x' \in X$ and $y, y' \in Y$. Its composite with $\sigma : Y \looparrowright Z$ is defined by

$$(\sigma \cdot \rho)(x, z) = \bigvee_{y \in Y} \rho(x, y) \otimes \sigma(y, z).$$

With the identity $\mathcal{V}$-distributor 1_X^* on X given by the structure of X, one obtains the category $\mathcal{V}$-Dist, together with the identity-on-objects functors

$$\mathcal{V}\text{-}\mathsf{Cat} \xrightarrow{\ (-)_*\ } \mathcal{V}\text{-}\mathsf{Dist} \xleftarrow{\ (-)^*\ } (\mathcal{V}\text{-}\mathsf{Cat})^{\mathrm{op}} ,$$

defined by $f_*(x, y) = Y(fx, y)$ and $f^*(y, x) = Y(y, fx)$ for every $\mathcal{V}$-functor $f : X \to Y$ and all $x \in X, y \in Y$. With the order of $\mathcal{V}$-distributors induced pointwise by the order of $\mathcal{V}$, we can regard $\mathcal{V}$-Dist as a 2-category, with 2-cells given by order. One then has $f_* \dashv f^*$, *i.e.*, $1_X^* \le f^* \cdot f_*$ and $f_* \cdot f^* \le 1_Y^*$.

Proposition 4.4. *Setting the* Hausdorff norm *of a $\mathcal{V}$-distributor $\rho : X \looparrowright Y$ of $\mathcal{V}$-categories as*

$$|\rho| = \bigwedge_{x \in X} \bigvee_{y \in Y} \rho(x, y)$$

one makes $\mathcal{V}$-Dist a $\mathcal{V}$-normed category such that every $\mathcal{V}$-functor f, represented as a $\mathcal{V}$-distributor f_* or f^*, becomes a k*-morphism. The function $|\text{-}|$ is monotone, thus making*

$$|\text{-}| : \mathcal{V}\text{-}\mathsf{Dist} \longrightarrow \mathcal{V}$$

of Facts 3.2(1) a lax 2-functor.

Proof. Given ρ and $\sigma : Y \looparrowright Z$ one has

$$\begin{aligned} |\rho| \otimes |\sigma| &= (\bigwedge_{x \in X} \bigvee_{y \in Y} \rho(x, y)) \otimes (\bigwedge_{y' \in Y} \bigvee_{z \in Z} \sigma(y', z)) \\ &\le \bigwedge_{x} \bigvee_{y} (\rho(x, y) \otimes \bigvee_{z} \sigma(y, z)) \\ &= \bigwedge_{x} \bigvee_{z} \bigvee_{y} \rho(x, y) \otimes \sigma(y, z) \\ &= \bigwedge_{x} \bigvee_{z} (\sigma \cdot \rho)(x, z) = |\sigma \cdot \rho| . \end{aligned}$$

Since one also has $\mathrm{k} \le \bigwedge_x X(x, x) = |1_X^*|$, this proves the principal assertion of the Proposition. The additional claim may also be verified easily. $\square$

Remark 4.5. Alternatively, for a $\mathcal{V}$-distributor $\rho : X \nrightarrow Y$ one may set (cp. [30])

$$\|\rho\| = \bigvee_{\varphi : X \to Y} \bigwedge_{x \in X} \rho(x, \varphi x)$$

to make the category $\mathcal{V}$-Dist $\mathcal{V}$-normed; here the join runs over *all* mappings $\varphi : X \to Y$. The choice-free proof of this claim proceeds similarly to the proof for $|\rho|$. But, under the Axiom of Choice, and if the complete lattice $\mathcal{V}$ is (constructively) completely distributive (see [40, 22]), one has in fact $\|\rho\| = |\rho|$ for all ρ. We note that the underlying lattices of all quantales considered in this paper so far are completely distributive; in particular $\mathsf{2}$ and $\mathcal{R}_+$ are.

For every $\mathcal{V}$-distributor $\rho : X \nrightarrow Y$ and all subsets $A \subseteq X$, $B \subseteq Y$, denoting their inclusion maps to X and Y by i_A and i_B, respectively, we define

$$(\mathcal{H}\rho)(A, B) := |\mathrm{i}_B^* \cdot \rho \cdot (\mathrm{i}_A)_*| = \bigwedge_{x \in A} \bigvee_{y \in B} \rho(x, y)$$

and use the abbreviation $\mathcal{H}X = \mathcal{H}1_X^*$. Applying the norm rules of Proposition 4.4 we now show how one easily concludes (some essential parts of) Theorem 3 in [4] on the Hausdorff monad on $\mathcal{V}$-Cat (identified in [36] as describing its Eilenberg-Moore algebras as the conically cocomplete $\mathcal{V}$-categories), and on the lax extension of that monad to $\mathcal{V}$-Dist:

Corollary 4.6. *The function $\mathcal{H}X$ makes the powerset of every $\mathcal{V}$-category X a $\mathcal{V}$-category, denoted again by $\mathcal{H}X$, such that $\mathcal{H}\rho : \mathcal{H}X \nrightarrow \mathcal{H}Y$ becomes a $\mathcal{V}$-distributor for every $\mathcal{V}$-distributor $\rho : X \nrightarrow Y$. This defines a $\mathcal{V}$-normed lax 2-functor $\mathcal{H}$, so that $|\rho| \leq |\mathcal{H}\rho|$, and it restricts to a (strict) endofunctor of $\mathcal{V}$-Cat which lifts the powerset functor of Set, so that one has the commutative diagram*

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathsf{Cat} \arrow[r, "(-)_*"] \arrow[d, "\mathcal{H}"'] & \mathcal{V}\text{-}\mathsf{Dist} \arrow[d, "\mathcal{H}"] & (\mathcal{V}\text{-}\mathsf{Cat})^{\mathrm{op}} \arrow[l, "(-)^*"'] \arrow[d, "\mathcal{H}^{\mathrm{op}}"] \\
\mathcal{V}\text{-}\mathsf{Cat} \arrow[r, "(-)_*"] & \mathcal{V}\text{-}\mathsf{Dist} & (\mathcal{V}\text{-}\mathsf{Cat})^{\mathrm{op}} \arrow[l, "(-)^*"']
\end{tikzcd}$$

Proof. For $\mathcal{V}$-distributors $\rho : X \nrightarrow Y$, $\sigma : Y \nrightarrow Z$ and all subsets $A \subseteq Y$ and $C \subseteq Z$, we have

$$\begin{aligned}
(\mathcal{H}\sigma \cdot \mathcal{H}\rho)(A, C) &= \bigvee_{B \subseteq Y} \mathcal{H}\rho(A, B) \otimes \mathcal{H}\sigma(B, C) \\
&= \bigvee_{B \subseteq Y} |\mathrm{i}_C^* \cdot \sigma \cdot (\mathrm{i}_B)_*| \otimes |\mathrm{i}_B^* \cdot \rho \cdot (\mathrm{i}_A)_*| \\
&\leq \bigvee_{B \subseteq Y} |\mathrm{i}_C^* \cdot \sigma \cdot (\mathrm{i}_B)_* \otimes \mathrm{i}_B^* \cdot \rho \cdot (\mathrm{i}_A)_*| \\
&\leq |\mathrm{i}_C^* \cdot \sigma \cdot 1_Y^* \cdot \rho \cdot (\mathrm{i}_A)_*| \\
&= \mathcal{H}(\sigma \cdot \rho)(A, C)
\end{aligned}$$

and $\mathrm{k} \le |1_A^*| \le |\mathrm{i}_A^* \cdot (\mathrm{i}_A)_*| = \mathcal{H}X(A,A)$. For $\rho = \sigma = 1_X^*$, this shows that $\mathcal{H}X$ is a $\mathcal{V}$-category. Choosing alternately only one of the $\mathcal{V}$-distributors to be identical will show that $\mathcal{H}\rho$ is a $\mathcal{V}$-distributor, while the general case confirms that $\mathcal{H}$ is a lax 2-functor of $\mathcal{V}$-Dist. It is $\mathcal{V}$-normed since

$$\mathcal{H}\rho(A,B) = |\mathrm{i}_B^* \cdot \rho \cdot (\mathrm{i}_A)_*| \ge |\mathrm{i}_B^*| \otimes |\rho| \otimes |(\mathrm{i}_A)_*| \ge \mathrm{k} \otimes |\rho| \otimes \mathrm{k} = |\rho| \ .$$

For a $\mathcal{V}$-functor $f : X \to Y$ also $\mathcal{H}f : \mathcal{H}X \to \mathcal{H}Y$ with $(\mathcal{H}f)(A) = f(A)$ is a $\mathcal{V}$-functor since

$$\mathcal{H}X(A,A') = |(\mathrm{i}_{A'})^* \cdot (\mathrm{i}_A)_*| \le |(\mathrm{i}_{A'})^* \cdot f^* \cdot f_* \cdot (\mathrm{i}_A)_*| = |(\mathrm{i}_{f(A')})^* \cdot (\mathrm{i}_{f(A)})_*| = \mathcal{H}Y(f(A'), f(A)) \ .$$

Finally, the left part of the diagram above commutes since

$$(\mathcal{H}f_*)(A,B) = |\mathrm{i}_B^* \cdot f_* \cdot (\mathrm{i}_A)_*| = |\mathrm{i}_B^* \cdot (\mathrm{i}_{f(A)})_*| = \mathcal{H}Y(f(A),B) = (\mathcal{H}f)_*(A,B) \ ,$$

and the commutativity of the right part follows by duality. □

Generalizing Example 3.5(4) from $\mathcal{R}_+$ to our general quantale $\mathcal{V}$, and trading the Hausdorff norm $|\text{-}|$ *à la* Lawvere for $\|\text{-}\|$ of Remark 4.5, one obtains the following corollary (without using Choice or assuming complete distributivity of $\mathcal{V}$!):

Corollary 4.7. *For every $\mathcal{V}$-category X, the subsets of X are the objects of a small $\mathcal{V}$-normed category $\mathbb{H}X$ whose morphisms are arbitrary maps $\varphi : A \to B$, normed by $|\varphi| = \bigwedge_{x \in A} X(x, \varphi x)$. The sum functor* s *of Facts 3.2(2) takes $\mathbb{H}X$ to the $\mathcal{V}$-category $\tilde{\mathcal{H}}X$ with $\tilde{\mathcal{H}}(A,B) = \bigvee_{\varphi : A \to B} |\varphi|$.*

As already noted in Examples 3.5(4), with Choice one has $\tilde{\mathcal{H}}X = \mathcal{H}X$. However, even then the passage $X \mapsto \mathbb{H}X$ (unlike $X \mapsto \tilde{\mathcal{H}}X$) generally fails to be functorial.

Remark 4.8. For further investigations on the functor $\mathcal{H}$ and various restrictions thereof we refer the reader to [20]. The question to which extent completeness properties of the object X get transferred to $\mathcal{H}X$, without the symmetrization of the structure and/or some restriction on the subsets of X to be considered, such as the (in some sense) compact subsets, remains open. This includes the Cauchy cocompleteness (of the normed category $\mathrm{i}X$) as introduced in Section 6.

5 Normed convergence and symmetry

In order to introduce the concept of normed convergence in a $\mathcal{V}$-normed category, we find it useful to remind ourselves how sequential colimits are formed in $\mathsf{Set}/\!/\mathcal{V}$. The following easily checked statement is an immediate consequence of $\mathsf{Set}/\!/\mathcal{V}$ being topological over Set, amended by an also easily verified second formula for the norm of the colimit object.

Proposition 5.1. *The colimit of a sequence $A_0 \to A_1 \to A_2 \dots$ in $\mathsf{Set}/\!/\mathcal{V}$ is formed by providing the colimit A of the underlying sequence in* Set *with the least norm that makes the colimit cocone $(A_N \xrightarrow{\kappa_N} A)_{N \in \mathbb{N}}$ live in $\mathsf{Set}/\!/\mathcal{V}$:*

$$|c| = \bigvee \{ |a| \mid a \in \bigcup_{N \in \mathbb{N}} \kappa_N^{-1} c \} = \bigwedge_{N \in \mathbb{N}} \bigvee_{n \ge N} \bigvee_{a \in \kappa_n^{-1} c} |a|$$

for all $c \in A$.

Definition 5.2. Let $s : \mathbb{N} \to \mathbb{X}$ be a sequence[2] in a $\mathcal{V}$-normed category $\mathbb{X}$, written as

$$s = \ (x_m \xrightarrow{s_{m,n}} x_n)_{m \le n \in \mathbb{N}} \ .$$

An object x is a *normed colimit* of s in $\mathbb{X}$ if

(C1) x is a colimit of s in the ordinary category $\mathbb{X}$, with a colimit cocone $(x_n \xrightarrow{\gamma_n} x)_{n \in \mathbb{N}}$ s. th.

(C2) for all objects y in $\mathbb{X}$, the canonical Set-bijections[3]

$$\mathrm{Nat}(s_{|N}, \Delta y) \xrightarrow{\ \ \kappa_N\ \ } \mathbb{X}(x,y), \qquad (f \cdot \gamma_n)_{n \ge N} \xleftarrow{\ \ \kappa_N^{-1}\ \ } f \qquad (N \in \mathbb{N})$$

form a colimit cocone in $\mathsf{Set}/\!/\mathcal{V}$, where $s_{|N}$ is the restriction of s to $\uparrow N = \{N, N+1, \dots\}$ and $\mathrm{Nat}(s_{|N}, \Delta y) = [\uparrow N, \mathbb{X}](s_{|N}, \Delta y)$ $(N \in \mathbb{N})$ is considered as a sequence in $\mathsf{Set}/\!/\mathcal{V}$, with all connecting maps given by restriction.

Keeping the notation of this definition, let us immediately analyze the meaning of (C2):

Proposition 5.3. *Condition* (C2) *says equivalently that, for all morphisms f in $\mathbb{X}$ with domain x, one must have*

$$|f| = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \ge N} |f \cdot \gamma_n| \, .$$

The "$\le$"-part of this equality is satisfied if, and only if,

$$\mathrm{k} \le \bigvee_{N \in \mathbb{N}} \bigwedge_{n \ge N} |\gamma_n| \, .$$

Proof. Trivially, the natural Set-bijections κ_N $(N \in \mathbb{N})$ form a colimit cocone in Set. In order to make it a colimit cocone in $\mathsf{Set}/\!/\mathcal{V}$, by Proposition 5.1 the norm on $\mathbb{X}(x,y)$ must satisfy

$$|f| = \bigvee \{ |\beta| \mid \beta = \Delta f \cdot \gamma_{|N} : s_{|N} \to \Delta y, \ N \in \mathbb{N} \}$$

for all $f : x \to y$ in $\mathbb{X}$, which, by the norm formula for natural transformations (Proposition 3.3), amounts to the claimed formula for $|f|$.

The second statement of the Proposition follows from the following more general lemma. $\square$

Lemma 5.4. *For any cocone $\alpha : s \to \Delta x$ over a sequence $s = (x_n)_{n \in \mathbb{N}}$ with vertex x in a $\mathcal{V}$-normed category $\mathbb{X}$, the following are equivalent:*

(i) $\mathrm{k} \le \bigvee_{N \in \mathbb{N}} \bigwedge_{n \ge N} |\alpha_n|$;

(ii) $|1_x| \le \bigvee_{N \in \mathbb{N}} \bigwedge_{n \ge N} |\alpha_n|$;

(iii) $|f| \le \bigvee_{N \in \mathbb{N}} \bigwedge_{n \ge N} |f \cdot \alpha_n|$, *for every morphism $f : x \to y$ in $\mathbb{X}$.*

[2] Here the ordered set $\mathbb{N}$ is treated as a category, discretely $\mathcal{V}$-normed with constant value $\perp$ for all non-identical arrows, so that the sequence s becomes a $\mathcal{V}$-normed functor $\mathbb{N} \to \mathbb{X}$.

[3] Note that a colimit x of s in the ordinary category $\mathbb{X}$ is also a colimit of every restricted sequence $s_{|N}$.

Proof. Trivially, one has (iii)$\Longrightarrow$(ii)$\Longrightarrow$(i). For (i)$\Longrightarrow$(iii), we note

$$|f| = |f| \otimes \mathrm{k} \leq |f| \otimes \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |\alpha_n| \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |f| \otimes |\alpha_n| \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |f \cdot \alpha_n|.$$

□

Extending the terminology used for morphisms in Facts 3.2(3), we call a cocone $\alpha : s \to \Delta x$ over a sequence $s = (x_n)_{n\in\mathbb{N}}$ in a normed category $\mathbb{X}$ a k-*cocone* if it satisfies condition (i) of Lemma 5.4. We conclude from Proposition 5.3:

Corollary 5.5. *An object x is a normed colimit of a sequence s in a $\mathcal{V}$-normed category $\mathbb{X}$ if, and only if, x is a colimit of s in the ordinary category $\mathbb{X}$ with a colimit cocone γ such that*

(C2a) $\mathrm{k} \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |\gamma_n|$, *i.e., γ is a* k*-cocone;*

(C2b) $|f| \geq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |f \cdot \gamma_n|$, *for every morphism $f : x \to y$ in $\mathbb{X}$.*

Corollary 5.6. *A normed colimit of a sequence in a $\mathcal{V}$-normed category $\mathbb{X}$ is uniquely determined up to a* k*-isomorphism, i.e., up to an isomorphism in $\mathbb{X}_\circ$.*

Proof. If $\gamma : s \to \Delta x$ and $\delta : s \to \Delta y$ are both colimit cocones representing x and y as normed colimits of s, respectively, then the canonical morphism $f : x \to y$ is not only an isomorphism in $\mathbb{X}$, but also satisfies

$$|f| = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |f \cdot \gamma_n| = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |\delta_n| \geq \mathrm{k}\ ,$$

and likewise $|f^{-1}| \geq \mathrm{k}$. Hence, f is an isomorphism in $\mathbb{X}_\circ$. □

Here is a sufficient, but not necessary, condition on the $\mathcal{V}$-normed category $\mathbb{X}$ (which will be discussed further in Facts 5.8) to make condition (C2b) of Corollary 5.5 redundant, as follows.

Corollary 5.7. *Let $\mathbb{X}$ be a $\mathcal{V}$-normed category satisfying the condition*

(S) $\qquad |f \cdot h| \otimes |h| \leq |f|$ *for all composable morphisms h and f.*

Then an object x is a normed colimit of a sequence s in $\mathbb{X}$ if, and only if, x is a colimit of s in the ordinary category $\mathbb{X}$, with a colimit cocone that is a k*-cocone.*

Proof. First we note that, in (S), the morphism h may be replaced equivalently by any cocone $\alpha : D \to \Delta x$, for some diagram $D : \mathbb{I} \to \mathbb{X}$ with $\mathbb{I} \neq \emptyset$, so that (S) then reads as $|\Delta f \cdot \alpha| \otimes |\alpha| \leq |f|$. Indeed, for all $i \in \mathbb{I}$, using (S) and $\mathbb{I} \neq \emptyset$, one has

$$|\Delta f \cdot \alpha| \otimes |\alpha| = \bigwedge_{i\in\mathbb{I}} |f \cdot \alpha_i| \otimes \bigwedge_{i\in\mathbb{I}} |\alpha_i| \leq \bigwedge_{i\in\mathbb{I}} |f \cdot \alpha_i| \otimes |\alpha_i| \leq \bigwedge_{i\in I} |f| = |f|\ .$$

Having (C2a) and this extended version of (S), we can now show (C2b) of Corollary 5.5, as follows, utilizing also the fact that the occurring joins are directed:

$$\begin{aligned}
\bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N}|f\cdot\gamma_n| &\leq (\bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N}|f\cdot\gamma_n|)\otimes(\bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N}|\gamma_n|)\\
&\leq \bigvee_{N\in\mathbb{N}}(\bigwedge_{n\geq N}|f\cdot\gamma_n|\otimes\bigwedge_{n\geq N}|\gamma_n|)\\
&\leq \bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N}|f\cdot\gamma_n|\otimes|\gamma_n|\\
&\leq |f|.
\end{aligned}$$

□

Facts 5.8. (1) Condition (S) is a (strong) *symmetry* condition on the normed category $\mathbb{X}$. Indeed, if $\mathbb{X}=\mathrm{i}X$ is given by a $\mathcal{V}$-category X as in Facts 3.2(2), then (S) means equivalently that X is *symmetric*, *i.e.*, that

$$X(x,y)=X(y,x)$$

holds for all $x,y\in X$. We call an arbitrary $\mathcal{V}$-normed category $\mathbb{X}$ satisfying (S) *forward symmetric*. The condition generally fails in $\mathsf{Set}||\mathcal{V}$, even for $\mathcal{V}=\mathcal{R}_+=([0,\infty],\geq,+,0)$. Indeed, considering $\mathbb{N}$ as an identically $\mathcal{R}_+$-normed set, then for the endomaps f and h which keep 0 fixed while $hn=n-1$ and $fn=n+1$ for all $n>0$, one has $|f|=1$ but $|f\cdot h|+|h|=0$.

(2) The dualization of (S) reads as

(S$^{\mathrm{op}}$) $\quad |g\cdot f|\otimes|g|\leq|f|$ for all composable morphisms f and g;

we call $\mathbb{X}$ *backwards symmetric* in this case. Indeed for $\mathbb{X}=\mathrm{i}X$ as in (1), condition (S$^{\mathrm{op}}$) again amounts to the $\mathcal{V}$-category X being symmetric, and again, it generally fails in $\mathsf{Set}||\mathcal{V}$. However, for arbitrary $\mathbb{X}$, conditions (S) and (S$^{\mathrm{op}}$) are far from being equivalent (as already the example in (3) shows). But, as noted for $\mathcal{V}=\mathcal{R}_+$ in Lemma 2.2 of [29], each of the two conditions implies the inverse of an isomorphism f in the ordinary category $\mathbb{X}$ to have the same norm as f; for example, with (S$^{\mathrm{op}}$) one has

$$|f|\geq|f^{-1}\cdot f|\otimes|f^{-1}|\geq\mathrm{k}\otimes|f^{-1}|=|f^{-1}|,$$

and likewise for $|f^{-1}|\geq|f|$. In particular, *if* (S) *or* (S$^{\mathrm{op}}$) *holds, a morphism in* $\mathbb{X}_\circ$ *that is an isomorphism in the ordinary category* $\mathbb{X}$ *is also an isomorphism in* $\mathbb{X}_\circ$.

(3) For $\mathcal{V}=\mathcal{R}_+$, in addition to our conditions on a normed category, Kubiś [29] includes condition (S$^{\mathrm{op}}$) as part of his definition of normed category, and then defines the normed convergence of s to x by requiring only conditions (C1) and (C2a), in their $\mathcal{R}_+$-versions. This, however, does not make the colimit unique up to a 0-isomorphism (here $0=\mathrm{k}$).

Indeed, the following simple witness appears already in [29]. Consider the category given by the preordered set $\mathbb{N}\cup\{a,b\}$ with new distinct elements a,b, and extend the natural order by $n\leq a\leq b$ and $n\leq b\leq a$ for all $n\in\mathbb{N}$; it gets (Kubiś-)normed by putting $|x\to y|=0$ whenever

$x \notin \{a, b\}$, and $|a \to b| = |b \to a| = \infty$. Hence, a and b are ordinary colimits of the sequence (n), both satisfying (C1) and (C2a), and the ambient category satisfies ($\mathrm{S^{op}}$). However, (S) is violated – not even (C2b) holds, which is why a and b fail to be 0-isomorphic.

If one modifies this example by declaring the norms of morphisms $n \to b$ to be 1, rather than 0, one still has a normed category satisfying ($\mathrm{S^{op}}$), but now the ordinary colimit b no longer satisfies (C2a) whilst a still does.

(4) A *$\mathcal{V}$-normed monoid* is simply a monoid $(A, \cdot, 1)$ which, considered as a one-object category $\mathbb{A}$ with A as its only hom-set, is $\mathcal{V}$-normed; that is: A comes with a function $|\text{-}| : A \to \mathcal{V}$ satisfying $\mathrm{k} \leq |1|$ and $|a| \otimes |b| \leq |ab|$ for all $a, b \in A$. In case $\mathcal{V} = \mathcal{R}_+$, such normed monoids are often called *semi-normed* [8], but here we will omit the prefix. Every *left-invariant* Lawvere metric on a monoid A makes A a normed monoid [29]. In fact, more generally, if a (multiplicatively written) monoid A carries a $\mathcal{V}$-category A structure such that, for all $a, b, c \in a$, one has $A(ca, cb) = A(a, b)$, then

$$|a| := A(1, a)$$

makes A a $\mathcal{V}$-normed monoid. Indeed, trivially one has $\mathrm{k} \leq |1|$ and

$$|a| \otimes |b| = X(1, a) \otimes X(1, b) = X(1, a) \otimes X(a, ab) \leq X(1, ab) = |ab| \;.$$

Conversely, if the $\mathcal{V}$-normed monoid A is, algebraically, a group, then the norm makes A a left-invariant $\mathcal{V}$-category, via

$$A(a, b) := |a^{-1}b| \;,$$

and this actually results into a one-one correspondence between $\mathcal{V}$-norms and left invariant $\mathcal{V}$-category structures on the given group A.

We note that the identity norm of the $\mathcal{V}$-normed commutative monoid $(\mathcal{V}, \otimes, \mathrm{k})$ of Remark 4.2 is induced by its $\mathcal{V}$-category structure, given by its internal hom [-,-], although this structure fails to be (left-) invariant, unless $\mathcal{V}$ is trivial. And, of course, $\mathcal{V}$ fails to be a group in all but the trivial case.

(5) Further to the case that the $\mathcal{V}$-normed monoid $(A, \cdot, 1)$ considered in (4) is actually a group, let us call A a *$\mathcal{V}$-normed group* if the additional condition $|a^{-1}| = |a|$ holds for all $a \in A$. (For $\mathcal{V} = \mathcal{R}_+$, this gives the standard notion of *normed group* as used in [8].) The induced $\mathcal{V}$-category structure of a $\mathcal{V}$-normed group A is symmetric, so that $A(a, b) = A(b, a)$ holds for all $a, b \in A$. Conversely, if the induced $\mathcal{V}$-category structure of A is symmetric, then the one-object $\mathcal{V}$-normed category $\mathbb{A}$ induced by A is forward symmetric, *i.e.*,

$$\text{(S)} \qquad |ab| \otimes |b| = A(1, ab) \otimes A(1, b) = A(1, ab) \otimes A(ab, a) \leq A(1, a) = |a|$$

holds. Moreover, as follows already from (2), condition (S) implies $|a^{-1}| = |a|$ for all $a \in A$ and thus makes A a $\mathcal{V}$-normed group. Consequently, for a $\mathcal{V}$-normed monoid A that, algebraically, is a group, the following conditions are equivalent:

(i) A is a $\mathcal{V}$-normed group;

(ii) the induced $\mathcal{V}$-category structure of A is symmetric;

(iii) the one-object $\mathcal{V}$-normed category $\mathbb{A}$ given by A is forward symmetric.

Caution: The one-object $\mathcal{V}$-normed category $\mathbb{A}$ must not be confused with the (generally) multi-object $\mathcal{V}$-normed category iA as considered in (1). In the latter category, conditions (S) and (S$^{\mathrm{op}}$) are equivalent, unlike in the former category, unless the group A is Abelian.

6 Cauchy cocompleteness

We now extend (the key) Definition 5.2 in the expected way:

Definition 6.1. For a $\mathcal{V}$-normed category $\mathbb{X}$, we say that

- a sequence $s = (x_m \xrightarrow{s_{m,n}} x_n)_{m\leq n\in\mathbb{N}}$ in $\mathbb{X}$ is *Cauchy* if $\mathrm{k} \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq m\geq N} |s_{m,n}|$, and
- $\mathbb{X}$ is *Cauchy (norm-)cocomplete* if every Cauchy sequence in $\mathbb{X}$ has a normed colimit in $\mathbb{X}$.

We will compare this notion for $\mathcal{V}$*-normed* categories with the well-established notion of *idempotent completeness* for *ordinary* categories (and $\mathcal{V}$-enriched categories, see [10]) in Corollary 7.3.

Facts 6.2. (1) Let us note first that the existence of a normed colimit for a sequence s does not necessitate s to be Cauchy, even when $\mathcal{V} = \mathcal{R}_+$. For example, considering the ordered set $\mathbb{N}\cup\{\infty\}$ of natural numbers with an added maximum as a category, normed by $|m \to n| = n - m$ and $|n \to \infty| = 0$ for all $m \leq n$ in $\mathbb{N}$, we obtain a normed category (satisfying (S), but not (S$^{\mathrm{op}}$)) such that ∞ is a normed colimit of the sequence $s = (n)_{n\in\mathbb{N}}$, although s badly fails to be Cauchy; in fact, here $\inf_{N\in\mathbb{N}} \sup_{n\geq m\geq N} |s_{m,n}| = \infty$.

(2) The notions of Definitions 5.2 and 6.1 are easily dualizable. For a $\mathcal{V}$-normed category $\mathbb{X}$, the dual $\mathbb{X}^{\mathrm{op}}$ of the ordinary category $\mathbb{X}$ becomes $\mathcal{V}$-normed when giving every morphism the same norm as in $\mathbb{X}$. Now, having an *inverse* sequence $s : \mathbb{N}^{\mathrm{op}} \to \mathbb{X}$, given by morphisms $s_{m,n} : x_n \to x_m$ in $\mathbb{X}$ for all $m \leq n \in \mathbb{N}$, the inverse sequence is said to be *Cauchy* in $\mathbb{X}$ if the sequence $s^{\mathrm{op}} : \mathbb{N} \to \mathbb{X}^{\mathrm{op}}$ is Cauchy in $\mathbb{X}^{\mathrm{op}}$. Furthermore, an object x is a *normed limit* of s in $\mathbb{X}$ if x is a normed colimit of s^{op} in $\mathbb{X}^{\mathrm{op}}$; this means that x is a limit of s in the ordinary category $\mathbb{X}$, with a limit cone $\lambda : \Delta x \to s$ such that

$$|f| = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |\lambda_n \cdot f|,$$

for all morphisms $f : y \to x$ in $\mathbb{X}$.

Examples 6.3. (1) For $\mathcal{V} = \mathcal{R}_+$, we repeat that the condition for a sequence s to be Cauchy in the $\mathcal{R}_+$-normed category $\mathbb{X}$ reads as

$$\inf_{N\in\mathbb{N}} \sup_{n\geq m\geq N} |s_{m,n}| = 0,$$

and for the ordinary colimit x with colimit cocone γ in $\mathbb{X}$ to be a normed colimit means that

$$\inf_{N\in\mathbb{N}}\sup_{n\geq N}|\gamma_n|=0 \quad \text{and} \quad |f|\leq \sup_{n\geq N}|f\cdot\gamma_n|,$$

for every morphism $f:x\to y$ in $\mathbb{X}$ and all $N\in\mathbb{N}$ (see Corollary 5.5).
In case $\mathbb{X}=\mathrm{i}X$ is induced by a (Lawvere) metric space X, the sequence $s=(x_n)$ is Cauchy if, and only if,

$$\inf_{N\in\mathbb{N}}\sup_{n\geq m\geq N} X(x_m,x_n)=0,$$

so that s must be *forward Cauchy* in the sense of [9]; furthermore, x is a normed colimit of s if, and only if,

$$X(x,y)=\inf_{N\in\mathbb{N}}\sup_{n\geq N}X(x_n,y)$$

for all $y\in X$, which means that x is a *forward limit* of s in the language of [9]. (Note that here the ordinary colimit condition for x comes for free since $\mathrm{i}X$ is a groupoid.) The notions of *backward Cauchy* sequence and *backward limit* in X come about by dualization according to Facts 6.2(2), *i.e.*, by interchanging the arguments of $X(\text{-},\text{-})$.

Of course, *if X is symmetric,* there is no need to distinguish between forward and backward, and *one obtains the standard notions of Cauchy sequence and sequential convergence.*

(2) Expanding on Examples 3.5(4), for $\mathcal{V}=1$ we have $\mathsf{Cat}/\!/\mathcal{V}\cong\mathsf{Cat}$, and every sequence in a category $\mathbb{X}$ is Cauchy, and $\mathbb{X}$ is Cauchy cocomplete if, and only if, $\mathbb{X}$ has colimits of sequences.

For $\mathcal{V}=2$, describing an object in $\mathsf{Cat}/\!/\mathcal{V}$ or $\mathsf{CAT}/\!/\mathcal{V}$ as a category $\mathbb{X}$ with a distinguished class $\mathcal{S}$ of morphisms satisfying $\mathsf{Id}(\mathbb{X})\subseteq\mathcal{S}$ and $\mathcal{S}\cdot\mathcal{S}\subseteq\mathcal{S}$, a sequence s in $\mathbb{X}$ is Cauchy if, and only if, eventually all of its connecting maps lie in $\mathcal{S}$; and $\mathbb{X}$ is Cauchy cocomplete if, and only if, every Cauchy sequence s has a colimit x with a colimit cocone $(\gamma_n)_n$ lying eventually in $\mathcal{S}$, such that any morphism $f:x\to y$ belongs to $\mathcal{S}$ as soon as eventually all morphisms $f\cdot\gamma_n$ do so.

(3) More generally, for $\mathcal{V}=\mathcal{P}M$ with a commutative monoid $(M,+,0)$ and $\mathsf{Cat}/\!/\mathcal{V}$ described as in Examples 3.5(5), *i.e.*, as containing all small categories $\mathbb{X}$ equipped with a class of morphisms "bounded by α" for every $\alpha\in M$, a sequence s in $\mathbb{X}$ is Cauchy if, and only if, eventually all connecting maps are bounded by 0; and $\mathbb{X}$ is Cauchy cocomplete if, and only if, every Cauchy sequence s has a colimit x with a colimit cocone $(\gamma_n)_n$ eventually bounded by 0 such that, for every $\alpha\in M$, any morphism $f:x\to y$ is bounded by α as soon as eventually all morphisms $f\cdot\gamma_n$ are bounded by α. The symmetry conditions (S) and (S$^{\mathrm{op}}$) of Corollary 5.7 and Facts 5.8(2) amount to the weak cancellations conditions

$$(\mathrm{S})\qquad f\cdot h\in\mathcal{S}\quad\&\quad h\in\mathcal{S}\Longrightarrow f\in\mathcal{S}\,,$$

$$(\mathrm{S}^{\mathrm{op}})\qquad g\cdot f\in\mathcal{S}\quad\&\quad g\in\mathcal{S}\Longrightarrow f\in\mathcal{S}\,,$$

which typically hold for classes of epimorphisms and classes of monomorphisms, respectively.

(4) For a $\mathcal{V}$-normed monoid $(A,\cdot,1)$, considered as a one-object $\mathcal{V}$-normed category $\mathbb{A}$ (as in Facts 5.8(4)), a sequence s in $\mathbb{A}$ is simply a sequence $(a_n)_n$ of elements in A, and a cocone α over s is given by elements $\alpha_n\in A$ satisfying $\alpha_{n+1}a_n=\alpha_n$ for all $n\in\mathbb{N}$. If A is a group, the

cocone is already determined by α_0, since necessarily $\alpha_n = \alpha_0 a_0^{-1} a_1^{-1} ... a_{n-1}^{-1}$ (where the product of an empty string of elements is 1). In particular, we may consider the cocone γ with $\gamma_0 = 1$ which, since every morphism in $\mathbb{A}$ is an isomorphism, presents the only object of $\mathbb{A}$ trivially as an ordinary colimit of s: the factorizing morphism induced by an arbitrary cocone α is simply α_0:

$$\begin{tikzcd}
* \arrow[r,"a_0"] \arrow[rrrrr, bend left, "\gamma_0=1"] \arrow[drrrr, "\alpha_0"'] & * \arrow[r,"a_1"] & * \arrow[r,"a_2"] & *......* \arrow[r,"a_{n-1}"] & * \arrow[r,"\gamma_n"] \arrow[d,"\alpha_n"'] & * \arrow[dl,"\alpha_0"] \\
& & & & * &
\end{tikzcd}$$

If the group A is actually a $\mathcal{V}$-normed group, so that $\mathbb{A}$ enjoys the symmetry condition (S) (as shown in Facts 5.8(5)), for γ to present $*$ as a normed colimit, by Corollary 5.7 it is necessary and sufficient that γ is k-cocone, which means (since $|a| = |a^{-1}|$ for all $a \in A$) that

$$\mathrm{k} \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |a_{n-1}....a_0|$$

holds. This, however, is a steep hurdle. For example, for $\mathcal{V} = \mathcal{R}_+$ consider the multiplicative group $\mathbb{Q}_{>0}$ of positive rationals, which becomes a normed group when one sets

$$|r| := \sum_p \max\{n_p, -n_p\} \text{ whenever } r = \prod_p p^{n_p} \text{ with } n_p \in \mathbb{Z},$$

where p runs through the set of prime numbers (and the products and sums are only nominally infinite). Note that $|r| = 0$ only if $r = 1$. The convergence condition for the sequence reads as $\inf_{N\in\mathbb{N}} \sup_{n\geq N} |a_{n-1}....a_0| = 0$, and since the norms are always integer valued, it is easy to see that this equivalently means that the sequence s must have some initial segment $a_0, ..., a_{N-1}$ of rational numbers whose product equals 1, followed by an infinite tail that is constantly 1. By contrast, for s to be Cauchy, an *arbitrary* initial segment is allowed, followed by a tail that is constantly 1. As a consequence, $\mathbb{Q}_{>0}$ fails to be Cauchy cocomplete.

7 A note on idempotent completeness

Instigated by Example 6.3(4), for a general $\mathcal{V}$-normed category $\mathbb{X}$ we briefly examine the question of whether constant sequences in $\mathbb{X}$ are Cauchy and have a normed colimit in $\mathbb{X}$. Here a sequence $s : \mathbb{N} \to \mathbb{X}$ is understood to be *constant* if $s_{m,n} = e : x \to x$ for all $m < n$ in $\mathbb{N}$. Such morphism e must necessarily be idempotent in the ordinary category $\mathbb{X}$, and every idempotent morphism defines a constant sequence. Recall that the idempotent e *splits* in $\mathbb{X}$ if $e = t \cdot r$ for some morphisms r, t with $r \cdot t = 1$ (which already exist when $\mathbb{X}$ has epi-mono factorizations, or equalizers, or coequalizers). Such factorization of e is unique, up to a uniquely determined isomorphism.

Lemma 7.1. *The constant sequence given by an idempotent $e : x \to x$ has a colimit in the ordinary category $\mathbb{X}$ if, and only if, e splits.*

Proof. Given a colimit cocone $\rho_n : x \to y \, (n \in \mathbb{N})$ of the constant sequence defined by e, one has

$$\rho_0 = \rho_1 \cdot e = \rho_2 \cdot e \cdot e = \rho_2 \cdot e = \rho_1$$

and, inductively, $\rho_1 = \rho_2 = \cdots =: r$. The colimit cocone makes this morphism epic. Furthermore, the cocone η with $\eta_n = e$ for all n corresponds to a morphism $t : y \to x$ with $t \cdot r = e$, which also satisfies $r \cdot t = 1_y$ since $r \cdot t \cdot r = r \cdot e = r$.

Conversely, given the splitting r, t of e, the cocone ρ with $\rho_n := r : x \to y$ for all n which we call *related to the splitting*, exhibits y as a colimit of the constant sequence e: since any cocone $\alpha : s \to \Delta z$ is easily seen to satisfy $\alpha_0 \cdot t = \alpha_n \cdot t$, we obtain $(\alpha_0 \cdot t) \cdot \rho_n = \alpha_n$ for all $n \in \mathbb{N}$, and furthermore, any morphism $f : y \to z$ with $\Delta f \cdot \rho = \alpha$ necessarily satisfies $f \cdot r = \alpha_0$, so that $f = \alpha_0 \cdot t$. ☐

Recall that $\mathbb{X}$ (as an ordinary category) is said to be *idempotent complete*[4] if all idempotents split in $\mathbb{X}$. Idempotent completeness of the category $\mathbb{X}_\circ$ suffices to provide an affirmative answer to the question raised at the beginning of this section. More precisely:

Proposition 7.2. *The constant sequence s in a $\mathcal{V}$-normed category $\mathbb{X}$ given by an idempotent morphism e is Cauchy precisely when e is a* k*-morphism. In this case, the constant cocone related to a given splitting $e = r \cdot t$ of e in $\mathbb{X}$ gives a normed colimit of s in $\mathbb{X}$ if, and only if, the morphisms r and t are both* k*-morphisms.*

Proof. The first claim is obvious. Also, trivially the constant cocone ρ related to the splitting r, t of e is a k-cocone if, and only if, r is a k-morphism. Since ρ is an ordinary colimit cocone, for the proof of the second claim, assuming $\mathrm{k} \leq |r|$, we just need to show that $\mathrm{k} \leq |t|$ holds if, and only if,

$$\bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} |f \cdot \rho_n| = |f \cdot r| \leq |f|$$

for all $f : y \to z$ in $\mathbb{X}$. Indeed, from $\mathrm{k} \leq |t|$ one obtains $|f \cdot r| \leq |f \cdot r| \otimes |t| \leq |f \cdot r \cdot t| = |f|$, and conversely, exploiting this inequality for $f = t$, from $\mathrm{k} \leq |e|$ we obtain $\mathrm{k} \leq |t \cdot r| \leq |t|$. ☐

Corollary 7.3. *For a $\mathcal{V}$-normed category $\mathbb{X}$, if the category $\mathbb{X}_\circ$ is idempotent complete, every constant Cauchy sequence in $\mathbb{X}$ has a normed colimit.*

[4] We adopt here the terminology used in the recent paper [19]. Other terms used in the literature are Karoubi complete, Lawvere complete or, most frequently, Cauchy complete. We avoid the latter term, not to risk confusion with the dualization of our term of Cauchy cocompleteness for normed categories which is far more directly modelled after Cauchy's original ideas than idempotent completeness (of any flavour) is. Besides, as a concept that gained its recognition through various important contributions in different contexts, it may indeed be difficult to attach just one person's name to idempotent completeness.

8 The principal example: semi-normed and normed vector spaces

When a norm function $\|\text{-}\|$ on a (real, say) vector space X satisfies the standard axioms for a norm, except that non-zero vectors are allowed to have norm 0, one usually calls X semi-normed. Here, in order to ensure that the category to be formed becomes Cauchy cocomplete, just like for the metric of a Lawvere metric space we should abandon not only the separation condition, but also the finiteness condition for norms. This then necessitates the extension of the real multiplication to ∞, so that we can maintain the norm axiom for scalar multiples of vectors with norm ∞. To that end we first introduce the quantale

$$\mathcal{R}_\times = ([0,\infty], \geq, \cdot, 1),$$

in such a way that the exponential function $e : \mathcal{R}_+ \to \mathcal{R}_\times$, extended by $e^\infty = \infty$, becomes a homomorphism of quantales, *i.e.*, a homomorphism of monoids which preserves infima (with respect to the natural order $\leq$ of the extended non-negative real line).

The monotonicity of the multiplication on $[0,\infty]$ in each variable certainly necessitates $\alpha\cdot\infty = \infty$ for $\alpha > 0$, and the preservation of infima then forces the equality

$$0 \cdot \infty = \inf\{\alpha \cdot \infty \mid \alpha > 0\} = \infty \ .$$

Since it extends the usual fractions in case $\alpha, \beta \notin \{0,\infty\}$, we denote the internal hom $[\beta, \alpha]$ in $\mathcal{R}_\times$ by $\frac{\alpha}{\beta}$ for all $\alpha, \beta \in [0,\infty]$. Hence, its value is given by the adjunction rule

$$\frac{\alpha}{\beta} \leq \gamma \iff \alpha \leq \beta \cdot \gamma \ ,$$

for all $\gamma \in [0,\infty]$, that is: $\frac{\alpha}{\beta} = \inf\{\gamma \in [0,\infty] \mid \alpha \leq \beta \cdot \gamma\}$. This gives in particular

$$\frac{0}{0} = 0, \quad \frac{\alpha}{0} = \infty \ (\alpha > 0), \quad \frac{\alpha}{\infty} = \frac{\infty}{\infty} = 0 \ .$$

The inf-preserving map $e : [0,\infty] \to [0,\infty]$ has a left adjoint, $\log^\circ$, whose values are ruled by adjunction, *i.e.*,

$$\log^\circ \alpha \leq \beta \iff \alpha \leq e^\beta \ ;$$

explicitly,

$$\log^\circ 0 = 0, \ \ \log^\circ \alpha = \max\{0, \log \alpha\} \ (0 < \alpha < \infty), \ \ \log^\circ \infty = \infty.$$

As a mapping $\log^\circ : \mathcal{R}_\times \to \mathcal{R}_+$ we obtain only a lax homomorphism of quantales, since the easily established inequality

$$\log^\circ(\alpha \cdot \beta) \leq \log^\circ \alpha + \log^\circ \beta$$

is generally strict.

We are ready to define our category of semi-normed vector spaces:

Definition 8.1. A *semi-norm* on a (real) vector space X is a function $\|\text{-}\| : X \to [0,\infty]$ satisfying

(N0) $\|0\| = 0$,

(N1) $\|ax\| = |a|\|x\|$,

(N2) $\|x+y\| \le \|x\| + \|y\|$,

for all $x \in X$ and $a \in \mathbb{R}, a \ne 0$. The thus defined *semi-normed vector spaces* are the objects of the category

$$\mathsf{SNVec}_\infty$$

whose morphisms are arbitrary linear maps. It contains as a full subcategory the category NVec_∞ mentioned in the Introduction; see Definition 8.5 for details.

We define the *logarithmic norm* of a morphism $f : X \to Y$ in SNVec_∞ by

$$|f| = \sup_{x\in X} \left(\log^\circ \frac{\|fx\|}{\|x\|}\right).$$

Here are some immediate consequences of this definition.

Lemma 8.2. *Let $f : X \to Y$ be a linear map of semi-normed vector spaces. Then:*

(1) *If X contains a vector x_0 with $\|x_0\| = 0$ and $\|fx_0\| \ne 0$, then $|f| = \infty$.*

(2) *If $\|x\| = 0$ always implies $\|fx\| = 0$, then $|f| = \sup_{\|x\|=1} (\log^\circ \|fx\|)$.*

(3) *For all $x \in X$ one has $\|fx\| \le e^{|f|}\,\|x\|$.*

(4) *One has $|f| = 0$ if, and only if, $\|fx\| \le \|x\|$ holds for all $x \in X$.*

Proof. (1) From $\log^\circ \frac{\|fx_0\|}{\|x_0\|} \le |f|$ one obtains $\infty = \frac{\|fx_0\|}{\|x_0\|} \le e^{|f|}$ and, hence, $|f| = \infty$.

(2) Trivially $t := \sup_{\|x\|=1} (\log^\circ \|fx\|) \le |f|$. For "$\ge$" consider any $x \in X$. If $\|x\| = 0$, also $\|fx\| = 0$ by hypothesis, and $\log^\circ \frac{\|fx\|}{\|x\|} = 0 \le t$ follows; likewise if $\|x\| = \infty$. In all other cases one considers $x_1 := \frac{1}{\|x\|}x$ in a standard manner.

(3) The claim follows from $\log^\circ \frac{\|fx\|}{\|x\|} \le |f|$ by the adjunction rules for fractions and for $\log^\circ \dashv e$.

(4) Again, by adjunction, $\|fx\| \le \|x\|$ implies $\frac{\|fx\|}{\|x\|} \le 1$ for all $x \in X$, so that "if" follows, while "only if" is obtained from (3). □

Theorem 8.3. *With its logarithmic norm, SNVec_∞ is a Cauchy-cocomplete normed category whose 0-morphisms are precisely the 1-Lipschitz linear maps: $(\mathsf{SNVec}_\infty)_\circ = \mathsf{SNVec}_1$.*

Proof. Let us first confirm that SNVec_∞ is indeed a normed category. By Lemma 8.2(4), one has $|\mathrm{id}_X| = 0$ for every semi-normed space X. For morphisms $f : X \to Y$ and $g : Y \to Z$ in SNVec_∞, we may assume that every $x \in X$ with $\|x\| = 0$ satisfies $\|g(fx)\| = 0$ since the existence of $x_0 \in X$ with $\|x_0\| = 0$ but $\|g(fx_0)\| \ne 0$ would, by Lemma 8.2(1), imply $|g| = \infty$ or $|f| = \infty$ (depending on whether $\|fx_0\| = 0$ or not), so that $\infty = |gf| \le |f| + |g|$ would hold trivially.

Consequently, by Lemma 8.2 (2),(3) we may restrict ourselves to considering only vectors $x \in X$ with $\|x\| = 1$ and obtain the required inequality from

$$\log^\circ \|g(fx)\| \le \log^\circ(e^{|g|} \cdot \|fx\|) \le \log^\circ e^{|g|} + \log^\circ \|fx\| \le |g| + \sup_{y \in X}(\log^\circ \frac{\|fy\|}{\|y\|}) = |g| + |f| \ .$$

Also, by Lemma 8.2(4), the 0-morphisms are precisely the 1-Lipschitz morphisms of SNVec_∞.

Let us now consider a Cauchy sequence $s = (\ X_m \xrightarrow{s_{m,n}} X_n\ )_{m \le n}$ in SNVec_∞, so that

$$\inf_{n \in \mathbb{N}} \sup_{n \ge m \ge N} |s_{m,n}| = 0 \ ,$$

and form its (ordinary) colimit X in the category Vec of (real) vector spaces and linear maps, with colimit cocone $\gamma = (\ X_n \xrightarrow{\gamma_n} X\ )_{n \in \mathbb{N}}$. First we define for all $x \in X$

$$\|x\| := \sup_{N \in \mathbb{N}} \inf_{n \ge N} \inf_{z \in \gamma_n^{-1} x} \|z\|_n \ ,$$

and will now show that this makes X a semi-normed vector space. (In what follows, we will no longer use subscripts to indicate where norms are being taken, as the context should give sufficient clarity.) We trivially have $\|0\| = 0$, as well as $\|x+y\| \le \|x\| + \|y\|$ whenever $x, y \in X$ are such that $\|x\| = \infty$ or $\|y\| = \infty$. Hence we may assume that the norms $\|x\|$ and $\|y\|$ are finite and will establish the triangle inequality for them by showing that, for all $N \in \mathbb{N}$ and any real ε with $0 < \varepsilon \le 1$, one has $\inf_{n \ge N} \inf_{z \in \gamma_n^{-1}(x+y)} \|z\| \le \|x\| + \|y\| + \varepsilon r$ with $r := \|x\| + \|y\| + 3$.

Putting $\eta := \log(1+\varepsilon) > 0$, the Cauchyness of s gives us some $M \in \mathbb{N}$ with $|s_{m,n}| \le \eta$ for all $n \ge m \ge M$. The definition of the norms of x and y guarantees the existence of $m, n \ge \max\{M, N\}$ and $u \in X_m, v \in X_n$ with $\gamma_m(u) = x,\ \gamma_n(v) = y$ and $\|u\| \le \|x\| + \varepsilon,\ \|v\| \le \|y\| + \varepsilon$. Without loss of generality we may assume $m \le n$ and, setting $w := s_{m,n}(u)$, have $\|w\| \le e^\eta \|u\| = \|u\| + \varepsilon\|u\|$. Consequently, $\gamma_n(w+v) = x + y$ and, with the triangle inequality holding in X_n, we obtain

$$\begin{aligned}\inf_{z \in \gamma_n^{-1}(x+y)} \|z\| \le \|w+v\| &\le \|w\| + \|v\| \\ &\le \|u\| + \varepsilon\|u\| + \|v\| \\ &\le \|x\| + \varepsilon + \varepsilon(\|x\| + \varepsilon) + \|y\| + \varepsilon \\ &\le \|x\| + \|y\| + \varepsilon r \ ,\end{aligned}$$

This concludes the proof of (N3). The validity of (N2), *i.e.*, of $\|ax\| = |a|\|x\|$ for all real $a \neq 0$ and $x \in X$, is an immediate consequence of the equivalence $(z \in \gamma_n^{-1}(ax) \iff w \in \gamma_n^{-1}x)$ whenever $z = aw$, and of the fact that multiplication in $[0, \infty]$ by the positive real number $|a|$ preserves both, infima and suprema.

We are left with having to confirm conditions (C2a) and (C2b) of Corollary 5.5.

(C2a) To show $\inf_N \sup_{m \ge N} |\gamma_m| = 0$, we consider any $\varepsilon > 0$ and, since s is Cauchy, obtain some $N \in \mathbb{N}$ with $\sup_{n \ge m \ge N} |s_{m,n}| \le \varepsilon$.

Claim: For all $m \geq N$ and $z \in X_m$ one has $\|\gamma_m z\| \leq \sup_{n\geq m}\|s_{m,n}z\|$. Indeed, by definition one has $\|\gamma_m z\| = \sup_{K\in\mathbb{N}} \phi(K)$ where $\phi(K) := \inf\{\|w\| \mid k \geq K, w \in X_k, \gamma_k w = \gamma_m z\}$. Since $\phi(K)$ is monotonely increasing in K, we can write

$$\|\gamma_m z\| = \sup_{n\geq m} \inf\{\|w\| \mid k \geq n, w \in X_k, \gamma_k w = \gamma_m z\} \leq \sup_{n\geq m} \|s_{m,n}z\|,$$

with the inequality holding since $s_{m,n}z$ is one of the admissible vectors w: $\gamma_n(s_{m,n}z) = \gamma_m z$.

A consequence of the Claim is that one has $\|\gamma_m z\| = 0$ whenever $\|z\| = 0$, since Lemma 8.2(3) implies

$$\|s_{m,n}z\| \leq e^{|s_{m,n}|}\|z\| \leq e^{\varepsilon}\|z\|$$

for all $n \geq m$. Therefore, with Lemma 8.2(2), we obtain

$$|\gamma_m| = \sup_{z\in X_m, \|z\|=1} \log^\circ \|\gamma_m z\| \leq \log^\circ(e^\varepsilon) = \varepsilon,$$

as desired.

(C2b) For every linear map $f : X \to Y$ to a semi-normed vector space Y, we must show $|f| \leq \inf_N \sup_{n\geq N} |f_n|$, where $f_n = f \cdot \gamma_n$.

Case 1: For every $x \in X$ with $\|x\| = 0$ we have $\|fx\| = 0$. Then Lemma 8.2(2) applies for the computation of $|f|$, so that it suffices for us to show $\log^\circ \|fx\| \leq \sup_{n\geq N} |f_n|$ whenever $\|x\| = 1$ and $N \in \mathbb{N}$; we may actually also assume $\|fx\| > 1$ since otherwise $\log^\circ \|fx\| = 0$. Given any $\varepsilon > 0$, the definition of $\|x\|$ guarantees the existence of some $m \geq N$ and $z \in \gamma_m^{-1}x$ with $\|z\| \leq 1 + \varepsilon$. Since $fx = f_m z$, the case $\|fx\| = \infty$ would imply $|f_m| = \infty$. Hence, it suffices to consider the case $\|fx\| < \infty$ and, without loss of generality, we may assume $0 < \varepsilon \leq \|fx\| - 1$. Then

$$|f_m| \geq \log^\circ \frac{\|f_m z\|}{\|z\|} \geq \log \frac{\|fx\|}{1+\varepsilon} = \log\|fx\| + \log\frac{1}{1+\varepsilon}$$

for all such ε, which implies the desired inequality.

Case 2: For some $x_0 \in X$ with $\|x_0\| = 0$ we have $\|fx_0\| \neq 0$. Then Lemma 8.2(1) gives $|f| = \infty$, and we must show $\inf_N \sup_{n\geq N} |f_n| \geq \infty$. Similarly to Case 1, given any $N \in \mathbb{N}$ and $\varepsilon > 0$, we have some $m \geq N$ and $z \in \gamma_m^{-1}x_0$ with $\|z\| \leq \varepsilon$, which implies

$$|f_m| \geq \log^\circ \frac{\|f_m z\|}{\|z\|} \geq \log^\circ \frac{\|fx_0\|}{\varepsilon}$$

for all such ε. Again, the desired inequality follows. □

Remark 8.4. The full normed subcategory NVec_∞ of SNVec_∞ as considered in the Introduction fails to be closed under the formation of normed colimits of Cauchy sequences. Even for a Cauchy sequence s of (strictly contractive) linear maps $s_{m,n} : X_m \to X_n$ of normed vector spaces (with all norms finite), the normed colimit in SNVec_∞ may fail to be a normed vector space. Indeed, consider the sequence already mentioned in the Introduction; that is: $X_n := \mathbb{R}$ normed by $\|1\|_n = \frac{1}{n}$ and $s_{n,m} = \mathrm{id}_\mathbb{R}$ for all $m \leq n$. The normed colimit of s in SNVec_∞ may again be formed by identity maps, $\gamma_n : X_n \to X = \mathbb{R}$, with the norm in X given by

$$\|1\| = \sup_N \inf_{n\geq N} \|1\|_n = 0,$$

i.e., all norms in X are 0, so that separation fails to the largest extent possible.

In general, the colimit maps $\gamma_n : X_n \to X$ presenting X as a normed colimit of the sequence $s = (X_m \to X_n)_{m \le n}$ in SNVec_∞ have an important extra property (which follows from the Claim formulated in the Proof of Theorem 8.3) that deserves to be named:

Definition 8.5. A linear map $f : X \to Y$ of semi-normed vector spaces is called a *zero-to-zero morphism* if $\|fx\| = 0$ holds for all $x \in X$ with $\|x\| = 0$. (We note that every bounded operator is a zero-to-zero morphism.) We denote by

$$\mathsf{SNVec}_{00}$$

the (non-full) subcategory of SNVec_∞ containing all semi-normed vector spaces and their zero-to-zero morphisms. The objects of the category

$$\mathsf{NVec}_\infty$$

mentioned in the Introduction are precisely the *separated* semi-normed vector spaces; *i.e.*, its objects X satisfy the separation condition

$$\|x\| = 0 \Longrightarrow x = 0$$

for all $x \in X$, and they fall short of being normed vector spaces in the classical sense only insofar as vectors are permitted to have infinite norms. The separation condition of its domain makes a morphism in NVec_∞, *i.e.*, an arbitrary linear map, automatically a zero-to-zero morphism. Therefore, NVec_∞ is a full subcategory of SNVec_{00}.

Proposition 8.6. *The normed category* NVec_∞ *is reflective in the normed category* SNVec_{00}*, as* $(\mathsf{Set}/\!/\mathcal{R}_+)$*-enriched categories.*

Proof. For $X \in \mathsf{SNVec}_{00}$ consider its subspace $X_0 := \{x \in X \mid \|x\| = 0\}$ and let $p : X \to X/X_0$ be the projection. Since $\|x\| = \|y + (x - y)\| \le \|y\| + \|x - y\|$ one has $(\|x - y\| = 0 \iff \|x\| = \|y\|)$ for all $x, y \in X$, so that $\|px\| := \|x\|$ makes X/X_0 a well-defined object of NVec_∞ and p a zero-to-zero morphism – in fact, an isometry. Furthermore, for all $Y \in \mathsf{NVec}_\infty$ we have the natural bijection

$$- \cdot p : \mathsf{NVec}_\infty(X/X_0, Y) \to \mathsf{SNVec}_{00}(X, Y),$$

whose surjectivity is guaranteed by our restriction to zero-to-zero morphisms (as opposed to all linear maps of semi-normed vector spaces). In fact, this bijection is a $(\mathsf{Set}/\!/\mathcal{V})$-isomorphism since, for every linear map $f : X/X_0 \to Y$, one has

$$|f| = \sup_{z \in X/X_0} (\log^\circ \frac{\|fz\|}{\|z\|}) = \sup_{x \in X} (\log^\circ \frac{\|f(px)\|}{\|px\|}) = \sup_{x \in X} (\log^\circ \frac{\|f(px)\|}{\|x\|}) = |f \cdot p| \ .$$

□

Corollary 8.7. *The normed category* NVec_∞ *has normed colimits of all those Cauchy sequences whose normed colimit in* SNVec_∞ *is a colimit in the ordinary category* SNVec_{00}*.*

Proof. Keeping the notation of the proof of Theorem 8.3, we consider a Cauchy sequence s in NVec_∞ and obtain its normed colimit X in SNVec_∞, with a colimit cocone γ formed by zero-to-zero morphisms. *A fortiori*, by hypothesis, γ is a colimit cocone in SNVec_{00}, so that any linear map $f : X \to Y$ must be a zero-to-zero morphism. To arrive at a colimit of s in NVec_∞, according to Proposition 8.6, we must apply the reflector to X and obtain the colimit cocone $(p \cdot \gamma_n : X_n \to X/X_0)_n$. As one easily checks (or formally derives with Proposition 12.1 proved below), since the adjunction of Proposition 8.6 is $(\mathsf{Set}/\!/\mathcal{R}_+)$-enriched, this cocone presents X/X_0 in fact as a normed colimit of s in NVec_{00}. $\square$

By contrast:

Corollary 8.8. *The normed category* NVec_∞ *is not Cauchy cocomplete.*

Proof. The sequence presented in Remark 8.4 is a Cauchy sequence in NVec_∞. If it had a normed colimit in NVec_∞, this would have to be also a colimit in the ordinary category of vector spaces and linear maps, so that we may assume that it is given by $\mathbb{R}$, with a cocone of identity maps and normed by $|1| = c$ for some $c > 0$. But then $|\gamma_n| = \log^\circ(nc)$ for all $n \in \mathbb{N}$, which would give $\sup_N \inf_{n \geq N} |\gamma_n| = \infty$, in contradiction to the normed colimit condition (C2a). $\square$

Let us also point out that existing normed colimits in NVec_∞ of Cauchy sequences of Banach spaces need not be Banach:

Example 8.9. Consider the sequence

$$\{0\} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \longrightarrow \ldots \longrightarrow \mathrm{colim}_n\, \mathbb{R}^n = \bigoplus_n \mathbb{R}^n$$

of isometric embeddings of Euclidean spaces whose normed colimit is given by the direct sum (in Vec) of its objects, normed accordingly. In the direct sum, we have the Cauchy sequence $(x_n)_n$, where the i-th component of x_n is $\frac{1}{i+1}$ for $i \leq n$, and 0 otherwise, but the sequence does not converge in $\bigoplus_n \mathbb{R}^n$.

9 Change of base for normed categories, metric spaces

Our next goal is to show that the category Met_∞ of all Lawvere metric spaces, with arbitrary maps between them as morphisms and normed as in Example 3.5(5), is Cauchy cocomplete. We will do so in three steps, by first giving conditions on our quantale $(\mathcal{V}, \leq, \otimes, \mathrm{k})$ guaranteeing that the normed category V-Lip of all small $\mathcal{V}$-categories with arbitrary maps as defined in Proposition 4.3 is Cauchy cocomplete. With the benefit of the methods used in [13, 14], the proof extends the "epsilon techniques" used in the proof of Theorem 8.3 to a fairly general quantalic context. Then we will briefly discuss how Cauchy cocompleteness for $\mathcal{V}$-normed categories fares under changing the "base" $\mathcal{V}$, before applying our findings to the adjunction $e \dashv \log^\circ : \mathcal{R}_\times \to \mathcal{R}_+$ to obtain the Cauchy cocompleteness of Met_∞.

Recall that for $u, v \in \mathcal{V}$ one says that u is *totally below* v, written as $u \ll v$, if $v \leq \bigvee W$ with $W \subseteq \mathcal{V}$ can hold only if $u \leq w$ for some $w \in W$. We say that v is *approximated from totally below* if $v = \bigvee \Downarrow v$, where $\Downarrow v = \{u \in \mathcal{V} \mid u \ll v\}$. Recall that the complete lattice $\mathcal{V}$ is *constructively completely distributive* [40, 22] if every element in $\mathcal{V}$ is approximated from totally below. (In the presence of Choice this property implies complete distributivity in the standard sense.) We say that the tensor product of $\mathcal{V}$ *preserves the totally below relation* if $u \ll v$ implies $u \otimes w \ll v \otimes w$ for all $w > \bot$ in $\mathcal{V}$. In the presence of this preservation property, and if $\mathrm{k} = \top$ is approximated from totally below, then $\mathcal{V}$ is constructively completely distributive.

Theorem 9.1. *Let the totally below relation in $\mathcal{V}$ be preserved by $\otimes$, and let the tensor-neutral element* k *be approximated from totally below, with the set $\Downarrow\mathrm{k}$ being up-directed in $\mathcal{V}$. Then the normed category $\mathcal{V}$-Lip is Cauchy cocomplete.*

Proof. For a given Cauchy sequence $s = (\ X_m \xrightarrow{s_{m,n}} X_n\ )_{m \leq n}$ in $\mathcal{V}$-Lip, we form its (ordinary) colimit X in the category Set with colimit cocone $\gamma = (\ X_n \xrightarrow{\gamma_n} X\ )_{n \in \mathbb{N}}$ and now want to define a $\mathcal{V}$-category structure on X by

$$X(x, y) := \bigwedge_{N \in \mathbb{N}} \bigvee \{X_n(x', y') \mid n \geq N, x' \in \gamma_n^{-1}x, y' \in \gamma_n^{-1}y\},$$

for all $x, y \in X$. Trivially, $\mathrm{k} \leq X(x, x)$. In order to establish the inequality $X(x, y) \otimes X(y, z) \leq X(x, z)$ in X, we may disregard the trivial case $X(x, y) = \bot$ or $X(y, z) = \bot$. We consider any $\varepsilon \in \mathcal{V}$ with $\bot < \varepsilon \ll \mathrm{k}$. Then the Cauchyness of s gives us some $M \in \mathbb{N}$ with $\varepsilon \leq |s_{m,\ell}|$ for all $\ell \geq m \geq M$. By the assumed preservation property we have $X(x, y) \otimes \varepsilon \ll X(x, y)$ and $X(y, z) \otimes \varepsilon \ll X(y, z)$. Therefore, for every $N \in \mathbb{N}$, the definitions of $X(x, y)$ and $X(y, z)$ let us pick $m, n \geq \max\{M, N\}$ and $x' \in \gamma_m^{-1}x,\ y' \in \gamma_m^{-1}y,\ y'' \in \gamma_n^{-1}y,\ z' \in \gamma_n^{-1}z$, such that

$$X(x, y) \otimes \varepsilon \leq X_m(x', y') \quad \text{and} \quad X(y, z) \otimes \varepsilon \leq X_n(y'', z')\ .$$

Now, since $\gamma_m y' = \gamma_n y''$, the construction of X as a directed colimit in Set allows us to find $\ell \geq m, n$ with $s_{m,\ell}y' = s_{n,\ell}y''$. Since, by the definition of the $\mathcal{V}$-norm in $\mathcal{V}$-Lip, the inequalities $\varepsilon \leq |s_{m,\ell}|$ and $\varepsilon \leq |s_{n,\ell}|$ imply $X_m(x', y') \otimes \varepsilon \leq X_\ell(s_{m,\ell}x', s_{m,\ell}y')$ and $X_n(y'', z') \otimes \varepsilon \leq X_\ell(s_{n,\ell}y'', s_{n,\ell}z')$, we obtain

$$\begin{aligned} X(x, y) \otimes \varepsilon \otimes \varepsilon \otimes X(y, z) \otimes \varepsilon \otimes \varepsilon &\leq X_\ell(s_{m,\ell}x', s_{m,\ell}y') \otimes X_\ell(s_{n,\ell}y'', s_{n,\ell}yz') \\ &\leq X_\ell(s_{m,\ell}x', s_{n,\ell}z') \\ &\leq X(x, z), \end{aligned}$$

with the last inequality holding since N was given arbitrarily. The desired inequality $X(x, y) \otimes X(y, z) \leq X(x, z)$ follows since, with the up-directed set $\Downarrow\mathrm{k}$, we have

$$\mathrm{k} = \mathrm{k} \otimes \mathrm{k} = (\bigvee_{\varepsilon \ll \mathrm{k}} \varepsilon) \otimes (\bigvee_{\varepsilon \ll \mathrm{k}} \varepsilon) = \bigvee_{\varepsilon \ll \mathrm{k}} \varepsilon \otimes \varepsilon$$

and then, likewise, $\mathrm{k} = \bigvee_{\varepsilon \ll \mathrm{k}} \varepsilon \otimes \varepsilon \otimes \varepsilon \otimes \varepsilon$.

(C2a) In order to show $\mathrm{k} \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |\gamma_n|$, let $\varepsilon \ll \mathrm{k}$ and, as above, pick $M \in \mathbb{N}$ with $\varepsilon \leq \bigwedge_{n\geq m\geq M} |s_{m,n}|$. For every $m \geq M$ and all $z, w \in X_m$, we have (similarly as in the proof of Theorem 8.3) $X(\gamma_m z, \gamma_m w) = \bigwedge_{K\in\mathbb{N}} \Phi(K)$, where

$$\Phi(K) = \bigvee\{X_\ell(z', w') \mid \ell \geq K, z', w' \in X_\ell, \gamma_\ell z' = \gamma_m z, \gamma_\ell w' = \gamma_m w\}$$

is monotonely decreasing in K. Therefore,

$$\begin{aligned} X(\gamma_m z, \gamma_m w) = \bigwedge_{n\geq m} \Phi(n) &\geq \bigwedge_{n\geq m} X_n(s_{m,n}z, s_{m,n}w) \\ &\geq \bigwedge_{n\geq m} (|s_{m,n}| \otimes X_m(z,w)) \\ &\geq (\bigwedge_{n\geq m} |s_{m,n}|) \otimes X_m(z,w). \end{aligned}$$

This gives

$$\varepsilon \leq \bigwedge_{n\geq m\geq M} |s_{m,n}| \leq \bigwedge_{m\geq M} [X_m(z,w), X(\gamma_m z, \gamma_m w)] \leq \bigwedge_{m\geq M} |\gamma_m| \leq \bigvee_N \bigwedge_{n\geq N} |\gamma_n|$$

and, with $\mathrm{k} = \bigvee_{\varepsilon\ll\mathrm{k}} \varepsilon$, the desired inequality.

(C2b) For any mapping $f : X \to Y$ of $\mathcal{V}$-categories, we must show

$$\bigvee_N \bigwedge_{n\geq N} |f \cdot \gamma_n| \;\leq\; |f| = \bigwedge_{x,y\in X} [X(x,y), Y(fx, fy)]\;;$$

equivalently, $\bigwedge_{n\geq N} |f\cdot\gamma_n| \otimes X(x,y) \leq Y(fx, fy)$, for all $N \in \mathbb{N}$ and $x, y \in X$. To this end, discarding the trivial case $X(x,y) = \bot$, we consider $\varepsilon \ll \mathrm{k}$ and have $X(x,y) \otimes \varepsilon \ll X(x,y)$ and may pick $m \geq N$ and $x' \in \gamma_m^{-1}x$, $y' \in \gamma_m^{-1}y$ with $X(x,y) \otimes \varepsilon \leq X_m(x', y')$. Therefore,

$$\bigwedge_{n\geq N} |f \cdot \gamma_n| \otimes X(x,y) \otimes \varepsilon \leq |f \cdot \gamma_m| \otimes X(x,y) \otimes \varepsilon \leq |f\gamma_M| \otimes X_m(x', y') \leq Y(fx, fy),$$

which implies the desired inequality. □

Examples 9.2. The quantales 2, $\mathcal{R}_+$ and $\mathcal{R}_\times = ([0,\infty], \geq, \cdot, 1)$ satisfy the hypotheses of Theorem 9.1. Therefore, the 2-normed category 2-Lip of preordered sets and arbitrary maps is Cauchy-cocomplete, and likewise for $\mathcal{R}_+$-Lip and $\mathcal{R}_\times$-Lip.

Recall that a monotone map $\varphi : \mathcal{V} \to \mathcal{W}$ to a quantale $(\mathcal{W}, \leq, \boxtimes, \mathrm{n})$ is a *lax homomorphism* of quantales if $\mathrm{n} \leq \varphi\mathrm{k}$ and $\varphi v \boxtimes \varphi v' \leq \varphi(v \otimes v')$ for all $v, v' \in \mathcal{V}$. Such lax homomorphism induces the change-of-base functor

$$\mathcal{B}_\varphi : \mathsf{CAT}/\!/\mathcal{V} \longrightarrow \mathsf{CAT}/\!/\mathcal{W}, \quad (\mathbb{X}, |\text{-}|) \longmapsto (\mathbb{X}, |\text{-}|_\varphi),$$

which regards a $\mathcal{V}$-normed category $\mathbb{X}$ as a $\mathcal{W}$-normed category via $|f|_\varphi := \varphi(|f|)$ for all morphisms f in $\mathbb{X}$, and which makes $\mathcal{V}$-normed functors become $\mathcal{W}$-normed.

Furthermore, if we also have a lax homomorphism $\psi : \mathcal{W} \to V$ which, as a monotone map, is right adjoint to φ, then we have the induced adjunction $\mathcal{B}_\varphi \dashv \mathcal{B}_\psi$. Indeed, a $\mathcal{V}$-normed functor $F : \mathbb{X} \to \mathcal{B}_\psi\mathbb{Y}$ may be considered equivalently as a $\mathcal{W}$-normed functor $F : \mathcal{B}_\varphi\mathbb{X} \to \mathbb{Y}$, since the adjunction $\varphi \dashv \psi$ facilitates, for all morphism f in $\mathbb{X}$, the equivalence

$$|f| \leq |Ff|_\psi \iff |f|_\varphi \leq |Ff| \, .$$

Proposition 9.3. *Let $\varphi \dashv \psi : \mathcal{W} \to \mathcal{V}$ be adjoint lax homomorphisms of quantales, with ψ preserving joins of monotone sequences in $\mathcal{W}$. Then, if the $\mathcal{W}$-normed category $\mathbb{Y}$ is Cauchy cocomplete, so is the $\mathcal{V}$-normed category $\mathcal{B}_\psi\mathbb{Y}$.*

Proof. Let $s = (s_{m,n})_{m\leq n}$ be a Cauchy sequence in $\mathcal{B}_\psi$. So, with $|\text{-}|$ denoting the norm in the given $\mathcal{W}$-normed category $\mathbb{Y}$, we have $\mathrm{k} \leq \bigvee_N \bigwedge_{n\geq m\geq N} |s_{n,m}|_\psi$ and, since the left adjoint φ preserves joins, obtain

$$\mathrm{n} \leq \varphi\mathrm{k} = \bigvee_N \varphi(\bigwedge_{n\geq m\geq N} \psi(|s_{m,n}|)) \leq \bigvee_N \bigwedge_{n\geq m\geq N} \varphi\psi(|s_{m,n}|) \leq \bigvee_N \bigwedge_{n\geq m\geq N} |s_{m,n}| \, .$$

So, s is Cauchy in $\mathbb{Y}$ and, hence, has a normed colimit x in $\mathbb{Y}$, with colimit cocone γ. We claim that x is also a normed colimit of s in $\mathcal{B}_\psi\mathbb{Y}$. Indeed, this is an immediate consequence of the assumed preservation of joins of monotone sequences in $\mathcal{W}$ and the preservation of all meets by the right adjoint ψ since, for all morphisms $f : x \to y$ in $\mathbb{Y}$, from $|f| = \bigvee_N \bigwedge_{n\geq N} |f \cdot \gamma_n|$ one obtains $|f|_\psi = \bigvee_N \bigwedge_{n\geq N} |f \cdot \gamma_n|_\psi$. □

Let us now specialize the adjunction $\varphi \dashv \psi$ of lax homomorphisms of quantales as in the Proposition to the adjunction

$$e \dashv \log^\circ : \mathcal{R}_\times = ([0, \infty], \geq, \cdot, 1) \longrightarrow \mathcal{R}_+ = ([0, \infty], \geq, +, 0) \, .$$

The exponential function is a (strict) homomorphism of quantales, and $\log^\circ$ is lax and preserves infima (w.r.t. the natural order). Since $\mathcal{R}_+$-Lip is Cauchy cocomplete by Theorem 9.1, the same is true for $\mathcal{B}_{\log^\circ}(\mathcal{R}_\times\text{-}\mathsf{Lip}) = \mathsf{Met}_\infty$, by the Proposition. Therefore:

Corollary 9.4. *The normed category Met_∞ is Cauchy cocomplete.*

10 Remarks on Cauchy cocompleteness for 2-normed categories

We consider the Boolean quantale $\mathcal{V} = 2$ and recall that 2-normed categories are ordinary categories $\mathbb{X}$ which come with a wide subcategory $\mathcal{S}$. In simplified form, the Cauchy cocompleteness of $(\mathbb{X}, \mathcal{S})$ as described in Examples 6.3(2) equivalently means that the colimit of any sequence s with all connecting morphisms in $\mathcal{S}$ exists in $\mathbb{X}$ and has a tail $s_{|N} = (s_{m,n})_{n\geq m\geq N}$ (with some $N \in \mathbb{N}$) which is actually a colimit in the subcategory $\mathcal{S}$. This, and some variations and generalizations thereof, is a concept that has been widely studied, primarily in the context of directed colimits and, dually, of inverse limits, rather than that of (co)limits of sequences. We limit ourselves to mentioning only a few instances and pointers to the literature.

Examples 10.1. *For the following categories $\mathbb{X}$ and their wide subcategories $\mathcal{S}$, the 2-normed category $(\mathbb{X}, \mathcal{S})$ is Cauchy cocomplete:*

(1) $\mathbb{X}$ is any locally finitely presentable category (in particular, any variety of finitary universal algebras), and $\mathcal{S}$ is the class of monomorphisms in $\mathbb{X}$: see Proposition 1.62 in [3]. The same source confirms that one may take for $\mathcal{S}$ also the class of regular monomorphisms whenever this class is closed under composition. But this latter provision frequently fails since, in any category with cokernel pairs and their equalizers, it means equivalently that all strong (or extremal) monomorphisms in $\mathbb{X}$ must be regular, or that $\mathbb{X}$ must have (epi, regular mono)-factorizations. For example, already since [25] the provision is known to fail in the category of semigroups; see Exercise 14I in [1] for a simple finite counter-example. However, various generalizations of the so-called amalgamation property guarantee in a general categorical context that extremal monomorphisms be regular: see Theorem 2 of [37].

(2) $\mathbb{X}$ is the category of topological spaces, and $\mathcal{S}$ is either the class of monomorphisms (injective continuous maps) or the class open embeddings: see Proposition 3.2 of [2]. However, we may *not* consider the class $\mathcal{S}^*$ of regular monomorphisms (arbitrary embeddings), even though $\mathcal{S}^*$ is trivially closed under composition and enjoys the property that the colimit cocone of any directed system with connecting maps in $\mathcal{S}^*$ consists of maps in $\mathcal{S}^*$ again (by Proposition 3.4 of [2]). Indeed, Example 3.5 of [2] gives a sequence of embeddings whose colimit in $\mathbb{X}$ is not a colimit in $\mathcal{S}^*$. For an extension of these observations from $\mathbb{X}$ to some of its full subcategories we also refer to [2].

(3) $\mathbb{X}$ is the *opposite* category of the categories of compact Hausdorff spaces, or of all Hausdorff spaces, and $\mathcal{S}$ is the class of epimorphisms in these two categories, *i.e.* the class of continuous surjections or of dense continuous maps, respectively: for compact Hausdorff spaces, see Corollary 2 of Section 9.6 in [11], and for Hausdorff spaces see Example 5 in [6]. The latter reference offers a wide array of categories from commutative algebra whose opposites provide further examples of Cauchy-cocomplete categories.

11 Presheaf categories are Cauchy cocomplete

We continue to work with a quantale $(\mathcal{V}, \leq, \otimes, \mathrm{k})$ and first consider an arbitrary sequence $s = (A_m \xrightarrow{s_{m,n}} A_n)_{m \leq n}$ in $\mathsf{Set}||\mathcal{V}$. So, while the sets A_n are $\mathcal{V}$-normed, the maps $s_{m,n}$ may not be. Still, with the forgetful functor $U : \mathsf{Set}||\mathcal{V} \to \mathsf{Set}$, we can form the colimit A of Us in Set, with cocone $(A_n \xrightarrow{\gamma_n} A)_n$. Trivially, *any* norm on A makes the resulting $\mathcal{V}$-normed set a colimit of s in $\mathsf{Set}||\mathcal{V}$, since there is no constraint on the morphisms in that category. But there is one norm on A that distinguishes itself by a special property, as follows.

$$|c| = \bigwedge_{N \in \mathbb{N}} \bigvee_{n \geq N} \bigvee_{a \in \gamma_n^{-1} c} |a|;$$

that is, we employ the same formula as the one established for colimits of sequences in $\mathsf{Set}/\!/\mathcal{V}$ (see Proposition 5.1), but now without any *a-priori* expectation that it would make the maps γ_n

$\mathcal{V}$-normed. We call the above norm on A the *γ-induced Cauchy norm* since it has the important property (C2b) (see Corollary 5.5):

Lemma 11.1. *Let the set A be provided with the γ-induced Cauchy norm as above. Then, for any mapping $f : A \to B$ to a $\mathcal{V}$-normed set B, one has*

$$|f| \geq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} |f \cdot \gamma_n| \,.$$

Proof. Since, in its first (contravariant) variable, the internal hom [-,-] of $\mathcal{V}$ transforms arbitrary joins into meets, we have:

$$\begin{aligned}
|f| = \bigwedge_{c\in A} [|c|, |fc|] &= \bigwedge_{c\in A} [\bigwedge_{N} \bigvee_{n\geq N} \bigvee_{a\in\gamma_n^{-1}c} |a|,\, |fc|] \\
&\geq \bigwedge_{c\in A} \bigvee_{N} [\bigvee_{n\geq N} \bigvee_{a\in\gamma_n^{-1}c} |a|,\, |fc|] \\
&\geq \bigvee_{N} \bigwedge_{c\in A} [\bigvee_{n\geq N} \bigvee_{a\in\gamma_n^{-1}c} |a|,\, |fc|] \\
&= \bigvee_{N} \bigwedge_{n\geq N} \bigwedge_{c\in A} \bigwedge_{a\in\gamma_n^{-1}c} [|a|,\, |fc|] \\
&= \bigvee_{N} \bigwedge_{n\geq N} \bigwedge_{a\in A_n} [|a|,\, |f(\gamma_n a)|] \\
&= \bigvee_{N} \bigwedge_{n\geq N} |f \cdot \gamma_n| \,.
\end{aligned}$$

□

In order to strengthen the assertion of Lemma 11.1 and show that, in fact, $\mathsf{Set}||\mathcal{V}$, and even all $\mathsf{Set}||\mathcal{V}$-valued presheaf categories, are Cauchy cocomplete, we need a small additional hypothesis on the $\otimes$-neutral element k of the quantale $\mathcal{V}$. Actually, we offer two alternative possibilities, (A) *or* (B), for suitably augmenting our general quantalic setting, as follows:

(A) k *is approximated from totally below* (see Section 9), that is:

$$\mathrm{k} = \bigvee\{\varepsilon \in \mathcal{V} \mid \varepsilon \ll \mathrm{k}\}.$$

(B) k *$\wedge$-distributes over arbitrary joins*, that is:

$$\mathrm{k} \wedge \bigvee_{i\in I} v_i = \bigvee_{i\in I} \mathrm{k} \wedge v_i \,.$$

Remarks 11.2. (1) Condition (A) certainly holds when the lattice $\mathcal{V}$ is *(constructively) completely distributive* in the sense of [40] since, in that case, *every* element in $\mathcal{V}$ is approximated from totally below by definition.

(2) Condition (B) trivially holds when the quantale $\mathcal{V}$ is *integral, i.e.,* when $\mathrm{k} = \top$, and also when the underlying lattice of the quantale $\mathcal{V}$ is a *frame* since, in the latter case *every* element

in $\mathcal{V}$ $\wedge$-distributes over arbitrary joins by definition, whilst in the former case the map $\mathrm{k} \wedge (-)$ is just the identity map on $\mathcal{V}$.

(3) All examples of quantales mentioned in this paper thus far satisfy *both* conditions, (A) and (B). We discuss the two conditions further in Section 15 and in particular confirm their logical independence.

We are now ready to prove the main general theorem of the paper.

Theorem 11.3. *When the quantale $\mathcal{V}$ satisfies condition* (A) or (B)*, then the $\mathcal{V}$-normed category $[\mathbb{X}, \mathsf{Set}||\mathcal{V}]$ is Cauchy cocomplete, for every small $\mathcal{V}$-normed category $\mathbb{X}$.*

Proof. Considering a Cauchy sequence $\sigma = (P_m \xrightarrow{\sigma_{m,n}} P_n)_{m \leq n \in \mathbb{N}}$ in the category $[\mathbb{X}, \mathsf{Set}||\mathcal{V}]$ (given by all $\mathcal{V}$-normed functors $\mathbb{X} \to \mathsf{Set}||\mathcal{V}$ and their natural transformations), with the forgetful functor $U : \mathsf{Set}||\mathcal{V} \to \mathsf{Set}$ we form the colimit P of $U\sigma$ in the ordinary functor category $\mathsf{Set}^{\mathbb{X}}$, with cocone $\gamma = (\, P_n \xrightarrow{\gamma_n} P \,)_n$. Then, for every object x in $\mathbb{X}$, the colimit Px of the sequence $(U\sigma^x_{m,n})_{m \leq n}$ in Set may be provided with the Cauchy norm induced by the cocone $(\gamma^x_n)_n$ (see Lemma 11.1), and in this way P is then considered as a $\mathsf{Set}||\mathcal{V}$-valued functor.

In order to establish P as a normed colimit of σ, by Corollary 5.5, we must show:

(C1) The functor $P : \mathbb{X} \to \mathsf{Set}||\mathcal{V}$ is $\mathcal{V}$-normed (so that it serves as a colimit of σ in the ordinary full subcategory $[\mathbb{X}, \mathsf{Set}||\mathcal{V}]$ of $(\mathsf{Set}||\mathcal{V})^{\mathbb{X}}$, formed by all $\mathcal{V}$-normed functors $\mathbb{X} \to \mathsf{Set}||\mathcal{V}$);

(C2a) γ is a k-cocone, *i.e.*, $\mathrm{k} \leq \bigvee_N \bigwedge_{n \geq N} |\gamma_n|$;

(C2b) $|\alpha| \geq \bigvee_N \bigwedge_{n \geq N} |\alpha \cdot \gamma_n|$, for every natural transformation $\alpha : P \to Q$.

For showing (C1) we use (C2a) (the proof of which is presented further below, independently of (C1)) and, since every P_n is $\mathcal{V}$-normed and every $\gamma_n = (\gamma^x_n)_{x \in \mathbb{X}} : P_n \to P$ is natural, obtain for all morphisms $f : x \to y$ in $\mathbb{X}$

$$\begin{aligned}
|f| = \mathrm{k} \otimes |f| &\leq (\bigvee_N \bigwedge_{n \geq N} |\gamma_n|) \otimes |f| \\
&\leq \bigvee_N (\bigwedge_{n \geq N} |\gamma^y_n| \otimes |f|) \\
&\leq \bigvee_N \bigwedge_{n \geq N} |\gamma^y_n| \otimes |P_n f| \\
&\leq \bigvee_N \bigwedge_{n \geq N} |\gamma^y_n \cdot P_n f| \\
&= \bigvee_N \bigwedge_{n \geq N} |Pf \cdot \gamma^x_n| \leq |Pf|\,,
\end{aligned}$$

with the last inequality following from the fact that Px carries the $(\gamma^x_n)_n$-induced Cauchy norm, so that Lemma 11.1 applies.

Using this last argument again for every $x \in \mathbb{X}$, and before turning to the more cumbersome proof of (C2a), we can immediately show that condition (C2b) holds, as follows:

$$|\alpha| = \bigwedge_{x\in X} |\alpha_x| \geq \bigwedge_{x\in\mathbb{X}} \bigvee_{N} \bigwedge_{n\geq N} |\alpha_x \cdot \gamma_n^x| \geq \bigvee_{N} \bigwedge_{n\geq N} \bigwedge_{x\in\mathbb{X}} |\alpha_x \cdot \gamma_n^x| = \bigvee_{N} \bigwedge_{n\geq N} |\alpha \cdot \gamma_n| \,.$$

For the proof of (C2a), we first calculate

$$\begin{aligned}
\bigvee_{N} \bigwedge_{n\geq N} |\gamma_n| &= \bigvee_{N} \bigwedge_{n\geq N} \bigwedge_{x\in\mathbb{X}} |\gamma_n^x| \\
&= \bigvee_{N} \bigwedge_{n\geq N} \bigwedge_{x\in\mathbb{X}} \bigwedge_{a\in P_n x} [|a|, |\gamma_n^x a|] \\
&= \bigvee_{N} \bigwedge_{x\in\mathbb{X}} \bigwedge_{c\in Px} \bigwedge_{n\geq N} \bigwedge_{a\in(\gamma_n^x)^{-1}c} [|a|, |c|] \\
&= \bigvee_{N} \bigwedge_{x\in\mathbb{X}} \bigwedge_{c\in Px} [\, \bigvee_{n\geq N} \bigvee_{a\in(\gamma_n^x)^{-1}c} |a|, \bigwedge_{M} \bigvee_{m\geq M} \bigvee_{b\in(\gamma_m^x)^{-1}c} |b| \,] \\
&= \bigvee_{N} \bigwedge_{M} \bigwedge_{x\in\mathbb{X}} \bigwedge_{c\in Px} [\, \|c\|_N, \|c\|_M \,] \,, \qquad (*)
\end{aligned}$$

where, for the last equality, we have used the abbreviation $\|c\|_N := \bigvee_{n\geq N} \bigvee_{a\in(\gamma_n^x)^{-1}c} |a|$ for all $N \in \mathbb{N}, x \in \mathbb{X}$, and $c \in Px$.

We now consider the alternative hypotheses (A) and (B) and finish the proof under each of them separately, as follows.

(A) Since the Cauchy sequence σ satisfies $\mathrm{k} \leq \bigvee_N \bigwedge_{n\geq m\geq N} |\sigma_{m,n}|$, for every $\varepsilon \ll \mathrm{k}$ in $\mathcal{V}$ we find an $N \in \mathbb{N}$ with $\varepsilon \leq \bigwedge_{n\geq m\geq N} |\sigma_{m,n}|$, *i.e.*,

$$\varepsilon \leq |\sigma_{m,n}^x|$$

for all $n \geq m \geq N$ and $x \in \mathbb{X}$. Now, given any $c \in Px$ and $M \in \mathbb{N}$, in the case $M \leq N$ we trivially have $\|c\|_N \leq \|c\|_M$ and obtain $\varepsilon \ll \mathrm{k} \leq [\|c\|_N, \|c\|_M]$, so certainly $\varepsilon \leq [\|c\|_N, \|c\|_M]$. If $M \geq N$, with $\ell := M - N$ we have

$$\begin{aligned}
\|c\|_M = \bigvee_{m\geq M} \bigvee_{b\in(\gamma_m^x)^{-1}c} |b| &\geq \bigvee_{n\geq N} \bigvee_{a\in(\gamma_n^x)^{-1}c} |\sigma_{n,n+\ell}^x\, a| \\
&\geq \bigvee_{n\geq N} \bigvee_{a\in(\gamma_n^x)^{-1}c} |a| \otimes |\sigma_{n,n+\ell}^x| \\
&\geq (\bigvee_{n\geq N} \bigvee_{a\in(\gamma_n^x)^{-1}c} |a|\,) \otimes u = \|c\|_N \otimes u \,,
\end{aligned}$$

which again implies $\varepsilon \leq [\|c\|_N, \|c\|_M]$. Consequently, since $\mathrm{k} = \bigvee\{\varepsilon \mid \varepsilon \ll \mathrm{k}\}$, with $(*)$ we obtain $\mathrm{k} \leq \bigvee_N \bigwedge_{n\geq N} |\gamma_n|$, as desired.

(B) Analyzing further the equality $(*)$,

$$\bigvee_{N} \bigwedge_{n\geq N} |\gamma_n| = \bigvee_{N} \bigwedge_{x\in\mathbb{X}} \bigwedge_{c\in Px} [\, \|c\|_N, \bigwedge_{M} \|c\|_M \,] \,,$$

we note that we have

$$
\begin{aligned}
[\,\|c\|_N, \bigwedge_M \|c\|_M\,] &= [\,\|c\|_N, \bigwedge_{M\le N} \|c\|_M \wedge \bigwedge_{M\ge N} \|c\|_M\,] \\
&= [\,\|c\|_N, \bigwedge_{M\le N} \|c\|_M\,] \;\wedge\; [\,\|c\|_N, \bigwedge_{M\ge N} \|c\|_M\,] \\
&\ge \mathrm{k} \wedge [\,\|c\|_N, \bigwedge_{M\ge N} \|c\|_M\,] \\
&= \bigwedge_{M\ge N} (\mathrm{k} \wedge [\,\|c\|_N, \|c\|_M\,])\,.
\end{aligned}
$$

Here, for $M \ge N$, as in part (A), setting $\ell = M - N$ one has

$$
\|c\|_M \ge \bigvee_{n\ge N} \bigvee_{a\in(\gamma^x_n)^{-1}c} |a| \otimes |\sigma^x_{n,n+\ell}| = \|c\|_N \otimes \bigvee_{n\ge N} |\sigma^x_{n,n+\ell}| \ge \|c\|_N \otimes |\sigma^x_{N,M}|
$$

and, hence, $[\,\|c\|_N, \|c\|_M\,] \ge |\sigma^x_{N,M}|$. Consequently, with hypothesis (B) and the Cauchyness of σ we obtain

$$
\begin{aligned}
\bigvee_N \bigwedge_{n\ge N} |\gamma_n| &\ge \bigvee_N \bigwedge_{x\in\mathbb{X}} \bigwedge_{M\ge N} (\mathrm{k} \wedge |\sigma^x_{N,M}|) \\
&= \bigvee_N (\mathrm{k} \wedge \bigwedge_{M\ge N} \bigwedge_{x\in\mathbb{X}} |\sigma^x_{N,M}|) \\
&= \mathrm{k} \wedge \bigvee_N \bigwedge_{M\ge N} |\sigma_{N,M}| \\
&\ge \mathrm{k} \wedge \mathrm{k} = \mathrm{k}\,,
\end{aligned}
$$

which concludes the proof. □

In conjunction with Remarks 11.2 we conclude:

Corollary 11.4. *For every small $\mathcal{V}$-normed category $\mathbb{X}$, the $\mathcal{V}$-normed category $[\mathbb{X}, \mathsf{Set}||\mathcal{V}]$ is Cauchy cocomplete under any of the following hypotheses:*

- *the quantale $\mathcal{V}$ is integral;*
- *the lattice $\mathcal{V}$ is a frame;*
- *the lattice $\mathcal{V}$ is (constructively) completely distributive.*

Problem 11.5. Is there a quantale $\mathcal{V}$ for which the $\mathcal{V}$-normed category $\mathsf{Set}||\mathcal{V}$ fails to be Cauchy cocomplete?

12 Normed colimits as weighted colimits

In this section we assume that the quantale $\mathcal{V}$ satisfies condition (A) *or* (B) so that we can apply Theorem 11.3. Under this condition, we show that normed colimits of Cauchy sequences can be equivalently described as *weighted* (formerly *indexd* colimits in the sense of [26], for an appropriate class of weights. By definition, a normed colimit of a sequence s in a $\mathcal{V}$-normed category $\mathbb{X}$ is given by an object x of $\mathbb{X}$ together with bijections

$$\mathrm{Nat}(s, \Delta z) \cong \mathbb{X}(x, z),$$

naturally in z, so that the induced cocone

$$(\mathrm{Nat}(s|_N, \Delta z) \longrightarrow \mathbb{X}(x, z))_{N \in \mathbb{N}}$$

is a colimit in $\mathsf{Set}/\!/\mathcal{V}$.

We start with the following observation.

Proposition 12.1. *Every left adjoint $\mathcal{V}$-normed functor $F\colon \mathbb{X} \to \mathbb{Y}$ preserves normed colimits of sequences.*

Proof. Let $G\colon \mathbb{Y} \to \mathbb{X}$ be right adjoint of F in $\mathsf{Cat}/\!/\mathcal{V}$, so that we have isomorphisms

$$\mathbb{X}(x, Gy) \to \mathbb{Y}(Fx, y),$$

in $\mathsf{Set}/\!/\mathcal{V}$, naturally in x and y. Therefore, for every sequence $s = (x_m \xrightarrow{s_{m,n}} x_n)_{m \leq n \in \mathbb{N}}$ in $\mathbb{X}$, every $N \in \mathbb{N}$ and every object y in $\mathbb{Y}$, we also have an isomorphism

$$\mathrm{Nat}(s|_N, \Delta Gy) \to \mathrm{Nat}(Fs|_N, \Delta y)$$

in $\mathsf{Set}/\!/\mathcal{V}$, naturally in s and y. Let x be a normed colimit of s in $\mathbb{X}$. For every object y in $\mathbb{Y}$, the diagram

$$\begin{array}{ccc}
\mathrm{Nat}(s|_N, \Delta Gy) & \xrightarrow{\sim} & \mathrm{Nat}(Fs|_N, \Delta y) \\
\downarrow & & \downarrow \\
\mathbb{X}(x, Gy) & \xrightarrow{\quad\sim\quad} & \mathbb{Y}(Fx, y)
\end{array}$$

commutes. Since the cocone $(\mathrm{Nat}(s|_N, \Delta Gy) \to \mathbb{X}(x, Gy))_{N \in \mathbb{N}}$ is a colimit in $\mathsf{Set}/\!/\mathcal{V}$, so is the cocone $(\mathrm{Nat}(Fs|_N, \Delta y) \to \mathbb{Y}(Fx, y))_{N \in \mathbb{N}}$. This proves that Fx is a normed colimit of Fs in $\mathbb{Y}$. ☐

Recall from [26] that, for $\mathcal{V}$-normed functors $F\colon \mathbb{A} \to \mathbb{X}$ and $\phi\colon \mathbb{A}^{\mathrm{op}} \to \mathsf{Set}|\!|\mathcal{V}$, a *$\phi$-weighted* colimit of F is given by an object x in $\mathbb{X}$ together with $\mathcal{V}$-normed isomorphisms

$$\mathbb{X}(x, y) \cong \mathrm{Nat}(\phi, \mathbb{X}(F-, y)),$$

naturally in y. In this context it is convenient to use the language of $(\mathsf{Set}||\mathcal{V})$-valued distributors $\mathbb{X} \nrightarrow \mathbb{Y}$ which, just like the $\mathcal{V}$-valued distributors in Section 4, are defined as $\mathcal{V}$-normed functors $\mathbb{X}^{\mathrm{op}} \otimes \mathbb{Y} \to \mathsf{Set}||\mathcal{V}$. Every $\mathcal{V}$-normed functor $F\colon \mathbb{X} \to \mathbb{Y}$ induces a pair of distributors

$$F_*\colon \mathbb{X} \nrightarrow \mathbb{Y}, \quad F_*(x,y) = \mathbb{Y}(Fx,y),$$
$$F^*\colon \mathbb{Y} \nrightarrow \mathbb{X}, \quad F^*(y,x) = \mathbb{Y}(y,Fx).$$

In particular, interpreting an object x in $\mathbb{X}$ as a $\mathcal{V}$-normed functor $x\colon \mathbb{E} \to \mathbb{X}$, one obtains

$$x_*\colon \mathbb{E} \nrightarrow \mathbb{X}, \quad x_* = \mathbb{X}(x,-) \quad \text{and} \quad x^*\colon \mathbb{X} \nrightarrow \mathbb{E}, \quad x^* = \mathbb{X}(-,x).$$

For distributors $\phi\colon \mathbb{X} \nrightarrow \mathbb{A}$ and $\psi\colon \mathbb{A} \nrightarrow \mathbb{X}$ and objects x in $\mathbb{X}$ and y in $\mathbb{Y}$, one considers

$$(\psi \cdot \phi)(x,y) \cong \int^{a\in\mathbb{A}} \psi(a,y) \otimes \phi(x,a)$$

whenever this coend exists (see [33, 32]). This is certainly the case when $\mathbb{A}$ is small (since $\mathsf{Set}||\mathcal{V}$, being equivalent to Set, is small-cocomplete), and then the formula above defines the composite distributor $\psi\cdot\phi\colon \mathbb{X} \nrightarrow \mathbb{Y}$. Another important case is $\phi = F_*$ for a $\mathcal{V}$-normed functor $F\colon \mathbb{X} \to \mathbb{A}$, since then one simply has

$$(\psi \cdot F_*)(x,y) \cong \psi(Fx,y)$$

for all objects x in $\mathbb{X}$ and y in $\mathbb{Y}$. Hence, the presheaf $\mathbb{X}(F-,y)$ can be written as the composite $\mathbb{X}(F-,y) = y^* \cdot F_*\colon \mathbb{A} \nrightarrow \mathbb{E}$. Moreover, for $\mathbb{A}$ small, any presheaf $\phi\colon \mathbb{A} \nrightarrow \mathbb{E}$ may be composed with $F^*\colon \mathbb{X} \nrightarrow \mathbb{A}$ to yield $\phi\cdot F^*\colon \mathbb{X} \nrightarrow \mathbb{E}$, and $-\cdot F^* \dashv -\cdot F_*$ is an adjunction between $[\mathbb{A}^{\mathrm{op}}, \mathsf{Set}||\mathcal{V}]$ and the higher-universe $\mathcal{V}$-normed category $[\mathbb{X}^{\mathrm{op}}, \mathsf{Set}||\mathcal{V}]$. Therefore we have natural isomorphisms

$$\mathrm{Nat}(\phi\cdot F^*, y^*) \cong \mathrm{Nat}(\phi, y^*\cdot F_*).$$

It follows that x is a ϕ-weighted colimit of F if, and only if, x is a $(\phi\cdot F^*)$-weighted colimit of the identity functor $\mathbb{X} \to \mathbb{X}$, and in that case we simply speak of a $(\phi\cdot F^*)$-weighted colimit in $\mathbb{X}$.

For a $\mathcal{V}$-normed category $\mathbb{X}$ we consider the Yoneda embedding

$$y_{\mathbb{X}}\colon \mathbb{X} \to [\mathbb{X}^{\mathrm{op}}, \mathsf{Set}||\mathcal{V}], \quad x \mapsto x^* = \mathbb{X}(\text{-},x),$$

whose codomain (irrespective of potentially having to be formed in a higher universe) is $\mathcal{V}$-normed again. Actually, $y_{\mathbb{X}}$ preserves norms since, for every $f : x \to y$ in $\mathbb{X}$, one has

$$|\, y_{\mathbb{X}} f| = |\mathbb{X}(\text{-},f)| = \bigwedge_{z\in\mathbb{X}} |\mathbb{X}(z,f)| = \bigwedge_{h:z\to x} [|h|, |f\cdot h|] = |f|.$$

We now let $\mathcal{P}\mathbb{X}$ denote the full $\mathcal{V}$-normed subcategory of $[\mathbb{X}^{\mathrm{op}}, \mathsf{Set}||\mathcal{V}]$ defined by all *accessible* presheaves (see [28]). By definition, these are the small-weighted colimits of representables. Viewing a presheaf $\phi\colon \mathbb{X}^{\mathrm{op}} \to \mathsf{Set}||\mathcal{V}$ as a distributor $\phi\colon \mathbb{X} \nrightarrow \mathbb{E}$, this means that ϕ belongs to $\mathcal{P}\mathbb{X}$ if, and only if, there is a fully faithful $\mathcal{V}$-normed functor $F\colon \mathbb{A} \to \mathbb{X}$ with $\mathbb{A}$ small and a distributor $\psi\colon \mathbb{A} \nrightarrow \mathbb{E}$ with $\phi = \psi\cdot F^*$. Of course, for $\mathbb{X}$ small, one has $\mathcal{P}\mathbb{X} = [\mathbb{X}^{\mathrm{op}}, \mathsf{Set}||\mathcal{V}]$.

Proposition 12.2. *For every $\mathcal{V}$-normed category $\mathbb{X}$, the $\mathcal{V}$-normed category $\mathcal{P}\mathbb{X}$ is Cauchy cocomplete. Moreover, for every $\mathcal{V}$-normed functor $F\colon \mathbb{X} \to \mathbb{Y}$, the $\mathcal{V}$-normed functor $\mathcal{P}F\colon \mathcal{P}\mathbb{X} \to \mathcal{P}\mathbb{Y}$ preserves normed colimits of sequences.*

Proof. Let $\sigma\colon \mathbb{N} \to \mathcal{P}\mathbb{X}$ be a Cauchy sequence in $\mathcal{P}\mathbb{X}$. Since $\mathbb{N}$ is a (countable) set, there is a small full $\mathcal{V}$-normed subcategory $\mathbb{A}$ of $\mathbb{X}$ (with inclusion functor $I\colon \mathbb{A} \to \mathbb{X}$) such that σ factors as

$$\begin{tikzcd} \mathbb{N} \arrow[d, "\sigma_0"'] \arrow[dr, "\sigma"] & \\ \mathcal{P}\mathbb{A} \arrow[r, "\mathcal{P}I = - \cdot I^*"'] & \mathcal{P}\mathbb{X}, \end{tikzcd}$$

and $\sigma_0\colon \mathbb{N} \to \mathcal{P}\mathbb{A}$ is Cauchy in $\mathcal{P}\mathbb{A}$. By Theorem 11.3, $\mathcal{P}\mathbb{A}$ is Cauchy cocomplete; we let Q be a normed colimit of σ_0 in $\mathcal{P}\mathbb{A}$. By Proposition 12.1, $\mathcal{P}I(Q)$ is a normed colimit of σ in $\mathcal{P}\mathbb{X}$. Finally, since $\mathcal{P}F \cdot \mathcal{P}I$ is left adjoint, $\mathcal{P}F(\mathcal{P}I(Q))$ is a normed colimit of $\mathcal{P}F \cdot \sigma$. $\square$

For a Cauchy sequence s in $\mathbb{X}$, we let ϕ_s denote the normed colimit of $y_{\mathbb{X}} \cdot s$ in $\mathcal{P}\mathbb{X}$. Then, for every object y in $\mathbb{X}$, the cocone

$$\mathrm{Nat}(s|_N, \Delta y) \cong \mathrm{Nat}(y_{\mathbb{X}} \cdot s|_N, \Delta\, y_{\mathbb{X}}\, y) \longrightarrow \mathrm{Nat}(\phi_s, y_{\mathbb{X}}\, y),\ N \in \mathbb{N},$$

is a colimit in $\mathsf{Set}/\!/\mathcal{V}$.

Proposition 12.3. *Let s be a Cauchy sequence in $\mathbb{X}$. Then $\mathbb{X}$ has a normed colimit of s if, and only if, $\mathbb{X}$ has a ϕ_s-weighted colimit.*

Proof. Assume first that x is a normed colimit of s. Then, for every object $y \in \mathbb{X}$, the cocone

$$\mathrm{Nat}(s|_N, \Delta y) \longrightarrow \mathbb{X}(x, y),\ N \in \mathbb{N},$$

is a colimit in $\mathsf{Set}/\!/\mathcal{V}$. Therefore we obtain an isomorphism

$$\mathbb{X}(x, y) \to \mathrm{Nat}(\phi_s, y_{\mathbb{X}}\, y), \tag{12.i}$$

naturally in y. This tells us that x is also a ϕ_s-weighted colimit.

Conversely, if x is a ϕ_s-weighted colimit so that there is a natural isomorphism (12.i), then also

$$\mathrm{Nat}(s|_N, \Delta y) \to \mathbb{X}(x, y),\ N \in \mathbb{N}$$

is a colimit cocone in $\mathsf{Set}/\!/\mathcal{V}$, and therefore x is a normed colimit of s in $\mathbb{X}$. $\square$

We conclude that normed colimits of Cauchy sequences are equivalently described as certain weighted colimits. Below we explain that it suffices to consider "countable diagram shapes". As usual, we call a ($\mathcal{V}$-normed) category $\mathbb{X}$ *countable* whenever the class of arrows of $\mathbb{X}$ is actually a countable set.

Facts 12.4. Let $\sigma\colon \mathbb{N} \to \mathbb{X}$ be a Cauchy sequence in a $\mathcal{V}$-normed category $\mathbb{X}$, and let $\phi\colon \mathbb{X} \looparrowright \mathbb{E}$ be a colimit of $y_{\mathbb{X}} \cdot \sigma$ in $\mathcal{P}\mathbb{X}$.

1. Consider the $\mathcal{V}$-normed subcategory $\mathbb{A}$ of $\mathbb{X}$ generated by the image of σ, with inclusion functor $I\colon \mathbb{A} \to \mathbb{X}$. By construction, $\mathbb{A}$ is countable. Moreover, with $\sigma_0\colon \mathbb{N} \to \mathbb{A}$ denoting the sequence in $\mathbb{A}$ with $I \cdot \sigma_o = \sigma$, also σ_0 is a Cauchy sequence in $\mathbb{A}$. Letting $\phi_0\colon \mathbb{A} \to\!\!\!\!\circ\, \mathbb{E}$ be the normed colimit of $y_{\mathbb{A}} \cdot \sigma_0$ in $\mathbb{A}$, we have $\phi = \phi_0 \cdot I^*$. Therefore, $\mathbb{X}$ has a ϕ-weighted colimit if, and only if, $\mathbb{X}$ has a ϕ_0-weighted colimit of $I\colon \mathbb{A} \to \mathbb{X}$.

2. Consider $\mathbb{N}$ just as an ordinary category (given by its order). By Proposition 3.3, $\mathbb{N}$ may be equipped with the initial normed structure with respect to the ordinary functor $\sigma\colon \mathbb{N} \to \mathbb{X}$ and the given norm of $\mathbb{X}$. Then, since σ is Cauchy in $\mathbb{X}$, the sequence
$$0 \leq 1 \leq 2 \dots$$
becomes Cauchy in $\mathbb{N}$ as well, and we can form the normed colimit ϕ_0 of $y_{\mathbb{N}}$ in $\mathcal{P}\mathbb{N}$. With ϕ defined as above, $\mathbb{X}$ has a ϕ-weighted colimit if, and only if, $\mathbb{X}$ has a ϕ_0-weighted colimit of $I\colon \mathbb{N} \to \mathbb{X}$.

All told, we have the following characterisation of Cauchy cocompleteness.

Corollary 12.5. *Let $\mathbb{X}$ be a $\mathcal{V}$-normed category. Then the following assertions are equivalent.*

(i) *$\mathbb{X}$ is Cauchy cocomplete.*

(ii) *$\mathbb{X}$ has all weighted colimits of normed colimits of Cauchy sequences of representables.*

(iii) *$\mathbb{X}$ has all weighted colimits of diagrams $F\colon \mathbb{A} \to \mathbb{X}$, $\Phi\colon \mathbb{A} \to\!\!\!\!\circ\, \mathbb{E}$, where $\mathbb{A}$ is countable and Φ is a normed colimit of a Cauchy sequence of representables in $\mathcal{P}\mathbb{A}$.*

(iv) *$\mathbb{X}$ has all weighted colimits of diagrams $F\colon \mathbb{A} \to \mathbb{X}$, $\Phi\colon \mathbb{A} \to\!\!\!\!\circ\, \mathbb{E}$, where the underlying category of $\mathbb{A}$ is $\mathbb{N}$ and Φ is a normed colimit of a Cauchy sequence of representables in $\mathcal{P}\mathbb{A}$.*

13 Cauchy cocompletion of $\mathcal{V}$-normed categories

In the previous section we have shown that normed colimits of Cauchy sequences can be equivalently described as weighted colimits, for a certain choice of weights – under the assumption that the quantale $\mathcal{V}$ satisfies condition (A) *or* (B), which we also make in this section. Following the nomenclature of [28, 5], for every small $\mathcal{V}$-normed category $\mathbb{A}$ we consider the class $\Phi[\mathbb{A}]$ of presheaves $\phi \in \mathcal{P}\mathbb{A}$ that are normed colimits of Cauchy sequences of representables, and put
$$\Phi = \sum_{\mathbb{A}\text{ small}} \Phi[\mathbb{A}].$$
A $\mathcal{V}$-normed category $\mathbb{X}$ is called Φ-*cocomplete* whenever, for all $\mathcal{V}$-normed functors $F\colon \mathbb{A} \to \mathbb{X}$ and $\phi\colon \mathbb{A}^{\mathrm{op}} \to \mathsf{Set}||\mathcal{V}$ with $\mathbb{A}$ small and $\phi \in \Phi[\mathbb{A}]$, the ϕ-weighted colimit of F in $\mathbb{X}$ exists. Moreover, a $\mathcal{V}$-normed functor is called Φ-*cocontinuous* whenever it preserves all weighted colimits with weight in Φ.

By Corollary 12.5, a normed category $\mathbb{X}$ is Cauchy cocomplete if, and only if, $\mathbb{X}$ is Φ-cocomplete. Furthermore, since the diagram

$$\begin{tikzcd} \mathbb{X} \arrow[r, "F"] \arrow[d, "y_{\mathbb{X}}"'] & \mathbb{Y} \arrow[d, "y_{\mathbb{Y}}"] \\ \mathcal{P}\mathbb{X} \arrow[r, "\mathcal{P}F"'] & \mathcal{P}\mathbb{Y} \end{tikzcd}$$

commutes (up to isomorphism) for every $\mathcal{V}$-normed functor $F\colon \mathbb{X} \to \mathbb{Y}$, using the notation of Proposition 12.3 and writing ncolim instead of just colim to stress the normedness of a colimit, for every Cauchy sequence $s\colon \mathbb{N} \to \mathbb{X}$ we have

$$PF(\phi_s) \cong PF(\mathrm{ncolim}(y_{\mathbb{X}} \cdot s)) \cong \mathrm{ncolim}(y_{\mathbb{Y}} \cdot F \cdot s) \cong \phi_{F \cdot s}.$$

Therefore, F preserves normed colimits of Cauchy sequences if, and only if, F is Φ-cocontinuous. We define

$$\Phi\text{-}\mathsf{Cocts}$$

to be the 2-category of Φ-cocomplete small $\mathcal{V}$-normed categories, Φ-cocontinuous $\mathcal{V}$-normed functors, and their natural transformations, and write

$$\Phi\text{-}\mathsf{COCTS}$$

for its higher universe counterpart. For every $\mathcal{V}$-normed category $\mathbb{X}$, we let $\Phi(\mathbb{X})$ denote the smallest replete full $\mathcal{V}$-normed subcategory of $\mathcal{P}\mathbb{X}$ containing $\mathbb{X}$ and closed under Φ-colimits. Then the Yoneda functor of $\mathbb{X}$ restricts to

$$y_{\mathbb{X}}\colon \mathbb{X} \to \Phi(\mathbb{X}),$$

and we have that $\Phi(\mathbb{X})$ is Φ-cocomplete and the inclusion functor $\Phi(\mathbb{X}) \to \mathcal{P}\mathbb{X}$ is Φ-cocontinuous. We now show that $\Phi(\mathbb{X})$ serves as a correct-size Cauchy cocompletion of the $\mathcal{V}$-normed category $\mathbb{X}$, both for small and large $\mathbb{X}$.

Lemma 13.1. *For each small normed category $\mathbb{X}$, the presheaf category $\mathcal{P}\mathbb{X}$ is small.*

Proof. Consider

$$\Phi_0 = \sum_{\mathbb{A} \text{ countable}} \Phi[\mathbb{A}],$$

hence Φ_0 is small. By Corollary 12.5, $\Phi(\mathbb{X}) = \Phi_0(\mathbb{X})$ for every normed category $\mathbb{X}$. By [5, Section 7], $\Phi_0(\mathbb{X})$ is small. □

Theorem 13.2 (Proposition 3.6 in [28]). *For every $\mathcal{V}$-normed category $\mathbb{X}$ and every Cauchy cocomplete $\mathcal{V}$-normed category $\mathbb{Y}$, the composition with $y_{\mathbb{X}}\colon \mathbb{X} \to \Phi(\mathbb{X})$ defines an equivalence*

$$\Phi\text{-}\mathsf{COCTS}(\Phi(\mathbb{X}), \mathbb{Y}) \to \mathsf{CAT}/\!/\mathcal{V}(\mathbb{X}, \mathbb{Y});$$

that is, $\Phi(-)$ provides a left biadjoint to the inclusion 2-functor $\Phi\text{-}\mathsf{Cocts} \to \mathsf{CAT}/\!/\mathcal{V}$. This equivalence restricts to

$$\Phi\text{-}\mathsf{Cocts}(\Phi(\mathbb{X}), \mathbb{Y}) \to \mathsf{Cat}/\!/\mathcal{V}(\mathbb{X}, \mathbb{Y}),$$

when $\mathbb{X}$ and $\mathbb{Y}$ are small.

14 The Banach Fixed Point Theorem for normed categories

At first, letting the quantale $\mathcal{V}$ remain general, but then specializing it to $\mathcal{V} = \mathcal{R}_+$, we consider a $\mathcal{V}$-normed category $\mathbb{X}$ and a $\mathcal{V}$-normed endofunctor F of $\mathbb{X}$ and give sufficient conditions guaranteeing the existence of an object x with $x \cong Fx$, in such a way that they reproduce Banach's Fixed Point Theorem when $\mathbb{X} = \mathrm{i}X$ for a metric space X. The following terminology makes precise what $x \cong Fx$ may mean in the $\mathcal{V}$-normed context.

Definition 14.1. For a $\mathcal{V}$-normed functor $F : \mathbb{X} \to \mathbb{X}$, we say that an object x in $\mathbb{X}$ is

- a *forward fixed point of* F if there is an isomorphism $f : x \to Fx$ of (the ordinary category) $\mathbb{X}$ with $\mathrm{k} \leq |f|$;
- a *backward fixed point of* F if there is an isomorphism $f : Fx \to x$ of (the ordinary category) $\mathbb{X}$ with $\mathrm{k} \leq |f|$;
- a *fixed point of* F if there is an isomorphism $f : x \to Fx$ in $\mathbb{X}_\circ$.

Facts 14.2. (1) Trivially, a fixed point of F is both, a forward fixed point and a backward fixed point of F. By Facts 5.8(2), if $\mathbb{X}$ is forward (backward) symmetric, every forward (backward, respectively) fixed point of F is already a fixed point of F.

(2) Here is a normed functor F of a normed category $\mathbb{X}$ in which every object is a forward fixed point of F, but which has no backward fixed point of F: consider $\mathbb{X} = \mathrm{i}X$ for $X = \{0, 1, 2, ...\}$ with $X(m, n) = 0$ for $m \leq n$ and $X(m, n) = 1$ otherwise, and let F be given by $Fn = n + 1$ for all n. (Note that F is even norm preserving.)

(3) For X as in (2), considering $\mathbb{X} = \mathrm{i}(X \otimes X^{\mathrm{op}})$ and its normed endofunctor $F \otimes F^{\mathrm{op}}$, we have (infinitely many) simultaneously forward and backward fixed points of $F \otimes F^{\mathrm{op}}$, but no fixed point.

For a $\mathcal{V}$-normed endofunctor $F : \mathbb{X} \to \mathbb{X}$, let us first consider any morphism $f : x \to Fx$ and form the *iteration sequence* s_f of f:

$$x \xrightarrow{f} Fx \xrightarrow{Ff} F^2x \xrightarrow{F^2f} F^3x \xrightarrow{F^3f} \dots .$$

Assuming that, at the ordinary category level, there is a colimit cocone $\gamma_f : s_f \to \Delta y$ in $\mathbb{X}$, we obtain a comparison morphism $\overline{f} : y \to Fy$ with $\Delta\overline{f}\cdot\gamma_f = F\gamma_f$, which is an isomorphism precisely when the (ordinary) functor F preserves the colimit. Assuming further that γ_f actually exhibits y as a *normed* colimit of s_f, since F is $\mathcal{V}$-normed, in the terminology of Lemma 5.4 not only γ_f must be k-cocone, but also $F\gamma_f$, so that with property (C2b) of Corollary 5.5 one concludes that $\overline{f}$ must be k-morphism:

$$|\overline{f}| \geq \bigvee_N \bigwedge_{n \geq N} |\overline{f} \cdot (\gamma_f)_n| = \bigvee_N \bigwedge_{n \geq N} |F(\gamma_f)_n| \geq \bigvee_N \bigwedge_{n \geq N} |(\gamma_f)_n| \geq \mathrm{k} \, .$$

Furthermore, if F preserves y as a normed colimit of s_f, then trivially also $\overline{f}^{-1}$ must be a k-morphism. This normed preservation of the colimit y is particularly guaranteed if $\mathbb{X}$ is forward or backward symmetric, since with Facts 5.8(2) and the proof of Corollary 5.7 we again obtain that $\overline{f}^{-1}$ is k-morphism.

In summary, we proved:

Proposition 14.3. *Let $F : \mathbb{X} \to \mathbb{X}$ be a $\mathcal{V}$-normed functor preserving ordinary colimits of sequences, and let $f : x \to Fx$ be a morphism for which the iterated sequence s_f has a normed colimit y in $\mathbb{X}$. Then y is a forward fixed point of F in $\mathbb{X}$, and it is even a fixed point if F preserves the colimit y as a normed colimit, in particular if $\mathbb{X}$ is forward or backward symmetric.*

We now consider $\mathcal{V} = \mathcal{R}_+$ and provide a sufficient condition *à la* Banach for the existence of a normed colimit of the iterated sequence of a morphism $x \to Fx$, for a *contractive* functor $F : \mathbb{X} \to \mathbb{X}$, so that there is a (non-negative) Lipschitz factor $L < 1$, *i.e.*, $|Fh| \leq L|h|$ for all morphisms h in $\mathbb{X}$.

Theorem 14.4. *Let $\mathbb{X}$ be a Cauchy cocomplete normed category, and let $F : \mathbb{X} \to \mathbb{X}$ be a contractive functor which preserves (ordinary) colimits of sequences. Then, if $\mathbb{X}$ contains any morphism $f : x \to Fx$ with $|f| < \infty$, then $\mathbb{X}$ contains a forward fixed point of F, and even a fixed point of F if F preserves normed colimits of Cauchy sequences; in particular, if $\mathbb{X}$ is forward or backward symmetric.*

Proof. In light of the Proposition, it suffices to show that the iterated sequence s_f of the given morphism f with $|f| < \infty$ is Cauchy. This, however, follows just like in the classical case of a contraction of a metric space from the Cauchyness of the geometric series given by L: indeed, for all $m \leq n$ one has

$$|(s_f)_{m,n} : F^m x \to F^n x| = |F^{n-1}f \cdot \ldots \cdot F^m f| \leq (L^{n-1} + \cdots + L^m)|f| \, .$$

□

Remarks 14.5. (1) The classical Banach Fixed Point Theorem for the contraction φ of a (non-empty) complete (classical) metric space X follows when we consider $\mathbb{X} = \mathrm{i}X$ and $F = \mathrm{i}\varphi$.

(2) One cannot expect the uniqueness statement for fixed points in the classical metric case to extend *verbatim* to normed categories, not even for Lawvere metric spaces: just consider the coproduct in Met_1 of two copies of the Euclidean line. However, the classical uniqueness is an obvious consequence of the following general statement: Suppose we are given a forward fixed point x and a backward fixed point y of the contraction $F : \mathbb{X} \to \mathbb{X}$, with the property that the minimum

$$\min\{|h| \mid h : x \to y \text{ in } \mathbb{X}\}$$

exists and is positive; then such minimal morphism h_0 must be a 0-isomorphism. Indeed, since we have isomorphisms $f : x \to Fx$ and $g : Fy \to y$ with $|f| = 0 = |g|$, the minimality of $|h_0|$ forbids $|h_0| > 0$, as this would imply

$$|h_0| \leq |g \cdot Fh_0 \cdot f| \leq |g| + |Fh_0| + |f| = |Fh_0| < |h_0| \, .$$

(3) Theorem 14.4 improves Kubiś's Corollary 4.2 in [29], since the normed sequential colimits considered there are not necessarily unique up to 0-isomorphism: see Facts 5.8(3). Actually, we have not been able to establish a valid proof of Kubiš's version of the Banach Fixed Point Theorem since, in the absence of Condition (C2b), one cannot argue as in our proof of Proposition 14.3.

15 Appendix: Condition A vs. Condition B

For a (unital and commutative) quantale $\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$, we show the logical independence of conditions (A) and (B) of Section 11.1, *i.e.*, of the conditions

$$\text{(A)} \quad \mathrm{k} = \bigvee\{u \in \mathcal{V} \mid u \ll \mathrm{k}\} \qquad \text{and} \qquad \text{(B)} \quad \mathrm{k} \wedge (-) : \mathcal{V} \to \mathcal{V} \quad \text{preserves arbitrary joins.}$$

(B) $\not\Rightarrow$ (A):

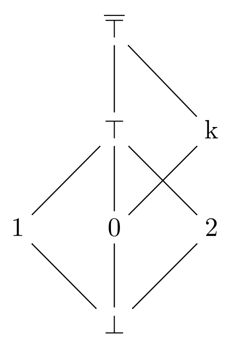
It suffices to find an integral quantale which does not satisfy (A) (see Remarks 11.2(2)). This is not hard; for example, for any infinite set X, consider the cofinite topology $\mathcal{O}(X)$ on X (so that a non-empty subset $U \subseteq X$ is open precisely when $X \setminus U$ is finite) as a quantale $(\mathcal{O}(X), \subseteq, \cap, X)$. Then any open set U with $U \ll X$ must be empty since, otherwise, the finiteness of $X \setminus U$ makes the infinite set X satisfy

$$X = \bigcup_{x \in U} X \setminus \{x\},$$

whereas no $x \in U$ allows $U \subseteq X \setminus \{x\}$. Consequently, (A) is violated in $\mathcal{O}(X)$.

(A) $\not\Rightarrow$ (B) (see [18]):

Consider the 3-element cyclic group $Z_3 = (\{0, 1, 2\}, +)$ as a discretely ordered set. Its MacNeille completion adds the top and bottom elements $\top$ and $\perp$ to it, giving the complete 5-element diamond lattice M_3, with atoms $0, 1, 2$. This lattice carries the operation $\otimes$ of a quantale which coincides with $+$ when restricted to Z_3, and which satisfies $\top \otimes \alpha = \top$ (and necessarily $\perp \otimes \alpha = \perp$) for all $\alpha \in M_3$, $\alpha \neq \perp$. Finally we extend the lattice M_3 by two new elements, k and $\overline{\top}$, to obtain the desired 7-element quantale $\overline{M}_3$ satisfying (A) but not (B). Its quantalic operation $\otimes$ extends the tensor product of M_3 and makes k a new tensor-neutral element in $\overline{M}_3$, above only 0 and $\perp$, while $\overline{\top}$ becomes a new top element in $\overline{M}_3$; tensoring by $\overline{\top}$ is defined by $\overline{\top} \otimes \overline{\top} = \overline{\top}$ and $\overline{\top} \otimes \alpha = \top$ for all $\alpha \in \overline{M}_3, \alpha \neq \perp$.



$\otimes$	$\overline{\top}$	k	$\top$	0	1	2	$\perp$
$\overline{\top}$	$\overline{\top}$	$\overline{\top}$	$\top$	$\top$	$\top$	$\top$	$\perp$
k	$\overline{\top}$	k	$\top$	0	1	2	$\perp$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\perp$
0	$\top$	0	$\top$	0	1	2	$\perp$
1	$\top$	1	$\top$	1	2	0	$\perp$
2	$\top$	2	$\top$	2	0	1	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

Remarks 15.1. (1) Following a question posed at the occasion of the second-named author's presentation of some of the results of this paper at the Portuguese Category Seminar in October 2023, a first witness for (A) $\not\Rightarrow$ (B) was communicated to the authors shortly afterwards by Javier Gutiérrez-García. His subsequent paper [18] with Ulrich Höhle comprehensively analyzes and characterizes many types of quantales with (A) $\not\Rightarrow$ (B), even in the context of not necessarily commutative or unital quantales. The above example of a quantale witnessing (A) $\not\Rightarrow$ (B) is smallest with that property, but there are other such 7-element quantales. Remarkably, it is also shown in [18] that the procedure of finding counterexamples as sketched here starting with the group Z_3 may in fact be followed with any group G of at least 3 elements instead.

(2) There are many infinite topological spaces X (other than those carrying the cofinite topology) such that $\mathcal{O}(X)$ witnesses (B) $\not\Rightarrow$ (A), but none of them can be Alexandroff (so that $\mathcal{O}(X)$ would be closed under arbitrary intersection). Indeed, one easily verifies that the quantale $\mathcal{O}(X)$ is completely distributive for every Alexandroff space X. Other types of integral quantales satisfying (B) but violating (A) include those complete MV-algebras whose underlying lattice fails to be completely distributive. Indeed, by Proposition 3.13 of [17], such MV-algebras must fail condition (A).

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