GENERALIZED INVERSES OF SPECIAL MATRICES PARTITIONED INTO BLOCKS

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ABSTRACT. It is known the connection between the best approximation and Moore-Penrose invertibility. In particular, the best approximation pair of two linear varieties is related to generalized invertibility of a block matrix of the form $\begin{bmatrix} I_n & A \\ B & I_m \end{bmatrix}$. In this paper, we address Moore-Penrose and group invertibility of a matrix of the form $\begin{bmatrix} I_n & A \\ B & D \end{bmatrix}$.

1. INTRODUCTION

Given a $m \times n$ complex matrix A, we denote by A^*, $R(A)$, $\ker(A)$ and $rk(A)$ the conjugate transpose, range (column space), kernel and rank, respectively, of A.

A solution X to the matrix equation $AXA = A$ always exists, and a particular solution can be obtained by the row-reduced echelon form or the Hermite normal form (see [2, Theorem 1, p. 41]), and therefore the set of solutions, denoted by $A\{1\} = \{X : AXA = A\}$, is non-empty. It should be noted that $A\{1\}$ is a singleton if and only if A is non-singular, and also there exists an invertible matrix in $A\{1\}$, for any choice of A. A von Neumann inverse, or (1)-inverse, of A is an element of $A\{1\}$, and a general element is denoted by A^- or by $A^{(1)}$.

We shall consider Moore-Penrose invertibility of a (possibly non-square) matrix A. Let $A^\dagger$ denote the Moore-Penrose inverse of A, i.e., for the unique matrix solution to the equations

$$
\begin{aligned}
&(1) \qquad AXA = A \\
&(2) \qquad XAX = X \\
&(3) \qquad AX = (AX)^* \\
&(4) \qquad XA = (XA)^*.
\end{aligned}
$$

It is easy to show that $R(A) = R((A^\dagger)^*)$ and that $rk(A) = rk(A^\dagger)$.

It is usual to refer a common solution to the equations (i_k) as a (i_k)-inverse. For instance, a (1)-inverse is a von Neumann inverse. A (1,2)-inverse is a common solution to equations (1) and (2), also known as a reflexive inverse. A (1,2,3,4)-inverse is exclusively identified as the Moore-Penrose inverse.

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The Moore-Penrose inverse of A can be computed by using the Singular Value Decomposition. Indeed, if $A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*$, where U, V are unitary and $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$, being σ_i the distinct positive singular values of A, then $A^\dagger = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*$.

Given a square $n \times n$ matrix A, we define the group inverse $A^\#$ of A as the unique solution, if it exists, to the matrix equations

$$AA^\# A = A, A^\# AA^\# = A^\#, AA^\# = A^\# A.$$

In contrast to Moore-Penrose invertibility, the group inverse may not exist. For instance, given a nilpotent matrix $N \neq 0$ such that $N^2 = 0$, it is easy to show, from the definition, that $N^\#$ does not exist. The existence of the group inverse of A is equivalent to $rk(A) = rk(A^2)$.

It should be noted that group invertibility is invariant to matrix similarity, and if $B = U^{-1}AU$ and $A^\#$ exists, then $B^\# = U^{-1}A^\# U$. Also, $diag(A_0, \ldots, A_k)$ is group invertible if and only if $A_i^\#$ all exist, in which case $diag(A_0, \ldots, A_k)^\# = diag(A_0^\#, \ldots, A_k^\#)$. So, considering the Jordan Normal Form of a matrix A, and since the Jordan blocks corresponding to the non-zero eigenvalues of A are invertible matrices, and hence group invertible, we can say $A^\#$ exists if and only if the Jordan blocks corresponding to the zero eigenvalue (if any) are group invertible. Since these are nilpotent, they must be all size 1×1, that is, the chains of generalized eigenvectors corresponding to the eigenvalue 0 are of length 1, which in turn is equivalent to A only having linear elementary divisors corresponding to the eigenvalue 0 (if A is singular). This means that the minimal polynomial of a singular group invertible matrix A is of the form $\psi_A(\lambda) = \lambda f(\lambda)$, where $f(0) \neq 0$.

For further results and references, the reader is referred to, for instance, [2, 6, 18].

A 2×2 block partitioned matrix of the form $F = \begin{bmatrix} I_n & A \\ B & I_m \end{bmatrix}$ plays a role when looking for the best approximation pair of two linear varieties, as in [7, 17]. A von Neumann inverse and the Moore-Penrose inverse of the matrix F are used while studying the best approximation pair of two linear varieties. Searching for an explicit formulae for the Moore-Penrose inverse $F^\dagger = \begin{bmatrix} I_n & A \\ B & I_m \end{bmatrix}^\dagger$ was motivated by the results presented in [7, Corollary 4.7], by using Remark 3.5 within, and in [17, Corollary 4.2], by using Corollary 3.4 within.

In this paper, not only we shall give an explicit formula for the Moore-Penrose inverse of $\begin{bmatrix} I_n & A \\ B & I_m \end{bmatrix}$, but we will provide a formula for a more general case of the Moore-Penrose inverse of $\begin{bmatrix} I_n & A \\ B & D \end{bmatrix}$, where the blocks vary freely, with conformal sizes. We complete our research by giving a necessary and sufficient condition for $\begin{bmatrix} I_n & A \\ B & D \end{bmatrix}^\#$ to exist.

We start with some preliminary results.

2. Preliminary results

It is well known the interplay between von Neumann inverses and Moore-Penrose invertibility. A very interesting result is presented in [20, p.132], where an explicit formula for the Moore-Penrose inverse $M^\dagger$ of a matrix M is given in terms of a von Neumann inverse, namely by $M^\dagger = M^*(M^*MM^*)^- M^*$.

In the present paper, we give another method for constructing the Moore-Penrose inverse $M^\dagger$ of a matrix M using a von Neumann inverse M^- of M. For such, we will use a particular factorization of M which is derived from the Singular Value Decomposition.

Lemma 2.1. [8, Corollary 6] Given a matrix A of rank r, then there exists a unitary matrix $U = (U^{-1})^*$ such that

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*,$$

where $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$, σ_i are the distinct positive singular values of A, and K, L are such that $KK^* + LL^* = I_r$.

We shall refer this decomposition as the Hartwig-Spindelböck factorization. It follows directly from Lemma 2.1 that $A^\dagger = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*$.

When dealing with the inclusion of subspaces, there are two main approaches: one may use generalized inverses as in [5, p. 255], [10, identity (1.8) p. 355], [9, identity (1.8) p. 74] and [16, pp. 21, 67]; or by using projections as in [14], [19, Theorems 4.30 and 4.31, pp. 82-83] and [3, pp. 152-153]. They are in fact relatable, as Moore-Penrose invertibility is strongly connected to orthogonal projections. Indeed, given a matrix M, then $MM^\dagger$ is the (orthogonal) projector onto $R(M)$, wheareas $M^\dagger M$ is the (orthogonal) projector onto $R(M^*)$. Furthermore, given two matrices A and B, then

(1) $B = BA^\dagger A$ if and only if $R(B^*) \subseteq R(A^*)$;
(2) $BB^\dagger A = A$ if and only if $R(A) \subseteq R(B)$;
(3) $R(A) \subseteq R(B)$ if and only if $P_{R(A)} = P_{R(A)}P_{R(B)} = P_{R(B)}P_{R(A)}$, where $P_{R(M)}$ stands for the (orthogonal) projector onto $R(M)$.

In same cases, a viable path in order to obtain the Moore-Penrose inverse involves a full-rank factorization of the matrix M. In fact, given a full rank factorization $M = FG$, then $M^\dagger = G^*(GG^*)^{-1}(F^*F)^{-1}F^*$.

As in the case of Moore-Penrose invertibility, a full rank decomposition may also presents as quite valuable in the scope of group invertibility. Indeed, given a full rank factorization $M = FG$, then M has a group inverse if and only if GF is invertible, in which case $M^\# = F(GF)^{-2}G$.

In this paper, we will not use a full rank factorization in order to approach Moore-Penrose and group invertibility of a matrix of the form $\begin{bmatrix} I & B \\ C & D \end{bmatrix}$. In fact, we will address this problem by studying associated non-singular matrices, and from there obtain the expressions for its Moore-Penrose and group inverses.

In order to do so, we need auxiliary results that concern matrix invertibility.

Lemma 2.2. Given an invertible matrix α, the block matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is invertible if and only if $Z = \delta - \gamma\alpha^{-1}\beta$ is invertible. In this case,

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^{-1} = \begin{bmatrix} \alpha^{-1}(\alpha + \beta Z^{-1}\gamma)\alpha^{-1} & -\alpha^{-1}\beta Z^{-1} \\ -Z^{-1}\gamma\alpha^{-1} & Z^{-1} \end{bmatrix}.$$

Proof. We take the Schur complement on the (1,1) block, using the factorization $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \gamma\alpha^{-1} & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \delta - \gamma\alpha^{-1}\beta \end{bmatrix} \begin{bmatrix} 1 & \alpha^{-1}\beta \\ 0 & 1 \end{bmatrix}$. □

We will also need a special case of the Woodbury identity:

Lemma 2.3. The matrix $I + XY$ is invertible if and only if the matrix $I + YX$ is invertible, in which case

$$(I + XY)^{-1} = I - X(I + YX)^{-1}Y.$$

The following result can be found in [2, Corollary 1, p. 52], and allows to characterize the set $A\{1\}$, given a particular von Neumann inverse of A.

Lemma 2.4. Given a von Neumann inverse A^- of A, then any von Neumann inverse of A is of the form $A^- + Z - A^-AZAA^-$, where Z is an arbitrary matrix. In particular, any von Neumann inverse of A is of the form $A^\dagger + Z - A^\dagger AZAA^\dagger$.

3. The Moore-Penrose inverse of the block matrix

From Lemma 2.2 as well as recovering [4, Fact 2.17.4, p.147], we remark that

$$\begin{aligned} \begin{bmatrix} I_n & A \\ B & I_m \end{bmatrix}^{-1} &= \begin{bmatrix} I_n + A(I_m - BA)^{-1}B & -A(I_m - BA)^{-1} \\ -(I_m - BA)^{-1}B & (I_m - BA)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (I_n - AB)^{-1} & -(I_n - AB)^{-1}A \\ -B(I_n - AB)^{-1} & I_m + B(I_n - AB)^{-1}A \end{bmatrix}. \end{aligned}$$

We wondered if a parallel expression could be used for the inverse of Moore-Penrose. We will show that it is not the case, although a similar expression arises for a special generalized inverse.

Theorem 3.1. Given a 2×2 block-matrix

$$F = \begin{bmatrix} I_n & A \\ B & I_m \end{bmatrix},$$

where $A \in \mathbb{M}_{n\times m}$ and $B \in \mathbb{M}_{m\times n}$, then

$$\begin{bmatrix} (I_n - AB)^- & -A(I_m - BA)^- \\ -(I_m - BA)^-B & (I_m - BA)^- \end{bmatrix} \in F\{1\}.$$

Proof. Factoring F as

$$F = \begin{bmatrix} I_n & A \\ B & I_m \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ B & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & I_m - BA \end{bmatrix} \begin{bmatrix} I_n & A \\ 0 & I_m \end{bmatrix},$$

we obtain that

$$\begin{bmatrix} I_n & -A \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & (I_m - BA)^- \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -B & I_m \end{bmatrix} = \begin{bmatrix} I_n + A(I_m - BA)^-B & -A(I_m - BA)^- \\ -(I_m - BA)^-B & (I_m - BA)^- \end{bmatrix}$$

is a von Neumann inverse of F, for any choice of $(I_m - BA)^-$. Using [12, Lemma 1.1], we know

$$I_n + A(I_m - BA)^-B \in (I_n - AB)\{1\} \tag{3.1}$$

and hence

$$\begin{bmatrix} (I_n - AB)^- & -A(I_m - BA)^- \\ -(I_m - BA)^- B & (I_m - BA)^- \end{bmatrix} \in F\{1\}.$$

□

We now consider a more general case where we allow the blocks to vary freely, except the (1,1) block that we fix as the identity matrix. That is, the block matrix is of the form $M = \begin{bmatrix} I & B \\ C & D \end{bmatrix}$, in which the free blocks have conformal sizes.

We prove, in the matrix setting, a result that relates Moore-Penrose invertibility and matrix inverses, recovering a result known in the ring setting ([12]).

Lemma 3.2. Given a matrix M and any (1)-inverse M^- of M, the matrix $MM^* + I - MM^-$ is invertible and $M^\dagger = ((MM^* + I - MM^-)^{-1}M)^*$.

Proof. Consider the Hartwig-Spindelböck decomposition of M as in Lemma 2.1; that is,

$$M = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*,$$

where $U = (U^{-1})^*$, K, L are such that $KK^* + LL^* = I_r$, with $r = \text{rank}(M)$, and $\Sigma = diag(\sigma_1 I_{r_1}, \ldots, \sigma_r I_{r_k})$ is the diagonal matrix whose diagonal blocks are given by the (positive) distinct singular values σ_i of M.

Using this factorization, it is easy to check that $M^\dagger = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*$. Furthermore, $MM^\dagger = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^*$ and $M^\dagger M = U \begin{bmatrix} K^*K & K^*L \\ L^*K & L^*L \end{bmatrix} U^*$.

Using Lemma 2.4, we know $MM^- = MM^\dagger + MZ - MZMM^\dagger$, for some matrix Z. Let us write $Z = U \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} U^*$. Now

$$\begin{aligned} MZMM^\dagger &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} \Sigma K Z_1 + \Sigma L Z_3 & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

Also, $I - MM^- = U \begin{bmatrix} 0 & -\Sigma K Z_2 - \Sigma L Z_4 \\ 0 & I \end{bmatrix} U^*$ and

$$MM^* = U \begin{bmatrix} \Sigma KK^*\Sigma + \Sigma LL^*\Sigma & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

which lead to

$$MM^* + I - MM^- = U \begin{bmatrix} \Sigma^2 & -\Sigma L Z_4 - \Sigma K Z_2 \\ 0 & I \end{bmatrix} U^* = K.$$

Since $K^{-1} = U \begin{bmatrix} \Sigma^{-2} & -\Sigma L Z_4 + \Sigma K Z_2 \\ 0 & I \end{bmatrix} U^*$, then

$$\begin{aligned} K^{-1}M &= U \begin{bmatrix} \Sigma^{-2} & \Sigma^{-1} L Z_4 + \Sigma^{-1} K Z_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} \Sigma^{-1} K & \Sigma^{-1} L \\ 0 & 0 \end{bmatrix} U^* \\ &= \left(U \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} U^* \right)^* \\ &= \left(M^\dagger\right)^* \end{aligned}$$

□

It should be noted that the invertibility of $MM^* + I - MM^-$ holds for *any* choice of von Neumann inverse M^- of M.

Theorem 3.3. Given a block matrix $M = \begin{bmatrix} I & B \\ C & D \end{bmatrix}$ in which the blocks are of conformal sizes, then

$$M^\dagger = \begin{bmatrix} \alpha^{-1}(\alpha + \beta Z^{-1}\gamma)\alpha^{-1} - \alpha^{-1}\beta Z^{-1} C & \alpha^{-1}(\alpha + \beta Z^{-1}\gamma)\alpha^{-1} B - \alpha^{-1}\beta Z^{-1} D \\ -Z^{-1}\gamma\alpha^{-1} + Z^{-1} C & Z^{-1} D - Z^{-1}\gamma\alpha^{-1} B \end{bmatrix}^*$$

where

$$\begin{aligned} \alpha &= I + BB^* \\ \beta &= C^* + BD^* \\ \gamma &= C + DB^* - (I - \zeta\zeta^-)C \\ \delta &= CC^* + DD^* + I - \zeta\zeta^- \\ \zeta &= D - CB \\ Z &= \delta - \gamma\alpha^{-1}\beta. \end{aligned}$$

Proof. We will use Lemma 3.2 in order to construct the Moore-Penrose inverse of M. For such, we will need a choice of von Neumann inverse of M. Since $M = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D - CB \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$ then we can take

$$M^- = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (D - CB)^- \end{bmatrix} \begin{bmatrix} I & 0 \\ -C & I \end{bmatrix},$$

which gives

$$MM^- = \begin{bmatrix} I & 0 \\ (I - \zeta\zeta^-)C & \zeta\zeta^- \end{bmatrix},$$

where $\zeta = D - CB$.

Then

$$\begin{aligned} MM^* + I - MM^- &= \begin{bmatrix} I & B \\ C & D \end{bmatrix} \begin{bmatrix} I & C^* \\ B^* & D^* \end{bmatrix} + I - MM^- \\ &= \begin{bmatrix} I + BB^* & C^* + BD^* \\ C + DB^* - (I - \zeta\zeta^-)C & CC^* + DD^* + I - \zeta\zeta^- \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \end{aligned}$$

Since BB^* is a positive semidefinite matrix, its spectrum is a subset of $\mathbb{R}_0^+$, and therefore 0 is not an eigenvalue of $\alpha = I + BB^*$, which makes α an invertible matrix. Now, the invertibility of $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is equivalent to the invertibility of $Z = \delta - \gamma\alpha^{-1}\beta$, by Lemma 2.2, and

$$(MM^* + I - MM^-)^{-1} = \begin{bmatrix} \alpha^{-1}(\alpha + \beta Z^{-1}\gamma)\alpha^{-1} & -\alpha^{-1}\beta Z^{-1} \\ -Z^{-1}\gamma\alpha^{-1} & Z^{-1} \end{bmatrix}.$$

Post multiplying by M, we obtain

$$\begin{aligned} &(MM^* + I - MM^-)^{-1}M = \\ &= \begin{bmatrix} \alpha^{-1}(\alpha + \beta Z^{-1}\gamma)\alpha^{-1} - \alpha^{-1}\beta Z^{-1}C & \alpha^{-1}(\alpha + \beta Z^{-1}\gamma)\alpha^{-1}B - \alpha^{-1}\beta Z^{-1}D \\ -Z^{-1}\gamma\alpha^{-1} + Z^{-1}C & Z^{-1}D - Z^{-1}\gamma\alpha^{-1}B \end{bmatrix}. \end{aligned}$$

□

As a corollary, we recover the Moore-Penrose of the block matrix F in which the (2,2) block is the identity matrix.

Corollary 3.4. Given a block matrix $F = \begin{bmatrix} I & B \\ C & I \end{bmatrix}$ in which the blocks are of conformal sizes, then

$$F^\dagger = \begin{bmatrix} \alpha^{-1}(\alpha + \beta Z^{-1}\gamma)\alpha^{-1} - \alpha^{-1}\beta Z^{-1}C & \alpha^{-1}(\alpha + \beta Z^{-1}\gamma)\alpha^{-1}B - \alpha^{-1}\beta Z^{-1} \\ -Z^{-1}\gamma\alpha^{-1} + Z^{-1}C & Z^{-1} - Z^{-1}\gamma\alpha^{-1}B \end{bmatrix}^*$$

where

$$\begin{aligned} \alpha &= I + BB^* \\ \beta &= C^* + B \\ \gamma &= C + B^* - (I - \zeta\zeta^-)C \\ \delta &= CC^* + 2I - \zeta\zeta^- \\ \zeta &= I - CB \\ Z &= \delta - \gamma\alpha^{-1}\beta. \end{aligned}$$

4. The group inverse of the block matrix

In order to address the group invertibility of the matrix $\begin{bmatrix} I & B \\ C & D \end{bmatrix}$, we need an auxiliary result that relates group invertibility to non-singular matrices. This result was primarily proved in the ring context, see [13, 15], but we give a matrix theoretical proof for the sake of completeness.

Lemma 4.1. The square $n \times n$ matrix A is group invertible if and only if $A + I_n - AA^-$ is invertible, for one and hence all choices of von Neumann inverse A^- of A, in which case $A^\# = (A + I_n - AA^-)^{-2}A$.

Proof. Consider the Jordan Normal Form of A, which is permutational similar to a matrix of the form $\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}$, where J is a direct sum of the Jordan blocks corresponding to the nonzero eigenvalues, and N is a direct sum of the Jordan blocks corresponding to the zero eigenvalue. That is, there exists an invertible matrix U such that $A = U \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} U^{-1}$. Obviously, the matrix J is invertible and N is a nilpotent matrix. We will consider $A^- = U \begin{bmatrix} J^{-1} & 0 \\ 0 & N^- \end{bmatrix} U^{-1}$ as a von Neumann inverse of A, where N^- is any von Neumann inverse of N. For this choice of A^-, we obtain $AA^- = U \begin{bmatrix} I & 0 \\ 0 & NN^- \end{bmatrix} U^{-1}$, from which $A+I_n-AA^- = U \begin{bmatrix} J & 0 \\ 0 & N + I_k - NN^- \end{bmatrix} U^{-1}$, were k is the algebraic multiplicity of the eigenvalue zero of A.

Now, $A + I_n - AA^-$ is invertible if and only if $N + I_k - NN^-$ is invertible. We can write $N + I_k - NN^- = \begin{bmatrix} I_k & N \end{bmatrix} \begin{bmatrix} I_k - NN^- \\ I_k \end{bmatrix} = XY$. The former being invertible means zero is *not* a root of the minimal polynomial $\psi_{XY}(\lambda)$ of XY, that is, λ does *not* divide $\psi_{XY}(\lambda)$. This implies that either $\psi_{YX}(\lambda) = \psi_{XY}(\lambda)$ and 0 is not a root of the minimal polynomial of YX, or $\psi_{YX}(\lambda) = \lambda\psi_{XY}(\lambda)$ and 0 is a simple root of $\psi_{YX}(\lambda)$. Since $YX = \begin{bmatrix} I_k - NN^- & 0 \\ I_k & N \end{bmatrix}$ is not invertible, since N is nilpotent, then 0 is an eigenvalue of YX, from which $\psi_{YX}(0) = 0$ and hence $\psi_{YX}(\lambda) = \lambda\psi_{XY}(\lambda)$. Hence, YX is group invertible, which implies, in particular, that N is group invertible. From the nilpotency of N, this can only occur if $N = 0$. From the above, the group invertibility of A is equivalent to $N = 0$, that is the Jordan blocks corresponding to the eigenvalue 0 are of size 1×1, or equivalently, the chains of generalized eigenvectors corresponding to the eigenvalue 0 are of length 1.

The equality $A^\# = U \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{-1} = (A + I_n - AA^-)^{-2}A$ is verified by performing simple calculations.

We are now left with the proof that we can take any von Neumann inverse of A.

For A^- fixed above, we know, from Lemma 2.4, that *any* von Neumann inverse $A^=$ of A is of the form $A^= = A^- + Z - A^-AZAA^-$, for some matrix Z. It should be noted that the invertibility of $U = A + I - AA^-$ is equivalent to the invertiblity of $V = A + I - AA^=$, from Lemma 2.3. Indeed, $U = I + AA^=(A - AA^-)$ is invertible if and only if $I + (A - AA^-)AA^= = A^2A^= + I - AA^=$ is invertible. We apply again Lemma 2.3 obtaining the equivalence between the invertibility of $I + (A - AA^=)AA^=$ and of $V = I + AA^=(A - AA^=)$.

We will now show that if U is invertible, and hence $A^\#$ exists, then $A^\# = V^{-2}A$. Let $Z = U \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} U^{-1}$, from which $AA^= = U \begin{bmatrix} I & JZ_2 \\ 0 & 0 \end{bmatrix} U^{-1}$ and

$$V = U \begin{bmatrix} J & -JZ_2 \\ 0 & I \end{bmatrix} U^{-1}, V^{-1} = U \begin{bmatrix} J^{-1} & Z_2 \\ 0 & I \end{bmatrix} U^{-1}$$

and

$$V^{-2} = U \begin{bmatrix} J^{-2} & J^{-1}Z_2 + Z_2 \\ 0 & I \end{bmatrix} U^{-1}.$$

Now, $V^{-2}A = U \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{-1} = A^{\#}$.

□

Theorem 4.2. Given a block matrix $M = \begin{bmatrix} I & B \\ C & D \end{bmatrix}$ in which the blocks are of conformal sizes, then M is group invertible if and only $Z = D + I - \zeta\zeta^{-}(I - CB)$ is an invertible matrix, where $\zeta = D - CB$, in which case

$$M^{\#} = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

where

$$\begin{aligned}
M_1 &= (I + BZ\gamma)(I + BZ\gamma - BZ^{-1}C) - BZ^{-2}(I - \zeta\zeta^{-})C \\
M_2 &= (I + BZ\gamma)(B + BZ\gamma B - BZ^{-1}D) + BZ^{-2}(\gamma B - D) \\
M_3 &= -Z^{-1}\gamma(I + BZ\gamma + BZ^{-1}C) + Z^{-2}(I - \zeta\zeta^{-})C \\
M_4 &= Z^{-1}\gamma(B + BZ\gamma B + BZ^{-1}D) + Z^{-2}(D - \gamma) \\
\gamma &= \zeta\zeta^{-}C.
\end{aligned}$$

Proof. Factoring $M = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D - CB \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$, we may take take

$$M^{-} = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (D - CB)^{-} \end{bmatrix} \begin{bmatrix} I & 0 \\ -C & I \end{bmatrix},$$

which gives

$$MM^{-} = \begin{bmatrix} I & 0 \\ (I - \zeta\zeta^{-})C & \zeta\zeta^{-} \end{bmatrix},$$

where $\zeta = D - CB$. Now, Lemma 4.1 guarantees that $M^{\#}$ exists if and only if $U = M + I - MM^{-}$ is a non singular matrix, that is,

$$\begin{aligned}
U &= M + I - MM^{-} \\
&= \begin{bmatrix} I & B \\ C - (I - \zeta\zeta^{-})C & D + I - \zeta\zeta^{-} \end{bmatrix} \\
&= \begin{bmatrix} I & B \\ \zeta\zeta^{-}C & D + I - \zeta\zeta^{-} \end{bmatrix} \\
&= \begin{bmatrix} I & B \\ \gamma & \delta \end{bmatrix},
\end{aligned}$$

is an invertible block matrix, where $\gamma = \zeta\zeta^{-}C$ and $\delta = D + I - \zeta\zeta^{-}$.

Applying Lemma 2.2, we obtain that U is invertible if and only if $Z = \delta - \gamma B$ is invertible, and

$$U^{-1} = \begin{bmatrix} I + BZ\gamma & -BZ^{-1} \\ -Z^{-1}\gamma & Z^{-1} \end{bmatrix}.$$

We are left to compute $M^{\#} = U^{-2}M$. □

We now obtain, as a corollary, the interesting case where B and C are prescribed.

Corollary 4.3. Given a block matrix $M = \begin{bmatrix} I & B \\ C & D \end{bmatrix}$ in which $R(B) \subseteq \ker(C)$, then M is group invertible if and only D is group invertible.

Proof. It is clear that $R(B) \subseteq \ker(C)$ is equivalent to $CB = 0$. Indeed, if $CB = 0$ then for any $y \in R(B)$ there exists x such that $y = Bx$, and hence $Cy = CBx = 0$ and $y \in \ker(C)$. Conversely, and writting $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix}$, since $b_i \in R(B) \subseteq \ker(C)$ then $Cb_i = 0$, and therefore $CB = C\begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} = 0$.

Using Theorem 4.2, the existence of $M^\#$ is equivalent to $Z = D + I - DD^-$ being invertible, for one and hence all choices of $D^- \in D\{1\}$. But this is equivalent by its turn, using Lemma 4.1, to $D^\#$ exists. □

Finally, we consider the case in which the $(2, 2)$ block is the zero matrix.

Corollary 4.4. Given a block matrix $M = \begin{bmatrix} I & B \\ C & 0 \end{bmatrix}$, then M is group invertible if and only CB is group invertible.

Proof. Using Theorem 4.2, we obtain $\zeta = -CB$ to which we may take $\zeta^- = -(CB)^-$. For this choice, $Z = I - CB(CB)^-(I - CB) = CB + I - (CB)(CB)^-$. Now, the invertibility of Z is equivalent to the group invertiblity of CB, from Lemma 4.1. □

5. Final remarks and questions

We close with some pertinent remarks and questions.

(1) We fixed the identity matrix in the $(1, 1)$ block. For the Moore-Penrose inverse case, it is irrelevant where the identity block is, since the new case would be unitarily equivalent to the one presented here. This follows from the fact that if $M = UKV^*$, with $U^{-1} = U^*, V^{-1} = V^*$, then $M^\dagger = VK^\dagger U^*$.

(2) For the group inverse, the results do not follow as simple as in the Moore-Penrose case as mentioned in the previous item, unless the identity block is in the $(2, 2)$ block, which makes the matrix similar to the one presented here.

(3) A matrix M is said to be range hermitian if $R(M) = R(M^*)$, or equivalently, $\ker(M) = \ker(M^*)$. This class of matrices contain normal (i.e. M and M^* commute) and hermitian matrices. It is known that range hermitian matrices are precisely the ones whose Moore-Penrose and group inverses coincide. It could be of interest to know when is $\begin{bmatrix} I & A \\ B & D \end{bmatrix}$ a range hermitian matrix.

(4) It is simple to show that if $I - AB$ and $I - BA$ are both non-singular, then $(I - AB)^{-1}A = A(I - BA)^{-1}$. A comparable equality for Moore-Penrose inverses, that is, $(I - AB)^\dagger A = A(I - BA)^\dagger$, does not hold in general. Take $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, for which $(I - AB)^\dagger A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^\dagger A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} A = 0$ and $A(I - BA)^\dagger = A\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}^\dagger = A\begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$. It could be of interest to give necessary and sufficient conditions for such an equality to hold.

(5) Given a matrix A for which $A^{\#}$ exists, the core inverse of A is the unique solution to the conditions $AX = AA^{\dagger}$ and $R(X) \subseteq R(A)$ (see, eg., [1, 11, 21]). It is known that such a solution, denoted by $A^{\textcircled{\#}}$, is related to group and Moore-Penrose inverses by $A^{\textcircled{\#}} = A^{\#}AA^{\dagger}$. From Theorem 3.3 and Theorem 4.2 we can find the core inverse of $\begin{bmatrix} I & B \\ C & D \end{bmatrix}$.

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