CONSTRUCTING POSITION VECTORS FOR THE INTERSECTION OF TWO AND THREE LINEAR VARIETIES

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ABSTRACT. Position vectors are much useful in several several fields, such as Differential Geometry, Mechanics and in Engineering, in particular in Dimensional Metrology. We generalize, for linear varieties in $\mathbb{R}^n$, the corresponding results referred to the Euclidean ordinary space. The Moore-Penrose inverse of matrices plays an important rôle in this paper. Generalizations for three linear varieties of the Anderson-Duffin formulae are presented. We establish several formulae for a position vector of the intersection of linear varieties. Some characterization of the position vector is done in terms of centres of spheres. Results, in the context of commuting projections, are given as well.

1. Introduction

The present paper is a natural continuation of two preceding papers on best approximation pair of two linear and the distance from a point to the intersection of spheres in linear varieties [5, 6]. With this paper, we conclude the tryptic about this subject.

Position vectors are much useful for practical purposes, namely in Dimensional Metrology, where all the measured parts are described in terms of position vectors. Moreover, when performing calibrations of measuring devices, the position vectors play a fundamental rôle (see [11, 12, 13] and the references therein). In fact, almost every measurement is done to obtain the coordinates of notable points (e.g. centres, vertices, etc.) of the measured objects, in addition to its dimensions.

We consider a linear variety $\mathscr{V} = v_0 + \mathscr{V}_0$, where v_0 is a position vector and $\mathscr{V}_0$ is the unique direction space. A position vector is not unique. In the present paper, we seek formulae for a position vector of given linear varieties. We are using the centres of spheres in order to characterize position vectors of the intersection of linear varieties.

Intersection of linear varieties have been considered in several papers [2, 19, 20, 10, 5]. Our characterization involves the intersection of subspaces (Proposition 6.1). Characterization in [2] and [5] involves the sum of subspaces.

The Moore-Penrose generalized inverse † plays an important rôle in the results we present. Also, it is well known the relation between the Moore-Penrose pseudoinverse and orthogonal projectors operators (see, for instance, [18]).

Generalizations of Anderson-Duffin formulae [5] are much used in the present paper, including an extension for the intersection of three linear varieties.

This paper is organized as follows. In section 2, we deal with the Moore-Penrose inverse and the projection onto linear varieties. The radical hyperplane of linear

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varieties is studied in section 3. Sections 4 and 5 refer to the position vectors of the intersection of three linear varieties and position vectors in terms of the commutativity of projectors, respectively. The characterization of position vectors in terms of centres of spheres and intersection of subspaces is treated in section 6. Finally, in section 7, we present some conclusions and remarks.

In the following, we set a list of results which will be used in the present paper.

Theorem 1.1.

(1) *Let* $S \subseteq \mathbb{R}^n$ *be a subspace. Then* $\mathbb{P} \in \mathbb{R}^n$ *is the orthogonal projection onto* S *if range* $(\mathbb{P}) = S$, $\mathbb{P}^2 = \mathbb{P}$, *and* $\mathbb{P}^T = \mathbb{P}$.

(2) *The vector* s *of the linear variety* $\mathscr{M} = m_0 + \mathscr{M}_0$ *is the orthogonal projection of a vector* $q \in \mathbb{R}^n$ *onto the variety* $\mathscr{M}$ *if and only if the vector* $q - s$ *is orthogonal to the associate subspace* $\mathscr{M}_0$ *of* $\mathscr{M}$.

(3) *If* $\mathbb{P}$ *is a projector, then* $\mathbb{P}^\dagger = \mathbb{P}$.

(4) *If* $\mathscr{N}$ *is the kernel of a matrix* $N \in M(m, n; \mathbb{R})$, *then*
$$\mathbb{P}_{\mathscr{N}} = I_n - N^\dagger N.$$

(5) *Let* $\mathscr{M}_0$ *be a vector subspace of* $\mathbb{R}^n$ *and* B *a matrix whose columns are a basis for* $\mathscr{M}_0$. *Then the orthogonal projector* $\mathbb{P}_{\mathscr{M}_0}$ *is given by*
$$\mathbb{P}_{\mathscr{M}_0} = B\left(B^T B\right)^{-1} B^T.$$

(6) *Let be the linear variety*
$$\mathscr{L} = \{x \in \mathbb{R}^n : Lx = l\} = L^\dagger l + \mathscr{N}(L),$$
where $L^\dagger l$ *is the element of minimal norm of* $Lx = l$. *Then*
$$\mathbb{P}_{\mathscr{L}}(x) = L^\dagger l + \mathbb{P}_{\mathscr{N}(L)}(x).$$

(7) *Let be the linear variety*
$$\mathscr{M} = \{x \in \mathbb{R}^n : Mx = m\} = M^\dagger m + \mathscr{N}(M),$$
where $M^\dagger m$ *is the minimal norm element of* $Mx = m$. *Then*
$$\mathbb{P}_{\mathscr{M}}(x) = M^\dagger m + (I - M^\dagger M)x.$$

(8) *Let* A *and* B *be two subspaces. Then* $A \subseteq B$ *if and only if* $P_A = P_A P_B = P_B P_A$, *where* P_A *stands for the projection onto the subspace* A.

(9) *Two linear varieties* $\mathscr{L} = L^\dagger l + \mathscr{L}_0$ *and* $\mathscr{M} = M^\dagger m + \mathscr{M}_0$ *are perpendicular if and only if the respective associated subspaces* $\mathscr{L}_0$ *and* $\mathscr{M}_0$ *are orthogonal.*

(10) *Let* $\mathscr{L}_0$ *and* $\mathscr{M}_0$ *be two vector subspaces of* $\mathbb{R}^n$. *Then the orthogonal projector onto the intersection* $\mathscr{L}_0 \cap \mathscr{M}_0$ *is given by*
$$\mathbb{P}_{\mathscr{L}_0 \cap \mathscr{M}_0} = 2\mathbb{P}_{\mathscr{L}_0}\left(\mathbb{P}_{\mathscr{L}_0} + \mathbb{P}_{\mathscr{M}_0}\right)^\dagger \mathbb{P}_{\mathscr{M}_0}.$$

(11) *Let* $\mathscr{L}$ *and* $\mathscr{M}$ *be two intersecting linear varieties in* $\mathbb{R}^n$ *given by*
$$\mathscr{L} = \{x \in \mathbb{R}^n : Lx = l\} = l_0 + \mathscr{N}(L) = l_0 + \mathscr{L}_0$$
and
$$\mathscr{M} = \{x \in \mathbb{R}^n : Mx = m\} = m_0 + \mathscr{N}(M) = m_0 + \mathscr{M}_0.$$
Let
$$\begin{bmatrix} L \\ M \end{bmatrix}^\dagger \begin{bmatrix} l \\ m \end{bmatrix}$$
be the minimum norm solution of the consistent system
$$\begin{bmatrix} L \\ M \end{bmatrix} x = \begin{bmatrix} l \\ m \end{bmatrix}.$$

Then

$$\mathbb{P}_{\mathscr{L}\cap\mathscr{M}}(x) = 2\mathbb{P}_{\mathscr{L}_0}\left(\mathbb{P}_{\mathscr{L}_0}+\mathbb{P}_{\mathscr{M}_0}\right)^{\dagger}\mathbb{P}_{\mathscr{M}_0}(x) + \begin{bmatrix} L \\ M \end{bmatrix}^{\dagger}\begin{bmatrix} l \\ m \end{bmatrix}.$$

(12) The linear varieties $\mathscr{L} = l_0 + \mathscr{L}_0$ and $\mathscr{M} = m_0 + \mathscr{M}_0$ have nonempty intersection if and only if

$$m_0 - l_0 \in \mathscr{L}_0 + \mathscr{M}_0.$$

(13) We say that the spheres $S_1 = S(c_1, r_1)$ and $S_2 = S(c_2, r_2)$ in $\mathbb{R}^n$ have nonempty intersection if and only if

$$|r_1 - r_2| < \|c_1 - c_2\| < r_1 + r_2.$$

Proof. For (1), (3), (4) and (5), see [5] and the references therein. For (2), see [8, Theorem 9.26, p. 215]. For (6), see [5, Remark 3.2] and [15, Corolário 3.4]. For (7), see [5, Remark 3.5]. For (8), see [19], [24, Theorems 4.30 and 4.31, pp. 82-83] and [9, p. 481]. For (9), see [7]. For (10), see [1]. For (11), see [5]. For (12), see [2, Theorem 5, p. 185]. □

2. Moore-Penrose inverse and projection onto linear varieties

A relation between the orthogonal projection onto the intersection of two linear subspaces and the orthogonal projectors on each subspace is given by the well known Anderson-Duffin formula [1]. An extension for two linear varieties is given in [5, Proposition 3.7].

We generalize these results to the intersection of three linear varieties, which is motivated by considering distances between a point and the intersection of spheres [6, Proposition 4.1]. In the latter reference, it is extended a formula concerning the distance between a point and the unit sphere in a subspace of $\mathbb{R}^n$ [4, p. 1428].

Proposition 2.1. *Let $\mathscr{K}$, $\mathscr{L}$ and $\mathscr{M}$ be three intersecting linear varieties in $\mathbb{R}^n$ given by*

$$\mathscr{K} = \{x \in \mathbb{R}^n : Kx = k\} = k_0 + \mathscr{N}(K) = k_0 + \mathscr{K}_0,$$
$$\mathscr{L} = \{x \in \mathbb{R}^n : Lx = l\} = l_0 + \mathscr{N}(L) = l_0 + \mathscr{L}_0$$

and

$$\mathscr{M} = \{x \in \mathbb{R}^n : Mx = m\} = m_0 + \mathscr{N}(M) = m_0 + \mathscr{M}_0.$$

Let

$$\begin{bmatrix} K \\ L \\ M \end{bmatrix}^{\dagger}\begin{bmatrix} k \\ l \\ m \end{bmatrix}$$

be the minimum norm solution of the consistent system

$$\begin{bmatrix} K \\ L \\ M \end{bmatrix} x = \begin{bmatrix} k \\ l \\ m \end{bmatrix}.$$

Then

(1)

$$\mathscr{K}\cap\mathscr{L}\cap\mathscr{M} = \begin{bmatrix} K \\ L \\ M \end{bmatrix}^{\dagger}\begin{bmatrix} k \\ l \\ m \end{bmatrix} + \mathscr{N}(K)\cap\mathscr{N}(L)\cap\mathscr{N}(M).$$

(2)

$$\mathbb{P}_{\mathscr{K}\cap\mathscr{L}\cap\mathscr{M}}(x) = 4\mathbb{P}_{\mathscr{K}_0}\left(\mathbb{P}_{\mathscr{K}_0}+\mathbb{P}_{\mathscr{L}_0}\right)^{\dagger}\mathbb{P}_{\mathscr{L}_0}\left[2\mathbb{P}_{\mathscr{K}_0}\left(\mathbb{P}_{\mathscr{K}_0}+\mathbb{P}_{\mathscr{L}_0}\right)^{\dagger}\mathbb{P}_{\mathscr{L}_0}+\mathbb{P}_{\mathscr{M}_0}\right]^{\dagger}\mathbb{P}_{\mathscr{M}_0}(x) + \begin{bmatrix} K \\ L \\ M \end{bmatrix}^{\dagger}\begin{bmatrix} k \\ l \\ m \end{bmatrix}.$$

Proof. For (1), see (6) of Theorem 1.1. For (2), notice that

$$\begin{aligned}
\mathscr{K}\cap\mathscr{L} &= \left\{x\in\mathbb{R}^n : \begin{bmatrix} K \\ L \end{bmatrix}x = \begin{bmatrix} k \\ l \end{bmatrix}\right\} \\
&= x_0 + (\mathscr{K}\cap\mathscr{L})_0 \\
&= x_0 + \mathscr{N}\left(\begin{bmatrix} K \\ L \end{bmatrix}\right) \\
&= x_0 + \mathscr{N}(K)\cap\mathscr{N}(L) \\
&= x_0 + \mathscr{K}_0\cap\mathscr{L}_0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{P}_{\mathscr{K}\cap\mathscr{L}\cap\mathscr{M}}(x) &:= \mathbb{P}_{(\mathscr{L}\cap\mathscr{K})\cap\mathscr{M}}(x) \\
&= \mathbb{P}_{(\mathscr{L}\cap\mathscr{K})_0\cap\mathscr{M}_0}(x) + \begin{bmatrix} K \\ L \\ M \end{bmatrix}^{\dagger}\begin{bmatrix} k \\ l \\ m \end{bmatrix} \\
&= \left[2\mathbb{P}_{(\mathscr{L}\cap\mathscr{K})_0}\left(\mathbb{P}_{(\mathscr{L}\cap\mathscr{K})_0}+\mathbb{P}_{\mathscr{M}_0}\right)^{\dagger}\mathbb{P}_{\mathscr{M}_0}\right](x) + \begin{bmatrix} K \\ L \\ M \end{bmatrix}^{\dagger}\begin{bmatrix} k \\ l \\ m \end{bmatrix}.
\end{aligned}$$

By the Anderson-Duffin formula (see (10) in Theorem 1.1),

$$\begin{aligned}
\mathbb{P}_{(\mathscr{L}\cap\mathscr{K})_0} &= \mathbb{P}_{\mathscr{K}_0\cap\mathscr{L}_0} \\
&= 2\mathbb{P}_{\mathscr{K}_0}\left(\mathbb{P}_{\mathscr{K}_0}+\mathbb{P}_{\mathscr{L}_0}\right)^{\dagger}\mathbb{P}_{\mathscr{L}_0}.
\end{aligned}$$

Hence,

$$\mathbb{P}_{\mathscr{K}\cap\mathscr{L}\cap\mathscr{M}}(x) = 4\mathbb{P}_{\mathscr{K}_0}\left(\mathbb{P}_{\mathscr{K}_0}+\mathbb{P}_{\mathscr{L}_0}\right)^{\dagger}\mathbb{P}_{\mathscr{L}_0}\left[2\mathbb{P}_{\mathscr{K}_0}\left(\mathbb{P}_{\mathscr{K}_0}+\mathbb{P}_{\mathscr{L}_0}\right)^{\dagger}\mathbb{P}_{\mathscr{L}_0}+\mathbb{P}_{\mathscr{M}_0}\right]^{\dagger}\mathbb{P}_{\mathscr{M}_0}(x) + \begin{bmatrix} K \\ L \\ M \end{bmatrix}^{\dagger}\begin{bmatrix} k \\ l \\ m \end{bmatrix}.$$

□

3. Radical hyperplane of linear varieties

The radical hyperplane of spheres plays an important rôle when studying the intersections of spheres in linear varieties. Properties of the radical hyperplane of spheres are dealt with in [6].

The concept of the power of a point with respect to a sphere and the concept of the radical hyperplane of two spheres are related.

For special spheres, we introduce the concept of radical hyperplane of linear varieties.

Definition 3.1. Let $S(c, r)$ be a sphere with centre $c \in \mathbb{R}^n$ and radius r. For each point $x \in \mathbb{R}^n$, the number

$$\|x-c\|^2 - r^2$$

is called the *power of* x with respect to $S(c,r)$.

Let $S_1 = S(c_1, r_1)$ and $S_2 = S(c_2, r_2)$ be two spheres in $\mathbb{R}^n$. The *radical hyperplane* of the spheres S_1 and S_2 is the set of points with the same power with respect to the two spheres,

$$\mathscr{H}\left(S_1, S_2\right) = \left\{x \in \mathbb{R}^n : \left\|x - c_1\right\|^2 - r_1^2 = \left\|x - c_2\right\|^2 - r_2^2\right\}.$$

Let the spheres $S_{\mathscr{L}}(c_1, r_1)$ and $S_{\mathscr{M}}(c_2, r_2)$ be subsets of the linear varieties $\mathscr{L}$ and $\mathscr{M}$ of $\mathbb{R}^n$. We will consider *the radical hyperplane of these spheres* as the radical hyperplane of the spheres $S_1 = S(c_1, r_1)$ and $S_2 = S(c_2, r_2)$ in $\mathbb{R}^n$,

$$\mathscr{H}\left(S_{\mathscr{L}}, S_{\mathscr{M}}\right) := \mathscr{H}\left(S_1, S_2\right).$$

The notation $\mathscr{H}$ stands either for $\mathscr{H}\left(S_1, S_2\right)$ or for $\mathscr{H}\left(S_{\mathscr{L}}, S_{\mathscr{M}}\right)$, depending on the context.

In the next result, we extend, for the radical hyperplane, some properties of the radical plane in the Euclidean space $\mathbb{R}^3$. See [6, Proposition 2.5] and the classical references therein.

Theorem 3.2. *Let $S(c_1, r_1)$ and $S(c_2, r_2)$ be two spheres in $\mathbb{R}^n$ with distinct centres and let $\mathscr{H}$ be their radical hyperplane. Then:*

(1) $\mathscr{H}$ is perpendicular to the line c_1c_2.

(2) If the spheres have non-empty intersection, then the radical hyperplane $\mathscr{H}$ contains the intersection of the two spheres and intersects the line c_1c_2 at the point

$$c^* = \frac{r_2^2 - r_1^2 + d^2}{2d^2} c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2} c_2,$$

where $d = \|c_1 - c_2\|$.

(3) If the spheres have empty intersection, then the radical hyperplane $\mathscr{H}$ intersects the line c_1c_2 at the point

$$c^* = \frac{r_2^2 - r_1^2 + d^2}{2d^2} c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2} c_2,$$

where $d = \|c_1 - c_2\|$.

(4) The radical hyperplane $\mathscr{H}$ is given by

$$\mathscr{H} = \{x \in \mathbb{R}^n : Hx = h\},$$

with $H = 2(c_2^T - c_1^T)$ and $h = \left\|c_2\right\|^2 - r_2^2 - \left\|c_1\right\|^2 + r_1^2$.

Proof. For (1) and (2). The point c^* belongs to the intersection of the line c_1c_2 and the radical hyperplane $\mathscr{H}$. Hence, regarding c^*, we consider the equations

$$\begin{cases} c^* = c_1 + \lambda\left(c_2 - c_1\right) \\ c^* = c_2 + \left(\lambda - 1\right)\left(c_2 - c_1\right) \\ \left\|c^* - c_1\right\|^2 - r_1^2 = \left\|c^* - c_2\right\|^2 - r_2^2 \end{cases},$$

from which we get, successively,

$$\lambda = \frac{r_1^2 - r_2^2 + d^2}{2d^2}$$

and

$$\begin{aligned} c^* &= c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2}\left(c_2 - c_1\right) \\ &= \frac{r_2^2 - r_1^2 + d^2}{2d^2} c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2} c_2. \end{aligned}$$

For (3). We walk the same path by considering an "imaginary" sphere which is the intersection of two non-intersecting spheres $S(c_1, r_1)$ and $S(c_2, r_2)$. □

Next, we present the concept of radical hyperplane of two linear varieties.

Definition 3.3. Let $\mathscr{L}$ and $\mathscr{M}$ be the linear varieties

$$\mathscr{L} = \{x \in \mathbb{R}^n : Lx = l\} = L^\dagger l + \mathscr{N}(L)$$

and

$$\mathscr{M} = \{x \in \mathbb{R}^n : Mx = m\} = M^\dagger m + \mathscr{N}(M),$$

such that $L^\dagger l \neq M^\dagger m$.

The radical hyperplane of the spheres $S(L^\dagger l\,,\, \|M^\dagger m\|)$ and $S(M^\dagger m\,,\, \|L^\dagger l\|)$ is called the *radical hyperplane of the linear varieties* $\mathscr{L}$ and $\mathscr{M}$ and is denoted by $\mathscr{H}(\mathscr{L},\mathscr{M})$.

As a consequence of Theorem 3.2, we have the following.

Proposition 3.4. *Let $\mathscr{L}$ and $\mathscr{M}$ be two intersecting linear varieties*

$$\mathscr{L} = \{x \in \mathbb{R}^n : Lx = l\} = L^\dagger l + \mathscr{N}(L)$$

and

$$\mathscr{M} = \{x \in \mathbb{R}^n : Mx = m\} = M^\dagger m + \mathscr{N}(M),$$

such that $L^\dagger l \neq M^\dagger m$.

Let be the intersecting spheres

$$S_{\mathscr{L}} = S(L^\dagger l\,,\, \|M^\dagger m\|) = \{x \in \mathscr{L} : \|x - L^\dagger l\| = \|M^\dagger m\|\}$$

and

$$S_{\mathscr{M}} = S(M^\dagger m\,,\, \|L^\dagger l\|) = \{x \in \mathscr{M} : \|x - M^\dagger m\| = \|L^\dagger l\|\}\,.$$

Then $\mathbb{P}_{\mathscr{L}\cap\mathscr{M}}(c^*)$ is a position vector of $\mathscr{L}\cap\mathscr{M}$, where

$$c^* = \frac{\|M^\dagger m\|^2 - \|L^\dagger l\|^2 + d^2}{2d^2} L^\dagger l + \frac{\|L^\dagger l\|^2 - \|M^\dagger m\|^2 + d^2}{2d^2} M^\dagger m,$$

with $d = \|L^\dagger l - M^\dagger m\|$.

Proof. Since $L^\dagger l \neq M^\dagger m$, these points cannot belong together to $\mathscr{L}\cap\mathscr{M}$. The centre c^* is an element of the line passing through $L^\dagger l$ and $M^\dagger m$, so it cannot belong to $\mathscr{L}\cap\mathscr{M}$. □

Corollary 3.5. *The hyperplane $\mathscr{H}(\mathscr{L},\mathscr{M})$ contains the point $M^\dagger m + L^\dagger l$.*

Corollary 3.6. *The radical hyperplane of two linear varieties $\mathscr{L}$ and $\mathscr{M}$ is a subspace of $\mathbb{R}^n$ if and only if $\mathscr{L}$ and $\mathscr{M}$ are at the same distance of the origin.*

Remark 3.7. For the particular spheres

$$S_{\mathscr{L}} = S\left(L^\dagger l\,,\, \|L^\dagger l\|\right)$$

and

$$S_{\mathscr{M}} = S\left(M^\dagger m\,,\, \|M^\dagger m\|\right),$$

the radical hyperplane $\mathscr{H}(\mathscr{L},\mathscr{M})$ is a subspace.

The next result is used in the proof of Corollary 3.11.

Corollary 3.8. *The element of $\mathscr{H}(\mathscr{L},\mathscr{M})$ with minimal norm is*

$$q^* = \frac{\|L^\dagger l\|^2 - \|M^\dagger m\|^2}{\|L^\dagger l - M^\dagger m\|^2}\left(L^\dagger l - M^\dagger m\right).$$

Proof. The radical hyperplane $\mathscr{H}(\mathscr{L},\mathscr{M})$ is given by

$$\mathscr{H} = \{x \in \mathbb{R}^n : Hx = h\},$$

where $H = (L^\dagger l)^T - (M^\dagger m)^T$ and $h = \left\|L^\dagger l\right\|^2 - \left\|M^\dagger m\right\|^2$.

Since the matrix $H = (L^\dagger l)^T - (M^\dagger m)^T$ has only one row, its Moore-Penrose inverse is given by

$$(L^\dagger l - M^\dagger m)\left\|L^\dagger l - M^\dagger m\right\|^{-2}$$

and the result follows. □

Corollary 3.9. *Suppose that $L^\dagger l$ and $M^\dagger m$ are collinear with the origin. Then the radical hyperplane $\mathscr{H}(\mathscr{L},\mathscr{M})$ is parallel to $\mathscr{L}$ and $\mathscr{M}$.*

Proof. If $L^\dagger l$ and $M^\dagger m$ are collinear with the origin, then the line passing through the points $L^\dagger l$ and $M^\dagger m$ is perpendicular to $\mathscr{L}$ and to $\mathscr{M}$. Hence, $\mathscr{L}$ and $\mathscr{M}$ must be parallel to the hyperplane $\mathscr{H}(\mathscr{L},\mathscr{M})$. □

In the next result, it is set a relationship between the intersection of linear varieties and the corresponding radical hyperplane. The radical hyperplane contains the intersection the spheres and also the intersection of the linear varieties that contains the spheres.

Proposition 3.10. *Let $\mathscr{L}$ and $\mathscr{M}$ be two intersecting linear varieties*

$$\mathscr{L} = \{x \in \mathbb{R}^n : Lx = l\} = L^\dagger l + \mathscr{N}(L)$$

and

$$\mathscr{M} = \{x \in \mathbb{R}^n : Mx = m\} = M^\dagger m + \mathscr{N}(M).$$

Then:

(1) If $L^\dagger l = M^\dagger m$, then $L^\dagger l$ is a position vector of $\mathscr{L} \cap \mathscr{M}$.

(2) If $L^\dagger l \neq M^\dagger m$, then $\mathscr{L} \cap \mathscr{M}$ is contained in the radical hyperplane of $\mathscr{L}$ and $\mathscr{M}$

$$\mathscr{L} \cap \mathscr{M} \subset \mathscr{H}(\mathscr{L},\mathscr{M}).$$

Proof. Let $v \in \mathscr{L} \cap \mathscr{M}$. Then $v - L^\dagger l \in \mathscr{N}(L)$ and $v - M^\dagger m \in \mathscr{N}(M)$. Therefore,

$$\left\|v - L^\dagger l\right\|^2 - \left\|M^\dagger m\right\|^2 = \|v\|^2 - \left\|L^\dagger l\right\|^2 - \left\|M^\dagger m\right\|^2.$$

In fact, we have

$$\begin{aligned}\left\|v - L^\dagger l\right\|^2 &= \left(v - L^\dagger l\right) \bullet \left(v - L^\dagger l\right)\\ &= \|v\|^2 - 2v \bullet L^\dagger l + \left\|L^\dagger l\right\|^2.\end{aligned}$$

As the point $L^\dagger l$ is orthogonal to the subspace $\mathscr{N}(L)$ (see (2) of Theorem 1.1), and hence orthogonal to the point $v - L^\dagger l$, we get

$$0 = \left(v - L^\dagger l\right) \bullet L^\dagger l = v \bullet L^\dagger l - \left\|L^\dagger l\right\|^2.$$

So,

$$v \bullet L^\dagger l = \left\|L^\dagger l\right\|^2.$$

Finally,

$$\left\|v - L^\dagger l\right\|^2 = \|v\|^2 - \left\|L^\dagger l\right\|^2.$$

Analogously, *mutatis mutandis*, we have

$$\left\|v - M^\dagger m\right\|^2 - \left\|L^\dagger l\right\|^2 = \|v\|^2 - \left\|M^\dagger m\right\|^2 - \left\|L^\dagger l\right\|^2.$$

This way, we see that v has the same power with respect to the spheres $S(L^\dagger l, \left\|M^\dagger m\right\|)$ and $S(M^\dagger m, \left\|L^\dagger l\right\|)$ and therefore we conclude that v belongs to their radical hyperplane $\mathscr{H}(\mathscr{L},\mathscr{M})$. □

The projection of the element of minimal norm of the radical hyperplane $\mathscr{H}(\mathscr{L},\mathscr{M})$ is the element of minimal norm of $\mathscr{L}\cap\mathscr{M}$.

Corollary 3.11. *Let $\mathscr{L}$ and $\mathscr{M}$ be two intersecting linear varieties*

$$\mathscr{L}=\{x\in\mathbb{R}^n : Lx=l\}=L^\dagger l+\mathscr{N}(L)$$

and

$$\mathscr{M}=\{x\in\mathbb{R}^n : Mx=m\}=M^\dagger m+\mathscr{N}(M),$$

where $L^\dagger l\neq M^\dagger m$.

Then

$$\mathbb{P}_{\mathscr{L}\cap\mathscr{M}}\left(\frac{\left\|L^\dagger l\right\|^2-\left\|M^\dagger m\right\|^2}{\left\|L^\dagger l-M^\dagger m\right\|^2}(L^\dagger l-M^\dagger m)\right)=\begin{bmatrix} L\\ M\end{bmatrix}^\dagger\begin{bmatrix} l\\ m\end{bmatrix}.$$

Proof. From Proposition 3.10, we have $\mathscr{L}\cap\mathscr{M}\subset\mathscr{H}(\mathscr{L},\mathscr{M})$. Setting

$$q^*=\frac{\left\|L^\dagger l\right\|^2-\left\|M^\dagger m\right\|^2}{\left\|L^\dagger l-M^\dagger m\right\|^2}\left(L^\dagger l-M^\dagger m\right),$$

we get, by using the Anderson-Duffin formula for linear varieties (see (11) in Theorem 1.1),

$$\mathbb{P}_{\mathscr{L}\cap\mathscr{M}}\left(q^*\right)=\mathbb{P}_{\mathscr{L}_0\cap\mathscr{M}_0}\left(q^*\right)+\begin{bmatrix} L\\ M\end{bmatrix}^\dagger\begin{bmatrix} l\\ m\end{bmatrix}.$$

As $\mathscr{L}\cap\mathscr{M}\subset\mathscr{H}(\mathscr{L},\mathscr{M})$, we obtain $\mathscr{L}_0\cap\mathscr{M}_0\subset\mathscr{H}_0(\mathscr{L},\mathscr{M})$. Hence, from (8) in Theorem 1.1 and [20, 24],

$$\begin{aligned}\mathbb{P}_{\mathscr{L}_0\cap\mathscr{M}_0}\left(q^*\right)&=\left(\mathbb{P}_{\mathscr{L}_0\cap\mathscr{M}_0}\right)\mathbb{P}_{\mathscr{H}_0}\left(q^*\right)\\&=\mathbb{P}_{\mathscr{L}_0\cap\mathscr{M}_0}\left(0\right)\\&=0,\end{aligned}$$

as $\mathscr{L}_0\cap\mathscr{M}_0$ is a subspace and $\mathbb{P}_S\left(x\right)=x$ if $x\in S$ and S is a subspace. This ends the proof. □

4. Position vectors of the intersection of three linear varieties

We consider spheres and their intersections with linear varieties. The case of the intersection of three linear varieties is dealt with by using a suitable generalization of the Anderson-Duffin formula.

Proposition 4.1. *Let $\mathscr{L}$ and $\mathscr{M}$ be two intersecting linear varieties*

$$\mathscr{L}=\{x\in\mathbb{R}^n : Lx=l\}=l_0+\mathscr{L}_0$$

and

$$\mathscr{M}=\{x\in\mathbb{R}^n : Mx=m\}=m_0+\mathscr{M}_0.$$

Let be two spheres with non-trivial intersection

$$S_{\mathscr{L}}(c_1,r_1)=\{x\in\mathscr{L} : \|x-c_1\|=r_1,\quad c_1\in\mathscr{L}\}$$

and

$$S_{\mathscr{M}}(c_2,r_2)=\{x\in\mathscr{M} : \|x-c_2\|=r_2,\quad c_2\in\mathscr{M}\}.$$

Let be the radical hyperplane

$$\begin{aligned}\mathscr{H}&:=\mathscr{H}\left(S_{\mathscr{L}},S_{\mathscr{M}}\right)\\&=\{x\in\mathbb{R}^n : Hx=h\}\\&=h_0+\mathscr{H}_0.\end{aligned}$$

Then a position vector of the intersection $\mathscr{H} \cap \mathscr{L} \cap \mathscr{M}$ of the three linear varieties $\mathscr{H}$, $\mathscr{L}$ and $\mathscr{M}$ is given by

$$c^{**} = 4\mathbb{P}_{\mathscr{H}_0}\left(\mathbb{P}_{\mathscr{H}_0} + \mathbb{P}_{\mathscr{L}_0}\right)^{\dagger}\mathbb{P}_{\mathscr{L}_0}\left[2\mathbb{P}_{\mathscr{H}_0}\left(\mathbb{P}_{\mathscr{H}_0} + \mathbb{P}_{\mathscr{L}_0}\right)^{\dagger}\mathbb{P}_{\mathscr{L}_0} + \mathbb{P}_{\mathscr{M}_0}\right]^{\dagger}\mathbb{P}_{\mathscr{M}_0}(c^*) + \begin{bmatrix} H \\ L \\ M \end{bmatrix}^{\dagger}\begin{bmatrix} h \\ l \\ m \end{bmatrix},$$

where c^* is

$$c^* = \frac{r_2^2 - r_1^2 + d^2}{2d^2}c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2}c_2,$$

with $d = \|c_1 - c_2\|$.

Proof. See Proposition 2.1 and [6, Proposition 2.1]. □

In the next result, we express, in another way, the projection of a point onto the intersection of three linear varieties.

Proposition 4.2. *Let be two intersecting linear varieties*

$$\mathscr{L} = \{x \in \mathbb{R}^n : Lx = l\} = l_0 + \mathscr{L}_0$$

and

$$\mathscr{M} = \{x \in \mathbb{R}^n : Mx = m\} = m_0 + \mathscr{M}_0.$$

Let be two spheres with non-trivial intersection

$$S_{\mathscr{L}}(c_1, r_1) = \{x \in \mathscr{L} : \|x - c_1\| = r_1, \quad c_1 \in \mathscr{L}\}$$

and

$$S_{\mathscr{M}}(c_2, r_2) = \{x \in \mathscr{M} : \|x - c_2\| = r_2, \quad c_2 \in \mathscr{M}\}.$$

Let $\mathscr{H}$ be the radical hyperplane given by

$$\begin{aligned} \mathscr{H} &:= \mathscr{H}\left(S_{\mathscr{L}}, S_{\mathscr{M}}\right) \\ &= \{x \in \mathbb{R}^n : Hx = h\} \\ &= h_0 + \mathscr{H}_0. \end{aligned}$$

Then a position vector of the intersection $\mathscr{H} \cap \mathscr{L} \cap \mathscr{M}$ of the three linear varieties $\mathscr{H}$, $\mathscr{L}$ and $\mathscr{M}$ is given by

$$g = \left(I - F^{\dagger}F\right)(c^*) + F^{\dagger}f,$$

where

$$c^* = \frac{r_2^2 - r_1^2 + d^2}{2d^2}c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2}c_2$$

and $d = \|c_1 - c_2\|$, with F and f expressed by

$$F = \begin{bmatrix} H \\ L \\ M \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} h \\ l \\ m \end{bmatrix}.$$

Proof. See (7) of Theorem 1.1 and [6, Proposition 2.6]. □

5. Position vectors in terms of commutativity of projectors

In this section we deal, in two settings (linear subspaces and linear varieties), with projectors that commute. Commutative projectors do constitute a subtle subject, and it is the object of a quantum-mechanical interpretation, as considered in [4, Remark 10.8, p. 1456], [21, 22] and [23].

Proposition 5.1. *(1) Let be two linear subspaces*

$$\mathscr{L}_0 = \{x \in \mathbb{R}^n : Lx = 0\}$$

and

$$\mathscr{M}_0 = \{x \in \mathbb{R}^n : Mx = 0\}.$$

Then the orthogonal projectors onto $\mathscr{L}_0$ and $\mathscr{M}_0$ commute, i.e.

$$\mathbb{P}_{\mathscr{L}_0}\mathbb{P}_{\mathscr{M}_0} = \mathbb{P}_{\mathscr{M}_0}\mathbb{P}_{\mathscr{L}_0}$$

if and only if

$$L^\dagger L M^\dagger M = M^\dagger M L^\dagger L.$$

(2) Let be two linear varieties

$$\mathscr{L} = \{x \in \mathbb{R}^n : Lx = l\}$$

and

$$\mathscr{M} = \{x \in \mathbb{R}^n : Mx = m\}.$$

Then the orthogonal projectors commute, i.e.

$$\mathbb{P}_{\mathscr{L}}\mathbb{P}_{\mathscr{M}} = \mathbb{P}_{\mathscr{M}}\mathbb{P}_{\mathscr{L}}$$

if and only if

$$L^\dagger L M^\dagger M = M^\dagger M L^\dagger L$$

and

$$M^\dagger M L^\dagger l = L^\dagger L M^\dagger m.$$

Proof. For (1). See [24, Theorem 4.30, pp. 82-83].

For (2). By (6) in Theorem 1.1, we see that the orthogonal projectors commute, i.e.

$$\mathbb{P}_{\mathscr{L}}\mathbb{P}_{\mathscr{M}} = \mathbb{P}_{\mathscr{M}}\mathbb{P}_{\mathscr{L}}$$

if and only if, for every $x \in \mathbb{R}^n$, we have

$$L^\dagger l + \mathbb{P}_{\mathscr{N}(L)}\left(M^\dagger m + \mathbb{P}_{\mathscr{N}(M)}(x)\right) = M^\dagger m + \mathbb{P}_{\mathscr{N}(M)}\left(L^\dagger l + \mathbb{P}_{\mathscr{N}(L)}(x)\right).$$

That is to say

$$\left(\mathbb{P}_{\mathscr{N}(M)}\mathbb{P}_{\mathscr{N}(L)} - \mathbb{P}_{\mathscr{N}(L)}\mathbb{P}_{\mathscr{N}(M)}\right)(x) = M^\dagger M L^\dagger l - L^\dagger L M^\dagger m, \quad x \in \mathbb{R}^n.$$

This is equivalent to

$$\mathbb{P}_{\mathscr{N}(M)}\mathbb{P}_{\mathscr{N}(L)} = \mathbb{P}_{\mathscr{N}(L)}\mathbb{P}_{\mathscr{N}(M)}$$

and

$$M^\dagger M L^\dagger l = L^\dagger L M^\dagger m.$$

□

In the case the projectors do commute, the projections of the minimal points of the linear varieties play an interesting rôle.

Proposition 5.2. *Let be two intersecting linear varieties*

$$\mathscr{L} = L^{\dagger}l + \mathscr{N}(L)$$

and

$$\mathscr{M} = M^{\dagger}m + \mathscr{N}(M).$$

Let

$$\mathbb{P}_{\mathscr{L}}\mathbb{P}_{\mathscr{M}} = \mathbb{P}_{\mathscr{M}}\mathbb{P}_{\mathscr{L}}.$$

Then $\mathbb{P}_{\mathscr{L}}\left(M^{\dagger}m\right) = \mathbb{P}_{\mathscr{M}}\left(L^{\dagger}l\right)$ *is a position vector of* $\mathscr{L} \cap \mathscr{M}$*.*

Proof. Using (7) of Theorem 1.1, we have

$$\mathbb{P}_{\mathscr{L}}(M^{\dagger}m) = L^{\dagger}l + M^{\dagger}m - L^{\dagger}LM^{\dagger}m$$

and

$$\mathbb{P}_{\mathscr{M}}(L^{\dagger}l) = L^{\dagger}l + M^{\dagger}m - M^{\dagger}ML^{\dagger}l.$$

Using Proposition 5.1, we get $L^{\dagger}LM^{\dagger}m = M^{\dagger}ML^{\dagger}l$. This ends the proof. □

6. Characterization of position vectors in terms of centres of spheres and intersection of subspaces

Finding a position vector of the intersection of linear varieties is useful when studying the projection of a point onto the intersection of two spheres and in the study of the distance between a point and a sphere [6, Propositions 3.4 and 4.1].

Next, we find the position vector of the intersection of two linear varieties in terms of intersecting spheres in $\mathbb{R}^n$.

Proposition 6.1. *Let be the intersecting spheres* $S_1 = S(c_1, r_1)$ *and* $S_2 = S(c_2, r_2)$ *in* $\mathbb{R}^n$*. Let be given two intersecting linear varieties*

$$\mathscr{L} = \{x \in \mathbb{R}^n : Lx = l\} = c_1 + \mathscr{L}_0$$

and

$$\mathscr{M} = \{x \in \mathbb{R}^n : Mx = m\} = c_2 + \mathscr{M}_0.$$

Then the centre c^* *of the sphere* $S_1 \cap S_2$*, given by*

$$c^* = \frac{r_2^2 - r_1^2 + d^2}{2d^2}c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2}c_2,$$

is a position vector of the linear variety $\mathscr{L} \cap \mathscr{M}$ *if and only if*

$$c_2 - c_1 \in \mathscr{L}_0 \cap \mathscr{M}_0,$$

where $d = \|c_1 - c_2\|$*.*

Proof. The centre c^* was achieved in Theorem 3.2 and in [6, Proposition 2.5]. Notice that $c^* \in \mathscr{L}$ if and only if $c^* - c_1 \in \mathscr{L}_0$. Since $c_2 - c_1$ and $c^* - c_1$ are linearly dependent, this is equivalent to $c_2 - c_1 \in \mathscr{L}_0$. Analogously, we conclude that $c^* \in \mathscr{M}$ if and only if $c_2 - c_1 \in \mathscr{M}_0$. Therefore, $c^* \in \mathscr{L} \cap \mathscr{M}$ if and only if $c_2 - c_1 \in \mathscr{L}_0 \cap \mathscr{M}_0$. □

Remark 6.2. By (12) of Theorem 1.1, the position vectors c_1 and c_2 of the linear varieties satisfy the relation $c_2 - c_1 \in \mathscr{L}_0 + \mathscr{M}_0$. In Proposition 6.1, these position vectors must satisfy simultaneously the relations $c_2 - c_1 \in \mathscr{L}_0 \cap \mathscr{M}_0$ and $c_2 - c_1 \in \mathscr{L}_0 + \mathscr{M}_0$.

7. Conclusions and Remarks

In this paper, we dealt mainly with linear varieties with non-void intersection.

The centre of the intersection of spheres plays a fundamental rôle in the most of the present paper.

We set results concerning vectors belonging both to the sum of subspaces and to the intersection of subspaces (Remark 6.2). This leads us to consider the Zassenhaus algorithm, which allows find simultaneously a basis for the intersection of subspaces and a basis for the sum of two subspaces [16, pp. 207-210].

Use was made of the subtle concept of commuting projectors. Some results were offered in this context, both to the intersection of subspaces and to the intersection of linear varieties. If, however, we intended to extend our preoccupations to the best approximation pair of two non-intersecting linear varieties [5, Corollary 4.17],

$$x^* = \left(L^\dagger L + M^\dagger M - L^\dagger L M^\dagger M\right)^\dagger \left(L^\dagger l + M^\dagger m - L^\dagger L M^\dagger m\right)$$

and

$$y^* = \left(L^\dagger L + M^\dagger M - M^\dagger M L^\dagger L\right)^\dagger \left(L^\dagger l + M^\dagger m - M^\dagger M L^\dagger l\right),$$

the use of commuting projectors will allow to drop the exponent † in the expressions of the referred to best approximation points. In fact, in the commutative context, the Boolean sum [17]

$$L^\dagger L \oplus M^\dagger M = L^\dagger L + M^\dagger M - L^\dagger L M^\dagger M$$

is a projector ([3, Fact 3.13.20], [9, p. 481] and [17, p. 126, Lemma 11.1]).

The radical hyperplane plays an important rôle in the present paper. It exists even when the linear varieties do not intersect. We observe, moreover, that the radical hyperplane usually is not parallel to the separating hyperplanes ([8, pp. 105-106] and [14]) of the non-intersecting linear varieties.

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