Solution of quaternion equations with imprecisely defined coefficients

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Abstract. This study examines quaternion polynomial equations with imprecisely defined coefficients for the first time. We define the closed quaternion ball to resolve these equations and demonstrate certain aspects pertaining to its arithmetic. Ultimately, we examine some specific equations and derive some pertinent results associated with them. The last equation will yield the de Moîvre formula for the n-th roots of a closed quaternion ball.

Keywords: Closed quaternion ball, algebraic structure, quaternion polynomial equation

1 Introduction

A quaternion polynomial equation is an equation of type $f(x) = g(x)$, where

$$f(x) = x^m + \boldsymbol{a}_{m-1}x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$$

and

$$g(x) = x^n + \boldsymbol{b}_{n-1}x^{n-1} + \cdots + \boldsymbol{b}_1 x + \boldsymbol{b}_0,$$

are quaternion polynomials. The solution set

$$S = \{x \in \mathbb{H} \ : \ f(x) = g(x)\}\,.$$

is composed by the zeros of the quaternion polynomial $p(x) = f(x) - g(x)$, which we know how to compute. This problem has received a lot of attention in the last years, see for example [13] for some historical background.

Motivated by [3–6, 11, 12, 15–18], we pondered the scenario in which some coefficients are defined with imprecision. What if we are uncertain about the precise values of certain coefficients, just aware that they are within a specific range? This set may be difficult to locate and represent. Therefore, we will select a closed quaternion ball that encompasses S. We will employ closed quaternion ball arithmetic to identify and represent this set. Until now, and as far as we know, no attention has been paid when the coefficients and the independent terms are uncertain in the quaternion case. The uncertainty can be expressed by an interval of the type $[a, b]$ for each component of the quaternion, i.e., considering a hypercube, or taking a quaternion as the center of a hypersphere. We will consider the second hypothesis.

In this work, operations on closed quaternion balls are considered in section 2. First, we start with a revision on quaternion algebra, focusing on quaternion polynomials and their zeros; this is given in Subsection 2.1. In subsection 2.2, we define closed quaternion balls and give two binary operations: addition and multiplication; some of their properties are established. Finally, in Section 3 we present some results related to some specific equations with imprecisely defined coefficients and independent term. Some examples are given.

2 Preliminaries

This section presents preliminary results on quaternion algebras and closed quaternion balls.

2.1 Quaternion Algebra

Let $\mathbb{H} = \{q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$ be the quaternion field, where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$, $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$, $\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}$, and $\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}$. For $\boldsymbol{q} = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} \in \mathbb{H}$, the conjugate of $\boldsymbol{q}$ is defined as $\overline{\boldsymbol{q}} = q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}$. Thus, q_0, the real part of $\boldsymbol{q}$, denoted by $Re(\boldsymbol{q})$, is given by $q_0 = (\boldsymbol{q} + \overline{\boldsymbol{q}})/2$ and $\boldsymbol{q}\overline{\boldsymbol{q}} = \overline{\boldsymbol{q}}\boldsymbol{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 \in \mathbb{R}$. The norm of $\boldsymbol{q}$, denoted by $|\boldsymbol{q}|$, is defined by $\sqrt{\boldsymbol{q}\overline{\boldsymbol{q}}}$. If $\boldsymbol{q} \neq \boldsymbol{0}$, then $\boldsymbol{q}$ has the inverse, and it is given by $\boldsymbol{q}^{-1} = \overline{\boldsymbol{q}}/\,|\boldsymbol{q}|$.

We list some basic properties and definitions.

Proposition 1. *For any* $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{H}$, $\overline{\boldsymbol{q}_1 + \boldsymbol{q}_2} = \overline{\boldsymbol{q}_1} + \overline{\boldsymbol{q}_2}$, *and* $\overline{\boldsymbol{q}_1\boldsymbol{q}_2} = \overline{\boldsymbol{q}_2}\ \overline{\boldsymbol{q}_1}$.

In $\mathbb{H}$ we introduce the equivalence relation of similarity.

Definition 1. *Given* $\boldsymbol{q}, \boldsymbol{q}' \in \mathbb{H}, \boldsymbol{q} \sim \boldsymbol{q}'$ *if there exists* $\sigma \in \mathbb{H}$ *such that* $\boldsymbol{q}' = \sigma\boldsymbol{q}\sigma^{-1}$.

If $\boldsymbol{q} \sim \boldsymbol{q}'$, then we say that $\boldsymbol{q}$ is **similar** to $\boldsymbol{q}'$. The relation $\sim$ in Definition 1 is an equivalence relation in $\mathbb{H}$. The **conjugacy class** of $\boldsymbol{q} \in \mathbb{H}$ is defined by $[\boldsymbol{q}] = \{x \in \mathbb{H} : x \sim \boldsymbol{q}\}$. All quaternions in the conjugacy class $[\boldsymbol{q}]$ satisfy the characterisitc polynomial $\Delta_{\boldsymbol{q}(x)} = x^2 - t(\boldsymbol{q})x + |\boldsymbol{q}|^2$.

Let $\mathbb{H}[x]$ denote the ring of unilateral left polynomials in one variable x over $\mathbb{H}$, where x commutes elementwise with $\mathbb{H}$. If $p(x) = \sum_{i=0}^{n} \boldsymbol{a}_i x^i \in \mathbb{H}[x]$ with $\boldsymbol{a}_n \neq 0$, we say that $p(x)$ is a polynomial of degree n. If $\boldsymbol{a}_n = 1$, we say that the polynomial $p(x)$ is monic. The conjugate of a quaternion polynomial $p(x) = \sum_{i=0}^{n} \boldsymbol{a}_i x^i$ is defined by $\overline{p}(x) = \sum_{i=0}^{n} \overline{\boldsymbol{a}}_i x^i$.

Given $p(x) \in \mathbb{H}[x]$ and an element $\boldsymbol{q} \in \mathbb{H}$, we define the **evaluation** of $p(x)$ at $\boldsymbol{q}$, to be the element $p(\boldsymbol{q}) = \sum_{i=0}^{n} \boldsymbol{a}_i \boldsymbol{q}^i \in \mathbb{H}$. To evaluate $p(\boldsymbol{q})$, we first have to express $p(x)$ as $\sum_{i=0}^{n} \boldsymbol{a}_i x^i$ and then substitute x by $\boldsymbol{q}$ (see p. 262 in [7]).

Addition and multiplication of polynomials in $\mathbb{H}[x]$ are defined as in the commutative case where the variable commutes with the coefficients. Note that although if $f(x), g(x) \in \mathbb{H}[x]$, then $p(x) = f(x)g(x) \in \mathbb{H}[x]$, but it does not generally follow that $p(\boldsymbol{q}) = f(\boldsymbol{q})g(\boldsymbol{q})$. If $\boldsymbol{q} \in \mathbb{H}$ is such that $p(\boldsymbol{q}) = 0$, then $\boldsymbol{q}$ is said to be a **zero** of $p(x)$. The set of all zeros of p is denoted by $Zero(p)$.

We remark that since $\boldsymbol{a}_n \neq 0$, $p(\boldsymbol{q}) = 0$ if and only if $\boldsymbol{a}_n^{-1}p(\boldsymbol{q}) = 0$. Thus, for the sake of simplicity, we can always assume that the polynomial is monic.

We now present some results regarding quaternion polynomials.

Let $p(x), q(x), r(x) \in \mathbb{H}[X]$. We define $r(x)$ as a **right divisor** of $p(x)$ if there exists a polynomial $r(x)$ such that $p(x) = q(x)r(x)$.

Theorem 1 (pp 262 in [7]). *Let $\boldsymbol{q} \in \mathbb{H}$ and $p(x) \in \mathbb{H}[x]$. Then $\boldsymbol{q} \in Zero(p)$ if and only if $x - \boldsymbol{q}$ is a right divisor of $p(x)$.*

Theorem 2 (pp 263 in [7]). *Let $\boldsymbol{q} \in \mathbb{H}$ and $p(x), q(x), r(x) \in \mathbb{H}[X]$ such that $p(x) = q(x)r(x)$. If $\boldsymbol{\gamma} = r(\boldsymbol{q}) \neq 0$, then*

$$p(\boldsymbol{q}) = q(\boldsymbol{\gamma}\boldsymbol{q}\boldsymbol{\gamma}^{-1})r(\boldsymbol{q}).$$

Theorem 3 (Wedderburn's Theorem [7]). *All non-constant quaternion polynomials can be factorized into a product of linear factors.*

Corollary 1. *All non-constant quaternion polynomial of degree m can be factorized into a product of m linear factors.*

Let $p(x) = \sum_{i=0}^{n} \boldsymbol{a}_i x^i$ a quaternion polynomial of degree n. The algorithm (see [**?**]) to compute the zeros of $p(x)$, is as follows:

1. compute the polynomial $n_p(x) = p(x)\overline{p}(x)$; this is a polynomial with coefficients in $\mathbb{R}$;
2. compute z_i, the complex zeros of $n_p(x)$, which came in conjugate pairs; the zeros of $p(x)$ will belong to the conjugacy class $[z_i]$;
3. use Niven's algorithm [10] to compute the zeros of $p(x)$. This is done by dividing polynomial $p(x)$ on the right side by the characteristic polynomials $[z_i]$, obtaining
$$p(x) = q(x)\Delta_{z_i}(x) + \boldsymbol{\alpha}x + \boldsymbol{\beta}.$$
   (a) if $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{0}$, then all quaternions in the conjugacy class are zeros of $p(x)$;
   (b) otherwise, the only zero in the class is $\boldsymbol{q} = -\boldsymbol{\alpha}^{-1}\boldsymbol{\beta}$.

2.2 Closed Quaternion Balls

The definitions provided in this part are adapted for closed quaternion balls from [2].

Definition 2. *Let $\boldsymbol{q} \in \mathbb{H}$ and let $r \in \mathbb{R}_0^+$. The closed ball in $\mathbb{H}$, called **a closed quaternion ball** is defined by*

$$\langle \boldsymbol{q}, r\rangle = \{\boldsymbol{x} \in \mathbb{H} : |\boldsymbol{x} - \boldsymbol{q}| \leq r\}.$$

*If $\boldsymbol{Q} = \langle \boldsymbol{q}, r\rangle$ then the **center** and the **radius** of $\boldsymbol{Q}$ are denoted by $C(\boldsymbol{Q}) = \boldsymbol{q}$ and $R(\boldsymbol{Q}) = r$, respectively.*

*The set of all closed quaternion balls is denoted by $\mathcal{H}$. The zero-radius elements in $\mathcal{H}$ are called **scalars** and correspond to the quaternions themselves.*

Definition 3. *Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{H}$. The closed quaternion balls $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are **concentric** if $C(\boldsymbol{Q}_1) = C(\boldsymbol{Q}_2)$. Moreover, if $R(\boldsymbol{Q}_1) = R(\boldsymbol{Q}_2)$, then $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are equal.*

Lemma 1. *Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{H}$ such that $\boldsymbol{Q}_1 = \langle \boldsymbol{q}_1, r_1\rangle$ and $\boldsymbol{Q}_2 = \langle \boldsymbol{q}_2, r_2\rangle$. Then $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_2$ if and only if $|\boldsymbol{q}_1 - \boldsymbol{q}_2| \leq r_2 - r_1$. In particular, if $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are concentric, then $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_2$ if and only if $r_1 \leq r_2$.*

Proof.

[$\Rightarrow$] Let $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_2$. The most distant $\boldsymbol{x} \in \boldsymbol{Q}_1$ from $\boldsymbol{q}_2$ is located at the boundary of $\boldsymbol{Q}_1$, along the line that extends through $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Consequently, $|\boldsymbol{x} - \boldsymbol{q}_2| = |\boldsymbol{q}_2 - \boldsymbol{q}_1| + |\boldsymbol{x} - \boldsymbol{q}_1| \leq r_2$, leading us to deduce that $|\boldsymbol{q}_2 - \boldsymbol{q}_1| \leq r_2 - r_1$.

[$\Leftarrow$] Suppose that $|\boldsymbol{q}_1 - \boldsymbol{q}_2| \leq r_2 - r_1$ and that $\boldsymbol{x} \in \boldsymbol{Q}_1$. Then

$$|\boldsymbol{x} - \boldsymbol{q}_2| = |\boldsymbol{x} - \boldsymbol{q}_1 + \boldsymbol{q}_1 - \boldsymbol{q}_2| \leq |\boldsymbol{x} - \boldsymbol{q}_1| + |\boldsymbol{q}_2 - \boldsymbol{q}_1| \leq r_1 + r_2 - r_1 = r_2.$$

Hence, $\boldsymbol{x} \in \boldsymbol{Q}_2$, and $\boldsymbol{Q}_1 \subset \boldsymbol{Q}_2$.

If the closed quaternion balls are concentric, then $\boldsymbol{q}_1 = \boldsymbol{q}_2$, leading to the conclusion that $\boldsymbol{Q}_1 \subset \boldsymbol{Q}_2$ if and only if $0 \leq r_2 - r_1$.

We dedicate the following subsection to operations on closed quaternion balls and their properties.

Addition

In the current subsection, results related to properties of the addition operation are established.

Definition 4. *The binary operation $+ : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, from now on referred to as **addition**, is defined by the equation*

$$\langle \boldsymbol{q}_1, r_1\rangle + \langle \boldsymbol{q}_2, r_2\rangle := \langle \boldsymbol{q}_1 + \boldsymbol{q}_2, r_1 + r_2\rangle\,.$$

Proposition 2. *Addition is both commutative and associative. Furthermore, $\langle \mathbf{0}, 0\rangle$ is the addition identity.*

Proof. Due to the commutativity and associativity of addition in $\mathbb{H}$ and in $\mathbb{R}$, it is evident that for all $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{H}$, the following holds: $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ and $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$. Considering the neutral elements of $\mathbb{H}$ and $\mathbb{R}$ with respect to their respective additions, it is also straightforward to demonstrate that $\langle \mathbf{0}, 0\rangle$ is the addition identity.

Corollary 2. *The elements in $\mathcal{H}$ that have an additive inverse are the zero-radius closed quaternion balls. Moreover, the reciprocal of $\langle \boldsymbol{q}, 0\rangle$ is $\langle -\boldsymbol{q}, 0\rangle$.*

Proof. A direct consequence of Definition 4.

Lemma 2. *Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{H}$. Then $\boldsymbol{Q}_1 + \boldsymbol{Q}_2 = \{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{Q}_1$ and $\boldsymbol{x}_2 \in \boldsymbol{Q}_2\}$.*

Proof. Let $\boldsymbol{y}_i \in \boldsymbol{Q}_i = \langle \boldsymbol{q}_i, r_i\rangle$, for $i \in \{1,2\}$. Then $\boldsymbol{y}_1 + \boldsymbol{y}_2 \in \{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{Q}_1$ and $\boldsymbol{x}_2 \in \boldsymbol{Q}_2\}$, and

$$\begin{aligned}
|\boldsymbol{y}_1 + \boldsymbol{y}_2 - (\boldsymbol{q}_1 + \boldsymbol{q}_2)| &= |\boldsymbol{y}_1 - \boldsymbol{q}_1 + \boldsymbol{y}_2 - \boldsymbol{q}_2| \\
&\leq |\boldsymbol{y}_1 - \boldsymbol{q}_1| + |\boldsymbol{y}_2 - \boldsymbol{q}_2| \\
&= r_1 + r_2.
\end{aligned}$$

Hence, $\boldsymbol{y}_1 + \boldsymbol{y}_2 \in \boldsymbol{Q}_1 + \boldsymbol{Q}_2$, and $\{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{Q}_1$ and $\boldsymbol{x}_2 \in \boldsymbol{Q}_2\} \subseteq \boldsymbol{Q}_1 + \boldsymbol{Q}_2$. Then

Now let $\boldsymbol{q} \in \boldsymbol{Q}_1 + \boldsymbol{Q}_2$, $\boldsymbol{y}_1 = \frac{r_1}{r_1+r_2}(\boldsymbol{q} - \boldsymbol{q}_2) + \frac{r_2}{r_1+r_2}\boldsymbol{q}_1$, and $\boldsymbol{y}_2 = \frac{r_2}{r_1+r_2}(\boldsymbol{q} - \boldsymbol{q}_1) + \frac{r_1}{r_1+r_2}\boldsymbol{q}_2$. Then

$$\begin{aligned}
|\boldsymbol{y}_1 - \boldsymbol{q}_1| &= \left|\frac{r_1}{r_1+r_2}(\boldsymbol{q} - \boldsymbol{q}_2) + \frac{r_2}{r_1+r_2}\boldsymbol{q}_1 - q_1\right| \\
&= \left|\frac{r_1}{r_1+r_2}(\boldsymbol{q} - \boldsymbol{q}_2) - \frac{r_1}{r_1+r_2}\boldsymbol{q}_1\right| \\
&= \frac{r_1}{r_1+r_2}\left|\boldsymbol{q} - \boldsymbol{q}_1 - \boldsymbol{q}_2\right| \\
&\leq \frac{r_1}{r_1+r_2}(r_1 + r_2) = r_1.
\end{aligned}$$

Consequently, $\boldsymbol{y}_1 \in \boldsymbol{Q}_1$. By swapping indices 1 and 2, we get $\boldsymbol{y}_2$ from $\boldsymbol{y}_1$, leading us to conclude that $\boldsymbol{y}_2 \in \boldsymbol{Q}_2$. Hence, $\boldsymbol{q} \in \{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{Q}_1$ and $\boldsymbol{x}_2 \in \boldsymbol{Q}_2\}$ and $\boldsymbol{Q}_1 + \boldsymbol{Q}_2 \subseteq \{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{Q}_1$ and $\boldsymbol{x}_2 \in \boldsymbol{Q}_2\}$.

Definition 5. *Let $\diamond : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be a binary operation in $\mathcal{B}$. The operation $\diamond$ is* ***inclusion monotonic*** *if, for all $\boldsymbol{a}_m, \boldsymbol{b}_m \in \mathcal{B}$ such that $\boldsymbol{a}_m \subseteq \boldsymbol{b}_m$, $m \in \{1,2\}$, $\boldsymbol{a}_1 \diamond \boldsymbol{a}_2 \subseteq \boldsymbol{b}_1 \diamond \boldsymbol{b}_2$.*

Proposition 3. *The addition operation is inclusion monotonic.*

Proof. Let $\boldsymbol{Q}_m, \boldsymbol{R}_m \in \mathcal{H}$ such that $\boldsymbol{Q}_m \subseteq \boldsymbol{R}_m$, $m \in \{1,2\}$. By Lemma 2, $\boldsymbol{Q}_1 + \boldsymbol{Q}_2 = \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in \boldsymbol{Q}_1$ and $\boldsymbol{y} \in \boldsymbol{Q}_2\} \subseteq \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in \boldsymbol{R}_1$ and $\boldsymbol{y} \in \boldsymbol{R}_2\} = \boldsymbol{R}_1 + \boldsymbol{R}_2$.

Multiplication

In the current subsection, results related to properties of the multiplication operation are established.

Definition 6. *The binary operation $*:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$, referred to as **multiplication**, is defined by*

$$\langle \boldsymbol{q}_1, r_1\rangle * \langle \boldsymbol{q}_2, r_2\rangle \equiv \langle \boldsymbol{q}_1, r_1\rangle \langle \boldsymbol{q}_2, r_2\rangle := \langle \boldsymbol{q}_1\boldsymbol{q}_2, r_1 |\boldsymbol{q}_2| + r_2 |\boldsymbol{q}_1| + r_1 r_2\rangle .$$

Multiplication by the quaternion $\boldsymbol{\alpha}\in\mathbb{H}$ is, by definition, given by the multiplication of the scalar $\langle \boldsymbol{\alpha}, 0\rangle$:

$$\boldsymbol{\alpha}\langle \boldsymbol{q}, r\rangle \equiv \langle \boldsymbol{\alpha}, 0\rangle \langle \boldsymbol{q}, r\rangle = \langle \boldsymbol{\alpha}\boldsymbol{q}, |\boldsymbol{\alpha}| r\rangle .$$

Although commutativity is not applicable, multiplication stays true for the following properties.

Proposition 4. *The identity element relative to multiplication is $\langle 1, 0\rangle$.*

Proof. Let $\langle \boldsymbol{q}, r\rangle \in\mathcal{H}$. Then we get $\langle \boldsymbol{q}, r\rangle\langle 1,0\rangle = \langle 1,0\rangle\langle \boldsymbol{q}, r\rangle = \langle \boldsymbol{q}, r\rangle$.

Definition 7. *The multiplication operation is **power-associative** if, for all $\boldsymbol{Q}\in\mathcal{H}$ and for all $m,s\in\mathbb{N}$, $\boldsymbol{Q}^s\boldsymbol{Q}^m=\boldsymbol{Q}^{s+m}$.*

Definition 8. *Let $\boldsymbol{Q}\in\mathcal{H}$. We define the **powers** of $\boldsymbol{Q}\neq\langle \boldsymbol{0},0\rangle$ by*

$$\boldsymbol{Q}^0=\langle 1,0\rangle \ \text{ and } \boldsymbol{Q}^k=\boldsymbol{Q}^{k-1}\boldsymbol{Q} \text{ for } k\in\mathbb{N}.$$

If $\boldsymbol{Q}=\langle \boldsymbol{0},0\rangle$, then $\boldsymbol{Q}^k=\langle \boldsymbol{0},0\rangle$, for all $k\in\mathbb{N}$.

Proposition 5. *The multiplication is power-associative.*

Proof. Let $\boldsymbol{Q}=\langle \boldsymbol{q}, r\rangle\in\mathcal{H}$. On one hand,

$$\begin{aligned}\boldsymbol{Q}^2\boldsymbol{Q} &= \left\langle \boldsymbol{q}^2, 2|\boldsymbol{q}| r + r^2\right\rangle \langle \boldsymbol{q}, r\rangle\\ &= \left\langle \boldsymbol{q}^3, 3|\boldsymbol{q}|^2 r + 3|\boldsymbol{q}| r^2 + r^3\right\rangle ,\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{Q}\boldsymbol{Q}^2 &= \langle \boldsymbol{q}, r\rangle \left\langle \boldsymbol{q}^2, 2|\boldsymbol{q}| r + r^2\right\rangle\\ &= \left\langle \boldsymbol{q}^3, 3|\boldsymbol{q}|^2 r + 3|\boldsymbol{q}| r^2 + r^3\right\rangle ,\end{aligned}$$

which implies $\boldsymbol{Q}^2\boldsymbol{Q}=\boldsymbol{Q}\boldsymbol{Q}^2$.

On the other hand, we have

$$\begin{aligned}\left(\boldsymbol{Q}^2\boldsymbol{Q}\right)\boldsymbol{Q} &= \left\langle \boldsymbol{q}^3, 3|\boldsymbol{q}|^2 r + 3|\boldsymbol{q}| r^2 + r^3\right\rangle \langle q, r\rangle\\ &= \left\langle \boldsymbol{q}^4, 4|\boldsymbol{q}|^3 r + 6|\boldsymbol{q}|^2 r^2 + |\boldsymbol{q}| r^3 + r^4\right\rangle\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{Q}^2\boldsymbol{Q}^2 &= \left\langle \boldsymbol{q}^2, 2|\boldsymbol{q}| r + r^2\right\rangle \left\langle \boldsymbol{q}^2, 2|\boldsymbol{q}| r + r^2\right\rangle\\ &= \left\langle \boldsymbol{q}^4, 4|\boldsymbol{q}|^3 r + 6|\boldsymbol{q}|^2 r^2 + |\boldsymbol{q}| r^3 + r^4\right\rangle .\end{aligned}$$

As $\boldsymbol{Q}^2\boldsymbol{Q}=\boldsymbol{Q}\boldsymbol{Q}^2$ and $\left(\boldsymbol{Q}^2\boldsymbol{Q}\right)\boldsymbol{Q}=\boldsymbol{Q}^2\boldsymbol{Q}^2$, invoking [1], the result follows.

Proposition 6. *Let* $\langle \boldsymbol{q}, r\rangle \in \mathcal{H}$. *Then, for all* $k \in \mathbb{N}$ *with* $k \geq 1$,

$$\langle \boldsymbol{q}, r\rangle^k = \left\langle \boldsymbol{q}^k, (|\boldsymbol{q}| + r)^k - |\boldsymbol{q}|^k \right\rangle.$$

Proof. Let $\langle \boldsymbol{q}, r\rangle \in \mathcal{H}$. We will prove by induction. For $k = 1$, the equation states

$$\langle \boldsymbol{q}, r\rangle = \langle \boldsymbol{q}, |\boldsymbol{q}| + r - |\boldsymbol{q}|\rangle = \langle \boldsymbol{q}, r\rangle,$$

which is clearly true. Suppose that the proposition is true for k. Then, for $k+1$ we have

$$\begin{aligned}
\langle \boldsymbol{q}, r\rangle^{k+1} &= \langle \boldsymbol{q}, r\rangle^k \langle \boldsymbol{q}, r\rangle \\
&= \left\langle \boldsymbol{q}^k, (|\boldsymbol{q}| + r)^k - |\boldsymbol{q}|^k \right\rangle \langle \boldsymbol{q}, r\rangle \\
&= \left\langle \boldsymbol{q}^{k+1}, |\boldsymbol{q}| \left((|\boldsymbol{q}| + r)^k - |\boldsymbol{q}|^k \right) + \left|\boldsymbol{q}^k\right| r + \left((|\boldsymbol{q}| + r)^k - |\boldsymbol{q}|^k \right) r \right\rangle \\
&= \left\langle \boldsymbol{q}^k, |\boldsymbol{q}| \left(|\boldsymbol{q}| + r\right)^k - |\boldsymbol{q}|^{k+1} + |\boldsymbol{q}|^k r + (|\boldsymbol{q}| + r)^k r - |\boldsymbol{q}|^k r \right\rangle \\
&= \left\langle \boldsymbol{q}^{k+1}, (|\boldsymbol{q}| + r)^{k+1} - |\boldsymbol{q}|^{k+1} \right\rangle.
\end{aligned}$$

By mathematical induction, it is proved that for all $k \geq 1$ the statement is true.

Proposition 7. *The multiplication in* $\mathcal{H}$ *is not distributive with respect to the addition.*

Proof. Let $\boldsymbol{Q}_1 = \langle \boldsymbol{i}, 1\rangle$, $\boldsymbol{Q}_2 = \langle \boldsymbol{j}, 1\rangle$, and $\boldsymbol{Q}_3 = \langle \boldsymbol{k}, 1\rangle$. Then

$$\boldsymbol{Q}_1 (\boldsymbol{Q}_2 + \boldsymbol{Q}_3) = \langle \boldsymbol{i}, 1\rangle \langle \boldsymbol{j} + \boldsymbol{k}, 2\rangle = \left\langle \boldsymbol{k} - \boldsymbol{j}, 4 + \sqrt{2} \right\rangle$$

and

$$\boldsymbol{Q}_1\boldsymbol{Q}_2 + \boldsymbol{Q}_1\boldsymbol{Q}_3 = \langle \boldsymbol{i}, 1\rangle \langle \boldsymbol{j}, 1\rangle + \langle \boldsymbol{i}, 1\rangle \langle \boldsymbol{k}, 1\rangle = \langle \boldsymbol{k}, 3\rangle + \langle -\boldsymbol{j}, 3\rangle = \langle \boldsymbol{k} - \boldsymbol{j}, 6\rangle.$$

Hence,

$$\boldsymbol{Q}_1 (\boldsymbol{Q}_2 + \boldsymbol{Q}_3) \neq \boldsymbol{Q}_1\boldsymbol{Q}_2 + \boldsymbol{Q}_1\boldsymbol{Q}_3.$$

The sum of two closed quaternion balls remains a closed quaternion ball.

The set $\{\boldsymbol{xy} : \boldsymbol{x} \in \boldsymbol{Q}_1 \text{ and } \boldsymbol{y} \in \boldsymbol{Q}_2\}$ is generally not a closed quaternion ball. Nonetheless, this set is bound up within an endless number of closed quaternion balls. Among the collection of closed quaternion balls containing the set $\{\boldsymbol{xy} : \boldsymbol{x} \in \boldsymbol{Q}_1 \text{ and } \boldsymbol{y} \in \boldsymbol{Q}_2\}$, we will demonstrate that the one with the minimal radius is the closed quaternion ball $\boldsymbol{Q}_1\boldsymbol{Q}_2$.

Proposition 8. *The set of all products of two quaternions from distinct closed quaternion balls is not necessarily a closed quaternion ball.*

Proof. It is sufficient to consider an example that fails. Let $\boldsymbol{Q}_1 = \langle -5 - 2\boldsymbol{i}, 2\rangle$ and $\boldsymbol{Q}_2 = \langle 5 + 2j, 4\rangle$. Then $\boldsymbol{Q}_1\boldsymbol{Q}_2 = \left\langle -25 - 10\boldsymbol{i} - 10\boldsymbol{j} - 4\boldsymbol{k}, 8 + 6\sqrt{29} \right\rangle$. Given that $|-25 - 10\boldsymbol{i} - 10\boldsymbol{j} - 4\boldsymbol{k}| \leq 8 + 6\sqrt{29}$, we deduce that $\boldsymbol{0} \in \boldsymbol{Q}_1\boldsymbol{Q}_2$, while $\boldsymbol{0} \notin \boldsymbol{Q}_1$ and $\boldsymbol{0} \notin \boldsymbol{Q}_2$.

Lemma 3. *Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{H}$. Then $\{\boldsymbol{x}\boldsymbol{y} : \boldsymbol{x} \in \boldsymbol{Q}_1$ and $\boldsymbol{y} \in \boldsymbol{Q}_2\} \subseteq \boldsymbol{Q}_1\boldsymbol{Q}_2$.*

Proof. Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{H}$, $\boldsymbol{x} \in \boldsymbol{Q}_1 = \langle \boldsymbol{q}_1, r_1\rangle$, and $\boldsymbol{y} \in \boldsymbol{Q}_2 = \langle \boldsymbol{q}_2, r_2\rangle$. Then

$$\begin{aligned}
|\boldsymbol{x}\boldsymbol{y} - \boldsymbol{q}_1\boldsymbol{q}_2| &= |\boldsymbol{x}\boldsymbol{y} - \boldsymbol{x}\boldsymbol{q}_2 + \boldsymbol{x}\boldsymbol{q}_2 - \boldsymbol{q}_1\boldsymbol{q}_2| \\
&= |\boldsymbol{x}(\boldsymbol{y} - \boldsymbol{q}_2) + (\boldsymbol{x} - \boldsymbol{q}_1)\boldsymbol{q}_2| \\
&= |\boldsymbol{x}(\boldsymbol{y} - \boldsymbol{q}_2) - \boldsymbol{q}_1(\boldsymbol{y} - \boldsymbol{q}_2) + \boldsymbol{q}_1(\boldsymbol{y} - \boldsymbol{q}_2) + (\boldsymbol{x} - \boldsymbol{q}_1)\boldsymbol{q}_2| \\
&= |(\boldsymbol{x} - \boldsymbol{q}_1)(\boldsymbol{y} - \boldsymbol{q}_2) + \boldsymbol{q}_1(\boldsymbol{y} - \boldsymbol{q}_2) + (\boldsymbol{x} - \boldsymbol{q}_1)\boldsymbol{q}_2| \\
&\leq |\boldsymbol{x} - \boldsymbol{q}_1|\,|\boldsymbol{y} - \boldsymbol{q}_2| + |\boldsymbol{q}_1|\,|\boldsymbol{y} - \boldsymbol{q}_2| + |\boldsymbol{x} - \boldsymbol{q}_1|\,|\boldsymbol{q}_2| \\
&= r_1 r_2 + |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1.
\end{aligned}$$

Hence, $\boldsymbol{x}\boldsymbol{y} \in \boldsymbol{Q}_1\boldsymbol{Q}_2$.

Proposition 9. *Given the closed quaternion balls $\boldsymbol{Q}_1 = \langle \boldsymbol{q}_1, r_1\rangle$ and $\boldsymbol{Q}_2 = \langle \boldsymbol{q}_2, r_2\rangle$, $\boldsymbol{Q}_1\boldsymbol{Q}_2$ is the closed quaternion ball centered at $\boldsymbol{q}_1\boldsymbol{q}_2$ with the smallest radius that contains $\{\boldsymbol{x}\boldsymbol{y} : \boldsymbol{x} \in \boldsymbol{Q}_1$ and $\boldsymbol{y} \in \boldsymbol{Q}_2\}$.*

Proof. If $\boldsymbol{Q}_1 = \langle \boldsymbol{q}_1, r_1\rangle$ and $\boldsymbol{Q}_2 = \langle \boldsymbol{q}_2, r_2\rangle$, then

$$\boldsymbol{Q}_1\boldsymbol{Q}_2 = \{\boldsymbol{x} \in \mathbb{H} \ : \ |\boldsymbol{x} - \boldsymbol{q}_1\boldsymbol{q}_2| \leq |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2\}\,.$$

It is easy to see that the quaternion

$$\boldsymbol{x} = \boldsymbol{q}_1\boldsymbol{q}_2 + \frac{|\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2}{|\boldsymbol{q}_1|\,|\boldsymbol{q}_2|}\boldsymbol{q}_1\boldsymbol{q}_2$$

belongs to the border of $\boldsymbol{Q}_1\boldsymbol{Q}_2$. Hence, by Lemma 3, we conclude the proposition.

Proposition 10. *The multiplication in $\mathcal{H}$ is inclusion monotonic.*

Proof. Let $\boldsymbol{Q}_1 = \langle \boldsymbol{q}_1, r_1\rangle$, $\boldsymbol{Q}_2 = \langle \boldsymbol{q}_2, r_2\rangle$, $\boldsymbol{Q}_3 = \langle \boldsymbol{q}_3, r_3\rangle$, and $\boldsymbol{Q}_4 = \langle \boldsymbol{q}_4, r_4\rangle$, such that $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_3$ and $\boldsymbol{Q}_2 \subseteq \boldsymbol{Q}_4$. Then

$$\boldsymbol{Q}_1\boldsymbol{Q}_2 = \langle \boldsymbol{q}_1\boldsymbol{q}_2, |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2\rangle\,.$$

Let $\boldsymbol{x} \in \boldsymbol{Q}_1\boldsymbol{Q}_2$. Then

$$\begin{aligned}
|\boldsymbol{x} - \boldsymbol{q}_3\boldsymbol{q}_4| &= |\boldsymbol{x} - \boldsymbol{q}_1\boldsymbol{q}_2 + \boldsymbol{q}_1\boldsymbol{q}_2 - \boldsymbol{q}_3\boldsymbol{q}_4| \\
&\leq |\boldsymbol{x} - \boldsymbol{q}_1\boldsymbol{q}_2| + |\boldsymbol{q}_1\boldsymbol{q}_2 - \boldsymbol{q}_3\boldsymbol{q}_4| \\
&\leq |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2 + |\boldsymbol{q}_1\boldsymbol{q}_2 - \boldsymbol{q}_1\boldsymbol{q}_4 + \boldsymbol{q}_1\boldsymbol{q}_4 - \boldsymbol{q}_3\boldsymbol{q}_4| \\
&\leq |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2 + |\boldsymbol{q}_1|\,|\boldsymbol{q}_4 - \boldsymbol{q}_2| + |\boldsymbol{q}_4|\,|\boldsymbol{q}_3 - \boldsymbol{q}_1|\,.
\end{aligned}$$

Since $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_3$ and $\boldsymbol{Q}_2 \subseteq \boldsymbol{Q}_4$, by Lemma 1,

$$\begin{aligned}
|\boldsymbol{x} - \boldsymbol{q}_3\boldsymbol{q}_4| &\leq |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2 + |\boldsymbol{q}_1|\,(r_4 - r_2) + |\boldsymbol{q}_4|\,(r_3 - r_1) \\
&= (|\boldsymbol{q}_2| - |\boldsymbol{q}_4|) r_1 + r_1 r_2 + |\boldsymbol{q}_1|\, r_4 + |\boldsymbol{q}_4|\, r_3 \\
&\leq |\boldsymbol{q}_4 - \boldsymbol{q}_2|\, r_1 + r_1 r_2 + |\boldsymbol{q}_1|\, r_4 + |\boldsymbol{q}_4|\, r_3.
\end{aligned}$$

Again, by Lemma 1,

$$\begin{aligned}|\boldsymbol{x} - \boldsymbol{q}_3\boldsymbol{q}_4| &\leq (r_4 - r_2)r_1 + r_1 r_2 + |\boldsymbol{q}_1|\, r_4 + |\boldsymbol{q}_4|\, r_3 \\ &= r_4 r_1 + |\boldsymbol{q}_1|\, r_4 + |\boldsymbol{q}_4|\, r_3.\end{aligned} \tag{1}$$

Since $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_3$, we know that $|\boldsymbol{q}_3 - \boldsymbol{q}_1| \leq r_3 - r_1$. Thus

$$|\boldsymbol{q}_1| - |\boldsymbol{q}_3| \leq |\boldsymbol{q}_3 - \boldsymbol{q}_1| \leq r_3 - r_1$$

from where we conclude that

$$|\boldsymbol{q}_1| \leq r_3 - r_1 + |\boldsymbol{q}_3|\,.$$

Substituting in equation (1), we obtain

$$\begin{aligned}|\boldsymbol{x} - \boldsymbol{q}_3\boldsymbol{q}_4| &\leq r_4 r_1 + (r_3 - r_1 + |\boldsymbol{q}_3|)r_4 + |\boldsymbol{q}_4|\, r_3 \\ &= |\boldsymbol{q}_4|\, r_3 + |\boldsymbol{q}_3|\, r_4 + r_3 r_4.\end{aligned}$$

Hence, $\boldsymbol{x} \in \boldsymbol{Q}_3\boldsymbol{Q}_4$, and $\boldsymbol{Q}_1\boldsymbol{Q}_2 \subseteq \boldsymbol{Q}_3\boldsymbol{Q}_4$.

3 Polynomial Equations in Closed Quaternion Balls

In this section, we will solve quaternion polynomial equations with coefficients and independent terms that are not precisely defined. This leads us to polynomial equations over closed quaternion balls.

Resolving equations within closed quaternion balls is not as straightforward as it may appear. For example, the solution of a simple equation as

$$\boldsymbol{A}_2 X^2 = \boldsymbol{A}_1 X + \boldsymbol{A}_0,$$

where $\boldsymbol{A}_2, \boldsymbol{A}_1, \boldsymbol{A}_0 \in \mathcal{H}$ is not the same as the solution of the equation

$$\boldsymbol{A}_2 X^2 - \boldsymbol{A}_1 X = \boldsymbol{A}_0.$$

As only scalars possess an additive inverse, only terms with a radius of zero may be transferred from the left-hand side to the right-hand side, and vice versa.

Considering this constraint, we can still derive interesting results for certain particular equations.

Proposition 11. *Let $X = \langle \boldsymbol{q}, r\rangle$ satisfy the equation*

$$\boldsymbol{A}_n X^n + \boldsymbol{A}_{n-1} X^{n-1} + \cdots + \boldsymbol{A}_1 X + \boldsymbol{A}_0 = \mathbf{0}_\alpha, \tag{2}$$

where $\boldsymbol{A}_i = \langle \boldsymbol{a}_i, r_i\rangle \in \mathcal{H}$, for $i = 0, \ldots, n$ and $\mathbf{0}_\alpha = \langle 0, \alpha\rangle \in \mathcal{H}$. Then equation (2) has solution if and only if

$$|\boldsymbol{q}|^n\, r_n + |\boldsymbol{q}|^{n-1}\, r_{n-1} + \cdots + |\boldsymbol{q}|\, r_1 + r_0 \leq \alpha, \tag{3}$$

where $\boldsymbol{q}$ is the zero of the quaternion polynomial

$$p(x) = \boldsymbol{a}_n x^n + \boldsymbol{a}_{n-1} x^{n-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0.$$

Besides, for each $\boldsymbol{q}$, there exists only one value for $r \geq 0$ that satisfies equation (2).

Proof. Let $X = \langle \boldsymbol{q}, r \rangle$ and $\boldsymbol{A}_k = \langle \boldsymbol{a}_k, r_k \rangle$. Calling upon Proposition 6, we have

$$\begin{aligned}
\boldsymbol{A}_k \boldsymbol{X}^k &= \langle \boldsymbol{a}_k, r_k \rangle \left\langle \boldsymbol{q}^k, (|\boldsymbol{q}| + r)^k - |\boldsymbol{q}|^k \right\rangle \\
&= \left\langle \boldsymbol{a}_k \boldsymbol{q}^k, \left|\boldsymbol{q}^k\right| r_k + (|\boldsymbol{a}_k| + r_k) \left((|\boldsymbol{q}| + r)^k - |\boldsymbol{q}|^k \right) \right\rangle \\
&= \left\langle \boldsymbol{a}_k \boldsymbol{q}^k, (|\boldsymbol{a}_k| + r_k) (|\boldsymbol{q}| + r)^k - \left|\boldsymbol{a}_k \boldsymbol{q}^k\right| \right\rangle .
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=0}^{n} \boldsymbol{A}_k \boldsymbol{X}^k &= \sum_{k=0}^{n} \left\langle \boldsymbol{a}_k \boldsymbol{q}^k, (|\boldsymbol{a}_k| + r_k) (|\boldsymbol{q}| + r)^k - \left|\boldsymbol{a}_k \boldsymbol{q}^k\right| \right\rangle \\
&= \left\langle \sum_{k=0}^{n} \boldsymbol{a}_k \boldsymbol{q}^k, \sum_{k=0}^{n} \left((|\boldsymbol{a}_k| + r_k) (|\boldsymbol{q}| + r)^k - \left|\boldsymbol{a}_k \boldsymbol{q}^k\right| \right) \right\rangle \\
&= \left\langle \sum_{k=0}^{n} \boldsymbol{a}_k \boldsymbol{q}^k, \sum_{k=0}^{n} \left(\sum_{i=0}^{k} (|\boldsymbol{a}_k| + r_k) |\boldsymbol{q}|^{k-i} r^i - \left|\boldsymbol{a}_k \boldsymbol{q}^k\right| \right) \right\rangle
\end{aligned}$$

Equaling $\sum\limits_{k=0}^{n} \boldsymbol{A}_k \boldsymbol{X}^k = \langle \mathbf{0}, \alpha \rangle$, we get from the center

$$p(\boldsymbol{q}) = 0, \tag{4}$$

where $p(x) = \sum\limits_{k=0}^{n} \boldsymbol{a}_k x^k$, and from the radius

$$\alpha = \sum_{k=0}^{n} \left(\sum_{i=0}^{k} (|\boldsymbol{a}_k| + r_k) |\boldsymbol{q}|^{k-i} r^i - |\boldsymbol{a}_k \boldsymbol{q}|^k \right) . \tag{5}$$

The RHS of this last equation is a polynomial in r where all the coefficients and the independent term are positive. The smallest value for r is naturally zero. Hence, taking all the terms in the RHS which correspond to $i = 0$ we obtain

$$\alpha \geq \sum_{k=0}^{n} |\boldsymbol{q}|^k r_k .$$

Besides, if this last condition is verified, for each quaternion zero of (4), equation (5) has all coefficients positive and a negative independent term. Hence, by Descartes' rule of signs, there exists only one positive solution.

We next consider second-degree equations given by the equality of two second-degree polynomials.

Proposition 12. *Consider the second-degree closed quaternion ball equation*

$$\langle \boldsymbol{a}_2, r_2 \rangle X^2 + \langle \boldsymbol{a}_1, r_1 \rangle X + \langle \boldsymbol{a}_0, r_0 \rangle = \langle \boldsymbol{b}_2, r_2 \rangle X^2 + \langle \boldsymbol{b}_1, r_1 \rangle X + \langle \boldsymbol{b}_0, r_0 \rangle , \tag{6}$$

where $|\boldsymbol{b}_2| \neq |\boldsymbol{a}_2|$, *and let* $\boldsymbol{q}$ *be a zero of the quaternion polynomial*

$$p(x) = (\boldsymbol{a}_2 - \boldsymbol{b}_2)x^2 + (\boldsymbol{a}_1 - \boldsymbol{b}_1)x + \boldsymbol{a}_0 - \boldsymbol{b}_0.$$

For each $\boldsymbol{q}$, *the solutions of* (6) *are the closed quaternion balls* $X = \langle \boldsymbol{q}, 0\rangle$, *and* $X = \langle \boldsymbol{q}, \boldsymbol{\beta} - 2\,|\boldsymbol{q}|\rangle\rangle$, *where*

$$\boldsymbol{\beta} = -\frac{|\boldsymbol{a}_1| - |\boldsymbol{b}_1|}{|\boldsymbol{a}_2| - |\boldsymbol{b}_2|}$$

provided that $\boldsymbol{\beta} \geq -2\,|\boldsymbol{q}|$.

Proof. Applying the closed quaternion ball arithmetic to both sides of the equation

$$\langle \boldsymbol{a}_2, r_2\rangle X^2 + \langle \boldsymbol{a}_1, r_1\rangle X + \langle \boldsymbol{a}_0, r_0\rangle = \langle \boldsymbol{b}_2, r_2\rangle X^2 + \langle \boldsymbol{b}_1, r_1\rangle X + \langle \boldsymbol{b}_0, r_0\rangle,$$

and putting $X = \langle \boldsymbol{x}, r\rangle$, we get two equations: one for the center

$$\boldsymbol{a}_2\boldsymbol{x}^2 + \boldsymbol{a}_1\boldsymbol{x} + \boldsymbol{a}_0 = \boldsymbol{b}_2\boldsymbol{x}^2 + \boldsymbol{b}_1\boldsymbol{x} + \boldsymbol{b}_0;$$

and another for the radius, which after canceling equal terms from both sides of the equation

$$|\boldsymbol{a}_2|\, R^2 + |\boldsymbol{a}_1|\, R - \left|\boldsymbol{a}_2\boldsymbol{q}^2\right| - |\boldsymbol{a}_1\boldsymbol{q}| = |\boldsymbol{b}_2|\, R^2 + |\boldsymbol{b}_1|\, R - \left|\boldsymbol{b}_2\boldsymbol{q}^2\right| - |\boldsymbol{b}_1\boldsymbol{q}|$$

where $R = |\boldsymbol{q}| + r$. Manipulating these two equations, we obtian

$$p(x) = \boldsymbol{\alpha}_2 x^2 + \boldsymbol{\alpha}_1 x + \boldsymbol{\alpha}_0$$

and

$$\boldsymbol{\alpha}_2(R^2 - |\boldsymbol{q}|^2) + \boldsymbol{\alpha}_1(R - |\boldsymbol{q}|) = 0,$$

where $\boldsymbol{\alpha}_i = |\boldsymbol{a}_i| - |\boldsymbol{b}|_i$.

From the second equation it follows

$$(\boldsymbol{\alpha}_2(R + |\boldsymbol{q}|) + \boldsymbol{\alpha}_1)(R - |\boldsymbol{q}|) = 0 \Longleftrightarrow (\boldsymbol{\alpha}_2(r + 2\,|\boldsymbol{q}|) + \boldsymbol{\alpha}_1)r = 0.$$

The two solutions are $r = 0$ and $r = -\boldsymbol{\alpha}_1/\boldsymbol{\alpha}_2 - 2\,|\boldsymbol{q}|$.

Proposition 13. *Consider the closed quaternion ball polynomial equation*

$$\boldsymbol{A}_n X^n + \cdots + \boldsymbol{A}_1 X + \boldsymbol{A}_0 = \boldsymbol{B}_n X^n + \cdots + \boldsymbol{B}_1 X + \boldsymbol{B}_0,$$

where $\boldsymbol{A}_i = \langle \boldsymbol{a}_i, r_i\rangle$ *and* $\boldsymbol{B} = \langle \boldsymbol{b}_i, r_i\rangle$, *for* $i = 0, \ldots, n$, *and the centers of* $\boldsymbol{A}_n$ *and* $\boldsymbol{B}_n$ *are not similar. Then the scalar* $X = \langle \boldsymbol{q}, 0\rangle$ *is a solution, where* $\boldsymbol{q}$ *the the zero of the quaternion polynomial*

$$p(x) = (\boldsymbol{a}_n - \boldsymbol{b}_n)x^n + \cdots + (\boldsymbol{a}_1 - \boldsymbol{b}_1)x + \boldsymbol{a}_0 - \boldsymbol{b}_0.$$

Besides that, if for each $\boldsymbol{q}$ *there exists another solution* $X \in \mathcal{H}$, *this solution doesn't depend on the radius of the coefficients.*

Proof. In the proof of the Proposition 12 we see that, from the closed quaternion ball polynomial equation, two equations are derived: one for the center, which provides the roots of the quaternion polynomial; and another for the radius. Since the radii of the coefficients of terms of equal degree are identical, they cancel each other, rendering the equation for the radius independent on them. Furthermore, it is evident that $r = 0$ constitutes a solution for every root $\boldsymbol{q}$ of the center equation. Since the radius equation does not depend on the r_i's, any other solution will not depend on them.

Finally, a classical problem associated with polynomial equations is the n-th root of an element. Indeed, an n-th root of a closed quaternion ball $\boldsymbol{A}$ is a closed quaternion ball $\boldsymbol{X}$ such that $\boldsymbol{X}^n = \boldsymbol{A}$. This equation leads to the de Moîvre's formula, from where we obtain the formula for the n-th root of a closed quaternion ball, as can be seen in the following result.

Proposition 14. *Let $\boldsymbol{a} \in \mathbb{H}$ be given in the polar form*

$$\boldsymbol{a} = |\boldsymbol{a}| \left(\cos\theta + \hat{\boldsymbol{u}}_{\boldsymbol{a}} \sin\theta\right),$$

with $\hat{\boldsymbol{u}}_{\boldsymbol{a}} \neq \boldsymbol{0}$. Then $\boldsymbol{A} = \langle \boldsymbol{a}, r_a \rangle \in \mathcal{H}$ has exactly n n-th roots given by

$$\sqrt[n]{\boldsymbol{A}} = \left\langle \sqrt[n]{|\boldsymbol{a}|} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + \hat{\boldsymbol{u}}_{\boldsymbol{a}} \sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \sqrt[n]{|\boldsymbol{a}| + r_a} - \sqrt[n]{|\boldsymbol{a}|} \right\rangle,$$

for $k = 0, \ldots, n-1$.

Proof. Let $\boldsymbol{X} = \langle \boldsymbol{q}, r \rangle$ and $\boldsymbol{A} = \langle \boldsymbol{a}, r_a \rangle$. The n-th roots of $\boldsymbol{A}$ can be obtained by the closed quaternion ball equation $\boldsymbol{X}^n = \boldsymbol{A}$. Substituting $\boldsymbol{X}$ and $\boldsymbol{A}$ in this equation, we obtain two equations:

$$\begin{cases} \boldsymbol{q}^n = \boldsymbol{a} \\ (|\boldsymbol{q}| + r)^n - |\boldsymbol{q}|^n = r_a \end{cases}.$$

The solution for the first equation is wellknown and can be found in [8], and is given by

$$\boldsymbol{q} = \sqrt[n]{|\boldsymbol{a}|} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + \hat{\boldsymbol{u}}_{\boldsymbol{a}} \sin\left(\frac{\theta + 2k\pi}{n}\right)\right),$$

for $k = 0, \ldots, n-1$.

For the second equation equation there is only one real positive solution, namely,

$$r = \sqrt[n]{|\boldsymbol{a}| + r_a} - \sqrt[n]{|\boldsymbol{a}|}.$$

Hence, the n solutions for the n-th roots of the closed quaternion balll $\boldsymbol{A}$ are given by

$$\sqrt[n]{\boldsymbol{A}} = \left\langle \sqrt[n]{|\boldsymbol{a}|} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + \hat{\boldsymbol{u}}_{\boldsymbol{a}} \sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \sqrt[n]{|\boldsymbol{a}| + r_a} - \sqrt[n]{|\boldsymbol{a}|} \right\rangle,$$

for $k = 0, \ldots, n-1$.

The next examples illustrate our findings.

Example 1. Let $X = \langle q, r \rangle$.

$$X^3 + \boldsymbol{a}_2 X^2 + \boldsymbol{a}_1 X = \boldsymbol{B}, \tag{7}$$

where

$$\begin{aligned} \boldsymbol{a}_2 &= -(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) \\ \boldsymbol{a}_1 &= 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k} \\ \boldsymbol{B} &= \langle -\boldsymbol{a}_0, 1 \rangle \\ \boldsymbol{a}_0 &= 6 \end{aligned}$$

From the closed quaternion ball polynomial equation, we obtain two equations, one for the center and the other for the radius:

$$\begin{cases} \boldsymbol{q}^3 + \boldsymbol{a}_2 \boldsymbol{q}^2 + \boldsymbol{a}_1 \boldsymbol{q} + \boldsymbol{a}_0 = 0 \\ r^3 + (3\,|\boldsymbol{q}| + \sqrt{14}) r^2 + (3\,|\boldsymbol{q}|^2 + 2\sqrt{14}\,|\boldsymbol{q}| + 7) r = 1 \end{cases}.$$

There are three zeros for the first equation,

$$\begin{aligned} \boldsymbol{q}_1 &= \frac{1}{25}(12\boldsymbol{i} - 16\boldsymbol{j} + 15\boldsymbol{k}) \\ \boldsymbol{q}_2 &= \frac{1}{13}(-10\boldsymbol{j} + 24\boldsymbol{k}) \\ \boldsymbol{q}_3 &= 3\boldsymbol{k}, \end{aligned}$$

where $\boldsymbol{q}_1 \in [\boldsymbol{i}]$, $\boldsymbol{q}_2 \in [2\boldsymbol{i}]$, and $\boldsymbol{q}_3 \in [3\boldsymbol{i}]$. Hence, $|\boldsymbol{q}_1| = 1$, $|\boldsymbol{q}_2| = 2$, and $|\boldsymbol{q}_3| = 3$. For each of these zeros, there exists only one radius. Solving the second equation, we obtain $r_1 \approx 0.055979$, $r_2 \approx 0.029195$, and $r_3 \approx 0.017644$, respectively. Thus, equation (7) has three solutions:

$$\begin{aligned} \boldsymbol{X}_1 &\approx \left\langle \frac{1}{25}(12\boldsymbol{i} - 16\boldsymbol{j} + 15\boldsymbol{k}), 0.055979 \right\rangle \\ \boldsymbol{X}_2 &\approx \left\langle \frac{1}{13}(-10\boldsymbol{j} + 24\boldsymbol{k}), 0.029195 \right\rangle \\ \boldsymbol{X}_3 &\approx \langle 3\boldsymbol{k}, 0.017644 \rangle. \end{aligned}.$$

Example 2. Let $\boldsymbol{X} = \langle \boldsymbol{q}, r \rangle$.

$$\boldsymbol{X}^3 + \boldsymbol{A}_2 \boldsymbol{X}^2 = -\boldsymbol{A}_1 X - \boldsymbol{B}, \tag{8}$$

where

$$\begin{aligned} \boldsymbol{A}_2 &= \langle -(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}), 0 \rangle \\ \boldsymbol{A}_1 &= \langle 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}, 0 \rangle \\ \boldsymbol{B} &= \langle -a_0, 1 \rangle \\ \boldsymbol{a}_0 &= 6 \end{aligned}$$

From the closed quaternion ball polynomial equation, we obtain two equations, one for the center and the other for the radius:

$$\begin{cases} \boldsymbol{q}^3 + \boldsymbol{a}_2\boldsymbol{q}^2 + \boldsymbol{a}_1 q + \boldsymbol{a}_0 = \mathbf{0} \\ r^3 + (3\,|\boldsymbol{q}| - \sqrt{14})r^2 + (3\,|\boldsymbol{q}|^2 - 2\sqrt{14}\,|\boldsymbol{q}| + 7)r - 1 = 0 \end{cases}.$$

The zeros of the first equation are the same as in Example 1. For each of these zeros, there exists only one radius. Solving the second equation, we obtain $r_1 \approx 0.41989$, $r_2 \approx 0.21859$, and $r_3 \approx 0.083365$, respectively. Thus, equation (8) has three solutions:

$$\begin{aligned} \boldsymbol{X}_1 &\approx \left\langle \frac{1}{25}(12\boldsymbol{i} - 16\boldsymbol{j} + 15\boldsymbol{k}), 0.41989 \right\rangle \\ \boldsymbol{X}_2 &\approx \left\langle \frac{1}{13}(-10\boldsymbol{j} + 24\boldsymbol{k}), 0.21859 \right\rangle \\ \boldsymbol{X}_3 &\approx \langle 3\boldsymbol{k}, 0.083365 \rangle . \end{aligned}.$$

Example 3. Let $\boldsymbol{X} = \langle \boldsymbol{q}, r \rangle$.

$$\boldsymbol{X}^3 + \boldsymbol{A}_2\boldsymbol{X}^2 + \boldsymbol{A}_1\boldsymbol{X} + \boldsymbol{A}_0 = \langle 0, 1 \rangle , \tag{9}$$

where

$$\begin{aligned} \boldsymbol{A}_2 &= \langle -(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}), 0.1 \rangle \\ \boldsymbol{A}_1 &= \langle 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}, 0.2 \rangle \\ \boldsymbol{A}_0 &= \langle 6, 0.1 \rangle \end{aligned}$$

From the closed quaternion ball polynomial equation, we obtain two equations, one for the center and the other for the radius:

$$\begin{cases} \boldsymbol{q}^3 + \boldsymbol{a}_2\boldsymbol{q}^2 + \boldsymbol{a}_1\boldsymbol{q} + \boldsymbol{a}_0 = \mathbf{0} \\ R^3 + (|\boldsymbol{a}_2| + r_2)R^2 + (|\boldsymbol{a}_1| + r_1)R + r_0 - 1 - \left|\boldsymbol{q}^3\right| - \left|\boldsymbol{a}_2\boldsymbol{q}^2\right| - |\boldsymbol{a}_1\boldsymbol{q}| = 0 \end{cases},$$

where $R = |\boldsymbol{q}| + r$.

The zeros of the first equation are the same as in Example 1. For $\boldsymbol{q}_3$, condition (3) in Proposition 11 is not verified. Therefore, there is no solution for (9). For $\boldsymbol{q}_2$ and q_1 the condition is verified. Solving the second equation, we obtain $r'_1 \approx 0.0331229$, and $r'_2 \approx 0.00289056$. Thus, equation (9) has two solutions:

$$\begin{aligned} \boldsymbol{X}_1 &\approx \left\langle \frac{1}{25}(12\boldsymbol{i} - 16\boldsymbol{j} + 15\boldsymbol{k}), 0.0331229 \right\rangle \\ \boldsymbol{X}_2 &\approx \left\langle \frac{1}{13}(-10\boldsymbol{j} + 24\boldsymbol{k}), 0.00289056 \right\rangle . \end{aligned}.$$

Example 4. Consider the equation

$$\langle \mathbf{2}, 2 \rangle X^2 + \langle \mathbf{1}, 10 \rangle X + \langle \mathbf{2}, 1 \rangle = \langle \mathbf{1}, 2 \rangle X^2 + \langle \mathbf{4}, 10 \rangle X + \langle \mathbf{5}, 1 \rangle .$$

From the closed quaternion ball polynomial equation, we obtain two equations, one for the center and the other for the radius:

$$\begin{cases} \boldsymbol{q}^2 - 3\boldsymbol{q} - 3 = \boldsymbol{0} \\ (r + 2\,|\boldsymbol{q}| - 3)r = 0 \end{cases}.$$

From the first equation, we get

$$\boldsymbol{q}_1 = \frac{3 - \sqrt{21}}{2} \quad \text{and} \quad \boldsymbol{q}_2 = \frac{3 + \sqrt{21}}{2}.$$

From Proposition 12, we have $\boldsymbol{\beta} = 3$. For $\boldsymbol{q}_1$, the quantity $\boldsymbol{\beta} - 2\,|\boldsymbol{q}_1| = 6 - \sqrt{2} > 0$. Hence, for this root, there are two solutions:

$$\boldsymbol{X}_1 = \langle \boldsymbol{q}_1, 0 \rangle \quad \text{and} \quad \boldsymbol{X}_2 = \left\langle \boldsymbol{q}_1, 6 - \sqrt{21} \right\rangle.$$

For $\boldsymbol{q}_2$, the quantity $\boldsymbol{\beta} - 2\,|\boldsymbol{q}_2| < 0$. Therefore, for this root, there is only one solution:

$$\boldsymbol{X}_3 = \langle \boldsymbol{q}_2, 0 \rangle\,.$$

Example 5. Let $\boldsymbol{A} = \langle 4(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}), 19 \rangle$. The quaternion $\boldsymbol{a} = 4(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$ written in polar form is given by $\boldsymbol{a} = 8\,(\cos\theta + \hat{\boldsymbol{u}}_{\boldsymbol{a}} \sin\theta)$, where $\theta = \frac{\pi}{3}$ and $\hat{\boldsymbol{u}}_{\boldsymbol{a}} = \frac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$. Applying the de Moîvre's formula, the cube roots of $\boldsymbol{A}$ are:

$$\begin{aligned} \boldsymbol{Q}_1 &= \left\langle 8\left(\cos\left(\frac{\pi}{9}\right) + \sin\left(\frac{\pi}{9}\right)\frac{\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{\sqrt{3}}\right), 1 \right\rangle \\ \boldsymbol{Q}_2 &= \left\langle 8\left(\cos\left(\frac{7\pi}{9}\right) + \sin\left(\frac{7\pi}{9}\right)\frac{\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{\sqrt{3}}\right), 1 \right\rangle \\ \boldsymbol{Q}_3 &= \left\langle 8\left(\cos\left(\frac{13\pi}{9}\right) + \sin\left(\frac{13\pi}{9}\right)\frac{\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{\sqrt{3}}\right), 1 \right\rangle. \end{aligned}$$

It can easily be checked that $\boldsymbol{Q}_i^3 = \boldsymbol{A}$, for $i = 1, 2, 3$.

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