REGULARITY ESTIMATES FOR PDES WITH NONLOCAL DEGENERACIES

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Abstract. We investigate a class of PDEs featuring nonlocal degeneracies arising from self-dependent regions determined by the solution's Hölder semi-norm. Notably, this framework unifies two classical settings: free boundary problems, where degeneracy occurs along the nodal set, and critical-point degenerate PDEs, both recast as extrema (local) problems within our formulation. For models where an elliptic PDE is only activated beyond a given positive threshold $\kappa > 0$, we establish the local Hölder continuity of solutions, which is the optimal regularity possible, and prove a result of Krylov-Safonov type for operators with coefficients, yielding universal continuity estimates for solutions, independent of coefficient regularity. In the globally degenerate case $\kappa = 0$, we develop a $C^{1,\beta}-$regularity theory, which sharply interpolates the known estimates for the extrema local problems. Beyond its intrinsic relevance, the framework developed in this paper provides new perspectives on the classical extrema models, offering insights that are not accessible through previously known approaches.

Contents

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1. Introduction

Degenerate diffusion problems have long fascinated both the mathematical community and researchers across various disciplines due to their dual significance: their critical role in modelling a wide range of applied phenomena and the deep, challenging mathematical questions they pose. These problems are especially intriguing when diffusion collapse occurs on sets determined by the solution itself, creating a complex interplay between the solution's geometry and the analytical behaviour of the governing equations.

Of particular interest are cases where ellipticity breaks down along the zero-level set of solutions, as in the classical theory of free boundary problems. Such phenomena arise, for instance, in the well-known Alt-Phillips problem, which involves minimizing a non-differentiable functional (see [27, 1]). The associated Euler-Lagrange equation takes the form

$$u^{1-\gamma}\Delta u = f(x)\chi_{\{u>0\}}, \tag{1.1}$$

for $\gamma \in (0,1)$ and f bounded and continuous. Solutions to this equation are generally non-unique and can be obtained either through variational methods or singular perturbation techniques, as in [6]. For additional references, see [5, 6, 18, 19, 20, 21, 25, 28].

In parallel with this theory, there has been a growing interest in models where ellipticity vanishes at critical points, such as in the fully nonlinear extension of the $p-$Laplace operator, namely,

$$|Du|^p\Delta u = f(x). \tag{1.2}$$

This class of problems has been systematically studied since the foundational works of Birindelli and Demengel [8, 9, 10, 11, 12]. Significant progress was later made by Imbert and Silvestre [23], who established gradient Hölder regularity, with optimality results subsequently achieved in [3]. For more recent developments, see [2, 4, 7, 13, 26].

In this paper, we explore for the first time the connection between the aforementioned problems, linking free boundary models where ellipticity vanishes along zeros of the solution, also known as the nodal set, and PDEs involving degeneracies at critical points. At first glance, one might question whether such a connection exists at all. A shred of evidence arises from the minimization process that leads to (1.1), which provides the key estimate

$$|Du|^2 \lesssim u^\gamma,$$

for the pertaining regularity program (see [27, Lemma III.1]). Thus, variational solutions of the Alt-Phillips equation do satisfy (1.2), for $f \in L^\infty$, with

$$p = \frac{2-2\gamma}{\gamma} > 0.$$

The reverse implication, *i.e.*, whether solutions to (1.2) conform to the same behaviour as implied by (1.1), does not seem to be true in general. This observation offers a compelling mathematical motivation for investigating the proper connection between both problems.

The approach we propose in this paper stems from applied considerations. We focus on diffusion models that are activated only in regions where the solutions' variation (potential energy, voltage, temperature gradient, deformation, etc.), $|u(x) - u(y)|$, greatly exceeds a *fractional* power of the displacement, $|x - y|^\alpha$. This framework naturally introduces a one-parameter family of operators $(F_\alpha)_{\alpha\in(0,1)}$, which, as we will show, offers an insightful way to continuously transition between the regimes of (1.1) (corresponding to $\alpha = 0$) and (1.2) (corresponding to $\alpha = 1$).

For each $0 < \alpha < 1$ fixed, the inherent degeneracy of the intermediary model is nonlocal in nature. As such, the bridge we establish between the two limiting (local) problems arises within a fundamentally nonlocal framework. This non-locality introduces significant challenges, particularly when developing a regularity theory that remains uniform with respect to the parameter α.

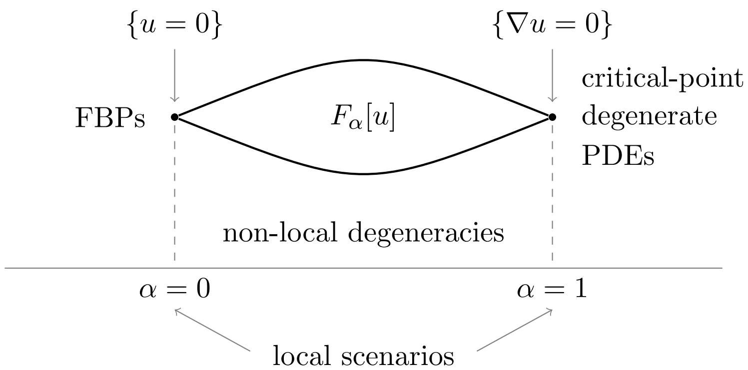

FIGURE 1. The diagram highlights the transition between different scenarios of degenerate problems. For $\alpha \in (0,1)$, the problem is purely nonlocal, linking the extremal (local) cases given by Free Boundary Problems (FPBs) and critical-point degenerate PDEs.

Specifically, for a parameter $\alpha \in (0,1)$ and $q \geq 0$, we study viscosity solutions to equations of the form

$$F_\alpha[u] := \big((1-\alpha)|u| + [u]_\alpha\big)^q F(D^2u) = f, \tag{1.3}$$

where $f \in L^\infty(B_1) \cap C(B_1)$ and $[\cdot]_\alpha$ is the $\alpha-$*bracket*, defined by

$$[v]_\alpha(x) := \sup_{y\in B_{\alpha(1-\alpha)}(x)\setminus\{x\}} \frac{|v(y) - v(x)|}{|y-x|^\alpha},$$

which can be interpreted as the pointwise $\alpha-$Hölder semi-norm of a function. We will show in this paper that viscosity solutions of (1.3), with F being $(\lambda, \Lambda)-$elliptic, are locally of class $C^{1,\beta}(B_1)$, with

$$\beta := \min\left(\alpha_F^-, \frac{1-q(1-\alpha)}{1+q}\right), \quad \text{provided} \quad q < \frac{1}{1-\alpha}.$$

This is the contents of Theorem 5.1. We explain that the number $0 < \alpha_F \leq 1$ denotes the maximal Hölder regularity exponent for the gradient of $F-$harmonic functions, *i.e.*, viscosity solutions of $F(D^2h) = 0$, e.g. [15]. In particular, if F is assumed to be concave or convex, one can take $\beta = (1-q(1-\alpha))/1+q$.

It is also crucial to emphasize that the key novelty of our result lies in the fact that the estimate we obtain is independent of the magnitude of the degeneracy quantity, $\left[(1-\alpha)|u| + [u]_\alpha\right]^q$, making it a foundational result to the analysis of those intermediary, nonlocal models. We expect this to be a significant research program set to offer a fresh perspective on the theory.

An appealing feature of this estimate is that the interplay between the degeneracy power q and the order α of the bracket determines the $C^{1,\beta}-$regularity theory that sharply interpolates between the known ones for the models in (1.1) and (1.2). Notably, when $\alpha = 0$, the degeneracy power q is confined to vary within the interval $(0, 1)$ and solutions are locally of class $C^{1,\frac{1-q}{1+q}}$, recovering the regularity theory for the Alt-Phillips problem, see [22]. At the other extreme, when $\alpha = 1$, the degeneracy power q can take any positive real value, and solutions are of class $C^{1,\frac{1}{1+q}}$, which aligns with the estimates from [3]. The proofs delivered here, however, differ quite substantially from those in the classical local settings. These differences not only highlight the unique analytical challenges of the nonlocal setting but also provide additional insights into the underlying mechanisms that ensure such critical estimates hold in the classical local problems.

We emphasize that for each intermediate $0 < \alpha < 1$, the operator F_α may be used to model problems related to certain anomalous diffusion, including interface dynamics and complex materials. In such applications, however, diffusion should only occur when the ratio between the solution's variation and a fractional power of the displacement exceeds a specified threshold. Mathematically, such considerations lead to the equation

$$([(1-\alpha)|v| + [v]_\alpha] - \kappa)_+^q \, F(D^2u) = f \in L^\infty(B_1) \cap C(B_1), \tag{1.4}$$

for some $\kappa > 0$. It is understood in (1.4) that no PDE is prescribed at points where $[(1-\alpha)|v| + [v]_\alpha] \leq \kappa$.

The problems described in the previous paragraphs, to which we shall establish $C^{1,\beta}-$regularity, correspond to the case $\kappa = 0$. However, for $\kappa > 0$, the best regularity one can expect is Hölder continuity. In fact, we will obtain the optimal regularity for solutions of (1.4), *viz.* solutions are locally $\alpha-$Hölder continuous, for the same parameter α in the operator F_α, see Proposition 3.1. That is, we show that there is no loss of regularity across the "*free boundary*"

$$\Gamma_{\alpha,\kappa} \coloneqq \{[(1-\alpha)|v| + [v]_\alpha] = \kappa\},$$

as one transitions from the PDE region to the dormant region.

We further investigate models that account for the complexity of the medium where the PDE operates, leading to problems involving variable coefficients. Similar to the classical Krylov-Safonov theory, our objective is to derive universal continuity estimates for solutions, independent of the regularity of the coefficients. Adopting the framework introduced by Caffarelli in [14], we reformulate the problem as viscosity inequalities and establish a universal $C^{0,\gamma}$ regularity estimate, see Theorem 4.1.

The remainder of the paper is organized as follows. We dedicate Section 2 to discuss the analysis regarding the $\alpha-$bracket $[\cdot]_\alpha$, and Section 3 to present the basic mathematical setup, including, in subsection 3.2, the optimal $C^{0,\alpha}-$regularity for solutions of (1.4). In Section 4, we establish an Hölder regularity result for solutions of (1.4) which is independent of the regularity of the coefficients. Finally, in Section 5, we prove the sharp gradient regularity for solutions of (1.3).

2. The $\alpha-$bracket

In this section, we introduce the feature that acts as the interpolating mechanism between the regimes discussed in the introduction. For $\alpha \in (0,1)$, consider the closed subset of B_1 defined by

$$\mathcal{N}_{\alpha(1-\alpha)}(B_1) \coloneqq \{x \in B_1 \colon \operatorname{dist}(x, \partial B_1) \geq \alpha(1-\alpha)\}.$$

If $x \in \mathcal{N}_{\alpha(1-\alpha)}(B_1)$, then clearly $B_{\alpha(1-\alpha)}(x) \subset B_1$.

Definition 2.1. *Let $v \in C(B_1)$. The $\alpha-$bracket of v is the function*

$$[v]_\alpha \colon \mathcal{N}_{\alpha(1-\alpha)}(B_1) \longrightarrow [0,\infty]$$

defined by

$$[v]_\alpha(x) \coloneqq \sup_{y \in B_{\alpha(1-\alpha)}(x)\setminus\{x\}} \frac{|v(y)-v(x)|}{|y-x|^\alpha}. \tag{2.1}$$

It is clear that the $\alpha-$bracket of a continuous function v at a point x_0 is the $\alpha-$Hölder semi-norm of v at the point x_0 in the domain $B_{\alpha(1-\alpha)}(x_0)$. The main difference with respect to the usual Hölder semi-norm is that the region where it is evaluated depends on the $\alpha-$Hölder exponent itself. This interdependence is the most important feature, allowing for the interpolation between the two degenerate regimes.

Lemma 2.1. *Let $v \in C^1(B_1)$ and $x_0 \in B_1$. Then*

$$[v]_\alpha(x_0) \to |Dv(x_0)| \quad \textit{as} \quad \alpha \to 1,$$

and

$$(1-\alpha)|v(x_0)| + [v]_\alpha(x_0) \to |v(x_0)| \quad \textit{as} \quad \alpha \to 0.$$

Proof. Observe that since α is either converging to 1 or to 0, it follows that $x_0 \in \mathcal{N}_{\alpha(1-\alpha)}(B_1)$ for α sufficiently close to 1 or to 0. To prove the first convergence, we first observe that if $x \in B_{\alpha(1-\alpha)}(x_0)$, then, by the mean value theorem, there exists $\theta \in [0,1]$ such that

$$\begin{aligned}\frac{|v(x)-v(x_0)|}{|x-x_0|^\alpha} &\leq |Dv(\theta x_0 + (1-\theta)x)||x-x_0|^{1-\alpha} \\ &\leq \sup_{z \in B_{\alpha(1-\alpha)}(x_0)} |Dv(z)|(\alpha(1-\alpha))^{1-\alpha}.\end{aligned}$$

This implies that

$$[v]_\alpha(x_0) \leq \sup_{z \in B_{\alpha(1-\alpha)}(x_0)} |Dv(z)|(\alpha(1-\alpha))^{1-\alpha}. \tag{2.2}$$

Taking the superior limit on both sides and using the continuity of Dv, we obtain

$$\limsup_{\alpha \to 1} [v]_\alpha(x_0) \leq |Dv(x_0)|,$$

where we used that

$$\lim_{\alpha \to 1} (1-\alpha)^{1-\alpha} = 1.$$

On the other hand, if we pick $x = x_0 + \alpha(1-\alpha)e$, for an arbitrary $e \in \partial B_1$, we have

$$\begin{aligned}[v]_\alpha(x_0) &\geq \frac{|v(x)-v(x_0)|}{|x-x_0|^\alpha} \\ &= \frac{|v(x_0+\alpha(1-\alpha)e) - v(x_0)|}{(\alpha(1-\alpha))^\alpha} \\ &= (\alpha(1-\alpha))^{1-\alpha} \frac{|v(x_0+\alpha(1-\alpha)e) - v(x_0)|}{\alpha(1-\alpha)}.\end{aligned}$$

Taking the inferior limit on both sides, we get

$$|Dv(x_0) \cdot e| \leq \liminf_{\alpha \to 1} [v]_\alpha(x_0).$$

Since this holds for every $e \in \partial B_1$, it follows that

$$|Dv(x_0)| \leq \liminf_{\alpha \to 1} [v]_\alpha(x_0).$$

As a consequence, both inferior and superior limits coincide, and the first convergence is proved. To prove the second convergence, taking advantage of inequality (2.2) again, it readily follows that $[v]_\alpha(x_0) \to 0$ as $\alpha \to 0$. $\square$

Remark 2.1. *Note that $[v]_\alpha(x_0) \to 0$ as $\alpha \to 0$ under the weaker assumption that the modulus of continuity of v satisfies*

$$\sup_{r\in[0,\alpha]} \frac{\omega_v(r)}{r^\alpha} \to 0 \quad \textit{as} \quad \alpha \to 0.$$

Remark 2.2. *The critical factor for the convergence of the $\alpha-$bracket as α approaches the interval endpoints is the shrinking of the domain to a single point as α reaches these extrema, motivating the use of balls with radius $\alpha(1-\alpha)$. Notably, this analysis would remain valid if we instead used a function $\tau(\alpha)$ as radius, where $\tau\colon \mathbb{R} \to \mathbb{R}$ satisfies $\tau(\alpha) > 0$ for all $\alpha \in (0,1)$,*

$$\tau(0) = \tau(1) = 0 \quad \textit{and} \quad \lim_{s\to 0} \omega_\tau^s(s) = 1,$$

with ω_τ denoting the modulus of continuity of τ.

There is an equivalent, useful way of defining the $\alpha-$bracket. Let

$$H_\alpha(v,x_0) := \inf\left\{C>0\colon \sup_{x\in\overline{B}_\rho(x_0)} |v(x)-v(x_0)| \le C\rho^\alpha,\ \forall\rho\in\big(0,\alpha(1-\alpha)\big]\right\}.$$

Lemma 2.2. *Let $v \in C(B_1)$ and $x_0 \in \mathcal{N}_{\alpha(1-\alpha)}(B_1)$. Then*

$$[v]_\alpha(x_0) = H_\alpha(v,x_0).$$

Proof. We prove only the case $[v]_\alpha(x_0) < \infty$ since the result is clear otherwise. By the definition of infimum, for each $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $C_\epsilon < H_\alpha(v,x_0) + \epsilon$ and

$$\sup_{x\in\overline{B}_\rho(x_0)} |v(x)-v(x_0)| \le C_\epsilon\rho^\alpha, \quad \forall\rho\in\big(0,\alpha(1-\alpha)\big].$$

Given $x \in B_{\alpha(1-\alpha)}(x_0)\backslash\{x_0\}$, we have $0 < |x-x_0| < \alpha(1-\alpha)$. It then follows that

$$\begin{aligned} |v(x)-v(x_0)| &\le \sup_{y\in\overline{B}_{|x-x_0|}(x_0)} |v(y)-v(x_0)| \\ &\le C_\epsilon|x-x_0|^\alpha \\ &< \left(H_\alpha(v,x_0)+\epsilon\right)|x-x_0|^\alpha. \end{aligned}$$

This implies that

$$[v]_\alpha(x_0) = \sup_{x\in B_{\alpha(1-\alpha)}(x_0)\backslash\{x_0\}} \frac{|v(x)-v(x_0)|}{|x-x_0|^\alpha} \le H_\alpha(v,x_0)+\epsilon,$$

for any $\epsilon > 0$. We now pass to the limit as $\epsilon \to 0$ to obtain

$$[v]_\alpha(x_0) \le H_\alpha(v,x_0).$$

To prove the other inequality, just observe that

$$|v(x)-v(x_0)| \le [v]_\alpha(x_0)\,|x-x_0|^\alpha,$$

for any $x \in B_{\alpha(1-\alpha)}(x_0)$. In particular, if $x \in \overline{B}_\rho(x_0)$, for $\rho \in \big(0, \alpha(1-\alpha)\big]$, then

$$|v(x) - v(x_0)| \leq [v]_\alpha(x_0)\, |x - x_0|^\alpha = [v]_\alpha(x_0)\, \rho^\alpha.$$

Since $H_\alpha(v, x_0)$ is the infimum, it follows that

$$H_\alpha(v, x_0) \leq [v]_\alpha(x_0).$$

□

In view of Lemma 2.2, it is possible to relate the notion of $\alpha-$bracket with the local Lipschitz constant of a function $v\colon B_1 \to \mathbb{R}$, which naturally arises in the calculus of variations in L^∞ (see, for instance, [16]), and is defined by

$$L(v, x_0) := \lim_{r \to 0^+} \mathrm{Lip}\,(v, B_r(x_0)), \qquad x_0 \in B_1,$$

where

$$\mathrm{Lip}\,(v, B_r(x_0)) := \inf \left\{ L > 0 \colon |u(x) - u(y)| \leq L|x - y|, \, \forall\, x, y \in B_r(x_0) \right\}.$$

For nonnegative functions, it can then be shown that

$$[v]_\alpha(x_0) = \mathrm{Lip}\,\left(v^{\frac{1}{\alpha}}, B_{\alpha(1-\alpha)}(x_0) \right).$$

The next result concerns a continuity property of the $\alpha-$bracket.

Lemma 2.3. *The $\alpha-$bracket of $v \in C(B_1)$ is a lower semicontinuous function in $\mathcal{N}_{\alpha(1-\alpha)}(B_1)$.*

Proof. Let $(x_k)_{k \in \mathbb{N}} \subset \mathcal{N}_{\alpha(1-\alpha)}(B_1)$ be such that

$$\lim_{k \to \infty} x_k = x_0.$$

Since $\mathcal{N}_{\alpha(1-\alpha)}(B_1)$ is closed and $x_k \to x_0$, it follows that $[v]_\alpha(x_0)$ is well-defined. We assume $x_0 = 0$, with no loss of generality, and let us first consider the case $[v]_\alpha(0) = \infty$. We will show

$$\lim_{k \to \infty} [v]_\alpha(x_k) = \infty,$$

meaning that given any $L > 0$, $[v]_\alpha(x_k) \geq L$, for k sufficiently large. Since $[v]_\alpha(0) = \infty$, it follows that, for $A > 0$ large enough such that

$$A > 2^\alpha \left(L + \frac{1}{2} \right),$$

we can find $x_A \in B_{\alpha(1-\alpha)}(0) \backslash \{0\}$ such that

$$\frac{|v(x_A) - v(0)|}{|x_A|^\alpha} \geq A.$$

For k large enough, we have $x_A \in B_{\alpha(1-\alpha)}(x_k)$, and so

$$
\begin{aligned}
[v]_\alpha(x_k) &\geq \frac{|v(x_A) - v(x_k)|}{|x_A - x_k|^\alpha} \\
&\geq \frac{|v(x_A) - v(0)|}{2^\alpha |x_A|^\alpha} - \frac{|v(x_k) - v(0)|}{2^\alpha |x_A|^\alpha} \\
&\geq \frac{A}{2^\alpha} - \frac{1}{2},
\end{aligned}
$$

since $x_k \to 0$ as $k \to \infty$. Then, by the choice of A, we have

$$
\frac{A}{2^\alpha} - \frac{1}{2} > L,
$$

from which it follows that $[v]_\alpha(x_k) \geq L$, as long as k is large enough.

Now, let us treat the case $[v]_\alpha(0) < \infty$. We first claim that there is $k_0 \in \mathbb{N}$ such that $[v]_\alpha(x_k) < \infty$, for every $k \geq k_0$. If this holds true, then for k large enough, we have $x \in B_{\alpha(1-\alpha)}(x_k)$, and so

$$
[v]_\alpha(x_k) \geq \frac{|v(x) - v(x_k)|}{|x - x_k|^\alpha}.
$$

Passing to the inferior limit as $k \to \infty$, we obtain

$$
\liminf_{k \to \infty} [v]_\alpha(x_k) \geq \frac{|v(x) - v(0)|}{|x|^\alpha}.
$$

Since $x \in B_{\alpha(1-\alpha)}(0)$ is arbitrary, this implies that

$$
\liminf_{k \to \infty} [v]_\alpha(x_k) \geq [v]_\alpha(0).
$$

Let us now prove the other inequality. By the definition of supremum, there is $y_k \in B_{\alpha(1-\alpha)}(0)$, such that

$$
\frac{|v(y_k) - v(x_k)|}{|y_k - x_k|^\alpha} \geq [v]_\alpha(x_k) - k^{-1}.
$$

Taking a subsequence if necessary, we assume $y_k \to y_\infty$. Passing to the inferior limit on both sides, we get

$$
\liminf_{k \to \infty} [v]_\alpha(x_k) \leq \frac{|v(y_\infty) - v(0)|}{|y_\infty|^\alpha} \leq [v]_\alpha(0).
$$

It then follows that

$$
\liminf_{k \to \infty} [v]_\alpha(x_k) = [v]_\alpha(0).
$$

We finally turn to the proof of the claim. If it is not true, we will find a subsequence $(x_{k_j})_{j \in \mathbb{N}}$, with $x_{k_j} \to 0$ as $j \to \infty$, with $[v]_\alpha(x_{k_j}) = \infty$, for every $j \in \mathbb{N}$. By the definition of supremum, for each large $L > 0$, we can find $y_{k_j} \in B_{\alpha(1-\alpha)}(x_{k_j})$ such that

$$
\frac{|v(y_{k_j}) - v(x_{k_j})|}{|y_{k_j} - x_{k_j}|^\alpha} \geq L.
$$

Taking a further subsequence if necessary, we can assume $y_{k_j} \to y_\infty$. Passing to the limit, we obtain

$$\frac{|v(y_\infty) - v(0)|}{|y_\infty|^\alpha} \geq L.$$

But since

$$\frac{|v(y_\infty) - v(0)|}{|y_\infty|^\alpha} \leq [v]_\alpha(0) < \infty,$$

we get a contradiction if we pick $L > [v]_\alpha(0)$. □

We conclude this section with a useful scaling property of the $\alpha-$bracket. We will state it as a lemma for further reference, introducing the notation

$$[v]_{\alpha,r}(x_0) := \sup_{x \in B_{\alpha(1-\alpha)r}(x_0)\setminus\{x_0\}} \frac{|v(x) - v(x_0)|}{|x - x_0|^\alpha}, \quad r > 0. \tag{2.3}$$

Lemma 2.4. *Let $v \in C(B_1)$ and $x_0 \in \mathcal{N}_{\alpha(1-\alpha)}(B_1)$. Given positive parameter λ and $\overline{\beta}$, the function $w(x) := \lambda^{-\overline{\beta}} v(x_0 + \lambda x)$ satisfies*

$$[v]_\alpha(x_0) = \lambda^{\overline{\beta}-\alpha}[w]_{\alpha,\lambda^{-1}}(0).$$

Proof. By direct computations, observe that

$$\begin{aligned} [w]_{\alpha,\lambda^{-1}}(0) &= \sup_{x \in B_{\lambda^{-1}\alpha(1-\alpha)}(0)\setminus\{0\}} \frac{|w(x) - w(0)|}{|x|^\alpha} \\ &= \sup_{x \in B_{\lambda^{-1}\alpha(1-\alpha)}(0)\setminus\{0\}} \frac{|v(x_0 + \lambda x) - v(x_0)|}{\lambda^{\overline{\beta}}|x|^\alpha}. \end{aligned}$$

Now if we replace $y = x_0 + \lambda x$, it follows that $y \in B_{\alpha(1-\alpha)}(x_0)$, and so

$$[w]_{\alpha,\lambda^{-1}}(0) = \sup_{y \in B_{\alpha(1-\alpha)}(x_0)\setminus\{x_0\}} \frac{|v(y) - v(x_0)|}{\lambda^{\overline{\beta}-\alpha}|y - x_0|^\alpha}.$$

The quantity in the right-hand side above is precisely $\lambda^{\alpha-\overline{\beta}}[v]_\alpha(x_0)$, from which the result follows. □

3. Problem formulation and preliminary findings

In this section, we discuss the basic setup to be used throughout this paper. First, since all the results are local, we may restrict ourselves to the case of the domain $\Omega = B_1$, the unit ball centred at the origin. We denote by $\mathrm{Sym}(n)$ the space of symmetric matrices of size $n \times n$ and, given constants $0 < \lambda \leq \Lambda$, we say that an operator $\mathcal{G}\colon \mathrm{Sym}(n) \to \mathbb{R}$ is $(\lambda, \Lambda)-$elliptic if it satisfies

$$\mathcal{M}^-_{\lambda,\Lambda}(M - N) \leq \mathcal{G}(M) - \mathcal{G}(N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M - N),$$

for all $M, N \in \mathrm{Sym}(n)$, where $\mathcal{M}^{+}_{\lambda,\Lambda}$ and $\mathcal{M}^{-}_{\lambda,\Lambda}$ stand for the *Pucci Extremal Operators* defined as

$$\begin{aligned}\mathcal{M}^{-}_{\lambda,\Lambda}(M) &\coloneqq \inf\{\mathrm{Tr}\,(AM) \,:\, \mathrm{spec}(A) \subseteq [\lambda,\Lambda]\}\\ \mathcal{M}^{+}_{\lambda,\Lambda}(M) &\coloneqq \sup\{\mathrm{Tr}\,(AM) \,:\, \mathrm{spec}(A) \subseteq [\lambda,\Lambda]\},\end{aligned}$$

where $\mathrm{spec}(A)$ denotes the set of eigenvalues of the matrix $A \in \mathrm{Sym}(n)$.

3.1. **Viscosity solutions.** To ease the notation, from this section onward, we define the function

$$[[v]]_\alpha \coloneqq (1-\alpha)|v| + [v]_\alpha.$$

We will briefly discuss the definition of viscosity solution to the problems treated here. It is enough to provide the definition for the more general model

$$([[u]]_\alpha - \kappa)^q_+ F(D^2 u) = f \in L^\infty(B_1) \cap C(B_1), \tag{3.1}$$

where F is a $(\lambda,\Lambda)-$elliptic fully nonlinear operator. We first bring out the concept of "touch from above" and "touch from below".

Definition 3.1. *We say that a function $\varphi \in C(B_1)$ touches u from below (above) at a point $x_0 \in B_1$, if there exists a neighborhood $V \subset B_1$ containing x_0 such that*

$$u \geq (\leq)\varphi \quad in \quad V\backslash\{x_0\} \quad and \quad u(x_0) = \varphi(x_0).$$

We say the touch is strict if we have instead the strict inequality.

We can now define the notion of viscosity solution to our problem (3.1).

Definition 3.2. *We say that $u \in C(B_1)$ is a $(\kappa,\alpha)-$viscosity supersolution (subsolution) to (3.1) if for every $\varphi \in C^2(B_2)$ touching u from below (above) at x_0, with $[[\varphi]]_\alpha(x_0) > \kappa$, we have*

$$([[\varphi]]_\alpha(x_0) - \kappa)^q\, F(D^2\varphi(x_0)) \leq (\geq) f(x_0).$$

We remark on the subtle importance of understanding objects in the viscosity sense, particularly because of the $\alpha-$bracket. This is to avoid pathological situations where stability may not be true, as in the sense described below.

Remark 3.1. *Let $u_k(x) = k^{-1}|x|^{\frac{\alpha}{2}}$. Observe that $[[u_k]]_\alpha(0) = \infty$ for every $k \in \mathbb{N}$, but u_k converges uniformly to $u_\infty \equiv 0$ that satisfies $[[u_\infty]]_\alpha(0) = 0$.*

On the other hand, if the objects are understood in the viscosity sense, then the $\alpha-$bracket will be stable under uniform limits. In particular, we have the following stability result.

Lemma 3.1. *Let $v_k \in C(B_1)$ be a sequence such that $v_k \to v_\infty$ uniformly in B_1. Assume*

$$([[v_k]]_\alpha - \kappa)_+^q F(D^2 v_k) = f_k,$$

in the viscosity sense in B_1, with $f_k \to f_\infty$ uniformly in B_1. Then, v_∞ solves

$$([[v_\infty]]_\alpha - \kappa)_+^q F(D^2 v_\infty) = f_\infty,$$

in the viscosity sense in B_1.

Proof. We prove only the subsolution side. Let ϕ be a C^2 function touching v_∞ from below at x_∞ in $B_r(x_\infty) \subset B_1$, with $[[\phi]]_\alpha(x_\infty) > \kappa$. By uniform convergence, we obtain $C_k \to 0$ such that $\phi_k := \phi + C_k$ will touch v_k from below at some point x_k. This sequence x_k will converge to x_∞. But then, since

$$[[\phi_k]]_\alpha(x_k) = (1-\alpha)|\phi(x_k) + C_k| + [\phi]_\alpha(x_k),$$

we have, from the proof of Lemma 2.3, that up to subsequence

$$[[\phi_k]]_\alpha(x_k) \to [[\phi]]_\alpha(x_\infty), \tag{3.2}$$

and so $[[\phi_k]]_\alpha(x_k) > \kappa$ for k sufficiently large. By assumption, there holds

$$([[\phi_k]]_\alpha(x_k) - \kappa)_+^q F(D^2\phi_k(x_k)) \le f_k(x_k).$$

Passing to the limit and using again (3.2), we get

$$([[\phi]]_\alpha(x_\infty) - \kappa)_+^q F(D^2\phi(x_\infty)) \le f_\infty(x_\infty).$$

The same could have been done if we had a touch from above, and this proves the lemma. □

3.2. $C^{0,\alpha}-$**Hölder regularity.** We will now discuss the $C^{0,\alpha}$ regularity of solutions to (3.1). Heuristically, this corresponds to the optimal regularity since any $\alpha-$Hölder continuous function u, with $[[u]]_\alpha \le \kappa$ at every point, automatically solves the equation with $f \equiv 0$. We prove this regularity in the following proposition.

Proposition 3.1. *Let u be a $(\kappa,\alpha)-$viscosity solution to* (3.1)*. Then, u is locally of class $C^{0,\alpha}$, and there exists a constant $C = C(n,\lambda,\Lambda,\alpha,\kappa)$ such that,*

$$\sup_{x,y\in B_{1/2}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\right).$$

Proof. Let L_1 and L_2 be positive constants to be chosen in the sequel and define

$$\Phi(x,y) := L_1\,\phi(|x-y|) + L_2(|x|^2+|y|^2),$$

where $\phi(r) = r^\alpha$. We will show that for large enough L_1 and L_2, the quantity

$$M := \max_{x,y\in B_{1/2}} \{u(x) - u(y) - \Phi(x,y)\}$$

is non-positive. Seeking a contradiction, we assume $M > 0$ and let $(x', y') \in B_{1/2} \times B_{1/2}$ be the points where the maximum is attained. First, observe that if we define

$$\begin{aligned} \Psi_+(x) &:= M + u(y') + \Phi(x, y') \\ \Psi_-(y) &:= -M + u(x') - \Phi(x', y), \end{aligned}$$

then Ψ_+ touches u from above at x' and Ψ_- touches u from below at y'. Moreover, we note that

$$L_2(|x'|^2 + |y'|^2) \leq \Phi(x', y') < u(x') - u(y'),$$

and so

$$\max\{|x'|, |y'|\} \leq \sqrt{\frac{2\|u\|_{L^\infty(B_1)}}{L_2}}.$$

Picking $L_2 > 0$ large enough, depending only on $\|u\|_{L^\infty(B_1)}$, we assure both x' and y' lies within the interior of $B_{1/2}$. Since $M > 0$, it also follows that $x' \neq y'$, and thus Φ is smooth in a neighborhood of (x', y'). Now, we choose L_1 large enough such that

$$L_1 > \frac{2\|u\|_{L^\infty(B_1)}}{(\alpha(1-\alpha))^\alpha}.$$

Since $M > 0$, it then follows that

$$|x' - y'| < \alpha(1 - \alpha).$$

The analysis now can be divided into two cases: if $[\Psi_+]_\alpha(x') \leq \kappa + 1$, then we have

$$\begin{aligned} L_1|x' - y'|^\alpha + L_2(|x'|^2 - |y'|^2) &= \Psi_+(x') - \Psi_+(y') \\ &\leq [\Psi_+]_\alpha(x') \\ &\leq (\kappa + 1)|x' - y'|^\alpha. \end{aligned}$$

Since $L_2(|x'|^2 - |y'|^2) \geq -2L_2|x' - y'|^\alpha$, we get

$$L_1|x' - y'|^\alpha \leq (\kappa + 1 + 2L_2)|x' - y'|^\alpha,$$

and so $L_1 \leq \kappa + 1 + 2L_2$. This leads to a contradiction once L_1 is chosen to be large enough, depending on L_2 and κ. The same contradiction would have been achieved if instead $[\Psi_-]_\alpha(y') \leq \kappa + 1$.

Now, we assume $[\Psi_+]_\alpha(x') > \kappa + 1$ and $[\Psi_-]_\alpha(y') > \kappa + 1$. In particular, the equation is available at those points since $[[\cdot]]_\alpha \geq [\cdot]_\alpha$. From Jensen-Ishii's Lemma [17, Theorem 3.2], given $\iota \in (0, 1)$, there exist $X, Y \in \mathcal{S}(n)$, such that

$$([[\Psi_+]]_\alpha(x') - \kappa)_+^q F(X) \geq f(x') \quad \text{and} \quad ([[\Psi_-]]_\alpha(y') - \kappa)_+^q F(Y) \leq f(y'),$$

which readily implies

$$F(X) \geq -\|f\|_{L^\infty(B_1)} \quad \text{and} \quad F(Y) \leq \|f\|_{L^\infty(B_1)}, \tag{3.3}$$

since

$$\min\left([[\Psi_+]]_\alpha(x'), [[\Psi_-]]_\alpha(y')\right) \geq 1 + \kappa.$$

In addition,

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + (2L_2 + \iota) I_{2n\times 2n}, \tag{3.4}$$

where $Z = L_1 D_x^2 \phi(|\cdot|)(\overline{x} - \overline{y})$. Estimate (3.4) applied to vectors (ξ, ξ), provides $spec(X - Y) \subset (-\infty, 4L_2 + 2\iota]$. On the other hand, now choosing $(\hat{\eta}, -\hat{\eta})$, for $\hat{\eta} = (\overline{x} - \overline{y})/|\overline{x} - \overline{y}|$, gives

$$\begin{aligned} (X - Y)\hat{\eta}\cdot\hat{\eta} &\leq 4Z\hat{\eta}\cdot\hat{\eta} + (4L_2 + 2\iota) \\ &= 4L_1\phi''(|\overline{x} - \overline{y}|) + 4L_2 + 2\iota. \end{aligned}$$

This implies that at least one eigenvalue of $(X - Y)$ should be less than

$$4L_1\phi''(|\overline{x} - \overline{y}|) + 4L_2 + 2\iota.$$

Therefore,

$$\begin{aligned} \mathcal{M}^+(X - Y) &\leq \Lambda(n-1)(4L_2 + 2\iota) + \lambda(4L_1\phi''(|\overline{x} - \overline{y}|) + 4L_2 + 2\iota) \\ &= n\Lambda(4L_2 + 2\iota) + 4\lambda L_1\phi''(|\overline{x} - \overline{y}|). \end{aligned}$$

From (3.3), we conclude $-2\|f\|_\infty \leq \mathcal{M}^+(X - Y)$ and so

$$L_1\phi''(|x' - y'|) \geq -C_0(n, \lambda, \Lambda, L_2),$$

for some $C_0 > 0$. Since $\phi(r) = r^\alpha$, we obtain

$$\alpha(1-\alpha)L_1 \leq \alpha(1-\alpha)L_1|x' - y'|^{\alpha-2} \leq C_0.$$

We get a contradiction by picking L_1 large enough, depending on C_0 and α. □

Remark 3.2. *The estimates established in this section depend on the governing operator F. Similar results can be derived for PDEs with coefficients, $F(x, D^2u)$, provided that $x \mapsto F(x, D^2u)$ is continuous. However, the estimates will inherently depend on the modulus of continuity of the operator. The primary goal of the next section is to achieve a universal $C^{0,\gamma}$ regularity estimate that remains independent of the smoothness of the coefficients.*

4. Uniform continuity estimates

The main contribution of this section is to establish a theorem of Krylov-Safonov type for differential inequalities of the form

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq C_0 \quad \text{and} \quad \mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq -C_0 \tag{4.1}$$

within $\{[u]_\alpha > \mu\} \cap B_1$, for some $C_0 > 0$ and $\mu > 0$. We remark that solutions will be understood in the viscosity sense, in the spirit of Definition

3.2, with $[[\cdot]]_\alpha$ replaced by $[\cdot]_\alpha$, which is a slightly weaker notion of solution since $[[\cdot]]_\alpha \geq [\cdot]_\alpha$. For simplicity, we will still refer to them as $(\mu, \alpha)-$viscosity solutions.

The estimates will depend solely on C_0, λ, Λ, the dimension, and μ, making them applicable to any uniformly elliptic fully nonlinear operator $F(x, D^2u)$, independently of the modulus of continuity of the coefficients.

4.1. **Scaling and approximation properties.** We start by discussing how scaling allows us to consider a normalized setting. Assuming u is a solution to (4.1), if we define, for $r > 0$,

$$v(x) = C_0^{-1}u(rx),$$

then v solves

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2v) \leq r^2 \quad \text{and} \quad \mathcal{M}^+_{\lambda,\Lambda}(D^2v) \geq -r^2,$$

within $\{[v]_{\alpha,r^{-1}} > \mu r^\alpha C_0^{-1}\} \cap B_{r^{-1}}$, where $[\cdot]_{\alpha,r^{-1}}$ is defined in (2.3) and we have used Lemma 2.4. We observe that, for $r < 1$ and $C_0 > 1$, the scaling v actually satisfies the inequalities in a larger set since

$$[v]_\alpha \leq [v]_{\alpha,r^{-1}}.$$

This scaling feature will play a key role when obtaining the Hölder estimate since the previous remarks allow us to deal with $(\nu, \alpha)-$viscosity solutions of

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq 1 \quad \text{and} \quad \mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq -1 \tag{4.2}$$

within $\{[u]_\alpha > \nu\} \cap B_1$, for a parameter ν small enough.

We next remark that this notion of solution is robust enough to allow the use of inf and sup convolutions, which are important tools in the viscosity theory.

Lemma 4.1. *Let u be a $(\nu, \alpha)-$viscosity solution to*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq 1 \quad in \quad \{[u]_\alpha > \nu\} \cap B_1.$$

Then the inf $-$*convolution u_ϵ given by*

$$u_\epsilon(x) := \inf_{y \in B_1} \left\{ u(y) + \frac{1}{2\epsilon}|y - x|^2 \right\},$$

is a $(\nu, \alpha)-$viscosity solution to

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u_\epsilon) \leq 1 \quad in \quad \{[u_\epsilon]_\alpha > \nu\} \cap V_\epsilon,$$

where $V_\epsilon := \{x \in B_1 \colon dist(x, \partial B_1) > 2\sqrt{\epsilon\|u\|_{L^\infty}}\}$. Similarly, if u is a $(\nu, \alpha)-$viscosity solution to

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq -1 \quad in \quad \{[u]_\alpha > \nu\} \cap B_1.$$

Then the $\sup -$*convolution* u^ϵ *given by*

$$u^\epsilon(x) := \sup_{y \in B_1} \left\{ u(y) - \frac{1}{2\epsilon}|y-x|^2 \right\},$$

is a $(\nu, \alpha)-$*viscosity solution to*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_\epsilon) \geq -1 \quad in \quad \{[u^\epsilon]_\alpha > \nu\} \cap V_\epsilon.$$

Proof. Let $\varphi \in C^2$ touch u_ϵ from below at $x_0 \in V_\epsilon$, with $[\varphi]_\alpha(x_0) > \nu$. Consider now the point $x_0^\star$ where $u_\epsilon(x_0)$ is attained. From the touch, we have

$$\varphi(x_0) = u_\epsilon(x_0) = u(x_0^\star) + \frac{1}{2\epsilon}|x_0 - x_0^\star|^2,$$

and, since u_ϵ is an infimum, we also have

$$\varphi(x) \leq u_\epsilon(x) \leq u(y) + \frac{1}{2\epsilon}|y - x_0|^2,$$

for every $x, y \in B_1$. The function Ψ, defined by

$$\Psi(x) := \varphi(x + x_0 - x_0^\star) - \frac{1}{2\epsilon}|x_0 - x_0^\star|^2,$$

touches u from below at $x_0^\star$. Moreover,

$$\frac{|\Psi(x) - \Psi(x_0^\star)|}{|x - x_0^\star|^\alpha} = \frac{|\varphi(x + x_0 - x_0^\star) - \varphi(x_0)|}{|x - x_0^\star|^\alpha} = \frac{|\varphi(x + x_0 - x_0^\star) - \varphi(x_0)|}{|x + x_0 - x_0^\star - x_0|^\alpha}.$$

Taking the supremum in $x \in B_{\alpha(1-\alpha)}(x_0^\star)$, we obtain

$$[\Psi]_\alpha(x_0^\star) = [\varphi]_\alpha(x_0),$$

and so $[\Psi]_\alpha(x_0^\star) > \nu$. By assumption, this implies

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2\Psi(x_0^\star)) \leq 1,$$

and the proof is done once we observe that $D^2\Psi(x_0^\star) = D^2\varphi(x_0)$. Similar arguments can be used to prove the second part of the lemma. □

4.2. **A positivity argument.** The program starts with a key positivity result: nonnegative continuous supersolutions that are large on a set of positive measure remain strictly positive in a smaller set. Building on this, a refinement of techniques from [24] is required to establish the result for classical C^2 supersolutions, while the extension to viscosity solutions is handled via approximation using inf and sup convolutions. In what follows, we bring the proof for C^2 functions, and the rest of the proof follows *mutatis mutandis* as in [24, Proposition 3.5], alongside with Lemma 4.1.

Lemma 4.2. *Let* u *be a nonnegative* $(\nu, \alpha)-$*viscosity solution to*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 u) \leq 1 \quad in \quad B_1.$$

There are positive constants ν_0, δ *and* M *such that, if* $\nu \leq \nu_0$ *and*

$$|\{u > M\} \cap B_1| > (1 - \delta)|B_1|,$$

then $u > 1$ *in* $B_{1/4}$.

Proof. As discussed before, we assume $u \in C^2$. Seeking a contradiction, assume that for all ν_0, δ, M, there is u satisfying the assumptions of the lemma, such that

$$u(x_0) \leq 1 \quad \text{for some} \quad x_0 \in B_{1/4}.$$

Let $V := \{u > M\} \cap B_{1/4}$ and consider $\psi(y) = \varpi |y|^{\alpha/2}$ for some constant ϖ satisfying

$$\varpi > \frac{1}{(3/4)^{\alpha/2} - (1/2)^{\alpha/2}}.$$

Given $x \in V$, slide $-\psi(y-x)$ until it touches u from below for the first time. In other words, let $y_x \in \overline{B}_1$ be the point where

$$\min_{y \in \overline{B}_1} \{u(y) + \psi(y - x)\}$$

is attained. Observe that since $u \geq 0$ and $x \in B_{1/4}$, it follows that

$$u(y) + \psi(y - x) \geq \varpi |y - x|^{\alpha/2} > \varpi (3/4)^{\alpha/2}, \quad \text{for} \quad y \in \partial B_1.$$

By the contradiction assumption, it follows that

$$u(x_0) + \psi(x_0 - x) \leq 1 + \varpi (1/2)^{\alpha/2},$$

and so

$$\min_{y \in \overline{B}_1} \{u(y) + \psi(y - x)\} \leq 1 + \varpi (1/2)^{\alpha/2}.$$

In particular, it follows that the point where the minimum is attained is in the interior since

$$1 + \varpi (1/2)^{\alpha/2} < \varpi (3/4)^{\alpha/2},$$

by the choice of ϖ. We denote it by y_x. Picking M large enough, it then follows that $u(y_x) < M$, which assures that $y_x \neq x$, since, by assumption, $u(x) > M$. As a consequence, $\psi(\cdot - x)$ is smooth near y_x.

Notice that from the C^2 assumption on u and the fact that y_x is an interior point where the minimum is attained, it follows that

$$D(u + \psi(\cdot - x))(y_x) = 0,$$

and so

$$Du(y_x) = -D\psi(y_x - x) \iff x = y_x + [D\psi]^{-1}(Du(y_x)). \tag{4.3}$$

This formula assures that the mapping $m \colon \mathcal{T} \to U$ given by $m(y_x) = x$ is well-defined, where

$$\mathcal{T} := \{y_x \in B_1 \colon \min\{u(y) + \psi(y - x)\} = u(y_x) - \psi(y_x - x), \quad \text{for } x \in V\},$$

is the set of points where the minimum is attained as we vary $x \in V$. From now on, for simplicity, we will drop the subscript when referring to points in the domain of m.

It follows from (4.3) that m is a Lipschitz map since $u \in C^2$. We also have that

$$D^2u(y) \succeq -D^2\psi(y - m(y)). \tag{4.4}$$

To ensure we can use the equation within $\mathcal{T}$, we need to guarantee that $[\psi]_\alpha(y - m(y))$ is large for every $y \in \mathcal{T}$. Since $y, m(y) \in B_1$, it is enough to show that

$$\inf_{z \in B_2} [\psi]_\alpha(z) > \nu_0. \tag{4.5}$$

To ease notation, let us define $r_\alpha = \alpha(1-\alpha)$ and consider $z \in B_2$. Notice that

$$\begin{aligned}
[\psi]_\alpha(z) &= \varpi \sup_{x \in B_{\alpha(1-\alpha)}(z)} \frac{||x|^{\alpha/2} - |z|^{\alpha/2}|}{|x-z|^\alpha} \\
&\geq \varpi \frac{||z + \frac{r_\alpha}{2}z|^{\alpha/2} - |z|^{\alpha/2}|}{|\frac{r_\alpha}{2}z|^\alpha} \\
&= \varpi |z|^{-\alpha/2} \frac{(1+\frac{r_\alpha}{2})^{\alpha/2} - 1}{(\frac{r_\alpha}{2})^\alpha} \\
&\geq 2^{-\alpha/2}\varpi \frac{(1+\frac{r_\alpha}{2})^{\alpha/2} - 1}{(\frac{r_\alpha}{2})^\alpha} = \mathcal{G}(\alpha) > 0,
\end{aligned}$$

where we used that if $z \in B_2$, then $(1 + r_\alpha/2)z \in B_{r_\alpha}(z)$. Picking $\nu_0 \leq \mathcal{G}(\alpha)$, we ensure (4.5) is true, and the equation is available within $\mathcal{T}$. Combining it with (4.4) and the fact that u is a supersolution, we obtain

$$|D^2u(y)| \leq C(1 + |D^2\psi(y-x)|),$$

for some constant C depending on λ, Λ and n. From (4.3), replacing $x = m(y)$, and applying the chain rule, it follows that

$$D^2u(y) = D^2\psi(y - m(y))(I - Dm(y)),$$

and so

$$|Dm(y)| \leq 1 + C\frac{1 + |D^2\psi(y-x)|}{|D^2\psi(y-x)|} \leq C_1.$$

By assumption, we estimate $(1 - 4^n\delta)|B_{1/4}| \leq |V|$. Moreover, it follows that

$$|V| = \int_V dx = \int_{\mathcal{T}} |\det Dm(y)| dy \leq C_1|\mathcal{T}|.$$

Since $\mathcal{T} \subset \{u < M\}$, we get

$$(1 - 4^n\delta)|B_{1/4}| \leq C_1|\mathcal{T}| \leq C_1\delta|B_{1/4}|.$$

This is a contradiction for δ small enough depending only on ellipticity constants and dimension. $\square$

4.3. **A doubling property and the $L^\epsilon-$estimate.** We will next prove a doubling property for supersolutions of our equations. This property is a type of positivity argument and begins with the construction of a special barrier function. As in [24], let $\phi(x) = |x|^{-p}$ for $p > 0$. By direct computations, if p is large enough, we have

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2\phi(x)) \geq p|x|^{-p-2} \quad \text{for} \quad x \neq 0. \tag{4.6}$$

This function also possesses a large $\alpha-$Hölder semi-norm. As we did before, for $z \in B_2$, we have

$$\begin{aligned} [\phi]_\alpha(z) &\geq \frac{||z+\alpha(1-\alpha)/2z|^{-p} - |z|^{-p}|}{|\alpha(1-\alpha)/2z|^\alpha} \\ &= |z|^{-p-\alpha}\frac{1-(1+r_\alpha/2)^{-p}}{r_\alpha^\alpha/2} \\ &\geq 2^{-p-\alpha}\left(\frac{1-(1+r_\alpha/2)^{-p}}{r_\alpha^\alpha/2}\right) \\ &=: g(\alpha), \end{aligned}$$

where $r_\alpha = \alpha(1-\alpha)$. Observe that $g(\alpha)$ is positive for every $\alpha > 0$, but, as $\alpha \to 1$, we also have $g(\alpha) \geq c$ for some universal constant (see the discussion at the end of this section).

Lemma 4.3. *Let u be a nonnegative $(\nu,\alpha)-$viscosity solution of*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq 1 \quad \textit{in} \quad B_2.$$

There are constants $M > 1$ and $\nu_0 > 0$ such that if $u > M$ in $B_{1/4}$ and $\nu \leq \nu_0$, then $u > 1$ in B_1.

Proof. Define

$$B(x) := M\frac{\phi(x) - 2^{-p}}{2\cdot 4^p},$$

where, by previous computations, $M > 1$ is chosen large enough and $\nu_0 > 0$ is chosen small enough so that $B > 1$ and $[B]_\alpha \geq \nu_0$ in B_1. We have $B = 0$ on ∂B_2 and $B < M$ on $\partial B_{1/4}$. Consequently, it follows that $B \leq u$ in the ring $\overline{B_2}\backslash B_{1/4}$. Indeed, if there exists $x_0 \in \overline{B_2}\backslash B_{1/4}$ such that $B(x_0) > u(x_0)$, then

$$\min_{\overline{B_2}\backslash B_{1/4}} \{u - B\} < 0,$$

and the minimum is attained in $B_2\backslash B_{1/4}$. Since $[B]_\alpha$ is large, the equation holds at the point of minimum, say x', implying that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2B)(x') \leq 1.$$

However, by (4.6), it follows that

$$p|x'|^{-p-2} \leq 1,$$

a contradiction for p large enough. Since $B > 1$, we obtain $u \geq B > 1$ in B_1. $\square$

The junction of both positivity lemmas leads to the following corollary.

Corollary 4.1. *Let u be a nonnegative $(\nu, \alpha)-$viscosity solution to*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq 1 \quad in \quad B_2.$$

There are positive constants ν_0, δ and $M > 1$ such that if $\nu \leq \nu_0$, and

$$|\{u > M\} \cap B_1| > (1 - \delta)|B_1|,$$

then $u > 1$ in B_1.

Proof. Indeed, let $M = M_1M_2$, where M_1 is from Lemma 4.2 and M_2 is from Lemma 4.3. It then follows that $v = u/M_2$ satisfies the assumption of Lemma 4.2, which implies $u > M_2$ in $B_{1/4}$. From Lemma 4.3, we will have $u > 1$ in B_1. $\square$

Scaling then leads to the next corollary.

Corollary 4.2. *Let u be a nonnegative $(\nu, \alpha)-$viscosity solution to*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq \eta \quad in \quad B_r,$$

for $r \leq 1$ and $\eta \geq 1$. There are positive constants ν_0, δ and $M > 1$ such that if $\nu \leq \nu_0$, and

$$|\{u > \eta M\} \cap B_{r/2}| > (1 - \delta)|B_{r/2}|,$$

then $u > \eta$ in $B_{r/2}$.

Proof. Consider the function u_r defined as

$$u_r(x) = \eta^{-1}u\left(\frac{r}{2}x\right).$$

From the scaling property (*cf.* subsection 4.1), this function satisfies

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u_r) \leq \left(\frac{r}{2}\right)^2 \quad \text{within} \quad \left\{[u_r]_{\alpha,\left(\frac{r}{2}\right)^{-1}} > \nu\left(\frac{r}{2}\right)^{\alpha}\eta^{-1}\right\},$$

and, in particular, within $\{[u_r]_\alpha > \nu\}$. We can then apply Corollary 4.1 to get $u_r > 1$ in B_1, from which the result follows. $\square$

A consequence of these results is the L^ϵ estimate, which we state in a scaled version for future purposes.

Lemma 4.4. *There exist small positive constants ν' and ϵ' such that if $\nu \leq \nu'$ and $\epsilon \leq \epsilon'$, and u is a nonnegative $(\nu, \alpha)-$viscosity solution to*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq \epsilon \quad in \quad B_{2r},$$

$$|\{u > r^\gamma\} \cap B_r| > \frac{1}{2}|B_r|,$$

for $r, \gamma \in (0, 1)$, then we have $u > \epsilon r^\gamma$ in B_r.

Proof. The proof is a consequence of the previous corollaries and follows the same lines as the proof of [24, Lemma 5.3]. □

It is then standard, see [24, Proof of Theorem 1.1], that this implies Hölder continuity, which we state for the more general problem (4.1).

Theorem 4.1. *Let u be a $(\mu,\alpha)-$viscosity solution to* (4.1)*. Then, there exists $\gamma \in (0,1)$ and a constant $C > 0$, depending only on dimension, ellipticity, α and μ, such that*

$$\|u\|_{C^{0,\gamma}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + C_0\right).$$

4.4. **Connection with the $\alpha-$bracket problem and uniform continuity.** We dedicate the final part of this section to explain how one connects the $\alpha-$bracket problem with the theory developed herein and briefly discuss uniform continuity in terms of the parameter α.

Let u be a viscosity solution of (3.1). Observe that if $\varphi \in C^2$ touches u from below at x_0, with $[\varphi]_\alpha(x_0) > \kappa + 1$, then, since $[[\cdot]]_\alpha \geq [\cdot]_\alpha$, we also have $[[\varphi]]_\alpha(x_0) > \kappa + 1$ and thus

$$([[\varphi]]_\alpha(x_0) - \kappa)^q F(x_0, D^2\varphi(x_0)) \leq f(x_0).$$

Therefore,

$$F(x_0, D^2\varphi(x_0)) \leq ([[\varphi]]_\alpha(x_0) - \kappa)^{-q}\|f\|_{L^\infty(B_1)} \leq \|f\|_{L^\infty(B_1)}.$$

Since F is $(\lambda,\Lambda)-$elliptic in the matrix variable, it follows that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2\varphi(x_0)) \leq \|f\|_{L^\infty(B_1)} + |F(x_0,0)|.$$

Likewise, if $\varphi \in C^2$ touches u from above with $[\varphi]_\alpha(x_0) > \kappa + 1$, we would have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi(x_0)) \geq -\left(\|f\|_{L^\infty(B_1)} + |F(\cdot,0)|_{L^\infty(B_1)}\right).$$

An immediate consequence is that a solution of (3.1) with bounded measurable coefficients also solves (4.1) with $C_0 = \|f\|_{L^\infty(B_1)} + |F(\cdot,0)|_{L^\infty(B_1)}$ and $\mu = \kappa + 1$.

We conclude with a remark on the stability of the estimates as the parameter α approaches its extrema. First, we observe that no uniformity in the estimates is expected as $\alpha \to 0$ since this limit corresponds to functions that solve a PDE only where $|u| > \kappa$, with no further information available for $|u| \leq \kappa$. However, our analysis reveals that uniform-in-α estimates can still be obtained, provided α remains bounded away from zero. In particular, stability is ensured as $\alpha \to 1$, and we briefly outline the reasoning behind this conclusion.

The heart of the matter lies within scaling, Lemma 4.2 and Lemma 4.3, which leads to the uniform-in-α L^ϵ estimate. In Lemma 4.2, the step that needs to be done carefully is precisely when estimating (4.5) from below.

Indeed, recall the notation $r_\alpha = \alpha(1-\alpha)$ and consider $z \in B_2$. In the proof of Lemma 4.2, we have shown that

$$[\psi]_\alpha(z) \quad \geq \quad 2^{-\alpha/2}\varpi\,\frac{(1+\frac{r_\alpha}{2})^{\alpha/2}-1}{(\frac{r_\alpha}{2})^\alpha} = \mathcal{G}(\alpha),$$

where

$$\varpi > \frac{1}{(3/4)^{\alpha/2}-(1/2)^{\alpha/2}}.$$

In the event that α is bounded from below, say $\alpha \geq \alpha_0$, the function $\mathcal{G}$ can also be bounded from below by a positive constant, uniformly in α. Indeed,

$$\begin{aligned} \mathcal{G}(\alpha) \quad &\geq \quad \frac{\sqrt{2}}{2}\varpi\frac{\left[(1+\frac{r_\alpha}{2})^{\alpha/2}-1\right]}{(\frac{r_\alpha}{2})^\alpha} \\ &\geq \quad \frac{\sqrt{2}}{2}\frac{1}{(3/4)^{\alpha_0/2}-(1/2)^{\alpha_0/2}}\left(\frac{(1+\frac{r_\alpha}{2})^{\alpha/2}-1}{(\frac{r_\alpha}{2})^\alpha}\right). \end{aligned}$$

The function on the right-hand side above is continuous in α, and none of the ingredients vanishes as long as $\alpha < 1$. Therefore, if $\alpha \in [\alpha_0, \alpha_1]$ for some α_1 close enough to 1, then

$$\mathcal{G}(\alpha) \geq c_1,$$

for some $c_1 = c_1(\alpha_0, \alpha_1)$. On the other hand, observe that by direct computations

$$\lim_{\alpha\to 1}\frac{\left[(1+\frac{r_\alpha}{2})^{\alpha/2}-1\right]}{(\frac{r_\alpha}{2})^\alpha} = \frac{1}{\sqrt{2}}\lim_{s\to 0}\left(\frac{\left(\frac{2}{1-s}+s\right)^{\frac{1-s}{2}}-\left(\frac{2}{1-s}\right)^{\frac{1-s}{2}}}{s}\right) = \frac{1}{4}.$$

Putting all together, we get that there exists a constant $c > 0$, depending on α_0, such that

$$\mathcal{G}(\alpha) \geq c,$$

for every $\alpha \in [\alpha_0, 1]$. The same reasoning applies to Lemma 4.3.

5. $C^{1,\beta}$–REGULARITY FOR $\kappa = 0$

In this section, we discuss the regularity theory of $(0,\alpha)$–viscosity solutions, that is functions that satisfy

$$[[u]]_\alpha^q F(D^2u) = f, \tag{5.1}$$

in the viscosity sense (*cf.* Definition 3.2). We will assume $F(0) = 0$ for simplicity. Our goal here is to seek gradient estimates that are independent of the magnitude of $[[u]]_\alpha$. We are also interested in their behaviour as $\alpha \to 1$.

Scaling analysis provides insight into the regularity regime that can be expected. For instance, if one considers $v(x) = u(\lambda x)\lambda^{-(1+\beta)}$ and picks the parameters so that the equation is invariant, then it can be seen that β has to satisfy

$$1+\beta \leq \frac{2+\alpha q}{1+q} = 1 + \frac{1-q(1-\alpha)}{1+q}.$$

Observe that this regularity regime interpolates the endpoint cases for which a local degenerate problem appears. Notice, as well, that we need to assume $1-q(1-\alpha) > 0$, *i.e.*,

$$q < \frac{1}{1-\alpha},$$

so we are in a regime where gradient regularity is available.

The program to prove gradient regularity for such solutions begins by seeking compactness estimates for solutions of an auxiliary problem with a shift in the zero-order term. More precisely, we define

$$w(x) = r^{-(1+\beta)}[u(rx) - \xi \cdot (rx)].$$

It can be checked that w solves

$$[[w + r^{-\beta}\xi \cdot (-)]]^q_{\alpha,r^{-1}} F_r(D^2 w) = f_r,$$

where $F_r(M) \coloneqq r^{1-\beta}F(r^{\beta-1}M)$, $f_r(x) = r^{q(-1-\beta+\alpha)+1-\beta}f(rx)$, and

$$[[v]]_{\alpha,r^{-1}}(x) = (1-\alpha)\left|v(r^{-1}x)\right| + [v]_{\alpha,r^{-1}}(x).$$

To obtain continuity for such solutions, we will have to consider two cases corresponding to when the shift term is large or small.

To ease notation, whenever it is clear that the shift in the zero-order term is a vector, we will write $[[w+\xi]]_\alpha$ instead of $[[w+\xi\cdot(-)]]_\alpha$.

Proposition 5.1. *Let $r \in (0,1)$, and $w \in C(B_{r^{-1}})$ be a viscosity solution to*

$$[[w+\vec{p}]]^q_{\alpha,r^{-1}} F(D^2 w) = f \quad \textit{in} \quad B_1.$$

If $|\vec{p}| \leq c_0$, for some $c_0 > 0$, then there exist $\gamma \in (0,1)$ and $C > 0$, depending only on dimension, ellipticity, α and c_0, such that

$$\|w\|_{C^{0,\gamma}(B_{1/2})} \leq C\left(\|w\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\right).$$

Proof. Define $\mu \coloneqq 1 + c_0(\alpha(1-\alpha))^{1-\alpha}$. We claim that w is a $(\mu,\alpha)-$viscosity solution to (4.1), with $C_0 = \|f\|_{L^\infty(B_1)}$. Indeed, if $[w]_\alpha > \mu$, then by the following chain of inequalities

$$[[w+\vec{p}]]_{\alpha,r^{-1}} \geq [[w+\vec{p}]]_\alpha \geq [w]_\alpha - c_0(\alpha(1-\alpha))^{1-\alpha},$$

it follows that $[[w+\vec{p}]]_{\alpha,r^{-1}} \geq 1$. Therefore,

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 w) \leq F(D^2 w) \leq \|f\|_{L^\infty(B_1)}[[w+\vec{p}]]^{-q}_{\alpha,r^{-1}} \leq C_0.$$

Likewise,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) \geq -C_0,$$

and thus w is entitled to Theorem 4.1 and the result follows. ∎

For large slopes, we have the following result.

Proposition 5.2. *Let $r \in (0,1)$, and $w \in C(B_{r^{-1}})$ be a viscosity solution to*

$$[[w+\vec{p}]]^q_{\alpha,r^{-1}} F(D^2 w) = f \quad in \quad B_1.$$

There exists $c_0 > 0$ large such that if $|\vec{p}| \geq c_0$, then

$$\|w\|_{C^{0,1}(B_{1/2})} \leq C\left(\|w\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\right).$$

Proof. We apply once more the Ishii-Lions method, as in the proof of Proposition 3.1, with the key difference that we do not need to localize further the points where the maximum is attained. Recall that the first argument, the one dealing with points where the maximum was attained in the region where no PDE was in place, made the estimate blow up when $\alpha \to 1$.

Let L_1 and L_2 be positive constants to be chosen in the sequel and define

$$\Phi(x,y) := L_1\,\phi(|x-y|) + L_2(|x|^2+|y|^2),$$

where $\phi(r) = r - \theta r^{3/2}$. We will show that for large enough constants L_1 and L_2, the quantity

$$M := \max_{x,y\in B_{1/2}} \{u(x) - u(y) - \Phi(x,y)\}$$

is non-positive. Seeking a contradiction, we assume $M > 0$ and let $(x',y') \in B_{1/2}\times B_{1/2}$ be the points where the maximum is attained. Observe that

$$L_2(|x'|^2+|y'|^2) \leq \Phi(x',y') < u(x') - u(y'),$$

and so

$$\max\{|x'|,|y'|\} \leq \sqrt{\frac{2\|u\|_{L^\infty(B_1)}}{L_2}}.$$

Picking $L_2 > 0$ large enough, depending only on $\|u\|_{L^\infty(B_1)}$, we assure both x' and y' lie within the interior of $B_{1/2}$. Since $M > 0$, it also follows that $x' \neq y'$, and thus Φ is smooth in a neighborhood of (x',y').

Notice that

$$\begin{aligned}
[[w+\vec{p}]]_{\alpha,r^{-1}}(y') &\geq [w+\vec{p}]_\alpha(y') \\
&\geq \frac{w\left(y' + \alpha(1-\alpha)\frac{\vec{p}}{|\vec{p}|}\right) - w(y')}{(\alpha(1-\alpha))^\alpha} + |\vec{p}|(\alpha(1-\alpha))^{1-\alpha} \\
&\geq \frac{\Phi\left(x', y' + \alpha(1-\alpha)\frac{\vec{p}}{|\vec{p}|}\right) - \Phi(x',y')}{(\alpha(1-\alpha))^\alpha} + c_0(\alpha(1-\alpha))^{1-\alpha} \\
&\geq \left(c_0 - [\Phi(x',-)]_{C^{0,1}}\right)(\alpha(1-\alpha))^{1-\alpha}.
\end{aligned}$$

Now, recalling the definition of Φ, one can explicitly calculate $[\Phi(x',-)]_{C^{0,1}}$ and get that $[[w+\vec{p}]]_{\alpha,r^{-1}}(y') \geq 1$ if c_0 is chosen large enough, depending on L_1, L_2. In particular, this choice is uniform as $\alpha \to 1$. The rest of the proof follows along the lines of the proof of Proposition 3.1. $\square$

Remark 5.1. *In Proposition 5.2, since the shift is large, we could divide both sides by $|\vec{p}|^q$ to get the following PDE*

$$[[\theta w + \vec{e}]]^q_{\alpha,r^{-1}} F(D^2 w) = |\vec{p}|^{-q} f,$$

where $\theta = |\vec{p}|^{-1}$. The reasoning to get the Lipschitz estimate would have been the same, except that the parameter θ would be small. This remark is useful for the reasoning in the following lemma.

In what follows, we prove a flatness lemma. To ease notation, we introduce

$$\mathcal{F}_{n,\lambda,\Lambda} := \left\{ u \in C(\overline{B_1}) \;\middle|\; \begin{array}{c} F(D^2u) = 0 \,\text{in the viscosity sense in}\, B_1 \,\text{for} \\ \text{some } (\lambda,\Lambda)\text{−elliptic operator}\, F\colon \mathrm{Sym}(n) \to \mathbb{R} \end{array} \right\}.$$

Lemma 5.1. *Let $r \in (0,1)$, and $w \in C(B_{r^{-1}})$ be a viscosity solution to*

$$[[w + \vec{p}]]^q_{\alpha,r^{-1}} F(D^2 w) = f \quad in \quad B_1.$$

Given $\delta > 0$, there exists $\epsilon > 0$ small such that if

$$\|w\|_{L^\infty(B_1)} \le 1 \quad and \quad \|f\|_{L^\infty(B_1)} \le \epsilon,$$

then we can find $h \in \mathcal{F}_{n,\lambda,\Lambda}$ such that

$$\|u - h\|_{L^\infty(B_1)} \le \delta.$$

Proof. Assume, seeking a contradiction, this is not true. There would be $\delta_0 > 0$ and a sequence $(r_k, w_k, \vec{p}_k, f_k, F_k)_{k\in\mathbb{N}}$ such that

$$[[w_k + \vec{p}_k]]^q_{\alpha,r_k^{-1}} F_k(D^2 w_k) = f_k,$$

with $\|f_k\|_{L^\infty(B_1)} \le k^{-1}$ and $\|w_k\|_{L^\infty(B_1)} \le 1$, but

$$\mathrm{dist}\,[w_k, \mathcal{F}_{n,\lambda,\Lambda}] \ge \delta_0, \tag{5.2}$$

for every $k \in \mathbb{N}$.

If we can extract a convergent subsequence of $\vec{p}_k$, then we also do it for w_k and F_k. This is possible by equicontinuity of F_k, which is due to the (λ,Λ)−ellipticity assumption and Proposition 5.1 or Proposition 5.2 applied to w_k, depending on how large is the vector sequence. Let us label their limits as $\vec{p}$, w and F.

We claim that w solves

$$F(D^2 w) = 0 \quad \text{in} \quad B_1, \tag{5.3}$$

in the viscosity sense. Indeed, let φ be a second-order polynomial touching w from below at x_0. If $[[\varphi]]_\alpha(x_0) = 0$, then $\varphi \equiv 0$ in $B_{\alpha(1-\alpha)}(x_0)$ and so

$$F(D^2\varphi(x_0)) = 0$$

trivially, since $F(0) = 0$. We may then assume $[[\varphi]]_\alpha(x_0) > 0$. By uniform convergence, there exists $C_k \to 0$ such that $\varphi = \varphi + C_k$ touches w_k from below at x_k with $\varphi_k \to \varphi$ and $x_k \to x_0$. By Lemma 2.3, $[[\cdot]]_\alpha$ is lower semi-continuous, from which follows that $[[\varphi_k]]_\alpha(x_k)$ is also positive and bounded from below. Using the equation for w_k, we get

$$\begin{aligned} F_k(D^2\varphi(x_k)) &\leq \|f_k\|_{L^\infty(B_1)}[[\varphi_k + \vec{p}_k]]^{-q}_{\alpha, r_k^{-1}}(x_k) \\ &\leq k^{-1}[[\varphi + \vec{p}_k]]^{-q}_\alpha. \end{aligned}$$

Passing to the limit as $k \to \infty$, we have

$$F(D^2\varphi(x_0)) \leq 0.$$

Reversing the inequalities, we obtain $F(D^2 w) \geq 0$ in the viscosity sense, and so (5.3) follows.

This leads to a contradiction for large k. Now, if we cannot extract a convergent subsequence of $\vec{p}_k$, this means that $|\vec{p}_k| \to \infty$. Dividing the equation by $\theta_k^q = 1/|\vec{p}_k|^q$ we write

$$[[\theta_k w_k + \vec{e}_k]]^q_{\alpha, r_k^{-1}} F_k(D^2 w_k) = \theta_k^q f_k.$$

The sequence w_k is Lipschitz by Remark 5.1, and passing to a subsequence if necessary, we get

$$F(D^2 w) = 0,$$

in the viscosity sense in B_1 by the very same arguments. This then leads to a contraction with (5.2) for k large enough. □

Now, gradient regularity for solutions to (5.1) follows by somewhat standard techniques (see, for instance, [13]), and thus, we omit the proof of our ultimate result.

Theorem 5.1. *Let u be a viscosity solution to* (5.1). *Then, $u \in C^{1,\beta}(B_{1/2})$, with*

$$\beta = \min\left(\alpha_F^-, \frac{1 - q(1-\alpha)}{1+q}\right),$$

where α_F denotes the maximal Hölder regularity exponent for the gradient of $F-$harmonic functions. Furthermore, there exists a constant $C = C(n, \lambda, \Lambda, \alpha, q)$, such that the following estimate holds

$$\|u\|_{C^{1,\beta}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|^{\frac{1}{1+q}}_{L^\infty(B_1)}\right).$$

Finally,

$$\lim_{\alpha \to 1} C(n, \lambda, \Lambda, \alpha, q) < \infty.$$

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References

[1] H.W. Alt and D. Phillips, *A free boundary problem for semilinear elliptic equations*, J. Reine Angew. Math. **368** (1986), 63–107.

[2] P. Andrade, D. Pellegrino, E. A. Pimentel and E. V. Teixeira, *C^1-regularity for degenerate diffusion equations*, Adv. Math. **409** (2022), Paper No. 108667, 34 pp.

[3] D.J. Araújo, G. Ricarte and E. V. Teixeira, *Geometric gradient estimates for solutions to degenerate elliptic equations*, Calc. Var. Partial Differential Equations **53** (2015), no. 3, 605-625.

[4] D.J. Araújo, A. Sobral and E. V. Teixeira, *Regularity in diffusion problems with gradient activation*, Preprint, arXiv:2401.07979 (2024).

[5] D.J. Araújo, A. Sobral, E. V. Teixeira and J.M. Urbano, *On free boundary problems shaped by oscillatory singularities*, Preprint, arXiv:2401.08071 (2024).

[6] D.J. Araújo and E. V. Teixeira, *Geometric approach to nonvariational singular elliptic equations*, Arch. Ration. Mech. Anal. **209** (2013), no. 3, 1019–1054.

[7] S. Baasandorj, S.-S. Byun, K.-A. Lee and S.-C. Lee, *$C^{1,\alpha}-$regularity for a class of degenerate/singular fully nonlinear elliptic equations*, Interfaces Free Bound. **26** (2024), no. 2, 189–215.

[8] I. Birindelli and F. Demengel, *Comparison principle and Liouville type results for singular fully nonlinear operators*, Ann. Fac. Sci. Toulouse Math. (6) **13** (2004), no. 2, 261–287.

[9] I. Birindelli and F. Demengel, *Eigenvalue, maximum principle and regularity for fully nonlinear homogeneous operators*, Commun. Pure Appl. Anal. **6** (2007), no. 2, 335–366.

[10] I. Birindelli and F. Demengel, *Eigenvalue and Dirichlet problem for fully-nonlinear operators in non-smooth domains*, J. Math. Anal. Appl. **352** (2009), no. 2, 822–835.

[11] I. Birindelli and F. Demengel, *Regularity and uniqueness of the first eigenfunction for singular fully nonlinear operators*, J. Differential Equations **249** (2010), no. 5, 1089–1110.

[12] I. Birindelli and F. Demengel, *Regularity for radial solutions of degenerate fully nonlinear equations*, Nonlinear Anal. **75** (2012), no. 17, 6237–6249.

[13] A.C. Bronzi, E.A. Pimentel, G.C. Rampasso and E.V. Teixeira, *Regularity of solutions to a class of variable-exponent fully nonlinear elliptic equations*, J. Funct. Anal. **279** (2020), no. 12, 108781, 31 pp.

[14] L.A. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. of Math. (2) **130** (1989), no. 1, 189-213.

[15] L.A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, 43, Amer. Math. Soc., Providence, RI, 1995.

[16] M. G. Crandall, *A visit with the $\infty-$Laplace equation*, in Calculus of variations and nonlinear partial differential equations, Lecture Notes in Math. 1927, Springer, Berlin (2008), 75–122.

[17] M. G. Crandall, H. Ishii and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 1–67.

[18] D. De Silva and O. Savin, *On certain degenerate one-phase free boundary problems*, SIAM J. Math. Anal. **53** (2021), no. 1, 649–680.

[19] D. De Silva and O. Savin, *The Alt-Philips functional for negative powers*, Bull. London Math. Soc. **55** (2023), 2749–2777.

[20] D. De Silva and O. Savin, *Uniform density estimates and* $\Gamma-$*convergence for the Alt-Phillips functional of negative powers*, Math. Eng. **5** (2023), no. 5, Paper No. 086, 27 pp.

[21] D. De Silva and O. Savin, *Compactness estimates for minimizers of the Alt-Phillips functional of negative exponents*, Adv. Nonlinear Stud. **23** (2023), no. 1, Paper No. 20220055, 19 pp.

[22] M. Giaquinta and E. Giusti, *Differentiability of minima of nondifferentiable functionals.* Invent. Math. **72** (1983), no. 2, 285-298.

[23] C. Imbert and L. Silvestre, $C^{1,\alpha}$ *regularity of solutions of some degenerate fully nonlinear elliptic equations*, Adv. Math. **233** (2013), no. 1, 196-206.

[24] C. Imbert and L. Silvestre, *Estimates on elliptic equations that hold only where the gradient is large*, J. Eur. Math. Soc. **18** (2016), no. 6, 1321-1338.

[25] A. Karakhanyan, *Regularity for the two-phase singular perturbation problems*, Proc. Lond. Math. Soc. (3) **123** (2021), no. 5, 433–459.

[26] K.-A. Lee, S.-C. Lee and H. Yun, $C^{1,\alpha}-$*regularity for solutions of degenerate/singular fully nonlinear parabolic equations*, J. Math. Pures Appl. (9) **181** (2024), 152–189.

[27] D. Phillips, *A minimization problem and the regularity of solutions in the presence of a free boundary*, Indiana Univ. Math. J. **32** (1983), no. 1, 1–17.

[28] Y. Wu and H. Yu, *On the fully nonlinear Alt-Phillips equation*, Int. Math. Res. Not. IMRN (2022), no.11, 8540–8570.

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