ENRICHED ASPECTS OF CALCULUS OF RELATIONS AND 2-PERMUTABILITY

MARIA MANUEL CLEMENTINO AND DIANA RODELO

ABSTRACT. The aim of this work is to further develop the calculus of (internal) relations for a regular Ord-category $\mathbb{C}$. To capture the enriched features of a regular Ord-category and obtain a good calculus, the relations we work with are precisely the *ideals* in $\mathbb{C}$. We then focus on an enriched version of the 1-dimensional algebraic 2-permutable (also called Mal'tsev) property and its well-known equivalent characterisations expressed through properties on ordinary relations. We introduce the notion of Ord-*Mal'tsev category* and show that these may be characterised through enriched versions of the above mentioned properties adapted to ideals. Any Ord-enrichment of a 1-dimensional Mal'tsev category is necessarily an Ord-Mal'tsev category. We also give some examples of categories which are not Mal'tsev categories, but are Ord-Mal'tsev categories.

INTRODUCTION

The notion of *regular category* [1] has been widely studied and explored in Category Theory over the past 50 years. Regular categories capture several nice exactness properties of abelian categories [4], one of notions in the genesis of Category Theory, but without requiring them to be additive. A handy exactness property of regular categories is the existence of (regular) images. This makes regular categories a good context to work with ordinary relations, since it is possible to define their composition and such composition is associative. The calculus of ordinary relations provides a well established and powerful tool for obtaining proofs in regular categories. Another good reason for the successful development of regular categories is the large number of examples. The category of sets, any elementary topos, abelian categories or any variety of universal algebras are all examples of Barr-exact categories [1], which are regular categories. The category of topological groups gives an example of a regular category which is not Barr-exact (see [9]).

A variety of universal algebras, of a certain type, is defined through its signature and axioms, i.e. its theory admits specific operations satisfying given identities. For example, a variety of universal algebras is called a 2-*permutable variety* [27] (they are also called *congruence permutable varieties* or *Mal'tsev varieties*) when its theory admits a ternary Mal'tsev operation p satisfying the identities $p(x,y,y) = x$ and $p(x,x,y) = y$. The variety Grp of groups is 2-permutable, where $p(x,y,z) = xy^{-1}z$. Sometimes it is possible to extract from the operations and identities equivalent properties involving homomorphic relations. The translation of these properties on homomorphic relations to an appropriate categorical setting could be used to define the categorical counterpart of such type of variety. For example, the existence of a Mal'tsev operation of a 2-permutable variety $\mathscr{V}$ is equivalent to the fact that the composition of any pair of congruences R, S on any algebra X of $\mathscr{V}$ is 2-permutable (=commutative): $RS \cong SR$ [20]. It was shown

2020 *Mathematics Subject Classification.* 18E08, 18E13, 18D20, 08C05, 18N10,

Key words and phrases. regular categories, categories enriched in the category of preorders, Mal'tsev categories, Mal'tsev objects, V-categories.

The authors acknowledge partial financial support by *Centro de Matemática da Universidade de Coimbra* (CMUC), funded by the Portuguese Government through FCT/MCTES, DOI 10.54499/UIDB/00324/2020.

in [19] that 2-permutable varieties can also be characterised by the fact that any homomorphic relation D from an algebra X to an algebra Y is *difunctional*:

$$[\,(x,y)\in D \wedge (u,y)\in D \wedge (u,v)\in D\,] \Rightarrow (x,v)\in D,$$

where $x,u\in X$ and $y,v\in Y$. The notion of 2-permutable variety was generalised to a categorical context in [6] (see also [5, 7, 2]). This was achieved by translating the characteristic properties on homomorphic relations of 2-permutable varieties into similar properties on ordinary relations for categories. A regular category C is called a *Mal'tsev category* when any pair of ordinary equivalence relations R,S on any object X in C is such that $RS\cong SR$. Without requiring any kind of exactness properties, a category C is a Mal'tsev category when any ordinary relation $D\colon X \nrightarrow Y$ in C is difunctional, i.e. the relation $\mathsf{C}(W,D)\colon \mathsf{C}(W,X)\nrightarrow \mathsf{C}(W,Y)$ in Set is difunctional, for every object W of C (see Definition 2.1). There are several alternative well-known characterisations of regular Mal'tsev categories given through other properties on ordinary relations, such as: every ordinary reflexive relation is an ordinary equivalence relation. They are recalled in Theorem 5.2.

The aim of this work is to explore 2-permutability in an Ord-enriched context and define, what we call, Ord-Mal'tsev category – Section 5. To do so we consider an enriched version of the property concerning the difunctionality of ordinary relations. The appropriate enriched version of an ordinary relation turns out to be that of *ideal* (see Definition 4.1). An Ord-category $\mathbb{C}$ is called an Ord-*Mal'tsev category* when every ideal $D\colon X \looparrowright Y$ in $\mathbb{C}$ satisfies the property: given morphisms $x,u,u'\colon A\to X$, $y,y',v\colon A\to Y$, the following implication holds

$$\big[\,(x,y)\in_A D \wedge y\preccurlyeq y' \wedge (u,y')\in_A D \wedge u\preccurlyeq u' \wedge (u',v)\in_A D\,\big] \Rightarrow (x,v)\in_A D.$$

Any Ord-enrichment of a Mal'tsev category is necessarily an Ord-Mal'tsev category. If an Ord-category $\mathbb{C}$ is regular, in the sense of [18, 28], then we obtain equivalent characterisations of regular Ord-Mal'tsev categories through properties on ideals (Theorem 5.9, which is the enriched version of Theorem 5.2).

A fundamental part of this work concerns the characterisations of regular Ord-Mal'tsev categories obtained in Theorem 5.9. This is achieved by developing an enriched calculus of relations for ideals in the context of regular Ord-categories – Section 4. We adapt the calculus of relations given in [28], which was done for regular Pos-categories, to regular Ord-categories and further explore the possible extensions of the known calculus of ordinary relations in the regular context (see [5]).

We give examples of categories which are not Mal'tsev categories, and provide them with an Ord-enrichment for which they are Ord-Mal'tsev categories – Section 6. The example concerning the category $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$ relies on an object-wise approach to Ord-Mal'tsev categories, which is developed in the Appendix.

1. Ord-enriched categories

Let $\mathbb{C}$ be an Ord-category, i.e. a category enriched in the category Ord of preordered sets (i.e. sets equipped with a reflexive and transitive relation) and monotone maps. This means that, for any objects X and Y of $\mathbb{C}$, $\mathbb{C}(X,Y)$ is equipped with a preorder such that (pre)composition preserves it. We will denote this preorder of morphisms by $\preccurlyeq$. If we consider in $\mathbb{C}$ the reverse preorder we obtain again an Ord-enriched category which we denote, as usual, by $\mathbb{C}^{\mathrm{co}}$. Any category C with the identity order on morphisms can be considered an Ord-category.

A morphism $m\colon X\to Y$ is said to be *full* when: given morphisms $a,a'\colon A\to X$ such that $ma\preccurlyeq ma'$, then $a\preccurlyeq a'$; equivalently, $ma\preccurlyeq ma'$ if and only if $a\preccurlyeq a'$. (Note that, in the

Ord-enriched context, all morphisms are faithful.) Such (mono)morphisms are also called *ff-(mono)morphisms*, where the "ff" stands for "fully faithful"; see [18, 28]. If the preorder $\leqslant$ is also antisymmetric, so that $\mathbb{C}$ is a Pos-category, then an ff-morphism is necessarily a monomorphism; this is not the case when $\mathbb{C}$ is an Ord-category. We denote ff-monomorphisms with arrows of the type $\rightarrowtail$. We have similar properties for ff-(mono)morphisms as those of monomorphisms in ordinary categories.

Lemma 1.1. *Let $m\colon X \to Y$ and $n\colon Y \to Z$ be morphisms in an* Ord*-category $\mathbb{C}$. Then:*

1. *if m and n are ff-(mono)morphisms, then nm is also an ff-(mono)morphism;*
2. *if nm is an ff-(mono)morphism, then m is an ff-(mono)morphism;*
3. *the 2-pullback of an ff-(mono)morphism is an ff-(mono)morphism.*

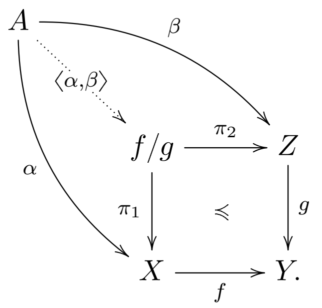
Definition 1.2. Given an ordered pair of morphisms $(f\colon X \to Y, g\colon Z \to Y)$ in an Ord-category $\mathbb{C}$ with common codomain, the *(strict) comma object* of (f,g) is defined by an object f/g and morphisms $\pi_1\colon f/g \to X$, $\pi_2\colon f/g \to Z$ (also called "projections") such that

(C1) $f\pi_1 \leqslant g\pi_2$;

(C2) it has the universal property: given morphisms $\alpha\colon A \to X$ and $\beta\colon A \to Z$ such that $f\alpha \leqslant g\beta$, there exists a unique morphism $\langle \alpha, \beta\rangle\colon A \to f/g$ such that $\pi_1\langle\alpha,\beta\rangle = \alpha$ and $\pi_2\langle\alpha,\beta\rangle = \beta$ (see diagram (1.i) below);

(C3) for morphisms $\alpha, \alpha'\colon A \to X$, $\beta, \beta'\colon A \to Z$ such that $f\alpha \leqslant g\beta$, $f\alpha' \leqslant g\beta'$, $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$, the corresponding unique morphisms $\langle\alpha,\beta\rangle, \langle\alpha',\beta'\rangle\colon A \to f/g$ verify $\langle\alpha,\beta\rangle \leqslant \langle\alpha',\beta'\rangle$;

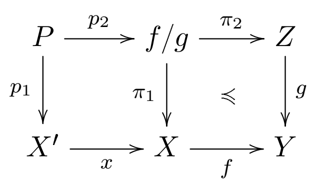
(1.i)



From conditions (C2) and (C3) we can deduce that (π_1, π_2) is *jointly ff-monomorphic*. If $\mathbb{C}$ admits 2-products, this translates into the fact that $\langle \pi_1, \pi_2\rangle\colon f/g \rightarrowtail X \times Z$ is an ff-monomorphism.

The following result combines comma objects and 2-pullbacks; a proof can be found, for instance, in [28]:

Lemma 1.3. *Let $\mathbb{C}$ be an* Ord*-category. Consider the diagram*

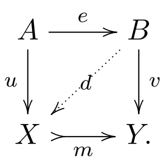


where the right square is a comma object and the left square is commutative. The outer rectangle is a comma object if and only if the left square is a 2-pullback.

We follow [18, 28] to adapt the notion of Pos-enriched regular category to that of Ord-enriched regular category. Recall that a 1-dimensional *regular category* $\mathbb{C}$ is a finitely complete category which admits a pullback-stable (regular epimorphism, monomorphism) factorisation system [1].

The 2-dimensional version of regularity is based on enriched versions for regular epimorphisms and monomorphisms which form a 2-pullback-stable factorisation system.

In the following $\mathbb{C}$ denotes an Ord-category. The Ord-enriched version of a monomorphism is that of an ff-monomorphism. The Ord-enriched version of a regular epimorphism is defined next. A morphism $e\colon A \to B$ is called *surjective on objects*, or *so-morphism*, when e is left orthogonal to every ff-monomorphism m, i.e. the usual diagonal fill-in property holds



If $\mathbb{C}$ has binary products, then every so-morphism is necessarily an epimorphism since it is left orthogonal to a class of monomorphisms. We denote so-morphisms with arrows of the type $\twoheadrightarrow$.

Lemma 1.4. *Let $e\colon A \to B$ and $f\colon B \to C$ be morphisms in an Ord-category $\mathbb{C}$. Then:*

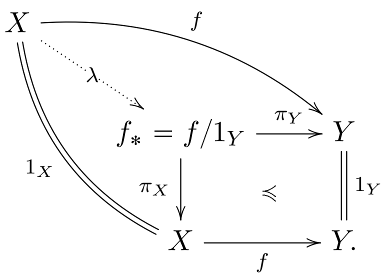
(1) *if e and f are so-morphisms, then fe is also an so-morphism;*
(2) *if fe is an so-morphism, then f is an so-morphism.*

Definition 1.5. An Ord-category $\mathbb{C}$ is called *regular* when:

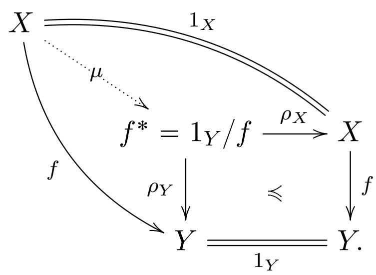
(R1) $\mathbb{C}$ has finite (weighted) limits;
(R2) $\mathbb{C}$ admits an (so-morphism, ff-monomorphism) factorisation system;
(R3) so-morphisms are stable under 2-pullbacks in $\mathbb{C}$;
(R4) every so-morphism is a coinserter.

The (so-morphism, ff-monomorphism) factorisation system is stable under 2-pullbacks in $\mathbb{C}$. Actually, in a regular Ord-category $\mathbb{C}$, so-morphisms are also stable under comma objects, as we show next.

Lemma 1.6. *Let $\mathbb{C}$ be an Ord-category which admits comma objects. Consider the comma objects $f_* = f/1_Y$, $f^* = 1_Y/f$ and the induced morphisms $\lambda = \langle 1_X, f\rangle$ and $\mu = \langle f, 1_X\rangle$*



and

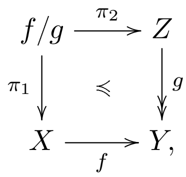


The projections π_X and ρ_X are split epimorphisms (thus, they are so-morphisms). If f is an so-morphism, then so are π_Y and ρ_Y.

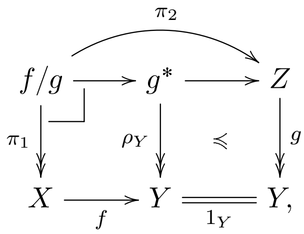
Proof. The proof is straightforward, and uses Lemma 1.4. □

Lemma 1.7. *Let $\mathbb{C}$ be a regular Ord-category. Then so-morphisms are stable under comma objects.*

Proof. Consider a comma object f/g



where g is an so-morphism. Consider the diagram



where the left side is a 2-pullback. The outer rectangle is the comma object of (f, g) by Lemma 1.3. Consequently, ρ_Y is an so-morphism by Lemma 1.6 and π_1 is an so-morphism, since $\mathbb{C}$ is regular (Definition 1.5(R3)). A similar proof holds for f and π_2. □

Remark 1.8. When $\mathbb{C}$ is an Ord-category with comma objects, ff-monomorphisms are not necessarily stable under comma objects in $\mathbb{C}$. This is easily seen by taking $g = 1_Y$, as in Lemma 1.6.

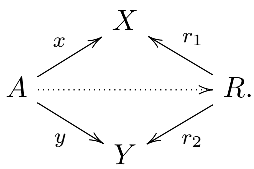
2. Relations in the 1-dimensional regular context

In this section we recall the basic definitions concerning (internal) relations in a 1-dimensional category, which shall be denoted by C (to distinguish it from the $\mathbb{C}$ which is used in an Ord-enriched context). We aim to extend some of those notions and results to the Ord-enriched context, which is one of the main goals of this work. To distinguish a relation in this usual sense from the one in the enriched context, we call the former an "ordinary relation".

Let C be an arbitrary category. An *ordinary relation* R from an object X to an object Y of C is a span $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$ such that (r_1, r_2) is jointly monomorphic. The *opposite* relation of R, denoted R°, is the span $Y \xleftarrow{r_2} R \xrightarrow{r_1} X$. If C admits binary products, then an ordinary relation as above can be viewed as a monomorphism $\langle r_1, r_2 \rangle \colon R \to X \times Y$. When $X = Y$, we simply say that R is an ordinary relation on X.

Any morphism $x \colon A \to X$ of C can be seen a "generalised element" of X. Given $x \colon A \to X$ and $y \colon A \to Y$, we write $(x, y) \in_A R$, or simply xRy (omitting the domain of the morphisms when this is not relevant), when there exists a commutative diagram

(2.i)



Definition 2.1. An ordinary relation $X \xleftarrow{d_1} D \xrightarrow{d_2} Y$ in C is called *difunctional* when the relation

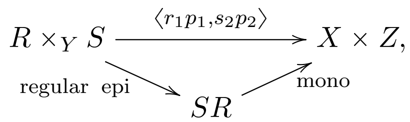
$$\text{(2.ii)} \qquad \mathsf{C}(W,X) \xleftarrow{\mathsf{C}(W,d_1)} \mathsf{C}(W,D) \xrightarrow{\mathsf{C}(W,d_2)} \mathsf{C}(W,Y)$$

in Set is difunctional [26], for every object W of C. More precisely, given morphisms $x, u\colon W \to X$, $y, v\colon W \to Y$, we have $(xDy \wedge uDy \wedge uDv) \Rightarrow xDv$. This can be pictured as

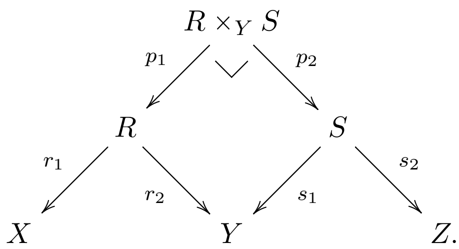
$$\begin{array}{ccc} x & D & y \\ u & D & y \\ u & D & v \\ \hline x & D & v. \end{array} \tag{2.iii}$$

The definition of a reflexive, symmetric, transitive, and equivalence ordinary relation in C is obtained similarly.

In order to define the composition of ordinary relations, the right setting is that of a regular category. Let C be a regular category and consider ordinary relations $\langle r_1, r_2\rangle\colon R \to X \times Y$ and $\langle s_1, s_2\rangle\colon S \to Y \times Z$. The composite ordinary relation $SR \to X \times Z$ is defined through the (regular epimorphism, monomorphism) factorisation of $\langle r_1p_1, s_2p_2\rangle$ in



given the pullback

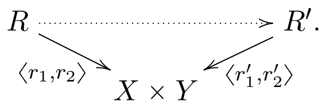


Lemma 2.2 ([5])**.** *Let C be a regular category. Consider ordinary relations $R \to X \times Y$, $S \to Y \times Z$, and generalised elements $x\colon A \to X$, $z\colon A \to Z$. Then $(x, z) \in_A SR$ if and only if there exists a regular epimorphism $b\colon B \to A$ and a morphism $y\colon B \to Y$ such that $(xb, y) \in_B R$ and $(y, zb) \in_B S$.*

This lemma allows one to prove that, in a regular category C, the composition of relations is associative. We get a bicategory Rel(C) of ordinary relations in C:

- a 0-cell in Rel(C) is an object of C;
- a 1-cell from X to Y is an ordinary relation $R \to X \times Y$, also denoted by $R\colon X \nrightarrow Y$;
- a 2-cell from R to R' is denoted by $R \subseteq R'$, and holds when R factors through R'

(2.iv)



  We write $R \cong R'$ when $R \subseteq R'$ and $R' \subseteq R$;
- the identity 1-cell on X is given by the *identity ordinary relation* $\Delta_X = \langle 1_X, 1_X\rangle\colon X \to X \times X$.

From [14], Rel(C) is a tabular allegory, with anti-involution given by taking the opposite ordinary relation. Freyd's *modular laws* hold: given ordinary relations $R\colon X \nrightarrow Y$, $S\colon Y \nrightarrow Z$ and $T\colon X \nrightarrow Z$ we have

$$SR \wedge T \subseteq S(R \wedge S^\circ T) \tag{2.v}$$

and

$$SR \wedge T \subseteq (S \wedge TR^\circ)R. \tag{2.vi}$$

Given an arbitrary category C, any morphism $f\colon X \to Y$ of C induces two ordinary relations $X \xleftarrow{1_X} X \xrightarrow{f} Y$, denoted by $f_\circ$, and $Y \xleftarrow{f} X \xrightarrow{1_X} X$, denoted by f°. If C is a regular category, for every morphism $f\colon X \to Y$ in C, $f_\circ$ is a *map* (in the sense of Lawvere) in $\mathsf{Rel}(\mathsf{C})$, meaning that it admits a right adjoint f°, i.e. $f_\circ \dashv f^\circ$, so that the inclusions $\Delta_X \subseteq f^\circ f_\circ$ and $f_\circ f^\circ \subseteq \Delta_Y$ hold in $\mathsf{Rel}(\mathsf{C})$. On the other hand, taking a map $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$ in $\mathsf{Rel}(\mathsf{C})$ guarantees that r_1 is a monomorphism and a regular epimorphism, which is necessarily an isomorphism; thus, $R \cong (r_2)_\circ$.

Remark 2.3. Let $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$ be an ordinary relation in a regular category C. It is easy to check that $R \cong (r_2)_\circ (r_1)^\circ$ and $R^\circ \cong (r_1)_\circ (r_2)^\circ$ ($R^\circ \cong ((r_2)_\circ (r_1)^\circ)^\circ \cong ((r_1)^\circ)^\circ (r_2)^\circ \cong (r_1)_\circ (r_2)^\circ$).

Remark 2.4. Let C be a regular category. Then an ordinary relation $D\colon X \nrightarrow Y$ is difunctional (Definition 2.1) when $DD^\circ D \subseteq D$. Since $D \subseteq DD^\circ D$ always holds, D is difunctional if and only $DD^\circ D \cong D$. Given any morphism $f\colon X \to Y$, one always has $f_\circ f^\circ f_\circ \cong f_\circ$ and $f^\circ f_\circ f^\circ \cong f^\circ$, which proves that $f_\circ$ and f° are examples of difunctional ordinary relations.

Remark 2.5. An ordinary relation $X \xleftarrow{r_1} R \xrightarrow{r_2} X$ in a category C is:

- reflexive when $(1_X, 1_X) \in_X R$, meaning that there exists a morphism $e\colon X \to R$ such that $r_1 e = 1_X = r_2 e$; equivalently $\Delta_X \subseteq R$;
- symmetric when $(r_2, r_1) \in_R R$, meaning that there exists a morphism $s\colon R \to R$ such that $r_1 s = r_2$ and $r_2 s = r_1$; also $R^\circ \subseteq R$ or, equivalently, $R^\circ \cong R$;

If C is a regular category, so that composition of ordinary relations exists, R is:

- transitive when $RR \subseteq R$;
- an ordinary equivalence relation when it is reflexive, symmetric, and transitive, so that $\Delta_X \subseteq R$, $R^\circ \cong R$ and $RR \cong R$ ($R \subseteq RR$ follows from the reflexivity of R).

There are many other properties concerning (the calculus of) ordinary relations which can be found in [5]. Instead of recalling them all here, we focus on their generalisations to the context of (regular) Ord-enriched categories next.

3. Relations in the Ord-enriched context

We extend the content of Section 2 to the enriched context. We shall use the same names and notation whenever it is possible. In this section $\mathbb{C}$ denotes an Ord-category.

A *relation* from an object X to an object Y of $\mathbb{C}$ is a span $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$ such that (r_1, r_2) is jointly ff-monomorphic. The *opposite* relation R° is the span $Y \xleftarrow{r_2} R \xrightarrow{r_1} X$. If $\mathbb{C}$ admits binary 2-products, then a relation is given by an ff-monomorphism $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$. When $X = Y$, we simply say that R is a relation on X.

Given morphisms $x\colon A \to X$ and $y\colon A \to Y$ of $\mathbb{C}$, we use the same notation $(x, y) \in_A R$, or xRy, when there exists a factorisation as in (2.i).

Example 3.1. 1. Any comma object in an Ord-category $\mathbb{C}$

$$\begin{array}{ccc} f/g & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\downarrow & \preccurlyeq & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y, \end{array}$$

gives a relation $X \xleftarrow{\pi_1} f/g \xrightarrow{\pi_2} Z$ since (π_1, π_2) is jointly ff-monomorphic by Definition 1.2. Moreover, given generalised elements $x\colon A \to X$, $z\colon A \to Z$, we have

$$(x, z) \in_A f/g \Leftrightarrow fx \leqslant gz. \tag{3.i}$$

2. When $f = g = 1_X$, we write $I_X = 1_X/1_X$ and denote its projections by $x_1, x_2\colon I_X \to X$. Given generalised elements $x, x'\colon A \to X$, we have $(x, x') \in_A I_X$ if and only if $x \leqslant x'$.

To define the composition of relations, we must assume $\mathbb{C}$ to be a regular Ord-category. Given relations $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$ and $\langle s_1, s_2\rangle\colon S \rightarrowtail Y \times Z$, the composite relation $SR \rightarrowtail X \times Z$ is defined through the (so-morphism, ff-monomorphism) factorisation of $\langle r_1p_1, s_2p_2\rangle$ in

$$\begin{array}{ccc} R \times_Y S & \xrightarrow{\langle r_1p_1, s_2p_2\rangle} & X \times Z, \\ & \text{so-morphism} \searrow \quad \nearrow \text{ff-monomorphism} & \\ & SR & \end{array}$$

given the 2-pullback

$$\begin{array}{ccccccc} & & & R \times_Y S & & & \\ & & {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} & & \\ & & R & & S & & \\ & {\scriptstyle r_1} \swarrow & & \searrow {\scriptstyle r_2} \quad {\scriptstyle s_1} \swarrow & & \searrow {\scriptstyle s_2} & \\ X & & & Y & & & Z. \end{array}$$

The Ord-enriched version of Lemma 2.2 holds in $\mathbb{C}$, with the difference that "regular epimorphism" is now "so-morphism".

Lemma 3.2 ([28]). *Let $\mathbb{C}$ be a regular Ord-category. Consider relations $R \rightarrowtail X \times Y$, $S \rightarrowtail Y \times Z$, and generalised elements $x\colon A \to X$, $z\colon A \to Z$. Then $(x, z) \in_A SR$ if and only if there exists an so-morphism $b\colon B \twoheadrightarrow A$ and a morphism $y\colon B \to Y$ such that $(xb, y) \in_B R$ and $(y, zb) \in_B S$.*

This lemma allows one to prove that, in a regular Ord-category $\mathbb{C}$, the composition of relations is associative. We get a bicategory $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$ of relations in $\mathbb{C}$:

- a 0-cell in $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$ is an object of $\mathbb{C}$;
- a 1-cell from X to Y is a relation $R \rightarrowtail X \times Y$;
- a 2-cell from R to R' is denoted by $R \subseteq R'$, and holds when R factors through R' as in (2.iv);
- the identity 1-cell on X is given by the *identity relation* $I_X = 1_X/1_X$, i.e. by the ff-monomorphism $\langle x_1, x_2\rangle\colon I_X \rightarrowtail X \times X$.

From [14], $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$ is a tabular allegory, with anti-involution given by taking the opposite relation. Freyd's *modular laws* still hold in $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$: see (2.v) and (2.vi).

4. Order ideals and their calculus of relations

The reasoning above shows that $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$ is not the right bicategory to consider when working with the enriched features of a regular Ord-category. In order to capture this enriched nature, we shall consider relations with a kind of "compatibility" condition. Such relations were called weakening-closed in [17, 28]. We prefer to follow [8] and call them (order) ideals.

Definition 4.1. A relation $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$ in an Ord-category $\mathbb{C}$ is called an *ideal* when, given generalised elements $x, x'\colon A \to X$, $y, y'\colon A \to Y$, we have

$$\big(x' \leqslant x \ \wedge\ (x, y) \in_A R \ \wedge\ y \leqslant y'\big) \Rightarrow (x', y') \in_A R. \tag{4.i}$$

Note that an ideal $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$ is a relation, by definition. So, (r_1, r_2) is jointly ff-monomorphic. We use the notation $R\colon X \looparrowright Y$ for ideals.

Example 4.2. Any comma object in an Ord-category $\mathbb{C}$

$$\begin{array}{ccc} f/g & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\downarrow & \leqslant & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y, \end{array}$$

gives an ideal $X \xleftarrow{\pi_1} f/g \xrightarrow{\pi_2} Z$. Indeed, we already know that it is a relation by Example 3.1. Also, given generalised elements $x, x'\colon A \to X$, $z, z'\colon A \to Z$ such that $x' \leqslant x$, $(x, z) \in_A f/g$ and $z \leqslant z'$, then

$$fx' \leqslant fx \overset{(3.i)}{\leqslant} gz \leqslant gz';$$

we get $(x', z') \in_A f/g$.

It is easy to check that a relation $R \rightarrowtail X \times Y$ in $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$ is an ideal if and only if $R \cong I_Y R I_X$. Consequently, $I_Y R I_X$ is always an ideal.

If $\mathbb{C}$ is a regular Ord-category, the composition of ideals is still an ideal. We denote by $\mathsf{Rel}_{\mathsf{idl}}(\mathbb{C})$ the bicategory of ideals in $\mathbb{C}$, where identities are the ideals I_X, for every object X of $\mathbb{C}$.

Given an Ord-category $\mathbb{C}$, any morphism $f\colon X \to Y$ of $\mathbb{C}$ can be seen as a relation $X \xleftarrow{1_X} X \xrightarrow{f} Y$, which is not necessarily an ideal. However, we can associate to any morphism $f\colon X \to Y$ two canonical ideals $f_* = f/1_Y\colon X \looparrowright Y$ and $f^* = 1_Y/f\colon Y \looparrowright X$. It is easy to check that $I_X \subseteq f^* f_*$ and $f_* f^* \subseteq I_Y$. So, any morphism $f\colon X \to Y$ gives rise to an adjunction $f_* \dashv f^*$ in $\mathsf{Rel}_{\mathsf{idl}}(\mathbb{C})$. The converse also holds, i.e. if $R\colon X \looparrowright Y$ has a right adjoint $\overline{R}\colon Y \looparrowright X$ in $\mathsf{Rel}_{\mathsf{idl}}(C)$, then there exists a unique morphism $f\colon X \to Y$ in $\mathbb{C}$ such that $R \cong f_*$ (see [28, Theorem 3.8.]).

The following results are easy to prove and most can be found in [28]. Those results which are provided with a proof are new. Results involving relations (which are not necessarily ideals) and ideals are meant to hold in $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$; this is explicitly added after the result. All the other results only involving ideals hold in $\mathsf{Rel}_{\mathsf{idl}}(\mathbb{C})$, without explicitly mentioning it.

Lemma 4.3. *Let $\mathbb{C}$ be a regular Ord-category. Consider morphisms $f, h\colon X \to Y$ and $g\colon Y \to Z$. Then:*

(1) $f_* \cong I_Y f_\circ$ *and* $f^* \cong f^\circ I_Y$ *in* $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$*;*
(2) $f_\circ \subseteq f_*$ *and* $f^\circ \subseteq f^*$ *in* $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$*;*
(3) $(gf)_* \cong g_* f_*$*;*
(4) $(gf)^* \cong f^* g^*$*;*
(5) $f \leqslant h \Leftrightarrow h_* \subseteq f_* \Leftrightarrow f^* \subseteq h^*$*;*
(6) $I_X \subseteq f^* f_*$*;*
(7) $f_* f^* \subseteq I_Y$*;*
(8) $f^* f_* f^* \cong f^*$ *and* $f_* f^* f_* \cong f_*$*.*

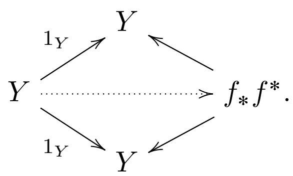
From (3), (4) and (5) we get Ord-enriched functors $(\ )_* : \mathbb{C}^{\mathrm{co}} \to \mathsf{Rel}_{\mathsf{idl}}(\mathbb{C})$ and $(\ )^* : \mathbb{C}^{\mathrm{op}} \to \mathsf{Rel}_{\mathsf{idl}}(\mathbb{C})$.

Lemma 4.4. *Let $\mathbb{C}$ be a regular Ord-category. Consider morphisms $f\colon X \to Y$, $g\colon Z \to Y$ and $h\colon X \to Z$. Then:*

(1) $f/g \cong g^* f_*$*;*
(2) *f is an ff-morphism if and only if* $I_X \cong f/f \cong f^* f_*$*;*

(3) *if $\leqslant$ is a partial order, then f is an ff-monomorphism if and only if $I_X \cong f/f \cong f^* f_*$;*
(4) *if f is an so-morphism, then $f_* f^* \cong I_Y$;*
(5) *if $\leqslant$ is a partial order, then f is an so-morphism if and only if $f_* f^* \cong I_Y$;*
(6) *$\langle f, h\rangle$ is an ff-morphism if and only if $f^* f_* \wedge h^* h_* \cong I_X$;*
(7) *if $\leqslant$ is a partial order, then $\langle f, h\rangle$ is an ff-monomorphism if and only if $f^* f_* \wedge h^* h_* \cong I_X$.*

Proof. (4) It is already known that $f_* f^* \subseteq I_Y$. Let $\langle y_1, y_2\rangle \colon I_Y \rightarrowtail Y \times Y$ represent the projections of $I_Y = 1_Y/1_Y$. Then, $y_1 \leqslant y_2$ (see (3.i)). It is easy to see that $(f, 1_X) \in_X f^*$ and $(1_X, f) \in_X f_*$, so that $(f, f) \in_X f_* f^*$. If f is an so-morphism, it follows that $(1_Y, 1_Y) \in_Y f_* f^*$, i.e. there exists a factorisation such as



Precomposing the dotted morphism with y_1, we get $(y_1, y_1) \in_{I_Y} f_* f^*$. Since $y_1 \leqslant y_2$ and $f_* f^*$ is an ideal, then $(y_1, y_2) \in_{I_Y} f_* f^*$; this means precisely that $I_Y \subseteq f_* f^*$.

(5) Since $(1_Y, 1_Y) \in_Y I_Y \cong f_* f^*$, there exist an so-morphism $z \colon Z \twoheadrightarrow Y$ and a morphism $x \colon Z \to X$ such that $(z, x) \in_Z f^*$ and $(x, z) \in_Z f_*$ (Lemma 3.2). Using (3.i), we get $z \leqslant fx$ and $fx \leqslant z$. Since $\leqslant$ is a partial order, we conclude that $z = fx$. From Lemma 1.4(2) we conclude that f is an so-morphism. $\square$

The following results generalise known ones concerning calculus of ordinary relations (see [5]).

Proposition 4.5. *Let $\mathbb{C}$ be a regular* Ord*-category. Consider ideals $R, S \colon X \nrightarrow Y$, $T \colon A \nrightarrow B$, and morphisms $g \colon B \to Y$, $f \colon A \to X$, $k \colon Y \to B$, $h \colon X \to A$. Then:*

(1) *$g^*(R \wedge S) \cong g^* R \wedge g^* S$;*
(2) *$(R \wedge S) f_* \cong R f_* \wedge S f_*$;*
(3) *$k_*(R \wedge S) \subseteq k_* R \wedge k_* S$;*
(4) *$(R \wedge S) h^* \subseteq R h^* \wedge S h^*$;*
(5) *$g_* T f^* \subseteq R \Leftrightarrow T \subseteq g^* R f_*$.*

Proof. (1) From $g^*(R \wedge S) \subseteq g^* R$ and $g^*(R \wedge S) \subseteq g^* S$, we conclude that $g^*(R \wedge S) \subseteq g^* R \wedge g^* S$.

Conversely, suppose that $(x, u) \in_U g^* R \wedge g^* S$. By Lemma 3.2, there exist so-morphisms $e \colon V \twoheadrightarrow U$, $e' \colon V' \twoheadrightarrow U$ and morphisms $y \colon V \to Y$, $y' \colon V' \to Y$ such that $(xe, y) \in_V R$, $(y, ue) \in_V g^*$, $(xe', y') \in_{V'} S$, $(y', ue') \in_{V'} g^*$. We use (3.i) and the fact that R and S are ideals to conclude that

$$((xe, y) \in_V R \text{ and } y \leqslant gue) \;\Rightarrow\; (xe, gue) \in_V R;$$
$$((xe', y') \in_{V'} S \text{ and } y' \leqslant gue') \;\Rightarrow\; (xe', gue') \in_{V'} S.$$

Since e and e' are so-morphisms, we get $(x, gu) \in_U R$ and $(x, gu) \in_U S$, so $(x, gu) \in_U R \wedge S$. Since $(gu, u) \in_U g^*$, we may conclude that $(x, u) \in g^*(R \wedge S)$.

The proofs of (2), (3) and (4) follow similar arguments. The fact that (3) and (4) are only inclusions is a consequence of the compatibility property of ideals (4.i) that does not apply for the other inclusions.

(5) Suppose that $g_* T f^* \subseteq R$. Since T is an ideal, then $T \cong I_B T I_A$. We have

$$T \cong I_B T I_A \overset{\text{Lemma 4.3(6)}}{\subseteq} g^* g_* T f^* f_* \overset{\text{assumption}}{\subseteq} g^* R f_*.$$

For the converse, suppose that $T \subseteq g^* R f_*$. Then

$$g_* T f^* \overset{\text{assumption}}{\subseteq} g_* g^* R f_* f^* \overset{\text{Lemma 4.3(7)}}{\subseteq} I_Y R I_X \cong R,$$

because R is an ideal. □

Given a relation $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$, we use the notation $R_* = (r_2)_*(r_1)^*$ in what follows.

Proposition 4.6. *Let $\mathbb{C}$ be a regular* Ord*-category. Given an ideal $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$, we have $R \cong R_*$.*

Proof. As a relation, $R \cong (r_2)_\circ(r_1)^\circ \subseteq (r_2)_*(r_1)^* = R_*$ (see Remark 2.3 and Lemma 4.3(2)). Conversely, suppose that the generalised elements $x\colon A \to X$ and $y\colon A \to Y$ are such that $(x, y) \in_A (r_2)_*(r_1)^*$. By Lemma 3.2, there exists an so-morphism $b\colon B \twoheadrightarrow A$ and a morphism $z\colon B \to R$ such that $(xb, z) \in_B (r_1)^*$ and $(z, yb) \in_B (r_2)_*$; we get $xb \leqslant r_1 z$ and $r_2 z \leqslant yb$ from (3.i). Since $(r_1 z, r_2 z) \in_B R$ and R is an ideal, then $(xb, yb) \in_B R$, which gives $(x, y) \in_A R$ since b is an so-morphism. □

Proposition 4.7. *Let $\mathbb{C}$ be a regular* Ord*-category. Given a relation $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$, then R_* is the smallest ideal containing R. Moreover, $R_* \cong I_Y R I_X$.*

Proof. We already know that $R \subseteq R_*$ from the proof of Proposition 4.6. Now, suppose that $S\colon X \looparrowright Y$ is an ideal such that $R \subseteq S$. Given generalised elements $x\colon A \to X$ and $y\colon A \to Y$, suppose that $(x, y) \in_A (r_2)_*(r_1)^*$. By Lemma 3.2, there exist an so-morphism $b\colon B \twoheadrightarrow A$ and a morphism $z\colon B \to R$ such that $(xb, z) \in_B (r_1)^*$ and $(z, yb) \in_B (r_2)_*$. This gives $xb \leqslant r_1 z$ and $r_2 z \leqslant yb$, by (3.i). Since $(r_1 z, r_2 z) \in_B R$ and $R \subseteq S$, then $(r_1 z, r_2 z) \in_B S$. Since S is an ideal, we have $(xb, yb) \in_B S$; consequently $(x, y) \in_A S$, because b is an so-morphism.

For the last statement, $I_Y R I_X \cong I_Y (r_2)_\circ (r_1)^\circ I_X \cong (r_2)_*(r_1)^*$, by Lemma 4.3(1). □

We have already mentioned above that, given an ideal $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$, the opposite relation $Y \xleftarrow{r_2} R \xrightarrow{r_1} X$ is not necessarily an ideal. When $\mathbb{C}$ is a regular Ord-category, we may write $R \cong (r_2)_\circ(r_1)^\circ$ and $R^\circ \cong (r_1)_\circ(r_2)^\circ$ (see Remark 2.3) in $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$. From Proposition 4.6, we know that $R \cong (r_2)_*(r_1)^* = R_*$. We denote by $R^* = (r_1)_*(r_2)^*$ the ideal which plays the role of the opposite (ideal) of R in $\mathsf{Rel}_{\mathsf{idl}}(\mathbb{C})$. In particular, if $R = f$, for a morphism $f\colon X \to Y$, then $R^* = (1_X)_*(f)^* = I_X f^* \cong f^*$.

Corollary 4.8. *Let $\mathbb{C}$ be a regular* Ord*-category. Given a relation $R \rightarrowtail X \times Y$, R^* is the smallest ideal containing R°. Moreover, $R^* \cong I_Y R^\circ I_X$.*

Proof. This is an immediate consequence of Proposition 4.7, since $R^* = (R^\circ)_*$. □

Proposition 4.9. *Let $\mathbb{C}$ be a regular* Ord*-category. If $R \rightarrowtail X \times Y$, $S \rightarrowtail X \times Y$ are relations such that $R \subseteq S$ (in $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$), then $R_* \subseteq S_*$ and $R^* \subseteq S^*$.*

Proof. This is an immediate consequence of Proposition 4.7 and Corollary 4.8. □

We can now state an Ord-enriched version for Freyd's modular laws. Indeed, it is straightforward to check that, if $\mathbb{C}$ is a regular Ord-category, then $\mathsf{Rel}_{\mathsf{idl}}(\mathbb{C})$ is a tabular allegory, with anti-involution given by $(\ )^*$.

Proposition 4.10. *Let $\mathbb{C}$ be a regular* Ord*-category. For ideals $R\colon X \looparrowright Y$, $S\colon Y \looparrowright Z$ and $T\colon X \looparrowright Z$ we have*

$$SR \wedge T \subseteq S(R \wedge S^* T) \tag{4.ii}$$

and

$$SR \wedge T \subseteq (S \wedge TR^*)R. \tag{4.iii}$$

Proof. These are immediate consequences of Freyd's modular laws (2.v) and (2.vi) (in $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$) and Corollary 4.8. □

Proposition 4.11. *Let $\mathbb{C}$ be a regular Ord-category and consider ideals $R\colon X ⇸ Y$, $S\colon Z ⇸ Y$ and $T\colon Z ⇸ W$ in $\mathbb{C}$. Then $TS^{\circ}R \cong TS^{*}R$. In particular, $TS^{\circ}R$ is an ideal.*

Proof. From Corollary 4.8 we have: $TS^*R \cong TI_Y S^{\circ} I_Z R \cong TS^{\circ}R$. □

Proposition 4.12. *Consider an ideal $R\colon X ⇸ Y$ and morphisms $f\colon U \to X$, $g\colon V \to Y$ in an Ord-category $\mathbb{C}$. The relation $S \rightarrowtail U \times V$ given by the 2-pullback of $R \rightarrowtail X \times Y$ along $f \times g$*

$$\begin{array}{ccc} S \cong g^{\circ}Rf_{\circ} & \xrightarrow{\ h\ } & R \\ {\scriptstyle \langle s_1,s_2\rangle}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \langle r_1,r_2\rangle} \\ U \times V & \xrightarrow[f\times g]{} & X \times Y, \end{array}$$

is an ideal. If $\mathbb{C}$ is a regular Ord-category, then $g^{\circ}Rf_{\circ} \cong g^*Rf_*$. In particular, the inverse image of an ideal $T\colon X ⇸ X$ by the morphism f, denoted $f^{-1}(T)$, is an ideal such that $f^{-1}(T) \cong f^*Tf_*$.

Proof. By Lemma 1.1(3) S is a relation, and it is easy to check that $S \cong g^{\circ}Rf_{\circ}$ in $\mathsf{Rel}_{\mathsf{ff}}(\mathbb{C})$. To see that it is an ideal, suppose we have morphisms $u, u'\colon A \to U$ and $v, v'\colon A \to V$ such that $(u,v) \in_A S$, $u' \leqslant u$ and $v \leqslant v'$. Then, there exists a morphism $\alpha\colon A \to S$ such that $\langle s_1, s_2\rangle\alpha = \langle u, v\rangle$. We get the factorisation $\langle r_1, r_2\rangle h\alpha = \langle fu, gv\rangle$, which shows that $(fu, gv) \in_A R$. Since $fu' \leqslant fu$ and $gv \leqslant gv'$, then $(fu', gv') \in_A R$, since R is an ideal. Consequently, there exists a morphism $\beta\colon A \to R$ such that $\langle r_1, r_2\rangle\beta = \langle fu', gv'\rangle = f \times g\langle u', v'\rangle$. The universal property of the above 2-pullback then gives a morphism $\gamma\colon A \to S$ such that $\langle s_1, s_2\rangle\gamma = \langle u', v'\rangle$. This proves that $(u', v') \in_A S$.

For the second statement, we always have $g^{\circ}Rf_{\circ} \subseteq g^*Rf_*$ by Lemma 4.3(2). For the converse, suppose that $u\colon A \to U$ and $v\colon A \to V$ are morphisms such that $(u,v) \in_A g^*Rf_*$. Applying Lemma 3.2 twice, there exist so-morphisms $b\colon B \twoheadrightarrow A$ and $c\colon C \twoheadrightarrow B$, and morphisms $x\colon B \to X$, $y\colon C \to Y$ such that $(ubc, xc) \in_C f_*$, $(xc, y) \in_C R$ and $(y, vbc) \in_C g^*$. We get $fubc \leqslant xc$ and $y \leqslant gvbc$ by (3.i); thus $(fubc, gvbc) \in_C R$, since R is an ideal. Using the fact that bc is an so-morphism (Lemma 1.4), we conclude that $(fu, gv) \in_A R$. To finish, we have $(u, fu) \in_A f_{\circ}$, $(fu, gv) \in_A R$ and $(gv, v) \in_A g^{\circ}$, which proves that $(u,v) \in_A g^{\circ}Rf_{\circ}$. □

Example 4.13. Let $\mathbb{C}$ be a regular Ord-category. Consider an ideal $S\colon U ⇸ V$ and morphisms $f\colon U \to X$, $g\colon V \to Y$. The (so-morphism, ff-monomorphism) factorisation of $f \times g\langle s_1, s_2\rangle$

$$\begin{array}{ccc} S & \overset{h}{\twoheadrightarrow} & R \\ {\scriptstyle \langle s_1,s_2\rangle}\big\downarrow & & \big\downarrow{\scriptstyle \langle r_1,r_2\rangle} \\ U \times V & \xrightarrow[f\times g]{} & X \times Y, \end{array}$$

gives a relation $R \rightarrowtail X \times Y$ which is not necessarily an ideal. In particular, when $U = V$ and $f = g$, such a factorisation gives the *direct image* of S under f. Consequently, the direct image of an ideal is not necessarily an ideal.

Consider the example where $S = I_U$, $f = 1_U$ and $g\colon U \to Y$ is any morphism in $\mathbb{C}$

$$\begin{array}{ccc} I_U & \xrightarrow{\;h\;} & R \cong g_\circ I_U \\ {\scriptstyle \langle u_1,u_2\rangle}\downarrow & & \downarrow{\scriptstyle \langle r_1,r_2\rangle} \\ U \times U & \xrightarrow[1_U \times g]{} & U \times Y. \end{array}$$

It is easy to see that $R \cong g_\circ I_U$. To have morphisms $u\colon A \to U$, $y\colon A \to Y$ such that $(u, y) \in_A g_\circ I_U$, means that there exist an so-morphism $b\colon B \twoheadrightarrow A$ and a morphism $\overline{u}\colon B \to U$ such that

$$\begin{array}{ccc} B & \overset{ub}{\underset{\overline{u}}{\rightrightarrows}}\ (\curlywedge) & U \\ {\scriptstyle b}\downarrow & \circlearrowleft & \downarrow{\scriptstyle g} \\ A & \xrightarrow[y]{} & Y; \end{array}$$

here $g\overline{u} = yb$ (see Lemma 2.2). If g is an ff-monomorphism, then the diagonal fill-in property gives a morphism $d\colon A \to U$ such that $y = gd$. As a consequence, y must factor through g. We want to show that $g_\circ I_U$ is not an ideal. Given another morphism $y'\colon A \to Y$ such that $y \leqslant y'$, there is no reason why y' should also factor through g. For example, take g to be the ff-monomorphism $\langle z_1, z_2\rangle\colon I_Z \rightarrowtail Z \times Z$ and $y = \langle y_1, y_2\rangle\colon A \to Z \times Z$. To have y factor through g implies that $(y_1, y_2) \in_A I_Z$, i.e. $y_1 \leqslant y_2$. If $y' = \langle y_1', y_2'\rangle\colon A \to Z \times Z$ is another morphism such that $y \leqslant y'$, then $y_1 \leqslant y_1'$ and $y_2 \leqslant y_2'$. This does not imply that $y_1' \leqslant y_2'$, so that y' may not factor through g.

A similar argument holds for $S = I_U$ and $f = g\colon U \to Y$. In that case $R \cong g_\circ I_U g^\circ$. To have $(u, y) \in_A g_\circ I_U g^\circ$ when g is an ff-monomorphism still implies that y factors through g. This shows that the direct image of an ideal is not necessarily an ideal.

5. Mal'tsev property in the Ord-enriched context

Recall from [6, 7, 5] that a Mal'tsev category C is defined through the property that every ordinary reflexive relation $R\colon X \nrightarrow X$ in C is an equivalence relation. The original definition asks for the base category C to be regular or just finitely complete. However, this property on ordinary reflexive relations can be stated through generalised elements without any assumption on C, as in Definition 2.1; see also (2.iii). Actually, the difunctionality of ordinary relations is another equivalent way to define a Mal'tsev category.

Definition 5.1 (see [6, 7, 5]). A category C is called a *Mal'tsev category* when every ordinary relation $D\colon X \nrightarrow Y$ is difunctional.

We recall Theorem 3.6 from [5], which gives several well-known characterisations of a regular Mal'tsev category.

Theorem 5.2. *Let C be a regular category. The following statements are equivalent, and characterise regular Mal'tsev categories:*

(i) *for any ordinary equivalence relations $R, S\colon X \nrightarrow X$ on an object X, RS is an ordinary equivalence relation on X;*
(ii) *$RS \cong SR$, for any ordinary equivalence relations $R, S\colon X \nrightarrow X$ on an object X;*
(iii) *$RS \cong SR$ for any ordinary effective equivalence relations (i.e. kernel pairs of some morphism in C) $R, S\colon X \nrightarrow X$;*
(iv) *every ordinary relation $D\colon X \nrightarrow Y$ is difunctional, i.e. $DD^\circ D \cong D$;*

(v) *every ordinary reflexive relation* $R\colon X \nrightarrow X$ *on an object* X *is an ordinary equivalence relation;*
(vi) *every ordinary reflexive relation* $R\colon X \nrightarrow X$ *on an object* X *is transitive;*
(vii) *every ordinary reflexive relation* $R\colon X \nrightarrow X$ *on an object* X *is symmetric.*

Definition 5.3. An Ord-category $\mathbb{C}$ is called an Ord-*Mal'tsev category* when every ideal $D\colon X \looparrowright Y$ satisfies the property: given morphisms $x, u, u'\colon A \to X$, $y, y', v\colon A \to Y$ such that $(x, y) \in_A D$, $y \preccurlyeq y'$, $(u, y') \in_A D$, $u \preccurlyeq u'$ and $(u', v) \in_A D$, then $(x, v) \in_A D$. This property may be pictured as

$$\begin{array}{ccc} x & D & y \\ & & \curlywedge \\ u & D & y' \\ \curlywedge & & \\ u' & D & v \\ \hline x & D & v. \end{array} \tag{5.i}$$

Proposition 5.4. *Let* C *be a Mal'tsev category. Then any* Ord-*enrichment of* C *is an* Ord-*Mal'tsev category.*

Proof. Any relation $D \rightarrowtail X \times Y$ is difunctional. If D is an ideal, with the relations given in the top part of (5.i), we get xDy' and uDv. Then

$$\begin{array}{ccc} x & D & y' \\ u & D & y' \\ u & D & v \\ \hline x & D & v \end{array}$$

from the difunctionality of D. □

If $\mathbb{C}$ is a regular Ord-category (so that we can compose relations in $\mathbb{C}$) then the property expressed in (5.i) has a similar interpretation as $DD^\circ D \cong D$, in the 1-dimensional regular context.

Proposition 5.5. *Let* $\mathbb{C}$ *be a regular* Ord-*category and consider an ideal* $\langle d_1, d_2\rangle\colon D \rightarrowtail X \times Y$ *in* $\mathbb{C}$. *Then* D *satisfies* (5.i) *if and only if* $D \cong DD^*D \cong DD^\circ D$.

Proof. We have $DD^*D \cong DD^\circ D$ from Proposition 4.11. Suppose now that D satisfies (5.i). We always have $D \subseteq DD^\circ D \subseteq DD^*D$, by Corollary 4.8. Next, consider $x\colon A \to X$ and $y\colon A \to Y$ such that $(x, y) \in_A DD^*D$. Applying Lemma 3.2 twice, there exist so-morphisms $b\colon B \twoheadrightarrow A$ and $c\colon C \twoheadrightarrow B$, and morphisms $\overline{x}\colon B \to X$, $\overline{y}\colon C \to Y$ such that $(xbc, \overline{y}) \in_C D$, $(\overline{y}, \overline{x}c) \in_C D^*$ and $(\overline{x}c, ybc) \in_C D$. Since $D^* = (d_1)_*(d_2)^*$, then there exists an so-morphism $w\colon W \twoheadrightarrow C$ and a morphism $z\colon W \to D$ such that $(\overline{y}w, z) \in_W (d_2)^*$ and $(z, \overline{x}cw) \in_W (d_1)_*$, again by Lemma 3.2. So, $\overline{y}w \preccurlyeq d_2 z$ and $d_1 z \preccurlyeq \overline{x}cw$ by (3.i). By assumption, we have

$$\begin{array}{ccc} xbcw & D & \overline{y}w \\ & & \curlywedge \\ d_1 z & D & d_2 z \\ \curlywedge & & \\ \overline{x}cw & D & ybcw \\ \hline xbcw & D & ybcw. \end{array}$$

Using the fact that bcw is an so-morphism (Lemma 1.4), we conclude that $(x, y) \in_A D$.

For the converse, suppose we have $DD^*D \subseteq D$. Consider generalised elements related as in (5.i). Using the fact that D is an ideal and Corollary 4.8, we have

$$\begin{array}{lcl} ((x,y) \in_A D \wedge y \preccurlyeq y') & \Rightarrow & (x,y') \in_A D; \\ (y',u) \in_A D^\circ \subseteq D^* & \Rightarrow & (y',u) \in D^*; \\ (u \preccurlyeq u' \wedge (u',v) \in_A D) & \Rightarrow & (u,v) \in_A D. \end{array}$$

So, $(x,v) \in_A DD^*D \subseteq D$; thus $(x,v) \in_A D$. □

Let $\mathbb{C}$ be a regular Ord-category. Next we show that the Ord-enriched version of Theorem 5.2 holds, thus giving several characterisations for regular Ord-Mal'tsev categories. To do so we must give the enriched counterpart of an ordinary (effective) equivalence relation.

Definition 5.6 ([18])**.** Let $\mathbb{C}$ be an Ord-category. An ideal $R\colon X \looparrowright X$ on an object X which is reflexive and transitive is called a *congruence* on X. A congruence R is called *effective* when $R \cong f/f \cong f^*f_*$, for some morphism $f\colon X \to Y$ in $\mathbb{C}$.

Lemma 5.7 (see [28])**.** *Let $\mathbb{C}$ be an Ord-category. An ideal $R\colon X \looparrowright X$ is reflexive if and only if $I_X \subseteq R$. Consequently, R is a congruence if and only if $I_X \subseteq R$ and R is transitive.*

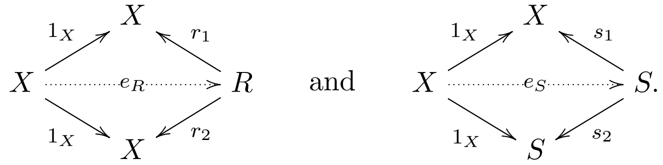
Proof. If R is reflexive, then $\Delta_X \subseteq R$. Hence, $I_X \cong I_X\Delta_X \subseteq I_XR \cong R$. For the converse, $\Delta_X \subseteq I_X \subseteq R$; thus R is reflexive (see Remark 2.5). □

Remark 5.8. The notion of a congruence $R\colon X \looparrowright X$ does not involve any sort of symmetry for R. Symmetry of R would mean that $R^\circ \cong R$, which is generally false for ideals (see Corollary 4.8). As a consequence, the Ord-enriched version of Theorem 5.2 does not include the statement (vii); also statements (v) and (vi) coincide. Actually, the symmetry of a reflexive and transitive ordinary relation comes for free when the base category is *n-permutable* [23] (see [15, 5] for the definitions of n-permutable variety and n-permutable category). This is the case of Mal'tsev categories, which are 2-permutable categories.

Theorem 5.9. *Let $\mathbb{C}$ be a regular Ord-category. Then the following statements are equivalent and characterise regular Ord-Mal'tsev categories:*

(i) *for any congruences $R, S\colon X \looparrowright X$ on an object X, RS is a congruence on X;*
(ii) *$RS \cong SR$, for any congruences $R, S\colon X \looparrowright X$ on an object X;*
(iii) *$RS \cong SR$, for any effective congruences $R, S\colon X \looparrowright X$ on an object X;*
(iv) *every ideal $X \xleftarrow{d_1} D \xrightarrow{d_2} Y$ is such that $DD^*D \cong D$;*
(v) *every reflexive ideal $R\colon X \looparrowright X$ on an object X is a congruence.*

Proof. Suppose that the reflexivity of R and S is given by the factorisations



(i) ⇒ (ii) If RS is a congruence, then it is transitive: $RSRS \subseteq RS$. We use Propositions 4.5(5) and 4.6, Lemma 4.3(3) and (4) and the fact that $(1_X)^* \cong I_X \cong (1_X)_*$ is the identity in $\mathsf{Rel}_{\mathsf{idl}}(\mathbb{C})$

to get the following

$$\begin{aligned} RSRS \subseteq RS \;&\Leftrightarrow\; (r_2)_*(r_1)^*SR(s_2)_*(s_1)^* \subseteq RS \\ &\Leftrightarrow\; (r_1)^*SR(s_2)_* \subseteq (r_2)^*RS(s_1)_* \\ &\Rightarrow\; (e_R)^*(r_1)^*SR(s_2)_*(e_S)_* \subseteq (e_R)^*(r_2)^*RS(s_1)_*(e_S)_* \\ &\Leftrightarrow\; (r_1e_R)^*SR(s_2e_S)_* \subseteq (r_2e_R)^*RS(s_1e_S)_* \\ &\Leftrightarrow\; SR \subseteq RS. \end{aligned}$$

Similarly, we can obtain $RS \subseteq SR$; thus $RS \cong SR$.
(ii) $\Rightarrow$ (iii) This implication is obvious.
(iii) $\Rightarrow$ (iv) We have $DD^*D \cong (d_2)_*(d_1)^*(d_1)_*(d_2)^*(d_2)_*(d_1)^*$, by Proposition 4.6 and the definition of D^*. Since $(d_1)^*(d_1)_*$ and $(d_2)^*(d_2)_*$ are effective congruences, their composition commutes. We get $DD^*D \cong (d_2)_*(d_2)^*(d_2)_*(d_1)^*(d_1)_*(d_1)^* \cong (d_2)_*(d_1)^* \cong D$ (see Lemma 4.3(8)).
(iv) $\Rightarrow$ (v) If R is a reflexive relation, then so is R^*; thus $I_X \subseteq R^*$, by Lemma 5.7. We have to prove that R is transitive: $RR \cong RI_XR \subseteq RR^*R \cong R$.
(v) $\Rightarrow$ (i) Since R and S are reflexive, then so is the composite RS. By assumption, RS is a congruence. ☐

6. Examples of Ord-Mal'tsev categories

Example 6.1. Any Ord-enrichment of a Mal'tsev category is an Ord-Mal'tsev category by Proposition 5.4. In particular, the varieties of (abelian) groups, rings, modules over a ring, Boolean algebras, Heyting algebras are such. As non-varietal examples, we have the dual of any elementary (pre)topos or the category of topological groups. These (and more) examples can be found in [5, 7, 2].

Example 6.2. Let Mon_{lc} denote the category of monoids with *left cancellation*. We use additive notation to denote such monoids, even though they are not necessarily abelian. By left cancellation we mean: $a + b = a + c \Rightarrow b = c$, for any elements a, b, c. It is easy to check that Mon_{lc} is not a Mal'tsev category. For example, the ordinary relation $\leqslant$ defined on $\mathbb{N}_0$ is not difunctional:

$$\begin{array}{ccc} 7 & \leqslant & 8 \\ 5 & \leqslant & 8 \\ 5 & \leqslant & 6 \end{array}$$

although $7 \nleqslant 6$. However, Mon_{lc} admits an Ord-enrichment that makes it an Ord-Mal'tsev category. We shall consider on each $\mathsf{Mon}_{lc}(X, Y)$ the preorder defined by:

$$f \preccurlyeq g \Leftrightarrow \forall x \in X, \exists(!) y_x \in Y : f(x) + y_x = g(x). \tag{6.i}$$

Note that:

- from the left cancellation property such y_x is unique, for each $x \in X$;
- the zero morphism $0 \colon X \to Y$ is such that $0 \preccurlyeq f$, for any $f \colon X \to Y$.

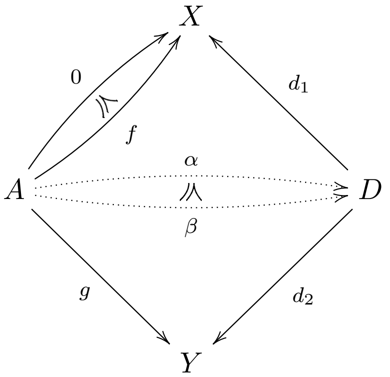
We denote this Ord-category by $\mathbb{M}\mathsf{on}_{lc}$.

Let $\langle d_1, d_2\rangle \colon D \rightarrowtail X \times Y$ be an ideal in $\mathbb{M}\mathsf{on}_{lc}$ and suppose we have morphisms $f, h, h' \colon A \to X$ and $g, g', k \colon A \to Y$ such that

$$\begin{array}{ccc} f & D & g \\ & & \curlyeqsucc \\ h & D & g' \\ \curlyeqsucc & & \\ h' & D & k \end{array} \tag{6.ii}$$

We want to prove that fDk. From $(f, g) \in_A D$ and $0 \preccurlyeq f$, then $(0, g) \in_A D$, since D is an ideal. Since $\langle d_1, d_2 \rangle$ is an ff-(mono)morphism, and given the factorisations

(6.iii)



(where $\langle d_1, d_2 \rangle \alpha = \langle 0, g \rangle$ and $\langle d_1, d_2 \rangle \beta = \langle f, g \rangle$), we conclude that $\alpha \preccurlyeq \beta$. By (6.i) this means that, for any element a of A, there exists an element δ_a in D such that $\alpha(a) + \delta_a = \beta(a)$. Let $\delta_a = (\delta_a^1, \delta_a^2)$, for each a. We obtain

$$\begin{cases} 0 + \delta_a^1 = f(a) \\ g(a) + \delta_a^2 = g(a). \end{cases}$$

It follows that $\delta_a^1 = f(a)$ and $\delta_a^2 = 0$ (by left cancellation), for each a. Consequently, $(f(a), 0) \in D$, for any element a in A. On the other hand, $(h', k) \in_A D$ and $0 \preccurlyeq h'$ gives $(0, k) \in_A D$, since D is an ideal. Then, $(0, k(a)) \in D$, for any element a in A. Since D is a submonoid of $X \times Y$, we conclude that $(f(a), 0) + (0, k(a)) = (f(a), k(a)) \in D$, for any element a in A. This proves that fDk, as desired.

We could also consider the preorder by $f \preccurlyeq g \Leftrightarrow \forall x \in X, \exists(!) y_x \in Y : f(x) = g(x) + y_x$. In that case $f \preccurlyeq 0$ for any morphism f. Similar arguments show that the category of monoids with right cancellation is an Ord-Mal'tsev category.

Example 6.3. Let GMon denote the category of gregarious monoids. A monoid $(X, +, 0)$ is called *gregarious* when:

$$\forall x \in X, \exists u_x, v_x \in X : u_x + x + v_x = 0.$$

Again, we use additive notation although the monoid is not necessarily abelian. We show that GMon is not a Mal'tsev category. Consider a monoid M generated by two elements x and y, which satisfy $x + y = 0$. It follows that $M = \{my + nx : m, n \in \mathbb{N}_0\}$. This gives an example of a gregarious monoid which is not a group. It is gregarious since

$$\forall my + nx \in M, \exists mx, ny \in M : mx + (my + nx) + ny = 0.$$

It is not a group since $y + x$, for instance, has no symmetric (see [2, Example 1.9.4]). We give an example of an ordinary relation D on M which is not difunctional. This example is due to Andrea Montoli. Consider the submonoid

$$D = \{(my + nx, my + nx) : m, n \in \mathbb{N}_0\} \cup \{(my + nx, (m+1)y + (n+1)x) : m, n \in \mathbb{N}_0\}$$

of $M \times M$. It is easy to check that it is a gregarious monoid, so that $D \rightarrowtail M \times M$ is indeed an ordinary relation in GMon. It is not difunctional:

$$\begin{array}{ccc} x & D & 2x \\ 2x & D & 2x \\ 2x & D & 3x \end{array}$$

although $x \not{D}\, 3x$.

We consider the same Ord-enrichment of Example 6.2; let $\mathbb{G}\mathsf{Mon}$ denote this Ord-category. We show that $\mathbb{G}\mathsf{Mon}$ is an Ord-Mal'tsev category next. Let $D\colon X \looparrowright Y$ be an ideal in $\mathbb{G}\mathsf{Mon}$ and suppose we have morphisms $f, h, h'\colon A \to X$ and $g, g', k\colon A \to Y$ such that the relations in (6.ii) hold. Since $(f, g) \in_A D$, then $(f(a), g(a)) \in D$, for all elements a in A. Being gregarious, there exist elements $u_a, v_a \in A$ such that $u_a + a + v_a = 0$, for each a. As in Example 6.2, we also know that $(0, g) \in_A D$. So, each $(0, g(u_a)), (0, g(v_a)) \in D$. We deduce that

$$(0, g(u_a)) + (f(a), g(a)) + (0, g(v_a)) = (f(a), g(u_a + a + v_a)) = (f(a), 0) \in D, \ \ \forall a \in A.$$

Using arguments similar to those of the final part of Example 6.2 we conclude that fDk.

Example 6.4. Consider the category OrdGrp of preordered groups and monotone group homomorphisms. Recall that a preordered group is a (not necessarily abelian) group $(X, +, 0)$ equipped with a preorder $\leqslant$ such that the group operation is monotone

$$x \leqslant y, u \leqslant v \ \ \Rightarrow \ \ x + u \leqslant y + v,$$

for any elements $x, y, u, v \in X$; their morphisms are the monotone group homomorphisms. Note that the preorder of a group $(X, +, 0)$ is completely determined by its *positive cone*, which is the submonoid of X, closed under conjugation, given by its positive elements, $P_X = \{x \in X : 0 \leqslant x\}$.

It was shown in [11] that OrdGrp is not a Mal'tsev category. However, we shall consider an Ord-enrichment for OrdGrp that makes it an Ord-Mal'tsev category. A similar study was done concerning the protomodularity and suitable Ord-enriched version of protomodularity for ordered (abelian) groups – see [12].

In OrdGrp the pointwise preorder on morphisms trivialises; that is, if one defines, for morphisms $f, g\colon X \to Y$, $f \leqslant g$ if, for all $x \in X$, $f(x) \leqslant g(x)$, then also $f(-x) \leqslant g(-x)$, and consequently, $\leqslant$ is symmetric. That is, $f \leqslant g$ only if $f \sim g$. The proof that the Ord-category given by this preorder is not an Ord-Mal'tsev category uses arguments similar to those used to prove that OrdGrp is not a Mal'tsev category.

We consider now the pointwise order restricted to positive elements, and define, for morphisms $f, g\colon X \to Y$ of OrdGrp,

$$f \leqslant g \ \ \Leftrightarrow \ \ \forall x \in P_X, f(x) \leqslant g(x). \tag{6.iv}$$

It is straightforward to check that (pre)composition preserves the preorder of $\mathsf{OrdGrp}(X, Y)$, for any preordered groups X and Y, and so this defines an Ord-category $\mathbb{O}\mathsf{rdGrp}$. As for the previous examples, we also have $0 \leqslant f$, for any morphism f in $\mathbb{O}\mathsf{rdGrp}$.

Any ideal $D\colon X \looparrowright Y$ in $\mathbb{O}\mathsf{rdGrp}$ is such that $D \cong X \times Y$, as a group. Indeed, consider the ordered group $A = (X \times Y, \{(0, 0)\})$ with only one positive element $(0, 0)$. The morphisms $0_X\colon X \times Y \to X$ and $0_Y\colon X \times Y \to Y$ are such that $(0_X, 0_Y) \in_A D$. The product projections $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ are such that $\pi_X \leqslant 0_X$ and $0_Y \leqslant \pi_Y$. Since D is an ideal, we conclude that $(\pi_X, \pi_Y) \in_A D$, i.e. $D \cong X \times Y$. Consequently, the relation D in $\mathbb{O}\mathsf{rdGrp}$ is given by the identity group homomorphism on $X \times Y$, which is also a monotone map

$$(D \cong X \times Y, P_D) \xrightarrow{1_{X \times Y}} (X \times Y, P_{X \times Y} = P_X \times P_Y).$$

Suppose we have morphisms $f, h, h'\colon (A, P_A) \to (X, P_X)$ and $g, g', k\colon (A, P_A) \to (Y, P_Y)$ such that the relations in (6.ii) hold. We want to prove that fDk. There is always a group homomorphism $\langle f, k\rangle\colon A \to D \cong X \times Y$. To have fDk, this group homorphism must also be a monotone map, i.e. for any positive element $a \in P_A$, we must prove that $(f(a), k(a)) \in P_D$.

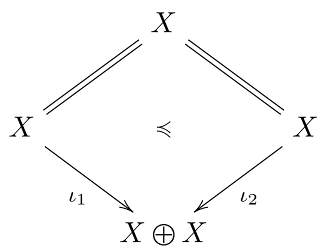
From diagram (6.iii) of Example 6.2, we know that $\langle 0, g\rangle \preccurlyeq \langle f, g\rangle$. This means that, for all positive elements $a \in P_A$, $(0, g(a)) \preccurlyeq (f(a), g(a))$; it follows that $(f(a), 0) \in P_D$, for each $a \in P_A$. We also know that $(0, k) \in_{(A,P_A)} D$, from which we conclude that $(0, k(a)) \in P_D$ for each $a \in P_A$. Since P_D is a submonoid of D, we get $(f(a), 0) + (0, k(a)) = (f(a), k(a)) \in P_D$, for each $a \in P_A$.

Note that in the three previous examples one does not use all the assumptions of (6.ii). From the definition of the preorder $\preccurlyeq$, we deduce a very strong property: $0 \preccurlyeq f$, for any $f\colon X \to Y$. This key property practically solves the issue on its own.

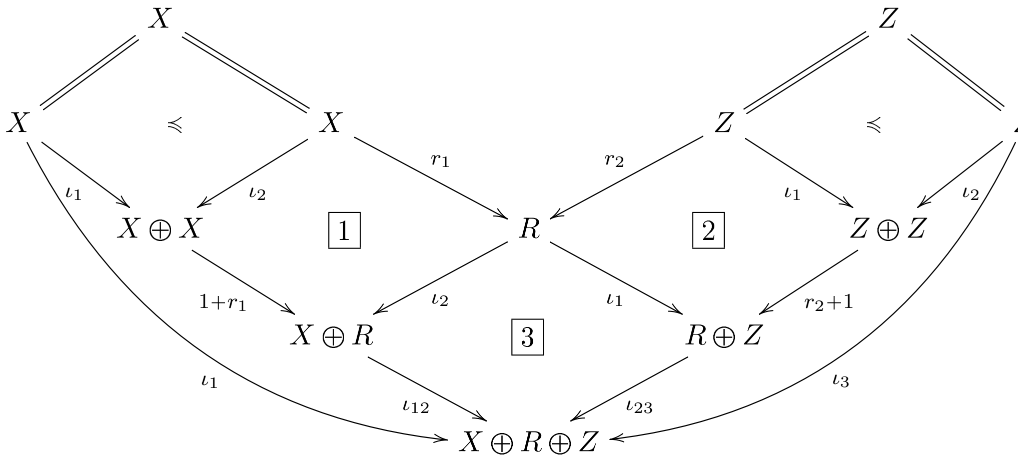
Example 6.5. We recall from [16, Corollary 5.1] (see also [21]) that a category with pullbacks and equalisers is *weakly Mal'tsev* if and only if every strong ordinary relation is difunctional. By strong ordinary relation we mean an ordinary relation $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$ such that (r_1, r_2) is jointly strongly monomorphic. Hence a weakly Mal'tsev category with an Ord-enrichment such that every ff-monomorphism is strong is automatically an Ord-Mal'tsev category.

Example 6.6. As shown in [22, Proposition 3], the category $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$, for a fixed unital and integral quantale $V = (V, \preccurlyeq, \otimes, k)$, is weakly Mal'tsev. It is also a quasivariety (see [25, 10]), hence in particular it is a regular category. Moreover, we have shown in [13] that the full subcategory of symmetric $V_\wedge$-categories is a Mal'tsev category. The category $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$ has a natural Ord-enrichment given, for every V-functor $f\colon X \to Y$, by $f \preccurlyeq g$ if, for all $x \in X$, $Y(f(x), g(x)) = k$. It is easy to check that ff-monomorphisms $f\colon X \to Y$ in $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$ are exactly surjective V-functors $Y \to X$, while a strong monomorphism is a final surjection, so that $X(x, x') = \bigvee \{Y(y, y'); y \in f^{-1}(x), y' \in f^{-1}(x')\}$. Therefore ff-monomorphisms do not need to be strong, and so one cannot conclude that $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$ is an Ord-Mal'tsev category. Indeed, using the results of the Appendix we show next that a V-category is an Ord-W-Mal'tsev object in $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$ if and only if it is a symmetric $V_\wedge$-category, showing this way that $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$ is not an Ord-Mal'tsev category.

First of all we should note that $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$ is a regular Ord-category. Here, in order to calculate R_* for a given relation R, we need to build the cocomma object of the identities on an object X:



It is straightforward to check that $X \oplus X$ has as underlying set $X + X = X \times \{1\} \cup X \times \{2\}$, where $X \oplus X((x, i), (x', j))$ is either $X(x, x')$ if $i \preccurlyeq j$, or $\bot$ (if $i = 2$ and $j = 1$). Hence, given a relation $X \xleftarrow{r_1} R \xrightarrow{r_2} Z$, R_* is defined by the following diagram



where $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ are pushouts. That is, the underlying set of $X \oplus R$ is $X + R$, with

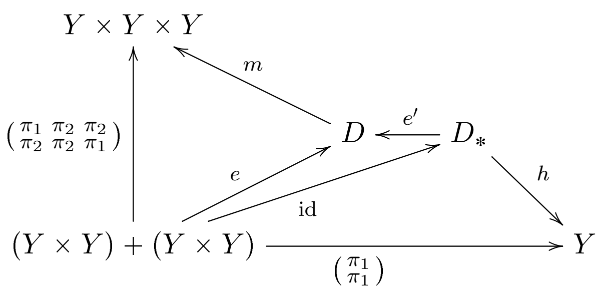
$$(X \oplus R)(w, w') = \begin{cases} X(w, w') & \text{if } w, w' \in X \\ R(w, w') & \text{if } w, w' \in R \\ R(r_1(w), w') & \text{if } w \in X,\ w' \in R \\ \bot & \text{if } w \in R, w' \in X. \end{cases}$$

Likewise for $R \oplus Z$ and $X \oplus R \oplus Z$. Now R_* is obtained via the (epimorphism, strong monomorphism)-factorisation of the morphism $\left(\begin{smallmatrix}\iota_1\\ \iota_3\end{smallmatrix}\right) : X + Z \to X \oplus R \oplus Z$. Therefore the underlying set of R_* is $X + Z$, with $R_*(w, w') = (X \oplus R \oplus Z)(w, w')$; that is,

$$R_*(w, w') = \begin{cases} X(w, w') & \text{if } w, w' \in X \\ Z(w, w') & \text{if } w, w' \in Z \\ R(r_1(w), r_2(w')) & \text{if } w \in X,\ w' \in Z \\ \bot & \text{if } w \in Z,\ w' \in X. \end{cases}$$

Proposition 6.7. *A V-category is an* Ord-*W-Mal'tsev object in* $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$ *if and only if it is a symmetric $V_\wedge$-category.*

Proof. To proof the claim we will make use of Proposition A.4. Let Y be a V-category and D be the relation defined in (A.i). Our aim is to check under which conditions the map $h \colon D_* \to Y$ making the following diagram commute, so that $h = \left(\begin{smallmatrix}\pi_1\\ \pi_1\end{smallmatrix}\right)$, is a V-functor:



Here $e'(y_1, y_2) = e(y_1, y_2) = (y_1, y_2, y_2)$ if (y_1, y_2) belongs to the first summand, and $e'(y_1, y_2) = e(y_1, y_2) = (y_2, y_2, y_1)$ if (y_1, y_2) belongs to the second one. Then h is a V-functor if and only if, for all $(y_1, y_2), (y_1', y_2')$ in $(Y \times Y) + (Y \times Y)$,

$$D_*((y_1, y_2), (y_1', y_2')) \leqslant Y(y_1, y_1'). \tag{6.v}$$

When (y_1, y_2) belongs to the first summand and (y_1', y_2') belongs to the second one this means, since $D_*((y_1, y_2), (y_1', y_2')) = D((y_1, y_2, y_2), (y_2', y_2', y_1'))$,

$$Y(y_1, y_2') \wedge Y(y_2, y_2') \wedge Y(y_2, y_1') \leqslant Y(y_1, y_1'). \tag{6.vi}$$

Taking $y_1 = y_2'$ this inequality translates to

$$Y(y_2, y_1) \wedge Y(y_2, y_1') \leqslant Y(y_1, y_1'),$$

which is equivalent to Y being a symmetric $V_\wedge$-category (see [13, Theorem 2.4]).

Conversely, to show that h is a V-functor provided that Y is a symmetric $V_\wedge$-category, we note that the inequality (6.v) is trivially satisfied in all cases but when (y_1, y_2) belongs to the first summand and (y_1', y_2') belongs to the second one. In this case we have to show that (6.vi) holds, for all $y_1, y_2, y_1', y_2' \in Y$: using first symmetry and then transitivity of the $V_\wedge$-category Y we obtain:

$$Y(y_1, y_2') \wedge Y(y_2, y_2') \wedge Y(y_2, y_1') = Y(y_1, y_2') \wedge Y(y_2', y_2) \wedge Y(y_2, y_1') \leqslant Y(y_1, y_1').$$

□

Appendix A. An object-wise approach to Ord-Mal'tsev categories

The authors of [24] explored several algebraic categorical notions at an object-wise level. One of those was the notion of Mal'tsev object. Their approach was inspired on the classification properties of the fibration of points studied in [3]. Independently, the author of [29] used the characterisation of a Mal'tsev category obtained through the difunctionality of ordinary relations to introduce a definition of Mal'tsev object (recall Definition 5.1 and Theorem 5.2). A comparison between both notions may be found in [13], where a Mal'tsev object in the sense of [29] was called a "W-Mal'tsev object"; we keep that designation in this work.

Definition A.1 ([29])**.** An object Y of a category C is called a *W-Mal'tsev object* when for every ordinary relation $X \xleftarrow{r_1} R \xrightarrow{r_2} Z$ in C, the Set-relation

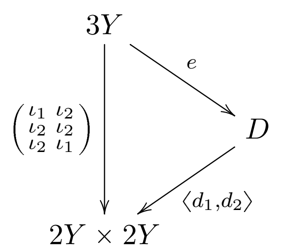
$$\mathsf{C}(Y, X) \xleftarrow{\mathsf{C}(Y, r_1)} \mathsf{C}(Y, R) \xrightarrow{\mathsf{C}(Y, r_2)} \mathsf{C}(Y, Z)$$

is difunctional.

It follows from Definitions 5.1 and A.1 that a category C is a Mal'tsev category if and only if all of its objects are W-Mal'tsev objects.

This definition does not impose any kind of assumption on the base category C. However, if C is regular and admits binary coproducts, the definition of a W-Mal'tsev object becomes easier to handle. Indeed, it allows the replacement of a property on all ordinary relations by a property on a specific ordinary relation defined on coproducts. As usual, for an object Y, we write $Y + Y = 2Y$, $Y + Y + Y = 3Y$, and $\iota_j \colon Y \to kY$ for the j-th coproduct coprojection.

Proposition A.2 ([29])**.** *Let* C *be a regular category with binary coproducts. An object* Y *is a W-Mal'tsev object in* C *if and only if, given the (regular epimorphism, monomorphism) factorisation in* C



(which guarantees that $(\iota_1, \iota_2) \in_Y D$, $(\iota_2, \iota_2) \in_Y D$, $(\iota_2, \iota_1) \in_Y D$*), we have* $(\iota_1, \iota_1) \in_Y D$.

We present the Ord-enriched versions of this approach next.

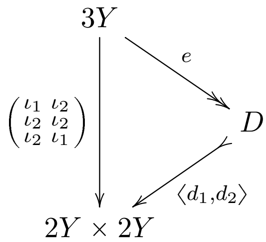
Definition A.3. An object Y of an Ord-category $\mathbb{C}$ is called an Ord-*W-Mal'tsev object* when every ideal $R\colon X \looparrowright Z$ satisfies

$$\begin{array}{ccc} x & R & z \\ & & \rotatebox{90}{$\leq$} \\ u & R & z' \\ \rotatebox{90}{$\leq$} & & \\ u' & R & v \\ \hline x & R & v \end{array}$$

for any generalised elements $x, u, u' \colon Y \to X$, $z, z', v\colon Y \to Z$.

It follows from Definitions 5.3 and A.3 that *an* Ord*-category* $\mathbb{C}$ *is an* Ord*-Mal'tsev category if and only if all of its objects are* Ord*-W-Mal'tsev objects.*

Proposition A.4. *Let* $\mathbb{C}$ *be a regular* Ord*-category with binary coproducts. An object* Y *is an* Ord*-W-Mal'tsev object in* $\mathbb{C}$ *if and only if, given the (so-morphism, ff-monomorphism) factorisation in* $\mathbb{C}$

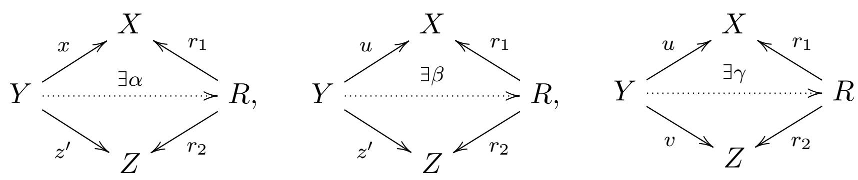
(A.i) 

(which guarantees that $(\iota_1, \iota_2) \in_Y D$, $(\iota_2, \iota_2) \in_Y D$, $(\iota_2, \iota_1) \in_Y D$*), we have* $(\iota_1, \iota_1) \in_Y D_*$.

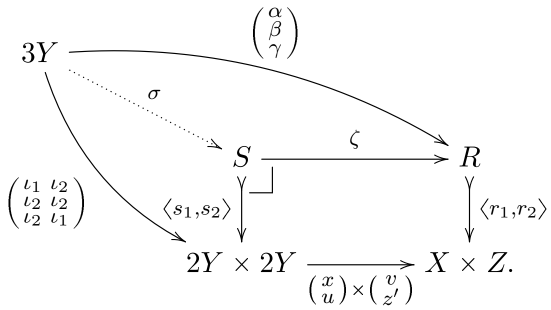
Proof. Suppose that Y is an Ord-W-Mal'tsev object. Diagram (A.i) tells us that $\iota_1 D \iota_2$, $\iota_2 D \iota_2$ and $\iota_2 D \iota_1$. We also know that $D \subseteq D_*$ by Corollary 4.8. We get

$$\begin{array}{ccc} \iota_1 & D_* & \iota_2 \\ & & \rotatebox{90}{$\leq$} \\ \iota_2 & D_* & \iota_2 \\ \rotatebox{90}{$\leq$} & & \\ \iota_2 & D_* & \iota_1 \\ \hline \iota_1 & D_* & \iota_1 \end{array}$$

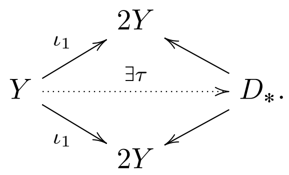
For the converse, consider an ideal $R\colon X \looparrowright Z$ and the relations as in Definition A.3. Since R is an ideal, we get $(x, z') \in_Y R$, and $(u, v) \in_Y R$; we have induced morphisms



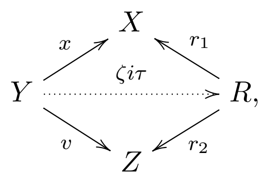
Now, consider the 2-pullback and the induced morphism in



From the (so-morphism,ff-monomorphism) factorisation (A.i), it follows that $D \subseteq S$. By Proposition 4.7, we get $D_* \subseteq S$; let i be the inclusion morphism $i\colon D_* \to S$. By assumption, we have $(\iota_1, \iota_1) \in_Y D_*$, meaning that there exists a factorisation



Finally, we get



which proves that $(x, v) \in_Y R$. □

References

[1] M. Barr, P.A. Grillet, D.H. van Osdol, *Exact categories and categories of sheaves*, Lecture Notes in Math. 236, Springer-Verlag (1971).

[2] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Math. Appl., vol. 566, Kluwer Acad. Publ., 2004.

[3] D. Bourn, *Mal'cev categories and fibration of pointed objects*, Appl. Categ. Structures **4** (1996) 307–327.

[4] D.A. Buchsbaum, *Exact categories and duality*, Trans. Amer. Math. Soc. **80** (1955) 1–34.

[5] A. Carboni, G. M. Kelly, M. C. Pedicchio, *Some remarks on Maltsev and Goursat categories*, Appl. Categ. Struct. 1**4** (1993) 385–421.

[6] A. Carboni, J. Lambek, M. C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra **69** (1991) 271–284.

[7] A. Carboni, M. C. Pedicchio, N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, Am. Math. Soc. for the Canad. Math. Soc., Providence, 1992, pp. 97–109.

[8] A. Carboni, R. Street, *Order ideals in categories*, Pacific Journal of Mathematics **124** No. 2 (1986) 275–288.

[9] M. M. Clementino, *An invitation to topological semi-abelian algebras.* In: New perspectives in Algebra, Topology and Categories, Coimbra Mathematical Texts, Vol. 1, Springer Nature and University of Coimbra (2021) 27–66.

[10] M. M. Clementino, C. Fitas, D. Hofmann, *A variety of co-quasivarieties*, arXiv 2404.05017

[11] M. M. Clementino, N. Martins-Ferreira, A. Montoli, *On the categorical behaviour of preordered groups*, J. Pure Appl. Algebra **223** (2019) 4226–4245.

[12] M. M. Clementino, A. Montoli, D. Rodelo, *On lax protomodularity for* Ord*-enriched categories*, J. Pure Appl. Algebra 227(8) (2023) 107348.

[13] M.M. Clementino, D. Rodelo, *A note on Mal'tsev objects*, Portugaliae Mathematica, to appear. arXiv 2311.17628.

[14] P. Freyd and A. Scedrov. *Categories, Allegories*, Elsevier, 1990.

[15] J. Hagemann, A. Mitschke, *On n-permutable congruences*, Algebra Universalis **3** (1973) 8–12.

[16] Z. Janelidze, N. Martins-Ferreira, *Weakly Mal'tsev categories and strong relations*, Theory Appl. Categ. **27** No.5 (2012) 65–79.

[17] A. Kurz, A. Moshier, A. Jung, *Stone Duality for Relations*, in Samson Abramsky on Logic and Structure in Computer Science and Beyond, Springer-Verlag (2023) 159–215.

[18] A. Kurz, J. Velebil, *Quasivarieties and varieties of ordered algebras: regularity and exactness*, Math. Struct. Comput. Sci. **27** (2017) 1153–1194.

[19] J. Lambek, *Goursat's theorem and the Zassenhaus lemma*, Canad. J. Math. **10** (1957) 45–56.

[20] A.I. Mal'tsev, *On the general theory of algebraic systems*, Mat. Sbornik N. S. **35**(77) (1954) 3–20.

[21] N. Martins-Ferreira, *Weakly Mal'tsev categories*, Theory Appl. Categ. **21**(6) (2008) 91–117.

[22] N. Martins-Ferreira, *New wide classes of weakly Mal'tsev categories*, Appl. Categ. Structures **23** (2015) 741–751.

[23] N. Martins-Ferreira, D. Rodelo, T. Van der Linden, *An observation on n-permutability*, Bull. Belg. Math. Soc. Simon Stevin **21** No. 2 (2014) 223–230.

[24] A. Montoli, D. Rodelo, T. Van der Linden, *Two characterisations of groups amongst monoids*, J. Pure Appl. Algebra **222** (2018) 747–777.

[25] Q. Pu and D. Zhang, *Categories enriched over a quantaloid: Algebras*, Theory Appl. Categ. **30**No. 21 (2015) 751–774.

[26] J. Riguet, *Relations binaires, fermetures, correspondances de Galois*, Bull. Soc. Math. France **76** (1948) 114-155.

[27] J. D. H. Smith, *Mal'cev varieties*, Lecture Notes in Math., vol. 554, Springer, 1976.

[28] V. Aravantinos-Sotiropoulos, *The exact completion for regular categories enriched in posets*, J. Pure Appl. Algebra 226(7) (2022) 106885.

[29] T. Weighill, *Mal'tsev objects, R_1-spaces and ultrametric spaces*, Theory Appl. Categ. **32** (2017), no. 42, 1485–1500.

CMUC, Department of Mathematics, University of Coimbra, 3000-143 Coimbra, Portugal
Email address: mmc@mat.uc.pt

Department de Mathematics, University of the Algarve, 8005-139 Faro, Portugal and CMUC, Department of Mathematics, University of Coimbra, 3000-143 Coimbra, Portugal
Email address: drodelo@ualg.pt