EFFECTIVE DESCENT MORPHISMS OF ORDERED FAMILIES

MARIA MANUEL CLEMENTINO AND RUI PREZADO

ABSTRACT. We present a characterization of effective descent morphisms in the lax comma category $\mathsf{Ord}//X$ when X is a locally complete ordered set with a bottom element.

INTRODUCTION

The role of lax comma 2-categories in [7], where the authors study properties of the lax change-of-base functor in the realm of Janelidze's Galois theory [9, 1] led Lucatelli Nunes and the first named author of this note to study the behaviour of the lax comma category $\mathsf{Ord}//X$ of ordered sets over a fixed ordered set X, in [6]. Objects of $\mathsf{Ord}//X$ are ordered sets A equipped with a monotone map $\alpha\colon A \to X$, which assigns to each element of A an X-value, and a morphism $f\colon (A,\alpha) \to (B,\beta)$ is a monotone map that does not increase X-values, so that $\alpha(a) \leqslant \beta(f(a))$ for every $a \in A$.

In particular, a study of the effective descent morphisms in $\mathsf{Ord}//X$ was carried out in [6], when X is a complete ordered set, locating them between two well-known classes of monotone maps, as stated in Theorem 1.3. Subsequently, these results were refined in [5], extending them to the case when X is locally complete (Theorem 1.4).

In this note, we obtain a complete characterization of the effective descent morphisms in $\mathsf{Ord}//X$ when X is locally complete, that is, $\downarrow x$ is complete for every $x \in X$, and has a bottom element. This is accomplished by reducing the problem to the study of effective descent morphisms in Ord – which were characterized in [10] – and in $\mathsf{Fam}(X)$ – which were characterized by the second author in [17].

We begin by recalling the necessary descent theoretical background, and by giving an overview of previously obtained results on effective descent morphisms in $\mathsf{Ord}//X$ in the prequels [6, 5].

In particular, it is well-understood that $\mathsf{Ord}//X \to \mathsf{Ord}$ preserves effective descent morphisms when X has a bottom element. Our main observation is that we can complete the characterization via effective descent conditions on morphisms in the category $\mathsf{Fam}(X)$. Thus, we recount the relevant details about such morphisms from [17], framed in our context. We also revisit the characterization of stable regular epimorphisms in $\mathsf{Ord}//X$ from [6] from the perspective of the work carried out in [17].

We conclude the paper by stating and proving our main result (Theorem 3.1), where we characterize the effective descent morphisms in $\mathsf{Ord}//X$ when X has a bottom element and is locally complete, that is, $\downarrow x$ is complete for all x.

Date: 11th July 2024.

2020 *Mathematics Subject Classification.* 06A07,18A25,18A30,18B35,18B50,18D30.

Key words and phrases. effective descent morphisms, Grothendieck descent theory, stable regular epimorphisms, lax comma categories, ordered families, free coproduct completion.

The authors acknowledge partial financial support by *Centro de Matemática da Universidade de Coimbra* (CMUC), funded by the Portuguese Government through FCT/MCTES, DOI 10.54499/UIDB/00324/2020.

1. State-of-the-art

In a category A with pullbacks, any morphism $p\colon A \to B$ induces a functor $p^*\colon \mathsf{A}/B \to \mathsf{A}/A$, by taking pullbacks along p. This functor has a left adjoint $p_!$, and this induces a monad T^p, so we may consider the factorization of p^* through the category of T^p-algebras (the *Eilenberg-Moore* factorization):

$$\begin{array}{ccc} \mathsf{A}/B & \xrightarrow{\quad p^* \quad} & \mathsf{A}/A \\ & {\scriptstyle K^p}\searrow \qquad \nearrow & \\ & T^p\text{-}\mathsf{Alg} & \end{array} \tag{1.i}$$

By the Bénabou-Roubaud theorem [2], the factorization (1.i) coincides with the *descent factorization* [13, 16] of p – a result which allows the aptly called the *monadic description* of descent [12].

We say that

- p is a descent morphism if K^p is fully faithful,
- p is an effective descent morphism if K^p is an equivalence.

In a category A with finite limits, the descent morphisms are exactly (pullback-)stable regular epimorphisms, which coincide with the effective descent morphisms when A is Barr-exact or locally cartesian closed (see [11] for details).

However, in an arbitrary category A with pullbacks, the identification of effective descent morphisms may be quite challenging – a notorious example is the characterization of effective descent morphisms in the category Top of topological spaces [19, 3].

A fruitful strategy to understand effective descent morphisms in an arbitrary category A with pullbacks is to find a category D with pullbacks for which the effective descent morphisms are well-understood, and a suitable embedding $F\colon \mathsf{A} \to \mathsf{D}$. Then, we may apply the following classical result:

Theorem 1.1. *Let A and D be categories with pullbacks, and $F\colon \mathsf{A} \to \mathsf{D}$ a fully faithful, pullback preserving functor. If $f\colon A \to B$ is a morphism in A such that $F(f)$ is effective for descent in D, then the following conditions are equivalent:*

(i) f is an effective descent morphism in A;
(ii) for every pullback diagram of the form

$$\begin{array}{ccc} F(C) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow {\scriptstyle g} \\ F(A) & \xrightarrow[F(f)]{} & F(B) \end{array}$$

we have $E \cong FD$ for some D in A.

This technique was used in [10] by G. Janelidze and M. Sobral to obtain the characterization of effective descent morphisms in the category Ord of *ordered sets* (that is, sets with a reflexive and transitive relation) and *monotone maps*:

Theorem 1.2 ([10])**.** *Given a morphism $f\colon A \to B$ in Ord:*

(1) f is a descent morphism, or, equivalently, a stable regular epimorphism, if:

$$\forall b_0 \leqslant b_1 \text{ in } B,\ \exists a_0 \leqslant a_1 \text{ in } A \ :\ f(a_0) = b_0,\ f(a_1) = b_1;$$

(2) f is an effective descent morphism if:

$$\forall b_0 \leqslant b_1 \leqslant b_2 \text{ in } B,\ \exists a_0 \leqslant a_1 \leqslant a_2 \text{ in } A \ : \ f(a_0) = b_0,\ f(a_1) = b_1,\ f(a_2) = b_2.$$

Moreover, Theorem 1.1 is also used in [6] and [5] to study the effective descent morphisms in $\mathsf{Ord}//X$. This result is also featured in the present note.

We note that, while the characterizations of Theorem 1.1 extend naturally to the comma categories Ord/X via the equivalence

$$(\mathsf{Ord}/X)/(B,\beta) \simeq \mathsf{Ord}/B,$$

this is not the case of the lax comma category $\mathsf{Ord}//X$, of which Ord/X is a wide subcategory (i.e. with the same objects but fewer morphisms).

In [6], the authors make use of Theorem 1.1 and of the fact that every monotone map $\alpha\colon A \to X$ induces naturally a functor $\Pi(\alpha)\colon X^{\mathsf{op}} \to \mathsf{Ord}$, so that a morphism $f\colon (A,\alpha) \to (B,\beta)$ induces a natural transformation $\Pi(\alpha) \to \Pi(\beta)$. Indeed:

- for a complete ordered set X, one defines an embedding

  $$\mathsf{Ord}//X \xrightarrow{\ \Pi\ } [X^{\mathsf{op}}, \mathsf{Ord}]$$

  with $\Pi(A,\alpha)(x) = \{a \in A\,;\, x \leqslant \alpha(a)\}$ and $\Pi(f)$ given by the (co)restriction of f to $\Pi(A,\alpha)(x) \to \Pi(B,\beta)(x)$: from $\alpha \leqslant \beta f$ it follows that if $x \leqslant \alpha(a)$ then $x \leqslant \beta(f(a))$;
- in $[X^{\mathsf{op}}, \mathsf{Ord}]$ a natural transformation $\eta\colon F \to G$ is effective for descent if and only if it is pointwise effective for descent, that is: for every $x \in X$, the monotone map $\eta_x\colon F(x) \to G(x)$ is effective for descent in Ord.

Theorem 1.3 ([6]). *Let X be a complete ordered set. Given a morphism $f\colon (A,\alpha) \to (B,\beta)$ in $\mathsf{Ord}//X$, consider the following conditions:*

(1) $f\colon A \to B$ and all $f_x\colon A_x \to B_x$ are effective descent morphisms in Ord;
(2) $f\colon (A,\alpha) \to (B,\beta)$ is effective for descent in $\mathsf{Ord}//X$;
(3) $f\colon A \to B$ is effective for descent in Ord.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Subsequently, in [5] the authors use the fact that every monotone map $\alpha\colon A \to X$ naturally defines a family $(A_x)_{x\in X}$ of subsets of A such that $A_x \subseteq A_{x'}$ whenever $x' \leqslant x$, and that every monotone map $f\colon (A,\alpha) \to (B,\beta)$ satisfies $f(A_x) \subseteq B_x$ for each $x \in X$. Considering the category C having

- as objects, pairs $(A,(A_x)_{x\in X})$, where A is an ordered set and $(A_x)_{x\in X}$ is a family of subsets of A such that $A_x \subseteq A_{x'}$ whenever $x' \leqslant x$,
- and as morphisms $f\colon (A,(A_x)) \to (B,(B_x))$, monotone maps $f\colon A \to B$ such that $f(A_x) \subseteq B_x$ for each $x \in X$,

one can apply Theorem 1.1 based on the following facts:

- the functor

  $$\mathsf{Ord}//X \xrightarrow{\ F\ } \mathsf{C},$$

  defined by $F(A,\alpha) = (A,(A_x = \{a \in A,\ x \leqslant \alpha(a)\})_x)$ and $F(f) = f$, is fully faithful and preserves pullbacks;
- a morphism $f\colon (A,(A_x)_x) \to (B,(B_x)_x)$ is effective for descent in C if and only if $f\colon A \to B$ and $f_x\colon A_x \to B_x$ for all $x \in X$ are surjective.

Theorem 1.4 ([5]). *Let X be a locally complete ordered set. For a morphism $f\colon (A,\alpha) \to (B,\beta)$ in $\mathsf{Ord}//X$, consider the following conditions:*

(1) In Ord*,* $f\colon A \to B$ *is effective for descent, and* $f_x\colon A_x \to B_x$ *is a descent morphism for all* $x \in X$*;*

(2) $f\colon (A,\alpha) \to (B,\beta)$ *is effective for descent in* $\mathsf{Ord}//X$*.*

Then (1) $\Rightarrow$ (2)*. If, in addition, for each* $x \in X$ *every subset of* $\downarrow x$ *has a largest element, then* (1) $\Leftrightarrow$ (2)*.*

Theorem 1.4 gives us, for a locally complete ordered set X, a sufficient condition for f to be effective for descent in $\mathsf{Ord}//X$ which is not necessary in general, as we show in the sequel. Indeed, in order to apply Theorem 1.1, we must start with a morphism whose F-image is an effective descent morphism in C, hence f and all f_x are a priori surjective, and this condition is not fulfilled by all effective descent morphisms in $\mathsf{Ord}//X$, as we show in Example 3.4.

2. Familial descent

One of the main insights behind our main result, Theorem 3.1, is that we can reduce the study of effective descent morphisms (respectively, stable regular epimorphisms) in $\mathsf{Ord}//X$ to the study of effective descent morphisms (respectively, stable regular epimorphisms) in $\mathsf{Fam}(X)$ and Ord. This latter problem in $\mathsf{Fam}(X)$ has been considered before in [17, Lemma 4.4] (see also [18, Lemma 3.17]), from which we proceed to recall the relevant details.

For a fixed ordered set X, we denote by $\mathsf{Fam}(X)$ the category of set-indexed *families of elements* in X. It consists of:

– Objects: families $(\alpha_j)_{j\in J}$ of elements $\alpha_j \in X$ indexed by a set J,

– Morphisms $(\alpha_j)_{j\in J} \to (\beta_k)_{k\in K}$: a function $f\colon J \to K$ such that $\alpha_j \leqslant \beta_{f(j)}$ for all $j \in J$.

We will assume that *locally* X *has binary meets*, that is, $\downarrow x$ has binary meets for all x. When X is seen as a thin category, this condition is equivalent to saying that X has pullbacks. Thus, it follows that $\mathsf{Fam}(X)$ is a category with pullbacks (see [1, Sections 6.2, 6.3]).

While the results of [17, 18] study (effective) descent morphisms in $\mathsf{Fam}(X)$ when X has a top element, this was due to the pertinence of the work carried out within, and the results plainly extend to the setting where X does not admit a top element.

Lemma 2.1 ([17, Lemma 4.4], [18, Lemma 3.17])**.** *Let* $f\colon (\alpha_j)_{j\in J} \to (\beta_k)_{k\in K}$ *be a morphism in* $\mathsf{Fam}(X)$*.*

(1) f *is a descent morphism if and only if, for all* $k \in K$ *and all* $z \leqslant \beta_k$*, we have*

$$z \cong \bigvee_{f(j)=k} z \wedge \alpha_j \tag{2.i}$$

(2) If X *is locally complete, then* f *is an effective descent morphism if and only if* f *is a descent morphism.*

Proof. We verify that having a top element is redundant.

By [11, Theorem 3.4(a)], we note that f is a descent morphism in $\mathsf{Fam}(X)$ if and only if it is a stable regular epimorphism in

$$\mathsf{Fam}(X)/(\beta_k)_{k\in K} \simeq \prod_{k\in K} \mathsf{Fam}(X)/\beta_k \simeq \prod_{k\in K} \mathsf{Fam}(X/\beta_k),$$

which is the case if and only if (2.i) holds. □

Lemma 2.1 on its own already allows us to smoothly extend the characterization of stable regular epimorphisms obtained in [6, Lemma 3.1, Proposition 3.2] for X complete and cartesian closed to our context.

Lemma 2.2. *Let X be locally complete ordered set with a bottom element, and let $f\colon (A,\alpha) \to (B,\beta)$ be a morphism in $\mathsf{Ord}//X$.*

(1) f is a regular epimorphism in $\mathsf{Ord}//X$ if and only if it is a regular epimorphism in Ord and

$$\forall b \in B \quad \beta(b) \cong \bigvee_{f(a) \leqslant b} \beta(b) \wedge \alpha(a).$$

(2) f is a stable regular epimorphism in $\mathsf{Ord}//X$ if and only if it is a stable regular epimorphism in Ord and

$$\forall b \in B \;\; \forall x \leqslant \beta(b) \;\; x \cong \bigvee_{f(a)=b} x \wedge \alpha(a). \tag{2.ii}$$

Proof. Statement (1) is precisely [6, Lemma 3.1], so we focus on (2).

If f is a stable regular epimorphism in $\mathsf{Ord}//X$, then, for each $b \in B$ and $x \leqslant \beta(b)$, we consider the pullback diagram

$$\begin{array}{ccc} \big(f^{-1}(b), \big(a \mapsto x \wedge \alpha(a)\big)\big) & \xrightarrow{\;u\;} & (b, x) \\ \downarrow & & \downarrow \\ (A,\alpha) & \xrightarrow[f]{} & (B,\beta) \end{array}$$

so that u is a regular epimorphism in $\mathsf{Ord}//X$, which entails (2.ii), as desired.

Conversely, if (2.ii) holds for all $b \in B$ and all $x \leqslant \beta(b)$, then for any pullback diagram

$$\begin{array}{ccc} \big(A \times_B C, \big((a,c) \mapsto \gamma(c) \wedge \alpha(a)\big)\big) & \xrightarrow{\;\pi_2\;} & (C,\gamma) \\ \downarrow & & \downarrow{\scriptstyle g} \\ (A,\alpha) & \xrightarrow[f]{} & (B,\beta) \end{array}$$

we claim that π_2 is a (stable) regular epimorphism. Indeed, for each $c \in C$ we have $\gamma(c) \leqslant \beta(g(c))$, so from (2.ii) we deduce that

$$\gamma(c) \cong \bigvee_{f(a)=g(c)} \gamma(c) \wedge \alpha(a) \cong \bigvee_{\substack{f(a')=g(c') \\ c' \leqslant c}} \gamma(c') \wedge \alpha(a')$$

which indeed confirms that π_2 is a regular epimorphism. ☐

Remark 2.3. We point out that condition (2.ii) can be interpreted in the category $\mathsf{Fam}(X)$ by considering the (faithful) forgetful functor

$$\mathsf{Ord}//X \longrightarrow \mathsf{Fam}(X)$$

which maps (A,α) to the family $(\alpha(a))_{a \in A}$. Thus, by Lemma 2.1, condition (2) can be restated as follows: $f\colon (A,\alpha) \to (B,\beta)$ *is a stable regular epimorphism in* $\mathsf{Ord}//X$ *if and only if the underlying morphisms in* Ord *and* $\mathsf{Fam}(X)$ *are stable regular epimorphisms.*

In fact, we can say more: since X is assumed to be locally complete, $U\colon \mathsf{Ord}//X \to \mathsf{Fam}(X)$ maps stable regular epimorphisms to effective descent morphisms. Therefore, when X is locally complete and has a bottom element, we conclude that both forgetful functors

$$\begin{array}{ccccc} & & \mathsf{Ord}//X & & \\ & \swarrow & & \searrow & \\ \mathsf{Ord} & & & & \mathsf{Fam}(X) \end{array} \tag{2.iii}$$

preserve effective descent morphisms.

3. The characterization

Having reviewed the necessary details, we may proceed to prove our main result:

Theorem 3.1. *Let X be a locally complete ordered set. A morphism $f\colon (A,\alpha) \to (B,\beta)$ is effective for descent in $\mathsf{Ord}//X$ if and only if*

(a) *$f\colon A \to B$ is effective for descent in Ord; that is*

$$\forall b_0 \leqslant b_1 \leqslant b_2 \text{ in } B\ \exists a_0 \leqslant a_1 \leqslant a_2 \text{ in } A\colon \quad f(a_0) = b_0,\ f(a_1) = b_1,\ f(a_2) = b_2.$$

(b) *for all $b_0 \leqslant b_1$ in B, and all $w \leqslant \beta(b_0)$ in X, we have*

$$w = \bigvee_{a_0 \in S_{b_0,b_1}} w \wedge \alpha(a_0),$$

where $S_{b_0,b_1} = \{a_0 \in f^{-1}(b_0) \mid \exists a_1 \in f^{-1}(b_1) \text{ with } a_0 \leqslant a_1\}$.

To prove this result, it is natural to consider the (pseudo)pullback diagram below (see [14], noting that $\mathsf{Fam}(X) \to \mathsf{Set}$ is an (iso)fibration):

$$\begin{array}{ccc} \mathsf{Ord} \times_{\mathsf{Set}} \mathsf{Fam}(X) & \xrightarrow{\rho_2} & \mathsf{Fam}(X) \\ {\scriptstyle\rho_1}\downarrow & & \downarrow \\ \mathsf{Ord} & \longrightarrow & \mathsf{Set} \end{array}$$

as well as the uniquely induced functor $\mathsf{Ord}//X \to \mathsf{Ord} \times_{\mathsf{Set}} \mathsf{Fam}(X)$ induced by the forgetful functors (2.iii).

Using either [5, Corollary 2.6] or [15, Corollary 9.6], we obtain:

Lemma 3.2. *Let X be a locally complete ordered set with a bottom element. A morphism f is effective for descent in $\mathsf{Ord} \times_{\mathsf{Set}} \mathsf{Fam}(X)$ if and only if:*

(1) *$\rho_1(f)$ is effective for descent in Ord.*
(2) *$\rho_2(f)$ is effective for descent in $\mathsf{Fam}(X)$.*

Now, the fully faithful and pullback preserving functor

$$\mathsf{Ord}//X \xrightarrow{\quad U \quad} \mathsf{Ord} \times_{\mathsf{Set}} \mathsf{Fam}(X)$$

and Theorem 1.1 give us the tools to characterize effective descent morphisms in $\mathsf{Ord}//X$. Before proceeding to the proof, we recall that the objects $\mathsf{Ord} \times_{\mathsf{Set}} \mathsf{Fam}(X)$ consist of pairs $(C, (\chi_c)_{c\in C})$ where C is an ordered set and $(\chi_c)_{c\in C}$ is a family of elements of X, that is, a map $\chi\colon C \to X$. Such a pair is in the (essential) image of U if and only if χ is monotone.

Proof of Theorem 3.1. Let $f\colon (A,\alpha) \to (B,\beta)$ be a morphism in $\mathsf{Ord}//X$.

If (b) holds, we note that for all $b_0 \leqslant b_1$ in B and all $w \leqslant \beta(b_0)$ in X, we have

$$w \cong \bigvee_{a_0 \in S_{b_0,b_1}} w \wedge \alpha(a_0) \leqslant \bigvee_{a_0 \in f^{-1}(b_0)} w \wedge \alpha(a_0) \leqslant w,$$

hence $\rho_2(U(f))$ is an effective descent morphism in $\mathsf{Fam}(X)$ by Lemma 2.1. Thus, if (a) also holds, $\rho_1(U(f))$ is an effective descent morphism in Ord, so $U(f)$ is an effective descent morphism in $\mathsf{Ord} \times_{\mathsf{Set}} \mathsf{Fam}(X)$.

Now, we apply Theorem 1.1: if we have a pullback diagram

$$\begin{array}{ccc} U(D,\delta) & \longrightarrow & (C,(\chi_c)_{c\in C}) \\ \big\downarrow & & \big\downarrow g \\ U(A,\alpha) & \xrightarrow[U(f)]{} & U(B,\beta) \end{array}$$

we want to show that $\chi\colon C \to X$ is monotone. Let $c_0 \leqslant c_1 \in C$, $b_i = g(c_i)$, $i = 0,1$, and let $S_{b_0,b_1} = \{a_0 \in f^{-1}(b_0) \mid \exists a_1 \in f^{-1}(b_1) \text{ with } a_0 \leqslant a_1\}$. Then $\chi(c_0) \leqslant \beta(b_0)$ and therefore, by condition (b),

$$\chi(c_0) \cong \bigvee_{a_0 \in S_{b_0,b_1}} \chi(c_0) \wedge \alpha(a_0) = \bigvee_{a_0 \in S_{b_0,b_1}} \delta(a_0,c_0) \leqslant \bigvee_{a \in f^{-1}(b_1)} \delta(a,c_1) \leqslant \chi(c_1),$$

as desired.

Conversely, if f is an effective descent morphism, then by Remark 2.3, it follows that both $\rho_1(U(f))$ and $\rho_2(U(f))$ are effective descent morphisms in Ord and $\mathsf{Fam}(X)$, respectively, from which we conclude that $U(f)$ is an effective descent morphism (by Lemma 3.2), and that (a) holds.

Again, we apply Theorem 1.1: we let $b_0 \leqslant b_1$ and $w \leqslant \beta(b_0)$, and we consider the pair $\big(\{b_0,b_1\},(\chi_{b_0},\chi_{b_1})\big)$, where

$$\chi_{b_0} = w, \qquad \chi_{b_1} = \bigvee_{a_0 \in S_{b_0,b_1}} w \wedge \alpha(a_0),$$

with $S_{b_0,b_1} = \{a_0 \in f^{-1}(b_0) \mid \exists a_1 \in f^{-1}(b_1) \text{ with } a_0 \leqslant a_1\}$. We also let

$$g\colon \big(\{b_0,b_1\},(\chi_{b_0},\chi_{b_1})\big) \to (B,\beta)$$

be the inclusion. Taking the pullback of $U(f)$ along g, we obtain

$$\begin{array}{ccc} \big(D,(\xi_d)_{d\in D}\big) & \longrightarrow & \big(\{b_0,b_1\},(\chi_{b_0},\chi_{b_1})\big) \\ \big\downarrow & & \big\downarrow g \\ U(A,\alpha) & \xrightarrow[U(f)]{} & U(B,\beta), \end{array}$$

where $D = \{(a_i,b_i) \mid f(a_i) = b_i,\ i = 0,1\}$, and ξ is given by

- $\xi_{(a_0,b_0)} = \alpha(a_0) \wedge w$ for each $a_0 \in A$ such that $f(a_0) = b_0$, and
- $\xi_{(a_1,b_1)} = \alpha(a_1) \wedge \chi_{b_1}$ for each $a_1 \in A$ such that $f(a_1) = b_1$.

Hence, if $(a_0,b_0) \leqslant (a_1,b_1)$, then $a_0 \leqslant a_1$ and $f(a_1) = b_1$, so that $a_0 \in S_{b_0,b_1}$. It follows that

$$\alpha(a_0) \wedge w \leqslant \chi_{b_1} \qquad \text{and} \qquad \alpha(a_0) \wedge w \leqslant \alpha(a_1),$$

and therefore $\xi_{a_0,b_0} \leqslant \xi_{a_1,b_1}$, confirming that ξ is monotone. Thus, χ must be monotone as well, so that $\chi_{b_0} \cong \chi_{b_1}$, confirming that (b) holds. □

We recall that an ordered set X with finite meets is said to be *cartesian closed* if there is an assignment $(y,z) \mapsto z^y$, which satisfies

$$x \wedge y \leqslant z \iff x \leqslant z^y$$

for every $x \in X$. When X is complete and the underlying order is antisymmetric, this is equivalent to X being a *frame*.

Corollary 3.3. *Let X be a cartesian closed, complete ordered set. A morphism $f\colon (A,\alpha) \to (B,\beta)$ in $\mathsf{Ord}//X$ is an effective descent morphism if and only if* *(a)* $f\colon A \to B$ *is effective for descent in* Ord*; that is*

$$\forall b_0 \leqslant b_1 \leqslant b_2 \text{ in } B \;\; \exists a_0 \leqslant a_1 \leqslant a_2 \text{ in } A\colon \quad f(a_0) = b_0,\; f(a_1) = b_1,\; f(a_2) = b_2.$$

(b) for all $b_0 \leqslant b_1$*, we have*

$$\beta(b_0) = \bigvee_{a_0 \in S_{b_0,b_1}} \alpha(a_0),$$

where $S_{b_0,b_1} = \{a_0 \in f^{-1}(b_0) \mid \exists a_1 \in f^{-1}(b_1) \text{ with } a_0 \leqslant a_1\}$.

Proof. Since X is cartesian closed, meets distributive over joins, hence we have

$$w \cong w \wedge \beta(b_0) \cong w \wedge \bigvee_{a_0 \in S_{b_0,b_1}} \alpha(a_0) \cong \bigvee_{a_0 \in S_{b_0,b_1}} w \wedge \alpha(a_0),$$

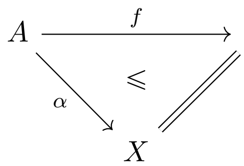
and we may apply Theorem 3.1. □

Examples 3.4. Let X be the interval $[0,1]$ with the usual order – we observe that X is a cartesian closed, complete ordered set.

Let $A = \{(x,y) \in X^2\,;\, y < x \text{ or } y = x = 0\}$, and write $\alpha = \pi_2$, $f = \pi_1$ for the projections.

(1) If we equip A with the product order, then both α and f are monotone, so that we have a morphism

(3.i)



in Ord$//X$ – indeed, we note that $\alpha(x,y) = y \leqslant f(x,y) = x$. Moreover,

– f is effective for descent in Ord:

$$\forall x_0 \leqslant x_1 \leqslant x_2 \text{ in } [0,1] \quad \exists (x_0,0) \leqslant (x_1,0) \leqslant (x_2,0) \text{ in } A \;:\; f(x_i,0) = x_i. \tag{3.ii}$$

– If $0 = x_0 \leqslant x_1$ then $(0,0) \leqslant (x_1,0)$ in A and $0 = \alpha(0,0)$; if $0 < x_0 \leqslant x_1$, then, for all $0 \leqslant y < x$, $(x_0,y) \leqslant (x_1,y)$ in A and clearly $x_0 = \bigvee\{\alpha(x_0,y) \,|\, 0 \leqslant y < x_0\}$,

and these respectively correspond to conditions (a) and (b) of Corollary 3.3. We conclude that f is an effective descent morphism in Ord$//X$.

(2) If we consider on A the order defined by

$$(x,y) \leqslant (x',y') \iff \quad (x,y) = (x',y') \quad \text{or} \quad x \leqslant x' \text{ and } y = y' = 0,$$

then, once again, both α and f are monotone, and f defines a morphism (3.i) in Ord$//X$.

Moreover, we note that f is an effective descent morphism in Ord, since (3.ii) still holds, and that f is a stable regular epimorphism in Ord$//X$, because we have $f^{-1}(x) = \{(x,y) \in A \mid 0 \leqslant y < x\}$, hence

$$\forall x \in X \quad x \cong \bigvee_{y<x} y.$$

While, if $0 < x < 1$, then $(x_0,y_0) \leqslant (1,y_1)$ in A only if $y_0 = y_1 = 0$, hence

$$S_{x_0,1} = \big\{(x_0,y_0) \in A \mid y_0 < x_0 \text{ and } \exists y_1 < 1 \text{ such that } (x_0,y_0) \leqslant (1,y_1)\big\} = \{(x_0,0)\},$$

and so f does not satisfy (b) of Corollary 3.3.

References

[1] F. Borceux, G. Janelidze. *Galois theories*. Volume 72 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2001. DOI: `https://doi.org/10.1017/CBO9780511619939`

[2] J. Bénabou, J. Roubaud. Monades et descente. *C. R. Math. Acad. Sci. Paris* 270(A):96–98, 1970.

[3] M.M. Clementino, D. Hofmann. Triquotient maps via ultrafilter convergence. *Proc. Amer. Math. Soc.*, 130(11):3423–3431, 2002. DOI: `https://doi.org/10.1090/S0002-9939-02-06472-9`

[4] M.M. Clementino, D. Hofmann. The rise and fall of V-functors. *Fuzzy Sets and Systems* 321:29–49, 2017. DOI: `https://doi.org/10.1016/j.fss.2016.09.005`

[5] M.M. Clementino, G. Janelidze. Effective descent morphisms of filtered preorders. *Order*, to appear.

[6] M.M. Clementino, F. Lucatelli Nunes. Lax comma categories of ordered sets. *Quaest. Math.* 46(S1):145–159, 2023. DOI: `https://doi.org/10.2989/16073606.2023.2247729`

[7] M.M. Clementino, F. Lucatelli Nunes. Lax comma 2-categories and admissible 2-functors. *Theory Appl. Categ.* 40(6):180–226, 2024.

[8] M.M. Clementino, F. Lucatelli Nunes, Rui Prezado. Lax comma categories: cartesian closedness, extensivity, topologicity, and descent. *Theory Appl. Categ.*, 41(16):516–530, 2024.

[9] G. Janelidze. Pure Galois theory in categories. *J. Algebra*, 132(2):270–286, 1990. DOI: `https://doi.org/10.1016/0021-8693(90)90130-G`

[10] G. Janelidze, M. Sobral. Finite preorders and topological descent I. *J. Pure Appl. Algebra*, 175:187–205, 2002. DOI: `https://doi.org/10.1016/S0022-4049(02)00134-2`

[11] G. Janelidze, M. Sobral, W. Tholen. Beyond Barr exactness: Effective descent morphisms. In *Categorical foundations*, pp. 359–405. Encyclopedia Math. Appl., 97. Cambridge Univ. Press, Cambridge, 2004. DOI: `https://doi.org/10.1017/CBO9781107340985.011`

[12] G. Janelidze, W. Tholen. Facets of Descent, I. *Appl. Categ. Structures*, 2(3):245–281, 1994. DOI: `https://doi.org/10.1023/A:1008697013769`

[13] G. Janelidze, W. Tholen. Facets of Descent, II. *Appl. Categ. Structures*, 5(3):229–248, 1997. DOI: `https://doi.org/10.1007/BF00878100`

[14] A. Joyal, R. Street. Pullbacks equivalent to pseudopullbacks. *Cah. Topol. Géom. Différ. Catég.*, 34(2):153–156, 1993.

[15] F. Lucatelli Nunes. Pseudo-Kan extensions and descent theory. *Theory Appl. Categ.* 33(15):390–444, 2018.

[16] F. Lucatelli Nunes. Descent data and absolute Kan extensions. *Theory Appl. Categ.*, 37(18):530–561, 2021.

[17] R. Prezado. On effective descent V-functors and familial descent morphisms. *J. Pure Appl. Algebra* 228(5):107597, 2024. DOI: `https://doi.org/10.1016/j.jpaa.2023.107597`

[18] R. Prezado. *Some aspects of descent theory and applications*. PhD Thesis, University of Coimbra and University of Porto, 2024. URI: `https://hdl.handle.net/10316/114685`

[19] J. Reiterman, W. Tholen. Effective descent maps of topological spaces. *Topology Appl.*, 57:53–69, 1994. DOI: `https://doi.org/10.1016/0166-8641(94)90033-7`

[20] R.J. Wood. Ordered Sets via Adjunction. In *Categorical foundations*, pp. 5–47. Encyclopedia Math. Appl., 97. Cambridge Univ. Press, Cambridge, 2004. DOI: `https://doi.org/10.1017/CBO9781107340985.004`

University of Coimbra, CMUC, Department of Mathematics, 3000-143 Coimbra, Portugal
Email address: `mmc@mat.uc.pt`

University of Coimbra, CMUC, Department of Mathematics, 3000-143 Coimbra, Portugal
Email address: `ruiprezado@gmail.com`