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Differentiability of solutions for a degenerate fully nonlinear free transmission problem

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Abstract

We study a fully nonlinear free transmission problem in the presence of general degeneracy terms. Under minimal conditions on the degeneracy of the model, we establish the existence of viscosity solutions for the associated Dirichlet problem. Once the existence of solutions has been established, we focus on their regularity estimates. By imposing a Dini-continuity condition on the degeneracy laws involved in the model, we prove that viscosity solutions are locally differentiable.

Keywords: Degenerate fully nonlinear free transmission problems; existence of solutions; differentiability of solutions; regularity theory.

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1 Introduction

We consider viscosity solutions of

$$\begin{aligned} \sigma_1(|Du|)F(D^2u) = f \quad &\text{in} \quad \Omega \cap \{u > 0\} \\ \sigma_2(|Du|)F(D^2u) = f \quad &\text{in} \quad \Omega \cap \{u < 0\}\,, \end{aligned} \tag{1}$$

where $F : S(d) \to \mathbb{R}$ is a uniformly elliptic operator, $f \in L^\infty(\Omega) \cap C(\overline{\Omega})$, and $\sigma_i : \mathbb{R}_+ \to \mathbb{R}_+$ are degeneracy rates. Here, $S(d) \sim \mathbb{R}^{\frac{d(d+1)}{2}}$ denotes the space of symmetric matrices of order d, whereas $\Omega \subset \mathbb{R}^d$ is open, bounded and connected.

We equip (1) with Dirichlet boundary data $g \in C(\partial\Omega)$ and prove the existence of viscosity solutions to the associated Dirichlet problem. Concerning the degeneracy rates σ_1 and σ_2, the existence of solutions only requires these functions to be monotone increasing and ordered. When it comes to regularity estimates, we impose a further condition. Namely, we require σ_2 to have an inverse σ_2^{-1} which is itself a Dini modulus of continuity. Under such a condition, we prove the differentiability of the solutions to (1).

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The model in (1) amounts to a degenerate fully nonlinear free transmission problem. In particular, (1) describes a diffusion process degenerating as a modulus of continuity of the gradient. Meanwhile, the degeneracy depends on the sign of the solution.

Discontinuous diffusions have been studied in the literature since the work of Mauro Picone, circa 1950; see [39]. The problem formulated in [39] finds its roots in the realm of elasticity theory. It attracted substantial attention from the mathematical community. Indeed, various authors have further developed the theory for that model, as well as proposed important variants. We refer the reader to [33, 31, 32, 44, 17, 18, 19, 42, 38, 45, 12, 24]; see also the monograph [13]. The main advances reported in these articles comprise the existence of solutions and their uniqueness and a priori estimates. After the establishment of a soundly based theory of the existence of solutions, the analysis of regularity properties took place. For example, we mention [30, 29, 4, 5].

More recently, a corpus of finer regularity results appeared in the context of transmission problems. In the interesting article [16], the authors examine the regularity of the solutions under *minimal* regularity conditions on the transmission interface. In that paper, the domain is split into two subdomains separated by a $C^{1,\alpha}$-regular interface. Under such an assumption, the authors first prove that solutions are locally of class $C^{0,\text{Log}-\text{Lip}}$ in the domain. It stems from the basic properties of harmonic functions. Then they show that solutions are $C^{1,\alpha}$-regular *up to the transmission interface*. Their findings rely on a new stability result relating hypersurfaces of class $C^{1,\alpha}$ with flat ones.

In [43], the authors pursue the program launched in [16] in the context of fully nonlinear equations. Indeed, they develop a fairly complete theory of viscosity solutions for transmission problems governed by fully nonlinear operators. Their contributions include the existence and the uniqueness of solutions, an Aleksandrov-Bakelman-Pucci estimate, and regularity results. The main novelty in [43] regards the regularity of the solutions up to the transmission interface. As before, they prove the optimal regularity of the solutions matches the regularity of the interface in Hölder spaces.

In recent years, the analysis of transmission problems started to account for diffusions with discontinuities across *solution-dependent* interfaces. That is, endogenously determined rather than prescribed a priori. The first analysis of a free transmission problems appeared recently in [1]. See also [37, 36].

In [20], the authors propose a variational free transmission problem modelled after the p-Laplace operator. They prove the existence of a minimiser u and obtain Hölder-continuity estimates. In addition, they examine the associated free boundary. Among their findings, we highlight a free boundary condition and the almost everywhere $C^{1,\alpha}$-regularity with respect to a p-harmonic measure. Finally, the authors verify the support of $\Delta_p u^+$ is of σ-finite $(d-1)$-dimensional

Hausdorff measure.

Free transmission problems driven by uniformly elliptic fully nonlinear operators have been addressed in [41] and [40]. In [41], the authors prove the existence of L^p-viscosity solutions and L^p-strong solutions for the problem

$$\begin{aligned} F_1(D^2u) = f \quad &\text{in} \quad \Omega \cap \{u > 0\} \\ F_2(D^2u) = f \quad &\text{in} \quad \Omega \cap \{u < 0\}\,, \end{aligned} \tag{2}$$

where $F_i : S(d) \to \mathbb{R}$ are uniformly elliptic operators. The regularity theory for (2) is the subject of [40]. In that paper, the authors prove that solutions to (2) are in $W^{2,\mathrm{BMO}}(\Omega)$, with estimates. Under a smallness assumption on the density of the negative phase, they obtain $C^{1,1}$-regularity estimates.

In the recent paper [28], the authors launch the analysis of fully nonlinear *parabolic* free transmission problems. Their main result concerns the regularity of the free boundary, as they verify that flat free boundaries are smooth.

Degenerate fully nonlinear free transmission problems are the subject of [23]. The authors consider

$$\begin{aligned} |Du|^{\theta_1} F(D^2u) = f \quad &\text{in} \quad \Omega \cap \{u > 0\} \\ |Du|^{\theta_2} F(D^2u) = f \quad &\text{in} \quad \Omega \cap \{u < 0\}\,, \end{aligned} \tag{3}$$

where F is a uniformly elliptic operator and $0 < \theta_1 < \theta_2$ are fixed constants. They prove the existence of L^p-viscosity solutions and an optimal regularity theory in Hölder spaces. The estimates in [23] depend explicitly on θ_1 and θ_2.

The model in (3) is motivated by the study of fully nonlinear equations degenerating as a power of the gradient. See [6, 7, 8, 9, 10, 22], to name just a few. The work-horse of this theory is the equation

$$|Du|^\theta F(D^2u) = f \quad \text{in} \quad \Omega, \tag{4}$$

where $\theta \in (-1, \infty)$ is a fixed constant, F is a uniformly elliptic operator, and $f \in L^\infty(\Omega) \cap C(\overline{\Omega})$. It is well-known that solutions to (4) are locally C^{1,α^*}-regular, where $\alpha^* \in (0,1)$ satisfies

$$\alpha^* < \min\left(\alpha_0, \frac{1}{1+\theta}\right),$$

and $\alpha_0 \in (0,1)$ is the universal exponent in the Krylov-Safonov theory available for $F = 0$. See [25, 11, 3]. Further developments include regularity estimates for equations holding in the regions where the gradient is large and the case of variable exponents; see also [26, 34, 14].

A generalisation of the degeneracy law $p \mapsto |p|^\theta$ appeared in [2]. In that paper, the authors propose an equation of the form

$$\sigma\left(|Du|\right)F(D^2u) = f \quad \text{in} \quad \Omega, \tag{5}$$

where $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity whose inverse σ^{-1} is itself a Dini-modulus of continuity. Working under such a condition on σ, the authors prove that solutions to (5) are locally of class C^1.

In the present paper, we extend the approach in [23] to a degenerate transmission problem governed by equations as in (5). We start with a result on the existence of solutions. Let $g \in C(\partial\Omega)$ and consider the Dirichlet problem

$$\begin{aligned} \sigma_1(|Du|)F(D^2u) = f \quad &\text{in} \quad \Omega \cap \{u > 0\} \\ \sigma_2(|Du|)F(D^2u) = f \quad &\text{in} \quad \Omega \cap \{u < 0\} \\ u = g \quad &\text{on} \quad \partial\Omega. \end{aligned} \tag{6}$$

Our first main result provides conditions on the degeneracy laws σ_1 and σ_2 ensuring the existence of viscosity solutions to (6). This is the content of the next theorem.

Theorem 1 (Existence of viscosity solutions)**.** *Let $\Omega \subset \mathbb{R}^d$ be a domain satisfying a uniform exterior sphere condition. Suppose F is a (λ, Λ)-uniformly elliptic operator, $f \in L^\infty(\Omega) \cap C(\overline{\Omega})$, and $g \in C(\partial\Omega)$. Suppose further that $\sigma_i \in C(\mathbb{R}_+)$ are monotone increasing, with $\sigma_2(t) \leq \sigma_1(t)$ for every $t > 0$, and $\sigma_i(t) \to 0$ as $t \to 0$. Then there exists a viscosity solution $u \in C(\Omega)$ to (6). Furthermore, u is a viscosity sub-solution to*

$$\min\left\{\sigma_1(|Du|)F(D^2u), \sigma_2(|Du|)F(D^2u)\right\} = \|f\|_{L^\infty(\Omega)} \quad \text{in} \;\; \Omega \tag{7}$$

and a viscosity super-solution to

$$\max\left\{\sigma_1(|Du|)F(D^2u), \sigma_2(|Du|)F(D^2u)\right\} = -\|f\|_{L^\infty(\Omega)} \quad \text{in} \;\; \Omega. \tag{8}$$

The proof of Theorem 1 relies on an approximate problem. Indeed, to the best of our knowledge, the comparison principle is not available for viscosity solutions of (6). This is expected since the dependence of the equation in (6) is not monotone. Therefore, there is no a priori reason to ensure properness of the operator. We design an approximate problem depending on a functional parameter $v \in C(\overline{\Omega})$ and a perturbation parameter $\varepsilon > 0$. For such a problem, the comparison principle is available; hence, we construct sub and super-solutions agreeing with g on $\partial\Omega$. An application of Perron's method yields the existence of a viscosity solution for v and ε fixed. Schauder's fixed point theorem eliminates the functional parameter, whereas the stability of viscosity solutions allows us to take the limit $\varepsilon \to 0$, ultimately establishing the Theorem 1.

We stress two interesting consequences of our arguments. First, a simple adjustment of our methods provides the existence of viscosity solutions to (5), completing the program launched in [2]. Also, by modifying the sub and supersolutions used in the proof of Theorem 1, we learn the solutions whose existence follows from the theorem are indeed Lipschitz continuous. We detail these remarks further in the paper.

Once the existence of solutions has been understood, we shift our focus to the regularity of the solutions. Here, we suppose that $\sigma_2 \leq \sigma_1$ and impose a Dini-continuity condition on σ_2^{-1}. Under those conditions, we prove that solutions to (1) are locally differentiable. Without loss of generality, we set $\Omega \equiv B_1$ when dealing with regularity estimates. Our second main theorem reads as follows.

Theorem 2 (Interior C^1-regularity estimates)**.** *Let $u \in C(B_1)$ be a viscosity solution to* (1)*. Suppose F is a (λ, Λ)-uniformly elliptic operator and $f \in L^\infty(\Omega) \cap C(\overline{\Omega})$. Suppose also σ_i are monotone increasing, with $\sigma_2(t) \leq \sigma_1(t)$ for every $t > 0$. Suppose further that σ_2^{-1} is a Dini-continuous modulus of continuity. Then $u \in C^1_{\text{loc}}(B_1)$. Moreover, there exists a modulus of continuity $\omega : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ depending only upon the dimension, ellipticity constants, σ_i, $\|u\|_{L^\infty(B_1)}$, and $\|f\|_{L^\infty(B_1)}$ such that*

$$|Du(x) - Du(y)| \leq \omega(|x - y|),$$

for every $x, y \in B_{1/4}$.

The proof of Theorem 2 relies on several ingredients. First, we notice that solutions to (1) satisfy two viscosity inequalities in the entire domain B_1. Then we resort to standard approximation methods to construct hyperplanes locally comparable with the solution, except for a prescribed error. Finally, the summability due to the Dini continuity of σ_2^{-1} ensures the convergence of such hyperplanes and concludes the proof. A by-product of our argument is an explicit modulus of continuity for Du.

We notice that it suffices to require σ_2^{-1} to be a Dini-continuous modulus of continuity, with no similar requirement on σ_1. This is because we resort to a characterisation of Dini-continuity based on the summability of a series governed by σ_2^{-1}. Therefore, the assumption that $\sigma_2 \leq \sigma_1$ builds upon the Dini-continuity of σ_2^{-1} to ensure that σ_1^{-1} is also Dini-continuous.

The remainder of this paper is organised as follows. Section 2.1 details our main assumptions, whereas Section 2.2 gathers definitions and former results used in the paper. The existence of solutions is the subject of Section 3, where we detail the proof of Theorem 1. In Section 4 we put forward the proof of Theorem 2, establishing the optimal regularity of the solutions to (1).

2 Preliminaries

In this section, we gather preliminary results and notions used throughout the paper. We start by stating our main assumptions.

2.1 Main assumptions

When studying the existence of solutions for the Dirichlet problem associated with (1), we impose a geometric condition on the boundary of Ω.

Definition 1 (Uniform exterior sphere condition)**.** *Let $\Omega \subset \mathbb{R}^d$ be a domain. We say that Ω satisfies a uniform exterior sphere condition if for any $x \in \partial\Omega$ there exists a ball $B \subset \mathbb{R}^d \setminus \Omega$ of radius $r > 0$, independent of x, such that $x \in \partial B$.*

Our first assumption concerns the geometry of $\partial\Omega$.

Assumption 1 (Uniform exterior sphere condition)**.** *We suppose the domain $\Omega \subset \mathbb{R}^d$ satisfies a uniform exterior sphere condition.*

We proceed by requiring the fully nonlinear operator F to be uniformly elliptic.

Assumption 2 (Uniform ellipticity)**.** *Fix $0 < \lambda \leq \Lambda$. We suppose that the operator $F : S(d) \to \mathbb{R}$ is (λ, Λ)-elliptic. That is, for every M, $N \in S(d)$ we have*

$$\lambda \|N\| \leq F(M+N) - F(M) \leq \Lambda \|N\|,$$

provided $N \geq 0$.

The existence of solutions relies on a monotonicity condition on the degeneracy rates. This is the content of our next assumption.

Assumption 3 (Monotonicity of degeneracy rates)**.** *The degeneracy rates σ_1, $\sigma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, monotone increasing, with*

$$\lim_{t \to 0} \sigma_i(t) = 0,$$

for $i = 1, 2$. In addition,

$$\sigma_1(t) \geq \sigma_2(t)$$

for every $t \in [0, 1]$. Furthermore, we assume without loss of generality that

$$\sigma_1(1) \geq \sigma_2(1) \geq 1.$$

When it comes to regularity estimates, additional conditions on σ_2 are necessary. Namely, one requires this degeneracy law to have an inverse which is a Dini-continuous modulus of continuity.

Assumption 4 (Dini continuity)**.** *The degeneracy rate* $\sigma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *has an inverse* σ_2^{-1} *that is a Dini-continuous moduli of continuity.*

A typical example of degeneracy law satisfying Assumptions 3 and 4 is

$$\sigma_1(t) := t^{p_1} \qquad \text{and} \qquad \sigma_2(t) := t^{p_2},$$

where $1 \leq p_2 < p_1$. They satisfy Assumption 3. Also, $\sigma_i^{-1}(t) = t^{1/p_i}$, which is Hölder continuous with exponent $1/p_i$. In what follows, we gather auxiliary results used in the manuscript.

2.2 Elementary notions and auxiliary results

We start with the notion of degenerate ellipticity. More general than the requirements in Assumption 2, this notion unlocks an important variant of the maximum principle, which we recall later.

Definition 2 (Degenerate ellipticity)**.** *We say the operator* $G : S(d) \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ *is degenerate elliptic if, for every* $(p, r, x) \in \mathbb{R}^d \times \mathbb{R} \times \Omega$ *and* $M, N \in S(d)$ *with* $M \leq N$*, we have*

$$G(M, p, r, x) \leq G(N, p, r, x).$$

Degenerate ellipticity can be regarded as an order-preserving property of a fully nonlinear operator. We note it follows from uniform ellipticity. For completeness, we recall the notion of viscosity solution.

Definition 3 (Viscosity solution)**.** *Let* $G : S(d) \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ *be a degenerate elliptic operator. We say that* $u \in \mathrm{USC}(\Omega)$ [resp. $u \in \mathrm{LSC}(\Omega)$] *is a viscosity sub-solution* [resp. *super-solution*] *to*

$$G(D^2u, Du, u, x) = 0 \qquad in \qquad \Omega \tag{9}$$

if, whenever $\varphi \in C^2(\Omega)$ *and* $u - \varphi$ *attains a maximum* [resp. *minimum*] *at* $x_0 \in \Omega$*, we have*

$$G(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq 0$$
$$[\text{resp. } G(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \geq 0].$$

If $u \in C(\overline{\Omega})$ *is both a viscosity sub-solution and a* C*-viscosity super-solution to* (9)*, we say that* u *is a viscosity solution to* (9)*.*

In case a viscosity solution $u \in C(\overline{\Omega})$ is such that $\|u\|_{L^\infty(\Omega)} \leq 1$, we refer to u as a *normalized* viscosity solution. We continue with the definition of the extremal Pucci operators. Indeed, for $0 < \lambda \leq \Lambda$, define $\mathcal{A}_{\lambda,\Lambda} \subset S(d)$ as

$$\mathcal{A}_{\lambda,\Lambda} := \left\{ A \in S(d) \mid \lambda|\xi|^2 \leq A\xi \cdot \xi \leq \Lambda|\xi|^2 \quad \text{for every} \quad \xi \in \mathbb{R}^d \right\}.$$

The extremal operators are defined as follows.

Definition 4 (Extremal Pucci operators)**.** *Let $0 < \lambda \leq \Lambda$ be fixed, though arbitrary. The extremal Pucci operator $\mathcal{M}^-_{\lambda,\Lambda} : S(d) \to \mathbb{R}$ is given by*

$$\mathcal{M}^-_{\lambda,\Lambda}(M) := \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{Tr}(AM).$$

Also, define $\mathcal{M}^+_{\lambda,\Lambda}(M) := -\mathcal{M}^-_{\lambda,\Lambda}(-M)$.

For properties of the extremal operators, we refer the reader to [15, Lemma 2.10]. Among other things, the extremal operators are useful in defining uniform ellipticity. Indeed, an operator $F : S(d) \to \mathbb{R}$ satisfies Assumption 2 if, for any M, $N \in S(d)$, we have

$$\mathcal{M}^-_{\lambda,\Lambda}(M - N) \leq F(M) - F(N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M - N).$$

In the study of interior regularity of (1), we resort to a sequential argument (usual in the setting of approximation methods). A fundamental ingredient in this realm is compactness. We continue by recalling auxiliary results used to produce compactness in our context.

Proposition 1 (Maximum principle)**.** *Let $H, G \in C(S(d) \times \mathbb{R}^d \times \Omega)$ be degenerate elliptic operators. Let $u \in \operatorname{USC}(\Omega)$ be a C-viscosity sub-solution to $G(D^2u, Du, x) = 0$ in Ω and $v \in \operatorname{LSC}(\Omega)$ be a C-viscosity super-solution to $H(D^2v, Dv, x) = 0$ in Ω. Define $w : \Omega \times \Omega \to \mathbb{R}$ as*

$$w(x, y) := u(x) - v(y).$$

Let $\Psi \in C^2(\Omega \times \Omega)$ and suppose $(\overline{x}, \overline{y}) \in \Omega \times \Omega$ is a local maximum point for $w - \Psi$. For every $\varepsilon > 0$ there exist X, $Y \in S(d)$ such that

$$G(X, D_x\Psi(\overline{x}, \overline{y}), \overline{x}) \leq 0 \leq H(Y, D_y\Psi(\overline{x}, \overline{y}), \overline{y}).$$

In addition,

$$-\left(\frac{1}{\varepsilon} + \|D^2\Psi(\overline{x}, \overline{y})\|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2\Psi(\overline{x}, \overline{y}) + \varepsilon \left[D^2\Psi(\overline{x}, \overline{y})\right]^2.$$

For a proof of Proposition 1, we refer the reader to [21, Theorem 3.2]. We continue with a variant of a regularity result for equations holding *only where the gradient is large.* See [27]; see also [35].

Proposition 2 (Hölder regularity)**.** *Fix $\gamma > 0$. Let $u \in C(\overline{\Omega})$ be a normalized viscosity solution to*

$$\begin{aligned} \mathcal{M}^-_{\lambda,\Lambda}(D^2u) &\le C_0 \quad &\text{in} \quad &\Omega \cap \{|Du| \ge \gamma\} \\ \mathcal{M}^+_{\lambda,\Lambda}(D^2u) &\ge -C_0 \quad &\text{in} \quad &\Omega \cap \{|Du| \ge \gamma\}\,. \end{aligned}$$

There exists $\beta \in (0,1)$ such that $u \in C^{\beta}_{\rm loc}(\Omega)$. Moreover, for $\Omega' \Subset \Omega$ there exists $C > 0$ such that

$$\|u\|_{C^\beta(\Omega')} \le CC_0.$$

Finally, $\beta = \beta(\lambda, \Lambda, d)$ and $C = C(\lambda, \Lambda, d, \gamma, \operatorname{diam}(\Omega), \operatorname{dist}(\Omega', \partial\Omega))$.

We close this section with a proposition relating the sequence spaces $\ell_1(\mathbb{R}^d)$ and $c_0(\mathbb{R}^d)$. See [2, Lemma 1].

Proposition 3. *Let $(a_j)_{j\in\mathbb{N}} \in \ell^1$ and take $\varepsilon, \delta > 0$, arbitrary. There exists a sequence $(c_j)_{j\in\mathbb{N}} \in c_0$, with $\max_{j\in\mathbb{N}} |c_j| \le \varepsilon^{-1}$, such that*

$$\left(\frac{a_j}{c_j}\right)_{j\in\mathbb{N}} \in \ell^1$$

and

$$\varepsilon\left(1 - \frac{\delta}{2}\right)\|(a_j)\|_{\ell_1} \le \left\|\left(\frac{a_j}{c_j}\right)\right\|_{\ell_1} \le \varepsilon(1+\delta)\,\|(a_j)\|_{\ell_1}\,.$$

3 The existence of solutions

The study of the existence of solutions for fully nonlinear free transmission problems entails genuine difficulties. For instance, fundamental properties - such as properness - are not available for the operator. As a consequence, the comparison principle for viscosity solutions and Perron's method are out of reach. To establish the existence of a viscosity solution to (1) equipped with a Dirichlet boundary condition, we resort to the approximation strategy devised in [41, 23]. In the sequel, we introduce an approximate equation.

Let $v \in C(\overline{\Omega})$. For $\varepsilon \in (0,1)$, define $g_{v,\varepsilon} : \mathbb{R}^d \to [0,1]$ as

$$g_{v,\varepsilon} := \begin{cases} \max\left(\min\left(\frac{v+\varepsilon}{2\varepsilon}, 1\right), 0\right) & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

The heuristic underlying the definition of $g_{v,\varepsilon}$ is the following. In $\{v > \varepsilon\}$, we have $g_{v,\varepsilon} = 1$, whereas $g_{v,\varepsilon} = 0$ in the set $\{v < -\varepsilon\}$. In the region $\{-\varepsilon \le v \le \varepsilon\}$ the function takes values between 0 and 1. The idea behind $g_{v,\varepsilon}$ is to build an indicator function outside of an ε-neighborhood of the free boundary. To smoothen $g_{v,\varepsilon}$, we define a weight function $h_{v,\varepsilon}$ as follows: for $x \in \Omega$, set

$$h_{v,\varepsilon}(x) := (g_{v,\varepsilon} * \eta_\varepsilon)(x).$$

Then we introduce the approximating degeneracy function $\sigma_{v,\varepsilon}$, given by

$$\sigma_{v,\varepsilon}(p,x) := \sigma_1(p)h_{v,\varepsilon}(x) + (1 - h_{v,\varepsilon}(x))\sigma_2(p).$$

Remark 1. *Because $0 \leq g_{v,\varepsilon} \leq 1$, we have $0 \leq h_{v,\varepsilon} \leq 1$. Therefore, $\sigma_{v,\varepsilon}$ is a convex combination of σ_1 and σ_2. Hence, we conclude $\sigma_1(p) \geq \sigma_{v,\varepsilon}(p,x) \geq \sigma_2(p)$ for every $p \in \mathbb{R}^d$ and every $x \in \Omega$.*

For $\varepsilon \in (0,1)$, consider

$$\sigma_{v,\varepsilon}(\varepsilon + |Du|)\left[\varepsilon u + F(D^2u)\right] = f \text{ in } \Omega. \tag{10}$$

Inspired by ideas in [23], we start by establishing a comparison principle for sub and super-solutions to (10).

Proposition 4 (Comparison Principle)**.** *Let $u \in \mathrm{USC}(\overline{\Omega})$ be a viscosity sub-solution to* (10)*, and $w \in \mathrm{LSC}(\overline{\Omega})$ be a viscosity super-solution to* (10)*. Suppose Ω is a bounded domain, F is degenerate elliptic, and $f \in L^\infty(\Omega) \cap C(\overline{\Omega})$. If $u \leq w$ on $\partial\Omega$, then, $u \leq w$ in Ω.*

Proof. As usual, the proof follows from a contradiction argument. Suppose the statement of the proposition is false. Then there is x_0 such that

$$(u - w)(x_0) = \max_{x \in \Omega} u - w > 0.$$

Set $\tau := (u - w)(x_0)$. For $\delta > 0$, define $\Phi : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ as

$$\Phi_\delta(x,y) := u(x) - w(y) - \frac{|x-y|^2}{2\delta}.$$

Denote with (x_δ, y_δ) the minimiser of $\Phi_\delta(x,y)$. Because

$$\lim_{\delta \to 0} \frac{|x_\delta - y_\delta|^2}{2\delta} = 0,$$

both x_δ and y_δ are in Ω, provided $\delta > 0$ is taken sufficiently small; see [21, Lemma 3.1]. From Proposition 1, there are X, Y such that

$$\left(\frac{x_\delta - y_\delta}{\delta}, X\right) \in \overline{J}_{2,+}u(x_\delta) \quad \text{and} \quad \left(\frac{x_\delta - y_\delta}{\delta}, Y\right) \in \overline{J}_{2,-}w(y_\delta),$$

with
$$-\frac{3}{\delta}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$
In particular, $X \leq Y$. Hence the degenerate ellipticity of F yields
$$\begin{aligned}\varepsilon\tau^2 &\leq \varepsilon(u(x_\delta) - w(y_\delta)) \\ &\leq \frac{f(x_\delta)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, x_\delta\right)} - \frac{f(y_\delta)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, y_\delta\right)}.\end{aligned}$$

Combining Remark 1 and Assumption 3 we get
$$\min\left(\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, x_\delta\right), \sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, y_\delta\right)\right) \geq \sigma_2(\varepsilon).$$

Hence,
$$\begin{aligned}\frac{f(x_\delta)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, x_\delta\right)} - \frac{f(y_\delta)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, y_\delta\right)} &\leq \frac{f(x_\delta) - f(y_\delta)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, x_\delta\right)} \\ &+ \frac{f(y_\delta)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, x_\delta\right)} - \frac{f(y_\delta)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, y_\delta\right)}\end{aligned}$$

Let ω_1 and ω_2 be the moduli of continuity of f and $\sigma_{v,\varepsilon}$ respectively. Then
$$\begin{aligned}&\frac{f(x_\delta) - f(y_\delta)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, x_\delta\right)} + \frac{f(y_\delta)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, x_\delta\right)} - \frac{f(y_\delta)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, y_\delta\right)} \\ &\leq \omega_1(|x_\delta - y_\delta|)\sigma_2(\varepsilon) + |f|_{L^\infty(\Omega)} \frac{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, y_\delta\right) - \sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, x_\delta\right)}{\sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, x_\delta\right) \cdot \sigma_{v,\varepsilon}\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}, y_\delta\right)} \\ &\leq \omega_1(|x_\delta - y_\delta|)\sigma_2(\varepsilon) + \|f\|_{L^\infty(\Omega)} \frac{\omega_2(|x_\delta - y_\delta|)}{\sigma_2(\varepsilon)^2}\end{aligned}$$

Because
$$\omega_1(|x_\delta - y_\delta|)\sigma_2(\varepsilon) + \|f\|_{L^\infty(\Omega)} \frac{\omega_2(|x_\delta - y_\delta|)}{\sigma_2(\varepsilon)^2} \longrightarrow 0$$
as we take the limit $\delta \to 0$, we conclude $\varepsilon\tau^2 \leq 0$, which yields a contradiction and completes the proof. The result is then obtained by letting δ go to zero. □

Once the comparison principle is available, we proceed by constructing explicit sub and super-solutions for (10). This is the subject of the next proposition.

Proposition 5 (Existence of global sub and super-solutions)**.** *Let Ω be a domain* *satisfying u uniform exterior sphere condition. Suppose F is a degenerate elliptic operator, $f \in L^\infty(\Omega) \cap C(\overline{\Omega})$ and $g \in C(\partial\Omega)$. Suppose further that Assumption 3 holds. Then there is a pair of functions $\overline{w}, \underline{w}$ in $C(\overline{\Omega})$ such that, for every $\varepsilon > 0$ and $v \in C(\overline{\Omega})$, $\overline{w}$, $\underline{w}$ are viscosity super and sub-solutions to* (10) *respectively.*

Proof. We will construct only a super-solution to (10) since one analogously obtains a sub-solution. Our argument follows exactly along the same lines as in [23]. In the sequel, we only mention the adjustments necessary to the present setting.

Define functions w_1 and $w_{y,\eta}$ as in [23, Lemma 2]. From the definition of w_1, we conclude

$$\sigma_{v,\varepsilon}(\varepsilon + |Dw_1|, x)[\varepsilon w_1 F(D^2 w_1(x))] \geq f(x), \tag{11}$$

for every $x \in \Omega$. Also, the definition of $w_{y,\eta}$ ensures that

$$\sigma_{v,\varepsilon}(\varepsilon + |Dw_{y,\eta}|)[\varepsilon w_{y,\eta}(x) + F(D^2 w_{y,\eta}(x)) \geq f(x),$$

for every $0 < \varepsilon, \eta < 1$, and $x \in \Omega$. Therefore, the functions $w_{y,\eta}$ are super-solutions of (10) for every $0 < \varepsilon, \eta < 1$ and $y \in \partial\Omega$. Thus, the functions

$$\widetilde{w}_{y,\eta}(x) := \min(w_{y,\eta}(x), w_1(x)) \tag{12}$$

are viscosity super-solutions of (10), since the minimum of viscosity super-solutions is a super-solution. In the same manner, we have at last that the function

$$\overline{w}(x) := \inf \{\widetilde{w}_{y,\eta}(x) \,|\, y \in \partial\Omega,\, 0 < \eta < 1\} \tag{13}$$

is a viscosity super-solution satisfying $\overline{w} = g$ on $\partial\Omega$. □

Corollary 1. *Let Ω be a domain satisfying a uniform exterior sphere condition. Suppose F is degenerate elliptic, $f \in L^\infty(\Omega) \cap C(\overline{\Omega})$ and $g \in C(\partial\Omega)$. Furthermore, assume Assumption 3 holds. Then for every $\varepsilon > 0$ and $v \in C(\overline{\Omega})$, there is a unique viscosity solution $u_{v,\varepsilon}$ in $C(\overline{\Omega})$, such that $\underline{w} \leq u_{v,\varepsilon} \leq \overline{w} \in \Omega$. There is also $\beta(d, \lambda, \Lambda) > 0$, such that for every $\Omega' \subset \Omega$,*

$$\|u_{v,\varepsilon}\|_{C^\beta(\Omega')} \leq C, \tag{14}$$

for some $C = (d, \lambda, \Lambda, \|u_{v,\varepsilon}\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}, \operatorname{dist}(\Omega', \partial\Omega))$

Proof. To prove the existence of $u_{v,\varepsilon}$, one combines the comparison principle (Proposition 4) and the existence of global sub and super-solutions (Proposition 5) and resort to Perron's method. To obtain (14) notice that $u_{v,\varepsilon}$ is a viscosity sub-solution of

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 u_{v,\varepsilon}) = \|f\|_{L^\infty(\Omega)} + \|u_{v,\varepsilon}\|_{L^\infty(\Omega)} \text{ in } \{|Du_{v,\varepsilon}| > 1\}$$

and a viscosity super-solution to

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_{v,\varepsilon}) = -\|f\|_{L^\infty(\Omega)} - \|u_{v,\varepsilon}\|_{L^\infty(\Omega)} \text{ in } \{|Du_{v,\varepsilon}| > 1\}. \tag{15}$$

A straightforward application of Proposition 2 completes the proof. □

Once we have established the existence of a unique viscosity solution $u_{v,\varepsilon}$ to (10) agreeing with g on $\partial\Omega$, we address the parameters $v \in C(\overline{\Omega})$ and $\varepsilon > 0$. To handle the functional parameter v, we resort to the Schauder fixed point theorem. Define $B \subset C(\overline{\Omega})$ as

$$B := \left\{ w \in C(\overline{\Omega}) \,|\, \underline{w} \leq w \leq \overline{w} \right\}, \tag{16}$$

where $\underline{w}$ and $\overline{w}$ are the sub and super-solutions constructed in Proposition 5. Define also the map $T : B \to C(\overline{\Omega})$ as $Tv := u_{v,\varepsilon}$. We prove there exists a fixed point $v^* \in B$ for T.

Proposition 6 (Properties of the map T)**.** *Let $B \subset C(\overline{\Omega})$ be defined as in* (16)*. Define $T : B \to C(\overline{\Omega})$ as $Tv := u_{v,\varepsilon}$. Then there exists $v^* \in B$ such that $Tv^* = v^*$.*

The proof of Proposition 6 follows along the same lines as in [23, Lemma 3] and is omitted. In what follows, we detail the proof of Theorem 1.

Proof of Theorem 1. Proposition 6 is tantamount to the existence of $u_\varepsilon \in B$ such that

$$\sigma_{u_\varepsilon,\varepsilon}(\varepsilon + |Du_\varepsilon|, x)[\varepsilon u_\varepsilon + F(D^2 u_\varepsilon)] = f \quad \text{ in } \quad \Omega, \tag{17}$$

with $u_\varepsilon = g$ on $\partial\Omega$.

Corollary 1 builds upon the Arzelà-Ascoli theorem to produce a subsequence $(u_{\varepsilon_n})_{n\in\mathbb{N}}$, with $\varepsilon_n < 1/n$, and a function $u \in B$ such that $u_{\varepsilon_n} \to u$ in $C(\Omega)$, as $n \to \infty$. Since $\sigma_{v,\varepsilon}$ converges to $\sigma_1\chi_{\{u>0\}} + \sigma_2\chi_{\{u<0\}}$ uniformly on compact subsets of $(\{u > 0\} \cup \{u < 0\}) \cap \Omega$, a standard argument on the stability of viscosity solutions allows us to conclude that u solves (1) in $(\{u > 0\} \cup \{u < 0\}) \cap \Omega$. Moreover, since $\sigma_1 \geq \sigma_{v,\varepsilon} \geq \sigma_2$, u is also a viscosity sub-solution to (7) and a viscosity super-solution to (8) in Ω. □

Remark 2 (Existence of solutions for the pure equation)**.** We notice the arguments above can be easily adjusted to provide an existence result for the problem

$$\begin{aligned} \sigma(|Du|)F(D^2u) = f \quad &\text{in} \quad \Omega \\ u = g \quad &\text{in} \quad \Omega, \end{aligned} \tag{18}$$

provided $\Omega \subset \mathbb{R}^d$ satisfies a uniform exterior sphere condition and $g \in C(\partial\Omega)$. The main modification is in the choice of the approximated problem. Indeed, we should consider

$$\sigma(\varepsilon + |Du|)\left[\varepsilon u + F(D^2u)\right] = f \quad \text{in} \quad \Omega,$$

and establish a comparison principle along the same lines as before. Then building sub and super-solutions also following the previous rationale, would allow one to evoke Perron's method and conclude the argument.

Remark 3 (Existence of globally Lipschitz-continuous solutions)**.** We believe it is possible to change the sub and super-solutions in Proposition 5 according to the model proposed in [11, Lemma 2.2]. Under such a construction, and resorting to the conclusion of Theorem 2, we conclude the existence of a *globally Lipschitz-continuous* viscosity solution to (1).

4 Local C^1-regularity estimates

In this section, we detail the proof of Theorem 2. Our strategy is based on a technique introduced in [41, 23], relating (1) with a pair of viscosity inequalities holding in the *entire* domain. Indeed, if $u \in C(\overline{\Omega})$ is a viscosity solution to (1), it solves

$$\min\left(\sigma_1(|Du|)F(D^2u), \sigma_2(|Du|)F(D^2u)\right) \leq C_0 \quad \text{in} \quad \Omega$$

and

$$\max\left(\sigma_1(|Du|)F(D^2u), \sigma_2(|Du|)F(D^2u)\right) \geq -C_0 \quad \text{in} \quad \Omega.$$

In the sequel, we prove that viscosity solutions to (1) are locally Hölder continuous, with estimates. For simplicity, and without loss of generality, we set $\Omega \equiv B_1$. The next result is a direct consequence of Proposition 2.

Proposition 7 (Hölder continuity)**.** *Let $u \in C(B_1)$ be a viscosity solution to*

$$\min\left(\sigma_1(|q + Du|)F(D^2u), \sigma_2(|q + Du|)F(D^2u)\right) \leq C_0 \quad in \quad B_1 \tag{19}$$

and

$$\max\left(\sigma_1(|q + Du|)F(D^2u), \sigma_2(|q + Du|)F(D^2u)\right) \geq -C_0 \quad in \quad B_1, \tag{20}$$

where $q \in \mathbb{R}^d$ is fixed, though arbitrary. Suppose Assumptions 2 and 3 are in force. Suppose further $|q| \leq A_0$, for some constant $A_0 > 1$. Then there exists $\beta \in (0,1)$ such that $u \in C^{\beta}_{\text{loc}}(B_1)$. In addition, for every $\tau \in (0,1)$ there exists $C_\tau > 0$ for which

$$\|u\|_{C^\beta(B_\tau)} \leq C_\tau.$$

The constant C_τ depends on λ, Λ, the dimension d, C_0 and $\sigma_i(A_0)$, for $i = 1$, 2.

Proof. Let p be such that $|p| > 2A_0$. Hence $|q+p| > A_0$. This inequality builds upon Assumption 3 to ensure

$$\overline{\sigma} := \min\left(\sigma_1(A_0), \sigma_2(A_0)\right) \leq \min\left(\sigma_1(|q+p|), \sigma_2(|q+p|)\right).$$

Hence, u solves

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq \frac{C_0}{\overline{\sigma}} \quad \text{in} \quad B_1 \cap \{|p| > 2A_0\}$$

and

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq -\frac{C_0}{\overline{\sigma}} \quad \text{in} \quad B_1 \cap \{|p| > 2A_0\}.$$

By setting $\gamma := 2A_0$ in Proposition 2 the result follows. □

We continue with an application of the maximum principle to produce Hölder continuity for a solution to (19)-(20) in case $|q| \geq A_0$. Our argument follows along the same lines as in [23, Proposition 5].

Proposition 8 (Hölder continuity)**.** *Let $u \in C(B_1)$ be a viscosity solution to (19)-(20). Suppose Assumptions 2 and 3 are in force. Then $u \in C^{\beta}_{\text{loc}}(B_1)$, for some universal constant $\beta \in (0,1)$. In addition, for every $\rho \in (0,1)$ there exists $C > 0$ such that*

$$\|u\|_{C^\beta(B_\rho)} \leq C.$$

Proof. We argue as in the proof of [23, Proposition 5]; in the sequel, we omit most of the details, stressing the main differences with respect to the argument in that paper. We split the proof into four steps.

Step 1 - Fix $0 < r < (1-\tau)/2$ and consider the modulus of continuity $\omega(t) := t - t^2/2$. Define the quantity

$$L := \sup_{x,y \in B_r(x_0)} \left(u(x) - u(y) - L_1\omega(|x-y|) - L_2\left(|x-x_0|^2 + |y-x_0|^2\right)\right).$$

As usual, our goal is to choose L_1 and L_2 such that $L \leq 0$ for every $x_0 \in B_\tau$. Suppose such a choice is not possible. It is tantamount to say there exists x_0 such that $L > 0$ regardless of the choice of L_1 and L_2. Arguing as in [23, Proposition 5], we find points (X, p_x, x), $(Y, p_y, y) \in S(d) \times \mathbb{R}^d \times B_r(x_0)$, and a constant $\iota > 0$, such that

$$\mathcal{M}^-_{\lambda,\Lambda}(X - Y) \geq 4\lambda L_1 - (\lambda + (d-1)\Lambda)(4L_2 + 2\iota), \tag{21}$$

$$\min\left(\sigma_1(|q + p_x|)F(X), \sigma_2(|q + p_x|)F(X)\right) \leq C_0, \tag{22}$$

$$\max\left(\sigma_1(|q + p_y|)F(Y), \sigma_2(|q + p_y|)F(Y)\right) \leq C_0, \tag{23}$$

and

$$F(X) \geq F(Y) + \mathcal{M}^-_{\lambda,\Lambda}(X - Y). \tag{24}$$

Combining (21)-(24), we get

$$4\lambda L_1 \leq (\lambda + (d-1)\Lambda)\,(4L_2 + 2\iota) + C_0 \left(\frac{1}{\sigma_i(|q + p_j|)} + \frac{1}{\sigma_k(|q + p_\ell|)} \right), \tag{25}$$

where $i, k \in \{1, 2\}$, and $j, \ell \in \{x, y\}$.

Step 2 - Because

$$|p_x|, |p_y| \leq L_1(1 + |x - y|) + 2L_2,$$

where $L_2 := (4\sqrt{2}/r)^2$, we conclude there exists $a > 0$ such that

$$|p_x|, |p_y| \leq aL_1.$$

Set $A_0 = 10aL_1$ and suppose $|q| > A_0$. For those choices, it is clear that $q \neq p_x$ and $q \neq p_y$. Moreover,

$$|q + p_j| \geq A_0 - \frac{A_0}{10} \geq \frac{9}{10}A_0,$$

for $j \in \{x, y\}$. Therefore,

$$\frac{1}{\sigma_i(|q + p_j|)} \leq \frac{1}{\sigma_i(|q + p_j|)} = \frac{1}{\sigma_i(9aL_1)},$$

for $i \in \{1, 2\}$. Hence, (25) becomes

$$4\lambda L_1 \leq (\lambda + (d-1)\Lambda)\,(4L_2 + 2\iota) + C(L_1)C_0,$$

where the constant $C(L_1)$ is monotone decreasing in L_1. By taking $L_1 > 0$ large enough, one gets a contradiction and proves that solutions to (19)-(20) are locally Lipschitz continuous if $|q| \geq A_0$.

Step 3 - In case $|q| \leq A_0$, Proposition 7 ensures the Hölder continuity of u. This fact builds upon the conclusion in Step 2 to complete the proof. $\square$

The compactness stemming from the former result unlocks an approximation lemma, instrumental in our analysis. This is the content of the next proposition.

Proposition 9 (Approximation Lemma)**.** *Let $u \in C(B_1)$ be a viscosity solution to* (19)-(20)*. Suppose Assumptions 2 and 3 are in force. Let $\alpha_0 \in (0,1)$ be the exponent for the Krylov-Safonov regularity theory available for $F = 0$. For every $\delta > 0$ there exists $\varepsilon > 0$ such that, if $C_0 \leq \varepsilon$ then one can find $h \in C^{1,\alpha_0}_{\mathrm{loc}}(B_{9/10})$ satisfying*

$$\|u - h\|_{L^\infty(B_{9/10})} \leq \delta.$$

In addition, there exists $C > 0$ for which

$$\|h\|_{C^{1,\alpha_0}(B_{8/9})} \leq C.$$

Finally, the constant $C = C(d, \lambda, \Lambda) > 0$ is independent of q.

Proof. The proof resorts to a contradiction argument. For ease of presentation, we split it into six steps.

Step 1 - Suppose the result does not hold. In this case, there are sequences $(\sigma_1^n)_{n\in\mathbb{N}}$, $(\sigma_2^n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$, $(F_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$ and $\delta_0 > 0$ such that:

1. The operator F_n satisfies Assumption 2, for every $n \in \mathbb{N}$;
2. The functions σ_1^n and σ_2^n are such that $\sigma_i^n(0) = 0$, $\sigma_i^n(1) \geq 1$ and, if $\sigma_i^n(a_n) \to 0$, then $a_n \to 0$;
3. The function $f_n \in L^\infty(B_1) \cap C(\overline{B_1})$ is such that
$$\|f_n\|_{L^\infty(B_1)} =: C_n \to 0 \qquad \text{as} \quad n \to \infty;$$
4. The following inequalities hold in the viscosity sense:
$$\min\left(\sigma_1^n\left(|Du_n + q_n|\right) F_n(D^2 u_n), \sigma_2^n\left(|Du_n + q_n|\right) F_n(D^2 u_n)\right) \leq C_n$$
and
$$\max\left(\sigma_1^n\left(|Du_n + q_n|\right) F_n(D^2 u_n), \sigma_2^n\left(|Du_n + q_n|\right) F_n(D^2 u_n)\right) \geq -C_n,$$
in the unit ball B_1;
5. We have
$$\sup_{x \in B_{7/8}} |u_n(x) - h(x)| > \delta_0$$
for every $n \in \mathbb{N}$, and every $h \in C^{1,\alpha_0}_{\mathrm{loc}}(B_{8/9})$.

To produce a contradiction, we use the compactness available for the sequence $(u_n)_{n\in\mathbb{N}}$ and the uniform ellipticity of F_n. This is the subject of the next steps.

Step 2 - Because of Proposition 8, there exists a subsequence, still denoted with $(u_n)_{n\in\mathbb{N}}$, converging uniformly to some $u_\infty \in C^\beta_{\text{loc}}(B_1)$. Also, Assumption 2 implies that $(F_n)_{n\in\mathbb{N}}$ is a sequence of uniformly Lipschitz-continuous operators. As a consequence, there exists F_∞ satisfying Assumption 2 such that F_n converges to F_∞, locally uniformly (through some subsequence if necessary). Our goal is to prove that u_∞ is a viscosity solution to $F_\infty(D^2w) = 0$ in B_1. We only show a sub-solution property, as its super-solution counterpart is entirely analogous. Consider the paraboloid $p(x)$ defined as

$$p(x) := u_\infty(y) + \mathrm{b}\cdot(x-y) + \frac{1}{2}(x-y)^T M(x-y).$$

Suppose p touches u_∞ from above in a vicinity of $y \in B_1$. Consider also the sequence $(x_n)_{n\in\mathbb{N}}$ such that p touches u_n from above at x_n and $x_n \to y$ as $n \to \infty$. We then have

$$\min\left(\sigma_1^n\left(|\mathrm{b}+q_n|\right)F_n(M), \sigma_2^n\left(|\mathrm{b}+q_n|\right)F_n(M)\right) \le C_n. \tag{26}$$

The proof is complete if we verify $F_\infty(M) \le 0$. To reach this conclusion we split the remainder of our argument into several cases, depending on the behaviour of $\mathrm{b}+q_n$.

Step 3 - Suppose the sequence $(q_n)_{n\in\mathbb{N}}$ does not admit a convergent subsequence. That is, $|q_n| \to \infty$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that

$$|\mathrm{b}+q_n| \ge 1$$

for $n > N$. In this case, (26) implies

$$F_n(M) \le \frac{C_n}{\sigma_1^n\left(|\mathrm{b}+q_n|\right)} \le C_n \qquad \text{or} \qquad F_n(M) \le \frac{C_n}{\sigma_2^n\left(|\mathrm{b}+q_n|\right)} \le C_n.$$

In any case, we get $F_n(M) \le C_n$. By taking the limit $n \to \infty$, one recovers $F_\infty(M) \le 0$ and completes the proof in this case. It remains to examine the case where $(q_n)_{n\in\mathbb{N}}$ is bounded.

Step 4 - If the sequence $(q_n)_{n\in\mathbb{N}}$ is bounded, at least through a subsequence it converges to some $q_\infty \in \mathbb{R}^d$. Suppose

$$|\mathrm{b}+q_\infty| =: \tau > 0.$$

Then one can find $N \in \mathbb{N}$ such that

$$|\mathrm{b} + q_n| > \frac{\tau}{2},$$

provided $n > N$. Hence, $\sigma_i^n\left(|\mathrm{b} + q_n|\right) > \sigma_i^n(\tau/2)$ for every $n > N$ and the previous argument easily adjusts to yield

$$F_n(M) \leq \frac{C_n}{\sigma_1^n\left(\tau/2\right)} \qquad \text{or} \qquad F_n(M) \leq \frac{C_n}{\sigma_2^n\left(\tau/2\right)}.$$

In any case, $F_n(M) \leq \overline{C_n}$, with $\overline{C_n} \to 0$ as $n \to \infty$. Once again we recover $F_\infty(M) \leq 0$. It remains to study the case $|\mathrm{b} + q_\infty| = 0$.

Step 5 - Without loss of generality we suppose $\mathrm{b} = 0$ and $y = 0$. Also, suppose M has $k \in \{1, \ldots, d\}$ strictly positive eigenvalues. Indeed, were all the eigenvalues of M non-positive, ellipticity would ensure $F_n(M) \leq 0$ for every $n \in \mathbb{N}$, leading immediately to the desired conclusion.

For $i = 1, \ldots, k$, denote with e_i the eigenvector associated with the i-th strictly positive eigenvalue of M. Define E as the subspace of $\mathbb{R}^d$ spanned by $e_1, \ldots, e_k$ and write $\mathbb{R}^d =: E \oplus G$. Finally, consider the test function

$$\varphi(x) := \kappa \sup_{e \in \mathbb{S}^{d-1}} \langle P_E x, e\rangle + \frac{1}{2} x^T M x,$$

where $0 < \kappa \ll 1$ is a fixed constant, P_E is the orthogonal projection into E and $\mathbb{S}^{d-1}$ is the unit sphere of dimension $(d-1)$. We notice that usual stability results ensure that φ touches u_n from above at some point x_n^κ, with $x_n^\kappa \to 0$ as $n \to \infty$. Note that φ is C^2 outside G. If $x_n^\kappa \in G$, we modify the function φ to consider

$$\varphi_e(x) := \kappa \langle P_E x, e\rangle + \frac{1}{2} x^T M x.$$

Hence,

$$\min\left(\sigma_1^n\left(|\kappa e + M x_n + q_n|\right) F_n(M), \sigma_2^n\left(|\kappa e + M x_n + q_n|\right) F_n(M)\right) \leq C_n. \quad (27)$$

Choosing,

$$e := \frac{M x_n}{|M x_n|}$$

and noticing that

$$\frac{\kappa}{2} \leq \kappa - |q_n| \leq |\kappa e + M x_n + q_n|,$$

for large enough $n \gg 1$, inequality (27) yields

$$F_n(M) \leq \frac{C_n}{\sigma_1^n\left(\kappa/2\right)} \qquad \text{or} \qquad F_n(M) \leq \frac{C_n}{\sigma_2^n\left(\kappa/2\right)}, \quad (28)$$

for every $n \in \mathbb{N}$.

Now, suppose $P_E x_n \neq 0$. Arguing as before, we get that either

$$\sigma_1^n\left(\left|Mx_n+\kappa\frac{P_Ex_n}{|P_Ex_n|}+q_n\right|\right)F_n\left(M+\kappa\left(Id+\frac{P_Ex_n}{|P_Ex_n|}\otimes\frac{P_Ex_n}{|P_Ex_n|}\right)\right)\leq C_n$$

or

$$\sigma_2^n\left(\left|Mx_n+\kappa\frac{P_Ex_n}{|P_Ex_n|}+q_n\right|\right)F_n\left(M+\kappa\left(Id+\frac{P_Ex_n}{|P_Ex_n|}\otimes\frac{P_Ex_n}{|P_Ex_n|}\right)\right)\leq C_n.$$

Notice that

$$\frac{\kappa}{2}\leq\kappa-|q_n|\leq\left|Mx_n+\kappa\frac{P_Ex_n}{|P_Ex_n|}+q_n\right|;$$

moreover,

$$\kappa\left(Id+\frac{P_Ex_n}{|P_Ex_n|}\otimes\frac{P_Ex_n}{|P_Ex_n|}\right)\geq 0,$$

in the sense of matrices. Ellipticity builds upon the monotonicity of σ_i^n to produce (28) also when $P_e x_n \neq 0$. By taking the limit $n \to \infty$ in (28), we get $F_\infty(M) \leq 0$. We have established that u_∞ is a viscosity solution to $F_\infty = 0$ in B_1.

Step 6 - Because $F_\infty(D^2 u_\infty) = 0$ in B_1, standard regularity results imply that $u_\infty \in C^{1,\alpha_0}_{loc}(B_1)$ with estimates, where $\alpha_0 \in (0,1)$ is the (universal) exponent stemming from the Krylov-Safonov theory. By taking $h := u_\infty$ one finds a contradiction and concludes the proof. □

We use Proposition 9 to construct a sequence of approximating hyperplanes. The difference between the solution and such approximating hyperplanes behave in a precise geometric fashion. Such a geometric control of the difference between the solution and a hyperplane is key to differentiability.

4.1 Proof of Theorem 2

We start by introducing $\gamma(t) := t\sigma_2(t)$; because $t \mapsto t\sigma_2(t)$ is a bijective map, it has an inverse. Set $\omega(t) := \gamma^{-1}(t)$. We use $\omega(\cdot)$ to examine an alternative.

For $\alpha_0 \in (0,1)$ as in Proposition 9, suppose first

$$\frac{t^{\alpha_0}}{\omega(t)} \longrightarrow 0, \qquad \text{as} \quad t \to 0,$$

we choose $r \in (0,1)$ and define $\mu_1 > 0$ such that

$$2Cr^{\alpha_0} = \omega(r) =: \mu_1, \tag{29}$$

where $C > 0$ is the universal constant in Proposition 9. On the contrary, suppose that

$$\frac{t^{\alpha_0}}{\omega(t)} \longrightarrow M, \qquad \text{as} \quad t \to 0,$$

where $M \in \mathbb{R}$ is some positive constant. In this case, set

$$\mu_1 := r^{\frac{\alpha_0}{2}} = \frac{1}{2C}, \tag{30}$$

where $\alpha_0 \in (0,1)$ and $C > 0$ are once again the constants in Proposition 9. We notice both (29) and (30) imply

$$2Cr^{1+\alpha_0} = \mu_1 r \tag{31}$$

We proceed by defining the constant $\theta = \frac{r}{\mu_1}$ and considering the sequence

$$(a_k)_{k\in\mathbb{N}} := \left(\sigma_2^{-1}(\theta^k)\right)_{k\in\mathbb{N}}.$$

Under Assumption 4, the inverse σ_2^{-1} is Dini-continuous; as a consequence, the sequence $\left(\sigma_2^{-1}(\theta^k)\right)_{k\in\mathbb{N}}$ is summable, and $(a_k)_{k\in\mathbb{N}} \in \ell^1$. Now, we resort to Proposition 3. For $0 < \delta < \frac{1}{4}$, we set $0 < \varepsilon < 1$ as

$$\varepsilon := \frac{1}{1+\delta}.$$

For these choices, an application of Proposition 3 yields a sequence $(c_k)_{k\in\mathbb{N}}$ such that

$$\frac{7}{10}\sum_{i=1}^{\infty}\sigma_2^{-1}(\theta^k) \le \sum_{i=1}^{\infty}\frac{\sigma_2^{-1}(\theta^k)}{c_k} \le \sum_{i=1}^{\infty}\sigma_2^{-1}(\theta^k).$$

Finally, we design two sequences of moduli of continuity $\left(\sigma_1^k(\cdot)\right)_{k\in\mathbb{N}}$ and $\left(\sigma_2^k(\cdot)\right)_{k\in\mathbb{N}}$ given by

$$\begin{aligned}
\sigma_i^0(t) &:= \sigma_i(t),\\
\sigma_i^1(t) &:= \frac{\mu_1}{r}\sigma_i(\mu_1 t),\\
\sigma_i^2(t) &:= \frac{\mu_1\mu_2}{r^2}\sigma_i(\mu_1\mu_2 t),\\
&\;\vdots\\
\sigma_i^k(t) &:= \frac{\prod_{j=1}^{k}\mu_j}{r^k}\sigma_i(\prod_{j=1}^{k}\mu_j t),
\end{aligned}$$

with $\mu_1 > r$ as defined and $(\mu_k)_{k\in\mathbb{N}}$ determined as follows. If

$$\frac{\mu_k \prod_{j=1}^{k-1} \mu_j}{r^k} \sigma_2 \left(\mu_k \prod_{j=1}^{k-1} \mu_j c_k \right) \geq 1,$$

then $\mu_k = \mu_{k-1}$. Otherwise $\mu_k < 1$ is chosen to ensure

$$\frac{\prod_{j=1}^{k} \mu_j}{r^k} \sigma_2 \left(\prod_{j=1}^{k} \mu_j c_k \right) = 1.$$

We notice Assumption 3 ensures that such a choice for the sequence $(\mu_k)_{k\in\mathbb{N}}$ implies

$$\frac{\prod_{j=1}^{k} \mu_j}{r^k} \sigma_1 \left(\prod_{j=1}^{k} \mu_j c_k \right) \geq 1.$$

Once these ingredients are available, we combine them with Proposition 9 to produce a sequence of affine functions whose difference with respect to u grows in a controlled fashion.

Proposition 10. *Let $u \in C(B_1)$ be a normalized viscosity solution to* (19)-(20)*. Suppose Assumptions 1-4 hold true. There exists $\varepsilon > 0$ such that, if $\|f\|_{L^\infty(B_1)} < \varepsilon$, one finds $a \in \mathbb{R}$ and $\mathrm{b} \in \mathbb{R}^d$ satisfying $|a| + \|b\| \leq C$, for some universal constant $C > 0$. In addition,*

$$\sup_{x\in B_r} |u(x) - (a + \mathrm{b}\cdot x)| \leq \mu_1 r,$$

where $\mu_1 > 0$ has been chosen in the algorithm described above.

Proof. We start by choosing the approximation parameter $\delta > 0$ in Proposition 9. Indeed, set

$$\delta := \frac{\mu_1 r}{2}$$

and let $\varepsilon > 0$ be the corresponding smallness regime ensuring the existence of $h \in C^{1,\alpha_0}_{\mathrm{loc}}(B_{9/10})$, with $\|h\|_{C^{1,\alpha_0}(B_{8/9})} \leq C$, such that

$$\sup_{x\in B_{8/9}} |u(x) - h(x)| \leq \frac{\mu_1 r}{2}, \tag{32}$$

where the inequality follows from (31). The regularity available for h implies

$$\sup_{x\in B_r} |h(x) - h(0) - Dh(0)\cdot x| \leq Cr^{1+\alpha_0}. \tag{33}$$

By combining (32) and (33), one finds

$$\sup_{x\in B_r} |u(x) - (a + \mathrm{b}\cdot x)| \leq \mu_1 r,$$

and completes the proof. □

Now we extrapolate the findings in Proposition 10 to arbitrary small scales, in a discrete scheme.

Proposition 11 (Oscillation control at discrete scales)**.** *Let $u \in C(B_1)$ be a normalized viscosity solution of* (19)-(20)*. Assume that Assumptions 1-4 hold and that $\|f\|_{L^\infty(B_1)} \leq \varepsilon$, where ε is the same as in Proposition 10. Then for every $n \in \mathbb{N}$, there are affine functions $(\phi_n)_{n\in\mathbb{N}}$ of the form*

$$\phi_n(x) := a_n + \mathrm{b}_n \cdot x \tag{34}$$

satisfying

$$\sup_{x\in B_{r_n}} |u(x) - \phi_n(x)| \leq \left(\prod_{i=1}^{n} \mu_i\right) r_n, \tag{35}$$

$$|a_{n+1} - a_n| \leq C\left(\prod_{i=1}^{n} \mu_i\right) r_n, \tag{36}$$

and

$$|\mathrm{b}_{n+1} - \mathrm{b}_n| \leq C\prod_{i=1}^{n} \mu_i. \tag{37}$$

Proof. The proof follows from an induction argument. The case $n = 1$ is covered by taking $\phi_1 := a + \mathrm{b}\cdot x$, where $a \in \mathbb{R}$ and $\mathrm{b} \in \mathbb{R}^d$ are as in Proposition 10. It accounts for the base case.

Suppose the case $n = k-1$ has already been established; it remains to verify the statement for $n = k$. To that end, consider the function

$$u_k(x) := \frac{u_{k-1}(rx) - \ell_{k-1}(rx)}{\mu_k \cdot r}, \tag{38}$$

where $r < \mu_1 \leq \mu_2 \ldots \leq \mu_k$ have been chosen earlier. Note that u_k solves

$$\min\left(F_1^k(Du_k, D^2u_k), F_2^k(Du_k, D^2u_k)\right) \leq \|f_k\|_{L^\infty(B_1)} \quad \text{in} \quad B_1$$

and

$$\max\left(F_1^k(Du_k, D^2u_k), F_2^k(Du_k, D^2u_k)\right) \geq -\|f_k\|_{L^\infty(B_1)} \quad \text{in} \quad B_1,$$

where

$$F_i^k(p, M) := \sigma_i^k\left(\left|p + \frac{D\phi_{k-1}}{\mu_{k-1}}\right|\right) F_k(M),$$

with

$$\sigma_i^k(t) := \frac{\prod_{j=1}^{k}\mu_j}{r^k}\sigma_i\left(\prod_{j=1}^{k}\mu_j t\right),$$

$$F_k(M) := r^k\left(\prod_{j=1}^{k}\mu_j\right)^{-1} F\left(\left(\prod_{j=1}^{k}\mu_j\right)\left(r^k\right)^{-1} M\right),$$

and

$$f_k(x) := f(r^k x).$$

The choice of $(\mu_j)_{j=1}^k$ and the construction of the degeneracies σ_i^k allows us to evoke Proposition 10 and obtain an affine function ϕ satisfying

$$\sup_{x\in B_r}|u_k(x) - \phi(x)| \leq \mu_1 r.$$

Now, rewriting the above in terms of u, we get

$$\sup_{x\in B_{r^k}}|u(x) - \phi_k(x)| \leq \left(\prod_{i=1}^{k}\mu_i\right) r^k,$$

where

$$\phi_k(x) := \phi_1(x) + \sum_{i=2}^{k-1}\phi_i(r^{-1}x)\prod_{i=1}^{k-1}(\mu_i)r^i = a_k + \mathrm{b}_k\cdot x.$$

In addition,

$$|a_k - a_{k-1}| \leq C\left(\prod_{i=1}^{k-1}\mu_i\right) r^{k-1},$$

and

$$|\mathrm{b}_k - \mathrm{b}_{k-1}| \leq C\left(\prod_{i=1}^{k-1}\mu_i\right),$$

which completes the proof. □

Next, we present the proof of Theorem 2, which stems from the choice of the sequence $(\mu_n)_{n\in\mathbb{N}}$ and the geometric decay in Proposition 11.

Proof of Theorem 2. We start by noticing two possibilities concerning the sequence $(\mu_n)_{n\in\mathbb{N}}$. Either the sequence repeats after some index $N \geq 2$ or we have $\mu_n < \mu_{n+1}$ for infinitely many indices $n \in \mathbb{N}$.

In the former case, it is well-known that $C^{1,\alpha}$-regularity estimates are available, for $0 < \alpha < \alpha_0$, where $\alpha_0 \in (0,1)$ is the (universal) exponent associated with the Krylov-Safonov theory for $F = 0$. The second possibility amounts to

$$\frac{\prod_{i=1}^{n+1} \mu_i}{r^{n+1}} \sigma_2 \left(\left[\prod_{i=1}^{n+1} \mu_i \right] c_{n+1} \right) = 1.$$

Here, the definition of μ_n implies

$$\frac{\prod_{i=1}^{n} \mu_i}{r^{n}} \sigma_2 \left(\prod_{i=1}^{n} \mu_i \, c_n \right) = 1.$$

Hence,

$$\prod_{i=1}^{n} \mu_i = \frac{1}{c_n} \sigma_2^{-1} \left(\frac{r^n}{\prod_{i=1}^{n} \mu_i} \right) \leq \frac{\sigma_2^{-1}(\theta_n)}{c_n}. \tag{39}$$

Define $(\tau_n)_{n\in\mathbb{N}}$ as

$$(\tau_n)_{n\in\mathbb{N}} := \left(\prod_{i=1}^{n} \mu_i \right)_{n\in\mathbb{N}}.$$

Because of (39), we conclude that $(\tau_n)_{n\in\mathbb{N}} \in \ell^1$, and its norm is bounded by $\sum_{i=1}^{\infty} \sigma_2^{-1}(\theta_i)$. The latter is finite due to the summable characterisation of Dini continuity and Assumption 4.

As a consequence, we infer that

$$\lim_{n\to\infty} \left(\prod_{i=1}^{n} \mu_i \right) = 0;$$

therefore, $(a_n)_{n\in\mathbb{N}}$ and $(\mathrm{b}_n)_{n\in\mathbb{N}}$ are Cauchy sequences and there exists $a_\infty \in \mathbb{R}$ and $\mathrm{b}_\infty \in \mathbb{R}^d$ such that

$$a_n \longrightarrow a_\infty \qquad \text{and} \qquad \mathrm{b}_n \longrightarrow \mathrm{b}_\infty,$$

as $n \to \infty$. Moreover, notice that

$$|a_\infty - a_n| \le C \left(\sum_{i=n}^{\infty} \tau_i \right) r^n \qquad \text{and} \qquad |\mathrm{b}_\infty - \mathrm{b}_n| \le C \left(\sum_{i=n}^{\infty} \tau_i \right). \tag{40}$$

Set $\varphi(x) := a_\infty + \mathrm{b}_\infty \cdot x$ and fix $0 < \rho \ll 1$. Let $n \in \mathbb{N}$ be such that $r^n < \rho \le r^{n+1}$. Combine Proposition 11 with the inequalities in (40) to obtain

$$\begin{aligned} \sup_{x \in B_\rho} |u(x) - \varphi(x)| &\le \sup_{x \in B_{r^n}} |u(x) - \phi_n(x)| + \sup_{x \in B_{r^n}} |\varphi(x) - \phi_n(x)| \\ &\le \frac{1}{r} C \left(\tau_n + \sum_{i=n}^{\infty} \tau_i \right) \rho \\ &\le C \left(\sum_{i=n}^{\infty} \tau_i \right) \rho. \end{aligned} \tag{41}$$

Because the parameter $n \in \mathbb{N}$ in the inequalities in (41) is arbitrary, one can choose $n = n(\rho)$ as

$$n(\rho) := \lfloor 1/\rho \rfloor .$$

Define $\sigma : [0, \infty) \to [0, \infty)$ as

$$\sigma(t) := \sum_{i=\lfloor 1/t \rfloor}^{\infty} \tau_i$$

if $t > 0$, with $\sigma(0) = 0$. We conclude that $\sigma(t)$ is a modulus of continuity. Hence, for every $0 < \rho \ll 1$, (41) becomes

$$\sup_{x \in B_\rho} |u(x) - \varphi(x)| \le C\sigma(\rho)\rho,$$

which completes the proof since φ is an affine function. □

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