

# Topological lax comma categories

Maria Manuel Clementino<sup>1</sup>, Dirk Hofmann<sup>2</sup>, and Rui Prezado<sup>3</sup>

<sup>1</sup>University of Coimbra, CMUC, Department of Mathematics, Portugal, mmc@mat.uc.pt

<sup>2,3</sup>University of Aveiro, CIDMA, Department of Mathematics, Portugal,  
{dirk,ruiprezado}@ua.pt

April 17, 2025

This paper investigates the interplay between properties of a topological space  $X$ , in particular of its natural order, and properties of the lax comma category  $\mathbf{Top} \Downarrow X$ , where  $\mathbf{Top}$  denotes the category of topological spaces and continuous maps. Namely, it is shown that, whenever  $X$  is a topological  $\wedge$ -semilattice, the canonical forgetful functor  $\mathbf{Top} \Downarrow X \rightarrow \mathbf{Top}$  is topological, preserves and reflects exponentials, and preserves effective descent morphisms. Moreover, under additional conditions on  $X$ , a characterisation of effective descent morphisms is obtained.

*Keywords:* exponentiability, descent theory, lax comma category, order topologies  
*2020 MSC:* 06B30, 06B35, 06F30, 18A25, 18F60, 18F20, 22A26, 54B30

**Acknowledgements.** We thank Matías Menni for calling our attention to the recent paper [1] and for raising us the question under which conditions the canonical forgetful functor  $\mathbf{Top} \Downarrow X \rightarrow \mathbf{Top}$  is topological.

This work was partially supported by UID/00324 – Centro de Matemática da Universidade de Coimbra, funded by the Portuguese Foundation for Science and Technology (FCT – Fundação para a Ciência e a Tecnologia), and by CIDMA (Center for Research and Development in Mathematics and Applications) under the FCT (Portuguese Foundation for Science and Technology) Multi-Annual Financing Program for R&D Units.

## Introduction

The recent study of the lax comma categories  $\mathbf{Ord} \Downarrow X$  and  $\mathbf{Cat} \Downarrow X$  in [8, 10, 7, 11], and the implicit use of  $\mathbf{Top} \Downarrow \mathbb{N}$  by [1], led the authors of this paper to investigate the behaviour of the lax comma category  $\mathbf{Top} \Downarrow X$  we describe below, namely its (co)completeness, exponentiability and descent. As expected, this behaviour depends essentially on the interplay between the order and the topology on  $X$ , and leads to the study of interesting properties of this order.

For a topological space  $X$ , we consider the *natural order* on  $X$  [19], with  $x \leq y$  whenever the principal filter  $\dot{x}$  converges to  $y$ :  $\dot{x} \rightsquigarrow y$ ; equivalently, whenever every open neighbourhood of  $y$  contains  $x$ :  $\mathcal{O}(y) \subseteq \mathcal{O}(x)$ . This relation is reflexive and transitive; it is anti-symmetric if and

only if  $X$  is a T0-space. Every continuous map  $f: X \rightarrow Y$  is a monotone map with respect to the natural orders on  $X$  and  $Y$ , and therefore this construction defines a functor  $\mathbf{Top} \rightarrow \mathbf{Ord}$  from  $\mathbf{Top}$  onto the category  $\mathbf{Ord}$  of ordered sets and monotone maps, and allows us to consider  $\mathbf{Top}$  as an ordered category where, for continuous maps  $f, g: X \rightarrow Y$ ,  $f \leq g$  if  $f(x) \leq g(x)$ , for all  $x \in X$ . We note that the order relation on a space considered in this paper is dual to the specialisation order [46] used by many authors. We find it convenient to think of the convergence of a topological space as a “generalised order”, and therefore take as underlying order of a space the “point shadow” of the convergence relation. This intuition makes also a better connection with the results of [8, 7, 11] on lax comma categories of ordered sets.

The category  $\mathbf{Top} \Downarrow X$ , called *lax comma category* over  $X$ , has as objects pairs  $(A, \alpha)$ , where  $A$  is a topological space and  $\alpha: A \rightarrow X$  is a continuous map, and as morphisms  $(A, \alpha) \rightarrow (B, \beta)$  continuous maps  $f: A \rightarrow B$  such that  $\alpha \leq \beta \cdot f$ :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \alpha & \swarrow \beta \\ & & X \end{array} \quad \begin{array}{c} \\ \leq \\ \end{array}$$

It has as a (wide) subcategory the comma category  $\mathbf{Top} \downarrow X$ , which has the same objects as  $\mathbf{Top} \Downarrow X$  but a morphism  $f: (A, \alpha) \rightarrow (B, \beta)$  must satisfy  $\alpha = \beta \cdot f$ . It is well-known that the category  $\mathbf{Top} \downarrow X$  is complete and cocomplete, and its canonical functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  preserves equalisers and colimits, but in general not products, nor exponentials; moreover, it trivially preserves and reflects (effective) descent morphisms.

If  $X$  is a T1-space, then its natural order is discrete, and therefore the lax comma category  $\mathbf{Top} \Downarrow X$  coincides with the comma category  $\mathbf{Top} \downarrow X$ . Furthermore, for any  $X$ , if we denote by  $\eta_X: X \rightarrow RX$  its T0-reflection, where  $\eta_X(x) = \eta_X(y)$  if and only if  $x \leq y$  and  $y \leq x$ , and  $RX$  has the quotient topology, it is easily checked that the functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top} \downarrow RX$  defined by composing with  $\eta_X$  is an equivalence of categories making the diagram

$$\begin{array}{ccc} \mathbf{Top} \downarrow X & \xrightarrow{\quad} & \mathbf{Top} \downarrow RX \\ & \searrow G & \swarrow G \\ & & \mathbf{Top} \end{array}$$

commute, where  $G$  denotes the canonical forgetful functor(s). Therefore, for simplicity, we may reduce our study to categories  $\mathbf{Top} \downarrow X$  with  $X$  a T0-space:

*From now on we assume that  $X$  is a topological T0-space.*

In Section 1 we investigate the existence of limits and colimits in  $\mathbf{Top} \downarrow X$ , starting by showing that sums and equalisers always exist, and that  $\mathbf{Top} \downarrow X$  is an (infinitely) extensive category. It is also shown that the existence of coequalisers and their preservation by  $G: \mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  is equivalent to the existence and preservation of products by  $G$ , it is in fact equivalent to  $G$  being a topological functor, and equivalent to  $X$  being sober and a topological  $\wedge$ -semilattice. These properties of  $X$  can be formulated in several different ways, as stated in Theorem 1.15, and in particular reveal  $X$  as an algebra for the lower Vietoris monad on  $\mathbf{Top}$ .

In Section 2 we study exponentiability in  $\mathbf{Top} \downarrow X$ , making use of the Special Adjoint Functor Theorem. In particular we generalise results obtained (using different techniques) in [34], showing that, if  $X$  is a topological Heyting  $\wedge$ -semilattice, then the forgetful functor  $G$  preserves and

reflects exponentiable objects and exponentials; moreover, reflection of exponentiable objects by  $G$  is equivalent to  $X$  being a topological Heyting  $\wedge$ -semilattice.

In the last section we study effective descent morphisms in  $\mathbf{Top} \Downarrow X$ . Here we follow the strategy of [11], considering the pseudopullback

$$\begin{array}{ccc} \mathbf{Top} \times_{\mathbf{Set}} \mathbf{Fam}(X) & \xrightarrow{\rho_2} & \mathbf{Fam}(X) \\ \rho_1 \downarrow & & \downarrow \\ \mathbf{Top} & \longrightarrow & \mathbf{Set} \end{array}$$

and the canonical functor

$$H: \mathbf{Top} \Downarrow X \longrightarrow \mathbf{Top} \times_{\mathbf{Set}} \mathbf{Fam}(X).$$

induced by the corresponding forgetful functors. This leads us to obtain, in Theorems 3.9 and 3.11, characterizations of effective descent morphisms in  $\mathbf{Top} \Downarrow X$ , for suitable  $X$ .

## 1 The category $\mathbf{Top} \Downarrow X$

In this section we investigate basic properties of  $\mathbf{Top} \Downarrow X$  and their relationship with properties of the base space  $X$ . As already observed by [34], the canonical forgetful functor  $\mathbf{Top} \Downarrow X \rightarrow \mathbf{Top}$  has a left adjoint if and only if  $X$  has a bottom element  $\perp$ ; in this case, the left adjoint  $\mathbf{Top} \rightarrow \mathbf{Top} \Downarrow X$  sends a space  $A$  to the pair consisting of  $A$  together with the constant map

$$\begin{array}{l} A \longrightarrow X \\ a \longmapsto \perp \end{array}$$

and leaves continuous maps unchanged. Likewise,  $\mathbf{Top} \Downarrow X \rightarrow \mathbf{Top}$  has a right adjoint if and only if  $X$  has a top element  $\top$ ; here the right adjoint  $\mathbf{Top} \rightarrow \mathbf{Top} \Downarrow X$  sends a space  $A$  to the pair consisting of  $A$  together with the constant map  $A \rightarrow X$ ,  $a \mapsto \top$ . Furthermore, the following properties are easy to verify:

**Proposition 1.1.** (1) *The category  $\mathbf{Top} \Downarrow X$  has sums: for a family  $(A_i, \alpha_i)_{i \in I}$  of objects of  $\mathbf{Top} \Downarrow X$ , its sum in  $\mathbf{Top} \Downarrow X$  is given by*

$$\left( \left( \sum_{i \in I} A_i, \alpha \right), \left( \iota_j: A_j \rightarrow \sum_{i \in I} A_i \right)_{j \in I} \right),$$

where  $(\sum_{i \in I} A_i, (\iota_j)_{j \in I})$  is the sum of  $(A_i)_{i \in I}$  in  $\mathbf{Top}$ , and  $\alpha: \sum_{i \in I} A_i \rightarrow X$  is the continuous map induced in the topological sum by the family  $(\alpha_i: A_i \rightarrow X)_{i \in I}$ .

(2) *The category  $\mathbf{Top} \Downarrow X$  has equalisers: for a parallel pair  $f, g: (A, \alpha) \rightarrow (B, \beta)$  of arrows in  $\mathbf{Top} \Downarrow X$ , its equaliser in  $\mathbf{Top} \Downarrow X$  is given by  $e: (E, \alpha \cdot e) \rightarrow (A, \alpha)$  where*

$$E \xrightarrow{e} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

is an equaliser in  $\mathbf{Top}$ .

Recall that a category  $\mathbf{C}$  with sums is called (*infinitely*) *extensive* (see [3]) whenever, for every small family  $(C_i)_{i \in I}$  of objects of  $\mathbf{C}$ , the canonical functor

$$\sum: \prod_{i \in I} (\mathbf{C} \downarrow C_i) \longrightarrow \mathbf{C} \downarrow \left( \sum_{i \in I} C_i \right)$$

is an equivalence. The category  $\mathbf{Top}$  is extensive, and this property is inherited by  $\mathbf{Top} \downarrow X$ :

**Proposition 1.2.** *The category  $\mathbf{Top} \downarrow X$  is extensive.*

*Proof.* The assertion follows immediately from the facts that  $\mathbf{Top}$  is extensive and that, for all small families  $(A_i)_{i \in I}$  of topological spaces, the canonical map

$$\prod_{i \in I} \mathbf{Top}(A_i, X) \longrightarrow \mathbf{Top}\left(\sum_{i \in I} A_i, X\right)$$

is an isomorphism of ordered sets (see also [30, Theorem 41]). □

Being extensive guarantees that a category has “well-behaved sums”; moreover, extensive categories with finite products are (infinitely) distributive:

**Corollary 1.3.** *Assume that all binary products exist in  $\mathbf{Top} \downarrow X$ . Then the functor*

$$(A, \alpha) \times -: \mathbf{Top} \downarrow X \longrightarrow \mathbf{Top} \downarrow X$$

*preserves sums.*

Further important (co)completeness properties of  $\mathbf{Top} \downarrow X$  are in direct relation with (co)completeness properties of the space  $X$ , as we explain next. We start by considering the existence of coequalisers. We point out that our assumption that the base space  $X$  is T0 guarantees uniqueness of infima and suprema.

**Lemma 1.4.** *Let  $f, g: (A, \alpha) \rightarrow (B, \beta)$  be arrows in  $\mathbf{Top} \downarrow X$  and let  $q: B \rightarrow Q$  be a coequaliser of  $f, g: A \rightarrow B$  in  $\mathbf{Top}$ . Then  $q: (B, \beta) \rightarrow (Q, \gamma)$  is a coequaliser of  $f, g: (A, \alpha) \rightarrow (B, \beta)$  in  $\mathbf{Top} \downarrow X$  if and only if  $\gamma: Q \rightarrow X$  is the left Kan extension  $\mathbf{Lan}_q(\beta): Q \rightarrow X$  of  $\beta: B \rightarrow X$  along  $q: B \rightarrow Q$ .*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{q} & Q \\ & \xrightarrow{g} & \downarrow \beta & & \downarrow \gamma \\ & \leq & X & \xleftarrow{\gamma = \mathbf{Lan}_q(\beta)} & \\ & \searrow \alpha & & & \end{array}$$

*Proof.* See [9, Theorem 9.3, version 1 of arXiv]. □

**Corollary 1.5.** *The category  $\mathbf{Top} \downarrow X$  has and the functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  preserves coequalisers if and only if, for every continuous map  $\beta: B \rightarrow X$  and every quotient map  $q: B \rightarrow Q$ , the left Kan extension  $\mathbf{Lan}_q(\beta): Q \rightarrow X$  of  $\beta$  along  $q$  exists.*

Recall that  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  is a left adjoint if and only if  $X$  has a top element, hence, under this condition,  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  preserves all existing colimits. Those spaces admitting left Kan extensions along arbitrary continuous maps are characterised in [13, Theorem 2.4]:

**Theorem 1.6.** *The T0-space  $X$  admits left Kan extensions along arbitrary continuous maps if and only if  $X$  is sober, complete with respect to the natural order, and the map  $\wedge: X \times X \rightarrow X$  is continuous.*

Below we analyse the conditions for the existence of left Kan extensions of the theorem above more in detail, starting with some notation. We call a T0-space  $X$  a *topological  $\wedge$ -semilattice* whenever  $X$  is a  $\wedge$ -semilattice with respect to the natural order and the map

$$\wedge: X \times X \longrightarrow X$$

is continuous. Similarly, we call  $X$  a *topological  $\bigwedge$ -semilattice* whenever the ordered set  $X$  is complete and, for every set  $I$ , the map  $\bigwedge: X^I \rightarrow X$  is continuous.

*Remark 1.7.* The space  $X$  has continuous infima of all families indexed by the set  $I$  if and only if the diagonal map  $\Delta_X: X \rightarrow X^I$  has a right adjoint in the ordered category  $\mathbf{Top}$ .

The following result is due to [20, Theorem 2.6] and [44, Section 6.3].

**Theorem 1.8.** *The following assertions are equivalent.*

- (i) *The topological space  $X$  is a topological  $\wedge$ -semilattice.*
- (ii) *The topological space  $X$  is sober and a topological  $\wedge$ -semilattice.*

*Proof.* Regarding (ii)  $\implies$  (i), just observe that, by [48, 20] (see also [26, Lemma II.1.9]), every sober space admits codirected infima with respect to the natural order, and, for every codirected ordered set  $(I, \leq)$ , the map

$$X^I \supseteq \{(x_i)_{i \in I} \in X^I \mid I \rightarrow X, i \mapsto x_i, \text{ is monotone}\} \xrightarrow{\wedge} X$$

is continuous, with the domain considered as a subspace of the product space  $X^I$ .  $\square$

With respect to the natural order of a space, every open subset is down-closed, likewise, every closed subset is up-closed. Given now an ordered set  $Z$ , there are various topologies on the set  $Z$  which induce the given order. Here we list three such topologies (see [16]):

- (1) the *lower topology* on  $Z$ , where the closed sets are generated by the sets of the form  $\uparrow a = \{z \in Z \mid a \leq z\}$ , with  $a \in Z$ ,
- (2) the *lower Scott topology* on  $Z$ , where  $A \subseteq Z$  is closed whenever  $A$  is up-closed and stable under codirected infima, and
- (3) the *lower Alexandroff topology* on  $Z$ , where  $A \subseteq Z$  is closed whenever  $A$  is up-closed.

Clearly, the lower topology is included in the lower Scott topology which in turn is included in the lower Alexandroff topology. Furthermore, the lower topology is the coarsest topology inducing the given order, while the lower Alexandroff topology is the largest such topology. A topological space is called an *Alexandroff space* whenever its topology coincides with the lower Alexandroff topology of its natural order; equivalently, whenever arbitrary unions of closed subsets are closed.

**Lemma 1.9.** *Assume that  $X$  is equipped with the lower topology with respect to its natural order.*

- (1) Given a space  $Y$  and a pair of adjoint maps  $(X \xrightarrow{f} Y) \dashv (Y \xrightarrow{g} X)$  in  $\mathbf{Ord}$ , with respect to the natural order of  $X$  and  $Y$ ,  $g$  is continuous.
- (2) If the natural order on  $X$  is complete, then  $X$  is a topological  $\wedge$ -semilattice; in particular,  $X$  is sober.

*Proof.* (1) Under these conditions, for every  $x \in X$ :

$$g^{-1}(\uparrow x) = \{y \in Y \mid x \leq g(y)\} = \{y \in Y \mid f(x) \leq y\} = \uparrow f(x).$$

(2) follows from (1), since, for every set  $I$ , the map  $\wedge: X^I \rightarrow X$  is right adjoint to the diagonal  $\Delta: X \rightarrow X^I$ .  $\square$

*Remark 1.10.* Our naming scheme for the various topologies on an ordered set  $Z$  reflects the fact that we declare certain down-closed (also called *lower*) subsets of  $Z$  to be open. There are also the somehow dual topologies on an ordered set  $Z$ , namely the upper topology, the upper Scott topology and the upper Alexandroff topology which are defined so that the upper (Scott, Alexandroff) topology of  $Z$  coincides with the lower (Scott, Alexandroff) topology of the dual ordered set  $Z^{\text{op}}$ . We note that these topologies have the dual order of  $Z$  as natural order, which of course means that the specialisation order of these topologies is the given order. For instance, a topological space carries the lower Scott topology with respect to the natural order if and only if it carries the upper Scott topology with respect to the specialisation order.

[46] has characterised the injective T0-spaces as precisely the continuous lattices with respect to the specialisation order equipped with the upper Scott topology; hence, the op-continuous lattices with respect to the natural order equipped with the lower Scott topology. We recall that an *op-continuous lattice* is a complete lattice  $Z$  so that every element is the meet of its way above elements – herein, we say  $x$  is *way above*  $y$ , denoted  $x \gg y$ , if, whenever  $S$  is a codirected subset of  $Z$  with  $y \geq \bigwedge S$ , there exists  $s \in S$  such that  $x \geq s$ . We recall that a topological (T0-)space  $Z$  is said to be *injective* if, for every embedding  $f: A \rightarrow B$  and every continuous map  $g: A \rightarrow Z$ , there exists a continuous map  $\bar{g}: B \rightarrow Z$  making the following diagram commute

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g & \swarrow \bar{g} \\ & & Z \end{array}$$

**Lemma 1.11.** *Assume that  $X$  is an injective space. Then  $X$  is a topological  $\wedge$ -semilattice (with respect to the natural order). Moreover, the map  $\vee: X \times X \rightarrow X$  is continuous.*

*Proof.* In [46, Proposition 2.7] it is shown that, for any injective space  $X$ , both maps  $\wedge: X \times X \rightarrow X$  and  $\vee: X \times X \rightarrow X$  are continuous. Moreover, every injective space is a retract of a product of the Sierpinski space, hence it is sober. By Theorem 1.8,  $X$  is a topological  $\wedge$ -semilattice as claimed.  $\square$

**Examples 1.12.** (1) Let  $X = \mathbb{S}$  be the Sierpinski space, with  $\{1\}$  the only non-trivial closed subset. Clearly  $0 \leq 1$  for the natural order, which is both a continuous and an op-continuous lattice. Here the three topologies considered above coincide.

The category  $\mathbf{Top} \Downarrow \mathbb{S}$  may be equivalently described as the category with objects pairs  $(A, A_0)$  where  $A$  is a topological space and  $A_0$  is a closed subset of  $A$ , and with morphisms  $f: (A, A_0) \rightarrow (B, B_0)$  those continuous maps  $f: A \rightarrow B$  with  $f(a) \in B_0$  for all  $a \in A_0$ , that is,  $f$  restricts to  $f_0: A_0 \rightarrow B_0$ .

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

- (2) Consider  $X = \mathbb{N}$  with the order  $\preceq$  defined by  $n \preceq m$  if  $m$  divides  $n$  (and we write  $m \mid n$ ). It is a complete lattice, with  $\top = 1$ ,  $\perp = 0$ , and  $\bigvee_{i \in I} m_i = \gcd(m_i)$ ,  $\bigwedge_{i \in I} m_i = \text{lcm}(m_i)$ . Then  $X$  is an op-continuous lattice but it is not a continuous lattice: it is easily seen that if  $p^\alpha \mid m$ , for  $p$  a prime number, then  $p^\alpha \gg m$ , while, if  $m$  is not a prime number, then  $n \ll m$  only if  $n = 0$ . Again the lower topology coincides with the lower Scott topology. We note that in [1] the category  $\mathbf{Top} \Downarrow \mathbb{N}$ , called there the category of  $a$ -spaces and  $a$ -maps, is used to obtain a generalization of Yosida-duality to norm-complete  $l$ -groups. The dual category is the full subcategory of  $\mathbf{Top} \Downarrow \mathbb{N}$  defined by compact Hausdorff spaces  $A$  with certain continuous maps  $A \rightarrow \mathbb{N}$ .

We point out that with the lower Alexandroff topology on  $\mathbb{N}$  we do not obtain a topological  $\bigwedge$ -semilattice: this space is not sober.

- (3) Let  $X$  be the interval  $[0, 1]$  with the lower topology for the natural order on  $[0, 1]$ ; that is,  $U \subseteq [0, 1]$  is closed if and only if  $U = \emptyset$  or  $U = [x, 1]$  for some  $x \in [0, 1]$ . Clearly,  $[0, 1]$  is both a continuous and an op-continuous lattice, and closed subsets are stable under (codirected) infima, hence  $X$  carries the lower Scott topology, and, consequently,  $X$  is an injective space. We point out that this topology does not coincide with the lower Alexandroff topology: the intervals  $[x, 1]$  are up-closed but not closed for the lower topology. Again,

$$[0, 1]$$

with the lower Alexandroff topology is not a sober space.

For  $X = [0, 1]$  with the lower topology (= Scott topology), the category  $\mathbf{Top} \Downarrow X$  admits a similar description as for  $X = \mathbb{S}$ . Firstly, for each set  $A$ , there is a bijection

$$\{\alpha: A \rightarrow [0, 1]\} \longrightarrow \{(A_u)_{u \in [0, 1]} \mid \text{for all } u \in [0, 1]: A_u \subseteq A \text{ and } A_u = \bigcap_{v < u} A_v\}$$

sending  $\alpha: A \rightarrow [0, 1]$  to the family  $(\alpha^{-1}(\uparrow u))_{u \in [0, 1]}$ . The inverse of the function above sends the family  $(A_u)_{u \in [0, 1]}$  to the map

$$\alpha: A \longrightarrow [0, 1], a \longmapsto \sup\{u \in [0, 1] \mid a \in A_u\}.$$

Furthermore, if  $A$  is a topological space, then a map  $\alpha: A \rightarrow [0, 1]$  is continuous if and only if the corresponding family  $(A_u)_{u \in [0, 1]}$  consists only of closed subsets of  $A$ . Given also continuous maps  $\beta: B \rightarrow [0, 1]$  (with corresponding family  $(B_u)_{u \in [0, 1]}$ ) and  $f: A \rightarrow B$ , then  $\alpha \leq \beta \cdot f$  if and only if, for all  $u \in [0, 1]$ ,

$$\forall a \in A. (a \in A_u \implies f(a) \in B_u); \tag{1.i}$$

that is,  $f$  restricts to a map  $f_u: A_u \rightarrow B_u$ .

$$\begin{array}{ccc} A_u & \xrightarrow{f_u} & B_u \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

Hence, we may consider  $\mathbf{Top} \downarrow [0, 1]$  as the category with objects pairs  $(A, (A_u)_{u \in [0, 1]})$  consisting of a topological space  $A$  and a family  $(A_u)_{u \in [0, 1]}$  of closed subsets of  $A$  so that, for all  $u \in [0, 1]$ ,  $A_u = \bigcap_{v < u} A_v$ , and with morphisms  $f: (A, (A_u)_{u \in [0, 1]}) \rightarrow (B, (B_u)_{u \in [0, 1]})$  those continuous maps  $f: A \rightarrow B$  satisfying (1.i) for all  $u \in [0, 1]$ .

- (4) Consider the complete lattice  $X = \mathbb{Z}_-^\infty = \{k \in \mathbb{Z} \mid k \leq 0\} \cup \{-\infty\}$ , with the usual order. Then it is easy to check that  $\mathbb{Z}_-^\infty$  is both a continuous and an op-continuous lattice. The way above relation is given by:  $k \gg j$  if  $k \geq j$  and  $k \gg -\infty$  for every  $k \in \mathbb{Z}_-$ . Again, its lower topology coincides with the lower Scott topology but it is weaker than its lower Alexandroff topology:  $\mathbb{Z}_-$  is up-closed but not closed in the lower topology.
- (5) Now let  $X = \mathbb{Z}_+^\infty = \{k \in \mathbb{Z} \mid k \geq 0\} \cup \{+\infty\}$ , with the usual order. It is again, of course, both a continuous and an op-continuous lattice. However the situation here is different from the previous example; indeed, every non-empty up-set has a minimum, hence the lower topology and the lower Alexandroff topology coincide.

*Remark 1.13.* Assume now that  $X$  is an injective space. For an embedding  $q: B \rightarrow C$  and a continuous map  $\beta: B \rightarrow X$ , as depicted in (1.ii):

$$\begin{array}{ccc} B & \xrightarrow{q} & C \\ & \searrow \beta & \swarrow \text{Lan}_q \beta \\ & & X \end{array} \quad (1.ii)$$

the extension  $\text{Lan}_q \beta: C \rightarrow X$  of  $\beta$  along  $q$  is given by

$$c \mapsto \bigwedge_{V \in \mathcal{O}(c)} \bigvee_{b \in q^{-1}(V)} \beta(b). \quad (1.iii)$$

By considering  $q = 1_B: B \rightarrow B$ , we obtain in particular that, for every  $b \in B$ ,

$$\beta(b) = \bigwedge_{V \in \mathcal{O}(b)} \bigvee_{b' \in V} \beta(b').$$

Furthermore, [14, Subsection 2.6] observed that the formula (1.iii) also defines the left Kan extension  $\text{Lan}_q(\beta): C \rightarrow X$  of  $\beta$  along an arbitrary continuous map  $q: B \rightarrow C$ .

Next we will expose a connection between topological  $\wedge$ -semilattices and two important constructions of topological spaces: the lower Vietoris space and the ultrafilter space.

The lower Vietoris functor  $V: \mathbf{Top} \rightarrow \mathbf{Top}$  is defined by  $VZ = \{A \subseteq Z \mid A \text{ closed}\}$  with the ‘‘hit topology’’, that is, the topology generated by the sets  $\{A \in VZ \mid A \cap W \neq \emptyset\}$  ( $W$  open in  $Z$ ), and by  $Vf(A) = \overline{f[A]}$ . Furthermore, this functor is part of a monad on  $\mathbf{Top}$ , with unit  $e: 1 \rightarrow V$  defined by  $e_Z: Z \rightarrow VZ$ ,  $z \mapsto \overline{\{z\}}$  and multiplication  $m: VV \rightarrow V$  given by union. This construction goes back to the work [47]; for more information we refer to [32, 44, 45].



We note that  $A \leq B$  in the natural order of  $VZ$  if and only if  $A \supseteq B$ , for all  $A, B \in VZ$ . Furthermore, a sub-basis for the closed subsets of  $VX$  is given by the sets

$$\{A \in VZ \mid B \supseteq A\},$$

for  $B \subseteq Z$  closed; that is, the lower Vietoris topology on  $VZ$  is the lower topology with respect to the containment order on  $VX$ .

For a topological space  $Z$ , we consider the set  $UZ$  of all ultrafilters on the set  $Z$  equipped with the topology generated by the basic open sets  $\{\mathfrak{r} \in UZ \mid B \in \mathfrak{r}\}$ , with  $B$  open in  $Z$ . This construction extends the ultrafilter monad  $(U, m, e)$  on  $\mathbf{Set}$  to a monad on  $\mathbf{Top}$ ; see [23], for instance. In particular, the unit

$$e_Z: Z \longrightarrow UZ, z \longmapsto \dot{z}$$

is continuous. The natural order of the space  $UZ$  is described by

$$\begin{aligned} \mathfrak{r} \leq \eta &\iff \forall B \subseteq Z \text{ open. } (B \in \eta \implies B \in \mathfrak{r}) \\ &\iff \forall A \subseteq Z \text{ closed. } (A \in \mathfrak{r} \implies A \in \eta) \\ &\iff \forall A \in \mathfrak{r}. \bar{A} \in \eta, \end{aligned}$$

for all  $\mathfrak{r}, \eta \in UZ$ . In particular, considering above  $\eta = \dot{z}$  the principal ultrafilter generated by  $z \in Z$ , then  $\mathfrak{r} \leq \dot{z}$  if and only if  $\mathfrak{r} \rightsquigarrow z$ . For a topological space  $Z$  and a map  $\sigma: UZ \rightarrow Z$ , one has  $e_Z \dashv \sigma$  in  $\mathbf{Ord}$  if and only if, for all  $z \in Z$  and  $\mathfrak{r} \in UZ$ ,

$$\mathfrak{r} \leq \dot{z} \iff \sigma(\mathfrak{r}) \leq z.$$

Therefore the monotone map  $e_Z: Z \rightarrow UZ$  has a left adjoint in  $\mathbf{Ord}$  if and only if every ultrafilter  $\mathfrak{r}$  on  $Z$  has a smallest convergence point with respect to the natural order of  $Z$ . Below we collect some elementary properties of such spaces.

**Lemma 1.14.** (1) *Assume that  $e_Z: Z \rightarrow UZ$  has a left adjoint  $\sigma: UZ \rightarrow Z$  in  $\mathbf{Ord}$ . Then also  $e_{Z \times Z}: Z \times Z \rightarrow U(Z \times Z)$  has a left adjoint in  $\mathbf{Ord}$ , given by*

$$U(Z \times Z) \longrightarrow UZ \times UZ \xrightarrow{\sigma \times \sigma} Z \times Z.$$

(2) *Let  $f: Y \rightarrow Z$  be a continuous map between  $T0$ -spaces and assume that both  $Y$  and  $Z$  have smallest convergence points of ultrafilters. If  $f$  has a right adjoint in  $\mathbf{Top}$ , then  $f$  preserves smallest convergence points of ultrafilters.*

*Proof.* The first assertion follows immediately from the fact that convergence in  $Z \times Z$  is component-wise. Regarding the second assertion, let  $g: Z \rightarrow Y$  be the right adjoint of  $f$  in  $\mathbf{Top}$ . Since the diagram

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ e_Z \downarrow & & \downarrow e_Y \\ UZ & \xrightarrow{Ug} & UY \end{array}$$

of right adjoints commutes, the diagram of the corresponding left adjoints in  $\mathbf{Ord}$  commutes as well.  $\square$

The algebras for the lower Vietoris monad are characterised as the “topological complete join-semilattices” (with respect to specialisation order, which is dual to the order considered in this paper) in [20, Theorem 2.6] and in [44, Section 6.3]. The theorem below summarises their results, and the connection with “smallest convergence points of ultrafilters” is described in [22, Proposition 6.4 and Example 6.5].

**Theorem 1.15.** *The following assertions are equivalent, for a  $T0$ -space  $Z$ .*

- (i)  $Z$  is an algebra for the lower Vietoris monad on  $\mathbf{Top}$ .
- (ii)  $Z$  is a topological  $\bigwedge$ -semilattice.
- (iii)  $Z$  is sober and a topological  $\bigwedge$ -semilattice.
- (iv) The ordered set  $Z$  is complete and the unit  $Z \rightarrow UZ$ ,  $z \mapsto \dot{z}$  has a left adjoint in  $\mathbf{Ord}$ .
- (v) The ordered set  $Z$  is finitely complete and the unit  $Z \rightarrow UZ$ ,  $z \mapsto \dot{z}$  has a left adjoint in  $\mathbf{Ord}$ .

*Remark 1.16.* Regarding the implication (v)  $\implies$  (iii) in Theorem 1.15, the fact that every ultrafilter has a smallest convergence point implies that every irreducible closed set is the closure of some point (because every irreducible closed set is the set of limit points of some ultrafilter), and that the map  $\bigwedge: Z \times Z \rightarrow Z$  is continuous (see [22, Example 6.5]).

We turn now our attention to the study of limits in  $\mathbf{Top} \downarrow X$ . The following result is essentially [34, Proposition 3.1].

**Proposition 1.17.** *Let  $I$  be a set. Then the following assertions are equivalent:*

- (i) With respect to the natural order,  $X$  has all infima of  $I$ -indexed families, and, moreover, the infima map  $\bigwedge: X^I \rightarrow X$  is continuous.
- (ii) The category  $\mathbf{Top} \downarrow X$  has products of families of objects indexed by  $I$ , and these products are preserved by  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$ .
- (iii) The category  $\mathbf{Top} \downarrow X$  has the  $I$ -powers of the object  $(X, 1_X: X \rightarrow X)$ , and this power is preserved by  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$ .

*Proof.* To see (i)  $\implies$  (ii), consider a family  $(A_i, \alpha_i)_{i \in I}$  of objects of  $\mathbf{Top} \downarrow X$ . Its product is given by

$$\left( \prod_{i \in I} A_i, \alpha \right)$$

where  $\alpha$  is the composite

$$\prod_{i \in I} A_i \xrightarrow{\langle \alpha_i \rangle_{i \in I}} X^I \xrightarrow{\bigwedge} X. \quad (1.iv)$$

The implication (ii)  $\implies$  (iii) is trivial. Finally, assuming (iii), we have necessarily

$$(X, 1_X: X \rightarrow X)^I = (X^I, \bigwedge: X^I \rightarrow X),$$

proving (i). □

*Remark 1.18.* In the sequel we denote the composite map (1.iv) as

$$\prod_{i \in I} \alpha_i: \prod_{i \in I} A_i \longrightarrow X.$$

As usual, in the binary case we write  $\alpha_1 \sqcap \alpha_2$  instead of  $\prod_{i \in \{1,2\}} \alpha_i$ .

As an immediate consequence of Propositions 1.17 and 1.1 and of Theorem 1.8 we obtain:

**Corollary 1.19.** (1) *The category  $\mathbf{Top} \downarrow X$  has and  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  preserves finite limits if and only if  $X$  is a topological  $\wedge$ -semilattice.*

(2) *The category  $\mathbf{Top} \downarrow X$  has and  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  preserves all limits if and only if  $X$  is a topological  $\wedge$ -semilattice.*

(3) *Assume that  $X$  is sober and has a bottom element. Then  $\mathbf{Top} \downarrow X$  is complete if and only if  $\mathbf{Top} \downarrow X$  is finitely complete.*

In fact, more can be said if  $X$  is a topological  $\wedge$ -semilattice.

**Theorem 1.20.** *The canonical forgetful functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  is topological if and only if  $X$  is a topological  $\wedge$ -semilattice.*

*Proof.* Clearly, if  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  is topological, then  $\mathbf{Top} \downarrow X$  is complete and  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  preserves limits. By Corollary 1.19,  $X$  is a topological  $\wedge$ -semilattice.

Assume now that  $X$  is a topological  $\wedge$ -semilattice. Consider a space  $A$  and (possibly large) families  $(A_i, \alpha_i)_{i \in I}$  of objects of  $\mathbf{Top} \downarrow X$  and  $(f_i: A \rightarrow A_i)_{i \in I}$  of continuous maps. Then the class

$$\mathcal{C} = \{\alpha_i \cdot f_i: A \rightarrow X \mid i \in I\}$$

is actually a set, so that we may consider it as

$$\mathcal{C} = \{\gamma_j: A \rightarrow X \mid j \in J\}$$

for a set  $J$ . By hypothesis,  $\gamma = \bigwedge_{i \in J} \gamma_j: A \rightarrow X$  is continuous and defines the initial lift  $(A, \gamma)$  of the cone  $(f_i: A \rightarrow A_i)_{i \in I}$  with respect to  $(A_i, \alpha_i)_{i \in I}$ .  $\square$

As we have already mentioned, the category  $\mathbf{Top} \downarrow X$  differs substantially from its wide subcategory  $\mathbf{Top} \downarrow X$ , whenever  $X$  is not a T1-space. This is particularly visible in the behaviour of products – as just shown, under suitable conditions on  $X$ , the forgetful functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  preserves products while  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  does not in general – and on the behaviour of exponentiability and descent, as detailed in the forthcoming sections.

## 2 Exponentiability

We recall that a category  $\mathbf{C}$  is said to be *cartesian closed* if it has finite products and, for each  $A \in \mathbf{C}$ , the functor  $A \times - : \mathbf{C} \rightarrow \mathbf{C}$  has a right adjoint. An object  $A$  of a category  $\mathbf{C}$  is said to be *exponentiable* if  $\mathbf{C}$  has products along  $A$  and the functor  $A \times -$  has a right adjoint, usually denoted by  $(-)^A : \mathbf{C} \rightarrow \mathbf{C}$ , with  $B^A$  said to be an *exponential*, for each object  $B$ . It is well-known that both  $\mathbf{Top}$  and  $\mathbf{Top} \downarrow X$  (for every non-empty topological space  $X$ ) are not cartesian closed categories, and that the forgetful functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  does not preserve exponentials in general. Their exponentiable objects were characterized in [12, 36, 35, 41, 42, 5, 43]. It is also well-known that the forgetful functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  preserves neither products nor exponentials (unless  $X$  is a singleton).

Exponentiability in  $\mathbf{Top} \downarrow X$ , for an Alexandroff space  $X$ , is extensively studied in [34]. Among other results, [34, Theorem 6.4] provides a characterisation of exponentiable objects in  $\mathbf{Top} \downarrow X$ , for particular Alexandroff spaces  $X$ . Below we state this result adapted to our notation, in particular it is formulated with respect to the natural order of a topological space.

**Theorem 2.1** ([34, Theorem 6.4]). *Assume that  $X$  is a complete lattice equipped with the lower Alexandroff topology and assume that the lower Alexandroff topology coincides with the lower topology on  $X$ . Then  $(A, \alpha)$  is exponentiable in  $\mathbf{Top} \downarrow X$  if and only if*

- (1)  $A$  is exponentiable in  $\mathbf{Top}$  and
- (2)  $\alpha(a) \wedge - : X \rightarrow X$  preserves suprema, for all  $a \in X$ .

In this section we go substantially beyond the result above and characterise, for suitable (not necessarily Alexandroff) spaces  $X$ , the exponentiable objects of  $\mathbf{Top} \downarrow X$ .

By Corollary 1.19 we know that the existence of finite limits in  $\mathbf{Top} \downarrow X$  and their preservation by the forgetful functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  is guaranteed exactly when  $X$  is a topological  $\wedge$ -semilattice. It is therefore not surprising that (see [34, Lemma 5.1]):

**Proposition 2.2.** *If  $X$  is a topological  $\wedge$ -semilattice with bottom element, and  $(A, \alpha)$  is an exponentiable object of  $\mathbf{Top} \downarrow X$ , then:*

- (1) *The topological space  $A$  is exponentiable in  $\mathbf{Top}$ .*
- (2) *The canonical forgetful functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  preserves exponentials, that is, the diagram*

$$\begin{array}{ccc} \mathbf{Top} \downarrow X & \xrightarrow{(-)^{(A, \alpha)}} & \mathbf{Top} \downarrow X \\ \downarrow & & \downarrow \\ \mathbf{Top} & \xrightarrow{(-)^A} & \mathbf{Top} \end{array}$$

*commutes up to isomorphism.*

*Proof.* Let  $G : \mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  denote the canonical forgetful functor. Since  $X$  has a top and a bottom element,  $G$  is both a left and a right adjoint. Let  $L : \mathbf{Top} \rightarrow \mathbf{Top} \downarrow X$  denote the left adjoint of  $G$ , which sends  $f : B \rightarrow C$  to  $f : (B, \perp) \rightarrow (C, \perp)$ . Regarding the first assertion, just

observe that the functor  $A \times - : \mathbf{Top} \rightarrow \mathbf{Top}$  is the composite  $G \cdot ((A, \alpha) \times -) \cdot L$  of left adjoints; regarding the second assertion, just observe that the diagram

$$\begin{array}{ccc} \mathbf{Top} & \xrightarrow{A \times -} & \mathbf{Top} \\ L \downarrow & & \downarrow L \\ \mathbf{Top} \Downarrow X & \xrightarrow{(A, \alpha) \times -} & \mathbf{Top} \Downarrow X \end{array}$$

of the corresponding left adjoints commutes up to isomorphism.  $\square$

When  $X$  is a topological  $\wedge$ -semilattice, the category  $\mathbf{Top} \Downarrow X$  satisfies the conditions of the Special Adjoint Functor Theorem (see [31]), that is,  $\mathbf{Top} \Downarrow X$  is cocomplete, well-copowered, with  $(1, \perp)$  being a generator. Hence the functor  $(A, \alpha) \times - : \mathbf{Top} \Downarrow X \rightarrow \mathbf{Top} \Downarrow X$  has a right adjoint if and only if it preserves coequalisers and sums. Since  $\mathbf{Top} \Downarrow X$  is infinitely distributive (see Corollary 1.3), the existence of a right adjoint to  $(A, \alpha) \times -$  is equivalent to the preservation of coequalisers.

We can therefore obtain the following criterion for exponentiability of objects in  $\mathbf{Top} \Downarrow X$ .

**Theorem 2.3.** *Assume that  $X$  is a topological  $\wedge$ -semilattice. The following assertions are equivalent, for an object  $(A, \alpha)$  in  $\mathbf{Top} \Downarrow X$ .*

- (i)  $(A, \alpha)$  is exponentiable in  $\mathbf{Top} \Downarrow X$ .
- (ii) The topological space  $A$  is exponentiable in  $\mathbf{Top}$  and, for every quotient map  $q: C \rightarrow Q$  in  $\mathbf{Top}$  and every continuous map  $\gamma: C \rightarrow X$ ,

$$\alpha \sqcap \mathbf{Lan}_q \gamma = \mathbf{Lan}_{1_A \times q}(\alpha \sqcap \gamma).$$

*Proof.* (i)  $\implies$  (ii): Given such  $q: C \rightarrow Q$  and  $\gamma: C \rightarrow X$ , and continuous maps  $f, g: B \rightarrow C$  such that  $q$  is their coequaliser in  $\mathbf{Top}$ , we build a coequaliser diagram in  $\mathbf{Top} \Downarrow X$

$$\begin{array}{ccccc} B & \xrightarrow{f} & C & \xrightarrow{q} & Q \\ & \searrow g & \downarrow \gamma & \searrow & \\ & & X & & \mathbf{Lan}_q(\gamma) \\ & \swarrow \beta & & \swarrow & \end{array}$$

with  $\beta = \gamma f \wedge \gamma g$ . Exponentiability of  $(A, \alpha)$  guarantees that its image under  $(A, \alpha) \times -$

$$\begin{array}{ccccc} A \times B & \xrightarrow{1_A \times f} & A \times C & \xrightarrow{1_A \times q} & A \times Q \\ & \searrow 1_A \times g & \downarrow \alpha \sqcap \gamma & \searrow & \\ & & X & & \alpha \sqcap \mathbf{Lan}_q(\gamma) \\ & \swarrow \alpha \sqcap \beta & & \swarrow & \end{array}$$

is a coequaliser diagram in  $\mathbf{Top} \Downarrow X$ ; that is

$$\alpha \sqcap \mathbf{Lan}_q \gamma = \mathbf{Lan}_{1_A \times q}(\alpha \sqcap \gamma). \quad (2.i)$$

Conversely, assertion (ii) guarantees that the functor  $(A, \alpha) \times -$  preserves coequalisers, and so it is a left adjoint.  $\square$

Remark 1.13 gives us a simple formula to compute the left Kan extension  $\mathbf{Lan}_q \gamma$  when  $X$  is an injective topological space, with  $\gamma: C \rightarrow X$  and  $q: C \rightarrow Q$  as above: for every  $y \in Q$ ,

$$\mathbf{Lan}_q \gamma(y) = \bigwedge_{V \in \mathcal{O}(y)} \bigvee_{c \in q^{-1}(V)} \gamma(c).$$

Using this formula we can replace equation (2.i) by the more handy equation (2.ii) below.

**Proposition 2.4.** *Assume that  $X$  is an injective topological space and let  $(A, \alpha)$  be an object of  $\mathbf{Top} \downarrow X$ . For every quotient map  $q: C \rightarrow Q$  in  $\mathbf{Top}$  and every continuous map  $\gamma: C \rightarrow X$ ,*

$$\alpha \sqcap \mathbf{Lan}_q \gamma = \mathbf{Lan}_{1_A \times q}(\alpha \sqcap \gamma)$$

if and only if, for every  $a \in A$  and  $y \in Q$ ,

$$\bigwedge_{V \in \mathcal{O}(y)} \left( \alpha(a) \wedge \bigvee_{c \in q^{-1}(V)} \gamma(c) \right) = \bigwedge_{V \in \mathcal{O}(y)} \bigvee_{c \in q^{-1}(V)} \alpha(a) \wedge \gamma(c). \quad (2.ii)$$

*Proof.* By definition, for every  $a \in A$  and  $y \in Q$ ,

$$\begin{aligned} \mathbf{Lan}_{1_A \times q}(\alpha \sqcap \gamma)(a, y) &= \bigwedge_{W \times V \in \mathcal{O}(a, y)} \bigvee \{ \alpha(a') \wedge \gamma(c) \mid (a', c) \in (1_A \times q)^{-1}(W \times V) \} \\ &= \bigwedge_{V \in \mathcal{O}(y)} \bigwedge_{W \in \mathcal{O}(a)} \bigvee_{a' \in W} \bigvee_{c \in q^{-1}(V)} \alpha(a') \wedge \gamma(c). \end{aligned}$$

Using Remark 1.13,

$$\bigwedge_{W \in \mathcal{O}(a)} \bigvee_{a' \in W} \left( \bigvee_{c \in q^{-1}(V)} \alpha(a') \wedge \gamma(c) \right) = \bigvee_{c \in q^{-1}(V)} \alpha(a) \wedge \gamma(c).$$

Therefore

$$\mathbf{Lan}_{1_A \times q}(\alpha \sqcap \gamma)(a, y) = \bigwedge_{V \in \mathcal{O}(y)} \bigvee_{c \in q^{-1}(V)} \alpha(a) \wedge \gamma(c).$$

Observing that

$$(\alpha \sqcap \mathbf{Lan}_q \gamma)(a, y) = \alpha(a) \wedge \left( \bigwedge_{V \in \mathcal{O}(y)} \bigvee_{c \in q^{-1}(V)} \gamma(c) \right) = \bigwedge_{V \in \mathcal{O}(y)} \left( \alpha(a) \wedge \bigvee_{c \in q^{-1}(V)} \gamma(c) \right),$$

we obtain the claimed assertion.  $\square$

**Theorem 2.5.** *Assume that  $X$  is an injective space. Then  $(A, \alpha)$  is exponentiable in  $\mathbf{Top} \downarrow X$  if and only if*

- (1)  $A$  is exponentiable in  $\mathbf{Top}$  and,
- (2) for every  $a \in A$ , the map  $\alpha(a) \wedge -: X \rightarrow X$  preserves suprema with respect to the natural order of  $X$ .

*Proof.* The “if-part” follows from Proposition 2.4. Assume now that  $(A, \alpha)$  is exponentiable in  $\mathbf{Top} \downarrow X$ . By Proposition 2.2,  $A$  is exponentiable in  $\mathbf{Top}$ . To see the second assertion, for a family  $\gamma: I \rightarrow X$  in  $X$ , consider in Proposition 2.4 the quotient  $q: I \rightarrow 1$  with  $I$  being a discrete space.  $\square$

*Remark 2.6.* Theorem 2.1 is a special case of our Theorem 2.5 since the assumption “the lower topology and the lower Alexandroff topology coincide” implies that the complete lattice  $X$  is actually an op-continuous lattice. In fact, under this assumption, the lower Scott topology and the lower Alexandroff topology coincide, hence every up-closed subset of  $X$  is also Scott-closed. Therefore, for every codirected subset  $C$  of  $X$ ,  $\bigwedge C \in \uparrow C$ , which tells us that this infimum is actually a minimum. As a consequence,  $x \gg x$  for every  $x \in X$ . This proves that  $X$  is an op-continuous lattice, even more,  $X$  is an op-algebraic lattice. Also note that every sequence

$$x_0 \geq x_1 \geq x_2 \geq \dots$$

in  $X$  is stationary.

Note that, for an injective space  $X$  and  $x \in X$ , the map  $x \wedge - : X \rightarrow X$  is continuous. Moreover,  $x \wedge -$  preserves suprema if and only if  $x \wedge -$  has a right adjoint  $x \Rightarrow - : X \rightarrow X$  in  $\mathbf{Ord}$ . Since  $x \Rightarrow -$ , being right adjoint, preserves in particular codirected infima, it is even a continuous map  $x \Rightarrow - : X \rightarrow X$  (see [46]). Therefore we can equivalently substitute condition (2) in Theorem 2.5 by

(2)' for every  $a \in A$ , the continuous map  $\alpha(a) \wedge - : X \rightarrow X$  has a right adjoint in  $\mathbf{Top}$ .

In fact, this condition guarantees exponentiability under milder conditions on  $X$ .

**Theorem 2.7.** *Assume that  $X$  is a topological  $\wedge$ -semilattice. An object  $(A, \alpha)$  of  $\mathbf{Top} \downarrow X$  is exponentiable provided that*

- (1)  $A$  is exponentiable in  $\mathbf{Top}$ ,
- (2)  $X$  has continuous infima for all families indexed by the underlying set of  $A$ , and
- (3) for every  $a \in A$ , the continuous map  $\alpha(a) \wedge - : X \rightarrow X$  has a right adjoint  $\alpha(a) \Rightarrow - : X \rightarrow X$  in  $\mathbf{Top}$ .

*Proof.* For  $(B, \beta)$  in  $\mathbf{Top} \downarrow X$ , put  $(B, \beta)^{(A, \alpha)} = (B^A, \alpha \Rightarrow \beta)$  where  $B^A$  is the exponential in  $\mathbf{Top}$ , that is,  $B^A = \{h : A \rightarrow B \mid h \text{ is continuous}\}$ , and

$$\begin{aligned} \alpha \Rightarrow \beta : B^A &\longrightarrow X \\ h &\longmapsto \bigwedge_{a \in A} (\alpha(a) \Rightarrow \beta(h(a))). \end{aligned}$$

This map is continuous because it is the composite of continuous maps as indicated in the following diagram

$$B^A \xrightarrow{\text{id}} B^{|A|} \xrightarrow{\beta^{|A|}} X^{|A|} \xrightarrow{\widehat{\alpha}} X^{|A|} \xrightarrow{\wedge} X$$

where  $|A|$  denotes the discrete space (with the same underlying set as  $A$ ), and the map  $\widehat{\alpha}$ , defined by  $\widehat{\alpha}((x_a)_{a \in A}) = (\alpha(a) \Rightarrow x_a)_{a \in A}$ , is continuous because it is induced in the product  $X^{|A|}$  by the family of continuous maps

$$(X^{|A|} \xrightarrow{p_a} X \xrightarrow{\alpha(a) \Rightarrow -} X)_{a \in A}.$$

It remains to be shown that the evaluation (continuous) map  $\text{ev} : (A, \alpha) \times (B^A, \alpha \Rightarrow \beta) \rightarrow (B, \beta)$  is a morphism in  $\mathbf{Top} \downarrow X$  and has the required universal property. For each  $(a, h) \in A \times B^A$ ,

$\alpha(a) \wedge (\alpha \Rightarrow \beta)(h) \leq \beta(h(a))$  by definition of  $\alpha \Rightarrow \beta$ , hence  $\text{ev}$  is in fact a morphism in  $\text{Top} \Downarrow X$ . Moreover, given a morphism  $g: (A, \alpha) \times (C, \gamma) \rightarrow (B, \beta)$ , there exists a unique continuous map  $\bar{g}: C \rightarrow B^A$  so that  $\text{ev} \cdot (1_A \times \bar{g}) = g$ ; we only need to show that such map is a morphism  $\bar{g}: (C, \gamma) \rightarrow (B^A, \alpha \Rightarrow \beta)$ : for each  $a \in A$  and  $c \in C$ ,  $\alpha(a) \wedge \gamma(c) \leq \beta(g(a, c))$ , and therefore  $\gamma(c) \leq \bigwedge_{a \in A} (\alpha(a) \Rightarrow \beta(g(a, c))) = (\alpha \Rightarrow \beta)(\bar{g}(c))$ .  $\square$

Let us call a topological  $\wedge$ -semilattice  $X$  a *topological Heyting  $\wedge$ -semilattice* if, for every  $x \in X$ , the continuous map  $x \wedge -: X \rightarrow X$  has a right adjoint in  $\text{Top}$ . Theorem 2.7, together with Proposition 2.2, gives:

**Corollary 2.8.** *If  $X$  is a topological Heyting  $\wedge$ -semilattice, then the following conditions are equivalent, for an object  $(A, \alpha)$  of  $\text{Top} \Downarrow X$ :*

- (i)  $(A, \alpha)$  is exponentiable in  $\text{Top} \Downarrow X$ .
- (ii)  $A$  is exponentiable in  $\text{Top}$ .

For a topological  $\wedge$ -semilattice  $X$  with bottom element, we analyse now the relationship between the existence of particular exponentials in  $\text{Top} \Downarrow X$  and properties of  $X$ . First recall that, by Proposition 2.2, for objects  $(A, \alpha)$  and  $(B, \beta)$  in  $\text{Top} \Downarrow X$  with  $(A, \alpha)$  exponentiable, the space  $A$  is exponentiable in  $\text{Top}$  and the exponential  $(B, \beta)^{(A, \alpha)}$  in  $\text{Top} \Downarrow X$  is of the form  $(B^A, \delta)$ , with  $B^A$  denoting the exponential in  $\text{Top}$ . Furthermore, for the co-unit  $\varepsilon_{(B, \beta)}: (A, \alpha) \times (B^A, \delta) \rightarrow (B, \beta)$  of the adjunction  $(A, \alpha) \times - \dashv (-)^{(A, \alpha)}$  at  $(B, \beta)$ , the continuous map  $\varepsilon_{(B, \beta)}: A \times B^A \rightarrow B$  is a co-unit for the adjunction  $A \times - \dashv (-)^A$  in  $\text{Top}$  at the space  $B$ . In fact, one easily verifies the required universal property using that, for every topological space  $C$ ,

$$\text{Top}(A \times C, B) = (\text{Top} \Downarrow X)((A, \alpha) \times (C, \perp), (B, \beta))$$

and

$$\text{Top}(C, B^A) = (\text{Top} \Downarrow X)((C, \perp), (B^A, \delta)).$$

We conclude that the co-unit  $\varepsilon_{(B, \beta)}: (A, \alpha) \times (B^A, \delta) \rightarrow (B, \beta)$  can be chosen as  $\varepsilon_{(B, \beta)}(a, h) = h(a)$ , for all  $a \in A$  and  $h \in B^A$ ; that is, it is – as in  $\text{Top}$  – the evaluation map. Given also  $(C, \gamma)$  in  $\text{Top} \Downarrow X$ , it follows immediately that the diagram

$$\begin{array}{ccc} (\text{Top} \Downarrow X)((C, \gamma), (B, \beta)^{(A, \alpha)}) & \longrightarrow & (\text{Top} \Downarrow X)((A, \alpha) \times (C, \gamma), (B, \beta)) \\ \downarrow & & \downarrow \\ \text{Top}(C, B^A) & \longrightarrow & \text{Top}(A \times C, B) \end{array}$$

commutes. Hence, for a continuous map  $f: C \rightarrow B^A$  and its mate  $\bar{f}: A \times C \rightarrow B$  (that is, the continuous map induced by the adjunction),  $f$  is a morphism  $(C, \gamma) \rightarrow (B, \beta)^{(A, \alpha)}$  in  $\text{Top} \Downarrow X$  if and only if  $\bar{f}$  is a morphism  $(A, \alpha) \times (C, \gamma) \rightarrow (B, \beta)$  in  $\text{Top} \Downarrow X$ .

**Theorem 2.9.** *Assume that  $X$  is a topological  $\wedge$ -semilattice with bottom element.*

- (1) *For every  $x \in X$ , the continuous map  $x \wedge -: X \rightarrow X$  has a right adjoint in  $\text{Top}$  if and only if  $(1, x)$  is exponentiable in  $\text{Top} \Downarrow X$ .*
- (2) *For every set  $I$ ,  $X$  has continuous infima indexed by  $I$  if and only if  $(I, \top)$  (with  $I$  considered as a discrete space) is exponentiable in  $\text{Top} \Downarrow X$ .*



*Proof.* To prove (1), consider  $x \in X$ . If  $x \wedge - : X \rightarrow X$  has a right adjoint in  $\mathbf{Top}$ , then  $(1, x)$  is exponentiable in  $\mathbf{Top} \downarrow X$  by Theorem 2.7 since every one-element space is exponentiable in  $\mathbf{Top}$ . Assume now that  $(1, x)$  is exponentiable and consider the exponential  $(X, 1_X)^{(1, x)} = (X, \gamma_x)$  in  $\mathbf{Top} \downarrow X$ . Then, for all  $y \in X$ ,

$$(\mathbf{Top} \downarrow X)((1, y), (X, 1_X)^{(1, x)}) \simeq (\mathbf{Top} \downarrow X)((1, x) \times (1, y), (X, 1_X)),$$

which is equivalent to

$$\forall z \in X. (y \leq \gamma_x(z) \iff x \wedge y \leq z).$$

Therefore  $x \wedge - \dashv \gamma_x$ , that is,  $\gamma_x = (x \Rightarrow -)$ .

Regarding (2), if  $X$  has continuous infima indexed by  $I$ , then  $(I, \top)$  is exponentiable by Theorem 2.7 since every discrete space is exponentiable in  $\mathbf{Top}$ . On the other hand, note that, for every  $(A, \alpha)$  in  $\mathbf{Top} \downarrow X$ , a map  $f: I \times A \rightarrow X$  is a morphism  $f: (I, \top) \times (A, \alpha) \rightarrow (X, 1_X)$  in  $\mathbf{Top} \downarrow X$  if and only if, for all  $i \in I$ ,  $f(i, -): (A, \alpha) \rightarrow (X, 1_X)$  is a morphism in  $\mathbf{Top} \downarrow X$ . Therefore  $(X, 1_X)^{(I, \top)}$  coincides with the power  $(X, 1_X)^I$  in  $\mathbf{Top} \downarrow X$ , and the assertion follows from Proposition 1.17.  $\square$

Combining Corollary 2.8 and Theorem 2.9, we obtain:

**Corollary 2.10.** *Assume that  $X$  is a topological  $\wedge$ -semilattice with bottom element. Then the following assertions are equivalent.*

- (i) *The canonical forgetful functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  reflects exponentiable objects.*
- (ii)  *$X$  is a topological Heyting  $\wedge$ -semilattice.*

**Examples 2.11.** Assume that the natural order on  $X$  makes it a complete lattice, and that  $X$  is equipped with the lower topology. For each  $x \in X$ , if  $x \wedge - : X \rightarrow X$  has a right adjoint  $x \Rightarrow -$  in  $\mathbf{Ord}$ , then, by Lemma 1.9,  $x \Rightarrow - : X \rightarrow X$  is a continuous map. Therefore, if the natural order on  $X$  makes it a frame, then  $X$  is a topological Heyting  $\wedge$ -semilattice, and consequently the functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top}$  preserves and reflects exponentiability and exponentials. This is the case when  $X = \mathbb{S}, \mathbb{N}, [0, 1], \mathbb{Z}_-^\infty, \mathbb{Z}_+^\infty$  of Examples 1.12, but also for  $X = VZ$  the lower Vietoris space of a topological space  $Z$ .

### 3 Descent

Let  $p: A \rightarrow B$  be a morphism in a category  $\mathbf{C}$  with pullbacks. In our fundamental setting, descent theory for  $p$  refers to the problem of studying whether bundles over  $A$  equipped with some additional algebraic structure (specified by  $p$ ) – called *descent data for  $p$*  – can be regarded as bundles over  $B$ .

More specifically, bundles over  $A$  equipped with such descent data for  $p$  form a category, denoted  $\mathbf{Desc}_{\mathbf{C}}(p)$ , via which the pullback change-of-base functor  $p^*: \mathbf{C} \downarrow B \rightarrow \mathbf{C} \downarrow A$  factors through, depicted as follows:

$$\begin{array}{ccc} \mathbf{C} \downarrow B & \xrightarrow{p^*} & \mathbf{C} \downarrow A \\ & \searrow \kappa^p & \nearrow \mathcal{U}^p \\ & \mathbf{Desc}_{\mathbf{C}}(p) & \end{array} \quad (3.1)$$

In Diagram (3.i), the functor  $\mathcal{U}^p$  forgets the descent data, and  $\mathcal{K}^p$  – the comparison functor – maps a bundle  $f: U \rightarrow B$  to the bundle  $p^*(f): p^*(U) \rightarrow A$  equipped with its canonical descent data.

By the Bénabou-Roubaud theorem [2] (see [28] for a modern account), Diagram (3.i) is equivalent to the Eilenberg-Moore factorization of  $p^*$  with respect to the adjunction  $p_! \dashv p^*$ , so that the category  $\text{Desc}_{\mathbb{C}}(p)$  can be regarded as the category of algebras for the induced monad.

We say  $p$  is an *effective descent morphism* if  $\mathcal{K}^p$  is an equivalence, that is, if  $p^*$  is monadic. Similarly,  $p^*$  is a *descent morphism* if  $\mathcal{K}^p$  is fully faithful, that is, if  $p^*$  is premonadic. If  $\mathbb{C}$  is finitely complete, then a morphism in  $\mathbb{C}$  is a descent morphism if and only if it is a pullback-stable regular epimorphism [25, 24].

The descent morphisms in  $\text{Top}$  are precisely the *biquotient* maps (also known as *limit lifting maps* and as *universal quotient maps*). They are those continuous maps  $f: A \rightarrow B$  such that, for every ultrafilter convergence  $\mathfrak{b} \rightsquigarrow b$  in  $B$ , there exists an ultrafilter convergence  $\mathfrak{a} \rightsquigarrow a$  in  $A$  with  $Uf(\mathfrak{a}) = \mathfrak{b}$  and  $f(a) = b$  (see [17, 18, 33, 12]). Effective descent morphisms in  $\text{Top}$  were first characterised by [40] via a lifting property on “2-chains of ultrafilter convergence”. For more information we refer to [4, 6]; however, in this paper we will not make explicit use of this characterisation.

In this section we give, for suitable spaces  $X$ , a characterisation of effective descent morphisms in  $\text{Top} \Downarrow X$ .

In what follows, we study the effective descent morphisms of  $\text{Top} \Downarrow X$ , for suitable spaces  $X$ .

*Throughout this section we assume that  $X$  is a topological  $\wedge$ -semilattice.*

We provide sufficient conditions for a morphism  $p: (A, \alpha) \rightarrow (B, \beta)$  in  $\text{Top} \Downarrow X$  to be effective for descent in Theorem 3.9, and in Theorem 3.11 we confirm that these conditions are necessary when  $X$  is a frame with respect to its natural order.

We leave the complete characterisation of the effective descent morphisms in  $\text{Top} \Downarrow X$  as an open problem. The obstacle in our approach lies in the preservation of effective descent morphisms by the functor  $\text{Top} \Downarrow X \rightarrow \text{Fam}(X)$ , which we analyse in Lemmas 3.3 and 3.4. This analysis is based on more general preservation techniques which are developed in [29].

We start with the following observation.

**Lemma 3.1.** *The canonical forgetful functor  $\text{Top} \Downarrow X \rightarrow \text{Top}$  preserves (effective) descent morphisms.*

*Proof.* As in [8, Theorem 3.3]. □

Besides  $\text{Top}$ , we shall also compare  $\text{Top} \Downarrow X$  with the category  $\text{Fam}(X)$  of *families in  $X$*  defined as follows: the objects of  $\text{Fam}(X)$  are families  $(x_i)_{i \in I}$  of elements of  $X$  indexed by a set  $I$ , and a morphism  $f: (x_i)_{i \in I} \rightarrow (y_j)_{j \in J}$  in  $\text{Fam}(X)$  is given by a map  $f: I \rightarrow J$  satisfying  $x_i \leq y_{f(i)}$ , for all  $i \in I$ . Identities and composition are inherited from  $\text{Set}$ .

**Lemma 3.2.** *The canonical forgetful functor  $\text{Top} \Downarrow X \rightarrow \text{Fam}(X)$  preserves:*

- (1) pullbacks;
- (2) regular epimorphisms of type  $(B, \beta) \rightarrow (1, \gamma)$ ;
- (3) descent morphisms.

*Proof.* We confirm (1) by noting that the functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Fam}(X)$  has a left adjoint, mapping each family  $(\alpha(a))_{a \in A}$  to the pair  $(A, \alpha)$  where  $A$  has the discrete topology: we simply note that, if  $(B, \beta)$  is an object in  $\mathbf{Top} \downarrow X$ , morphisms  $(A, \alpha) \rightarrow (B, \beta)$  in  $\mathbf{Top} \downarrow X$  are in a natural bijective correspondence with morphisms  $(\alpha(a))_{a \in A} \rightarrow (\beta(b))_{b \in B}$ .

Regarding (2), observe that  $q: (B, \beta) \rightarrow (1, \gamma)$  is a regular epimorphism in  $\mathbf{Top} \downarrow X$  if and only if  $B \neq \emptyset$  and

$$\gamma = \bigvee \{\beta(b) \mid b \in B\}.$$

To see (3), let now  $q: (B, \beta) \rightarrow (C, \gamma)$  be a pullback-stable regular epimorphism in  $\mathbf{Top} \downarrow X$ . Then, for every  $c \in C$ ,

$$\gamma(c) = \bigvee \{\beta(b) \mid b \in B, f(b) = c\}$$

since in the pullback diagram

$$\begin{array}{ccc} (B_c, \beta) & \xrightarrow{q} & (1, \gamma(c)) \\ \downarrow & & \downarrow c \\ (B, \beta) & \xrightarrow{q} & (C, \gamma) \end{array}$$

the top row is a regular epimorphism in  $\mathbf{Top} \downarrow X$ . We conclude that  $q: (\beta(b))_{b \in B} \rightarrow (\gamma(c))_{c \in C}$  is a regular epimorphism in  $\mathbf{Fam}(X)$ . It is indeed a pullback-stable regular epimorphism in  $\mathbf{Fam}(X)$  because for every pullback diagram

$$\begin{array}{ccc} (\delta(d))_{d \in D} & \xrightarrow{\pi_2} & (\alpha(a))_{a \in A} \\ \pi_1 \downarrow & & \downarrow g \\ (\beta(b))_{b \in B} & \xrightarrow{q} & (\gamma(c))_{c \in C} \end{array}$$

in  $\mathbf{Fam}(X)$  we may consider the discrete topology on  $A$  and the pullback diagram

$$\begin{array}{ccc} (D, \delta) & \xrightarrow{\pi_2} & (A, \alpha) \\ \pi_1 \downarrow & & \downarrow g \\ (B, \beta) & \xrightarrow{q} & (C, \gamma) \end{array}$$

in  $\mathbf{Top} \downarrow X$ . Then  $\pi_2: (D, \delta) \rightarrow (A, \alpha)$  is a pullback-stable regular epimorphism in  $\mathbf{Top} \downarrow X$  and therefore  $\pi_2: (\delta(d))_{d \in D} \rightarrow (\alpha(a))_{a \in A}$  is a regular epimorphism in  $\mathbf{Fam}(X)$ .  $\square$

**Lemma 3.3.** *Let  $q: (B, \beta) \rightarrow (C, \gamma)$  be an effective descent morphism in  $\mathbf{Top} \downarrow X$ , and consider the following pullback diagram for each  $c \in C$ :*

$$\begin{array}{ccc} (\beta^{-1}(c), \beta|_{\beta^{-1}(c)}) & \xrightarrow{q|_{\beta^{-1}(c)}} & (1, \gamma(c)) \\ \downarrow & & \downarrow c \\ (B, \beta) & \xrightarrow{q} & (C, \gamma) \end{array} \tag{3.ii}$$

*We have that  $q|_{\beta^{-1}(c)}$  is an effective descent morphism in  $\mathbf{Fam}(X)$  for all  $c$  if and only if  $q$  is an effective descent morphism in  $\mathbf{Fam}(X)$ .*

*Proof.* We note that, in an extensive category, coproducts of effective descent morphisms are effective for descent, and the underlying morphism  $q: (B, \beta) \rightarrow (C, \gamma)$  in  $\mathbf{Fam}(X)$  is the coproduct of the family of effective descent morphisms  $(q|_{\beta^{-1}(c)})_{c \in C}$ .

The converse holds by pullback stability of effective descent morphisms.  $\square$

Lemma 3.3 says that the canonical forgetful functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Fam}(X)$  preserves the effective descent morphisms  $q|_{\beta^{-1}(c)}$  for all  $c$  if and only if it preserves  $q$ . Hence, in order to study the preservation of effective descent morphisms by  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Fam}(X)$ , it is sufficient to analyse whether it preserves effective descent morphisms of the form  $(B, \beta) \rightarrow (1, x)$ .

**Lemma 3.4.** *Let  $(B, \beta) \rightarrow (1, x)$  be an effective descent morphism in  $\mathbf{Top} \downarrow X$ . The following are equivalent:*

- (i)  $p: (B, \beta) \rightarrow (1, x)$  is an effective descent morphism in  $\mathbf{Fam}(X)$ .
- (ii) For all families of points  $(\theta(b))_{b \in B}$  in  $X$  with  $\theta(b) \leq \beta(b)$  for all  $b$  satisfying  $\beta(b') \wedge \theta(b) = \theta(b') \wedge \beta(b)$  for all pairs  $b, b' \in B$ , we have  $\theta: B \rightarrow X$  continuous.

*Proof.* First, we note that the following diagram commutes, up to isomorphism:

$$\begin{array}{ccc}
 \mathbf{Top} \downarrow X \downarrow (1, x) & \xrightarrow{\mathcal{K}^p} & \mathbf{Desc}_{\mathbf{Top} \downarrow X}(p) \\
 H \downarrow & & \downarrow \tilde{H} \\
 \mathbf{Fam}(X) \downarrow (1, x) & \xrightarrow{\mathcal{K}^p} & \mathbf{Desc}_{\mathbf{Fam}(X)}(p)
 \end{array} \tag{3.iii}$$

Moreover, we know that  $H$  is essentially surjective.

Second, we note that a family of points  $(\theta(b))_{b \in B}$  in  $X$  with  $\theta(b) \leq \beta(b)$  for all  $b$  satisfying  $\beta(b') \wedge \theta(b) = \theta(b') \wedge \beta(b)$  is precisely a *connected* descent datum  $\text{id}: (B, \theta) \rightarrow (B, \beta)$  in  $\mathbf{Fam}(X)$  for  $p$  in the sense of [37, 38].

Let us assume (i) holds, in which case  $\tilde{H}$  must be essentially surjective. We note that descent data for  $p$  in  $\mathbf{Top} \downarrow X$  must be of the form  $\pi_2: (B \times C, \gamma) \rightarrow (B, \beta)$ , where  $\pi_2$  is the projection  $B \times C \rightarrow B$ . Thus, given descent data  $\text{id}: (B, \theta) \rightarrow (B, \beta)$  for  $p$  in  $\mathbf{Fam}(X)$ , the only topology on the underlying set of  $B$  for which  $\text{id}: (B, \theta) \rightarrow (B, \beta)$  still defines descent data for  $p$  in  $\mathbf{Top} \downarrow X$  is the topology on  $B$  itself, meaning  $\theta$  must be continuous, thereby confirming (ii).

Conversely, if (ii) holds – that is, for any connected descent data  $\text{id}: (B, \theta) \rightarrow (B, \beta)$  for  $p$  in  $\mathbf{Fam}(X)$  we have  $\theta$  continuous – we have that  $\text{id}: (B, \theta) \rightarrow (B, \beta)$  determines descent data for  $p$  in  $\mathbf{Top} \downarrow X$ . As  $p$  is effective for descent in  $\mathbf{Top} \downarrow X$ , we conclude that this descent data corresponds to the bundle  $(1, \bigvee_{b \in B} \theta(b)) \rightarrow (1, x)$ , whose pullback along  $p$  guarantees that  $\theta(b) = \beta(b) \wedge \bigvee_{b' \in B} \theta(b')$  for all  $b \in B$ . This confirms that  $p$  is effective for descent in  $\mathbf{Fam}(X)$ .  $\square$

**Corollary 3.5.** *Let  $(B, \beta)$  in  $\mathbf{Top} \downarrow X$  so that the image of  $\beta: B \rightarrow X$  has a maximum. Then  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Fam}(X)$  preserves all effective descent morphisms  $(B, \beta) \rightarrow (1, x)$ .*

*Proof.* Let  $b_0 \in B$  so that  $\beta(b) \leq \beta(b_0)$ , for all  $b \in B$ . Let  $(\theta(b))_{b \in B}$  be a family of elements of  $X$  as in Lemma 3.4. Then, for every  $b \in B$ ,

$$\theta(b) = \beta(b_0) \wedge \theta(b) = \theta(b_0) \wedge \beta(b);$$

hence,  $\theta: B \rightarrow X$  is continuous, because  $X$ , by our general assumption, is a topological  $\wedge$ -semilattice.  $\square$

Following [11], we consider the (pseudo)pullback

$$\begin{array}{ccc} \mathbf{Top} \times_{\mathbf{Set}} \mathbf{Fam}(X) & \xrightarrow{\rho_2} & \mathbf{Fam}(X) \\ \rho_1 \downarrow & & \downarrow \\ \mathbf{Top} & \longrightarrow & \mathbf{Set} \end{array}$$

and the canonical functor

$$H: \mathbf{Top} \downarrow X \longrightarrow \mathbf{Top} \times_{\mathbf{Set}} \mathbf{Fam}(X).$$

induced by the forgetful functors

$$\mathbf{Top} \downarrow X \longrightarrow \mathbf{Top} \quad \text{and} \quad \mathbf{Top} \downarrow X \longrightarrow \mathbf{Fam}(X).$$

**Proposition 3.6.** (1) *A morphism  $f$  in  $\mathbf{Top} \times_{\mathbf{Set}} \mathbf{Fam}(X)$  is effective for descent if and only if  $\rho_2(f)$  is effective for descent in  $\mathbf{Fam}(X)$  and  $\rho_1(f)$  is effective for descent in  $\mathbf{Top}$ .*

(2) *The functor  $\mathbf{Top} \downarrow X \rightarrow \mathbf{Top} \times_{\mathbf{Set}} \mathbf{Fam}(X)$  is fully faithful and preserves pullbacks.*

*Proof.* Regarding (1), both functors  $\rho_1$  and  $\rho_2$  preserve effective descent morphism by [7, Corollary 2.3]. On the other hand, if  $\rho_1(f)$  and  $\rho_2(f)$  are effective for descent in  $\mathbf{Top}$  and  $\mathbf{Fam}(X)$ , respectively, then either [7, Corollary 2.6] or [27, Corollary 9.6] guarantee that  $f$  is effective for descent in  $\mathbf{Top} \times_{\mathbf{Set}} \mathbf{Fam}(X)$ . The assertion (2) is clear.  $\square$

**Corollary 3.7.** *Let  $f: (A, \alpha) \rightarrow (B, \beta)$  be a morphism in  $\mathbf{Top} \downarrow X$  so that  $f: (\alpha(a))_{a \in A} \rightarrow (\beta(b))_{b \in B}$  is effective for descent in  $\mathbf{Fam}(X)$ . Then  $f$  is effective for descent in  $\mathbf{Top} \downarrow X$  if and only if  $f: A \rightarrow B$  is effective for descent in  $\mathbf{Top}$  and, for every pullback*

$$\begin{array}{ccc} H(D, \delta) & \xrightarrow{\pi_2} & (C, (\gamma_c)_{c \in C}) \\ \pi_1 \downarrow & & \downarrow g \\ H(A, \alpha) & \xrightarrow{Hf} & H(B, \beta) \end{array} \tag{3.iv}$$

in  $\mathbf{Top} \times_{\mathbf{Set}} \mathbf{Fam}(X)$ , the family  $\gamma = (\gamma_c)_{c \in C}$  is a continuous map  $\gamma: C \rightarrow X$ .

*Proof.* First note that  $f$  being effective for descent in  $\mathbf{Top} \downarrow X$  implies that  $f$  is effective descent in  $\mathbf{Top}$  (by Lemma 3.1) and, by Proposition 3.6 (1), it is also effective for descent in  $\mathbf{Top} \times_{\mathbf{Set}} \mathbf{Fam}(X)$ . The assertion follows now from [25, Proposition 2.6] using Proposition 3.6.  $\square$

Recall from Section 1 that  $e_X: X \rightarrow UX$  has a left adjoint in  $\mathbf{Ord}$  if and only if every ultrafilter in  $X$  has a smallest convergence point with respect to the natural order in  $X$ . We say that *binary infima preserve smallest convergence points in  $X$*  whenever  $e_X: X \rightarrow UX$  has a left adjoint  $\sigma: UX \rightarrow X$  in  $\mathbf{Ord}$  and the diagram

$$\begin{array}{ccc} U(X \times X) & \xrightarrow{U\wedge} & UX \\ \text{can} \downarrow & & \downarrow \sigma \\ UX \times UX & & \\ \sigma \times \sigma \downarrow & & \downarrow \\ X \times X & \xrightarrow{\wedge} & X \end{array}$$

commutes.

**Lemma 3.8.** *Assume that binary infima preserve smallest convergence points in  $X$ . In the pullback (3.iv), let  $\mathfrak{w} \in UD$ , and put  $\mathfrak{a} = U\pi_1(\mathfrak{w})$  and  $\mathfrak{c} = U\pi_2(\mathfrak{w})$ . Then*

$$\sigma(U\delta(\mathfrak{w})) = \sigma(U\alpha(\mathfrak{a})) \wedge \sigma(U\gamma(\mathfrak{c})).$$

*Proof.* Observe that in the diagram

$$\begin{array}{ccccccc} UD & \longrightarrow & U(A \times C) & \xrightarrow{U(\alpha \times \gamma)} & U(X \times X) & \xrightarrow{U\wedge} & UX \\ & & \downarrow & & \downarrow & & \downarrow \sigma \\ & & UA \times UC & \xrightarrow{U\alpha \times U\gamma} & UX \times UX & & \\ & & & & \sigma \times \sigma \downarrow & & \\ & & & & X \times X & \xrightarrow{\wedge} & X \end{array}$$

both squares commute, the top row represents  $U\delta: UD \rightarrow UX$  and the map corresponding to the bottom path sends  $\mathfrak{w} \in UD$  to  $\sigma(U\alpha(\mathfrak{a})) \wedge \sigma(U\gamma(\mathfrak{c}))$ .  $\square$

**Theorem 3.9.** *Assume that binary infima preserve smallest convergence points in the topological  $\wedge$ -semilattice  $X$ . Let  $f: (A, \alpha) \rightarrow (B, \beta)$  be a morphism in  $\mathbf{Top} \downarrow X$  so that  $f: (\alpha(a))_{a \in A} \rightarrow (\beta(b))_{b \in B}$  is effective for descent in  $\mathbf{Fam}(X)$ . Then  $f: (A, \alpha) \rightarrow (B, \beta)$  is effective for descent in  $\mathbf{Top} \downarrow X$  if and only if*

- (1)  $f: A \rightarrow B$  is effective for descent in  $\mathbf{Top}$ , and
- (2) for all ultrafilter convergences  $\mathfrak{b} \rightsquigarrow b$  in  $B$  and for all  $w \in X$  with  $w \leq \sigma(U\beta(\mathfrak{b}))$ , we have

$$w = \bigvee_{\substack{\mathfrak{a} \rightsquigarrow a \\ Uf(\mathfrak{a})=\mathfrak{b}, \\ f(a)=b}} w \wedge \sigma(U\alpha(\mathfrak{a})). \quad (3.v)$$

*Proof.* We show first that conditions (1) and (2) are sufficient for  $f$  to be effective for descent in  $\mathbf{Top} \downarrow X$ . To do so, consider a pullback in  $\mathbf{Top} \times_{\mathbf{Set}} \mathbf{Fam}(X)$  as in (3.iv). We want to show that  $\gamma$  is continuous. For  $\mathfrak{c} \rightsquigarrow c$  an ultrafilter convergence in  $C$ , we show that  $U\gamma(\mathfrak{c}) \rightsquigarrow \gamma(c)$ . Put  $\mathfrak{b} = Ug(\mathfrak{c})$  and  $b = g(c)$ . Then, since the functor  $U: \mathbf{Top} \rightarrow \mathbf{Top}$  is locally monotone,

$\sigma: UX \rightarrow X$  is monotone and we have  $\sigma(U\gamma(\mathbf{c})) \leq \sigma(U\beta(\mathbf{b}))$  (in order to apply  $U$  to  $\gamma$  we consider  $C$  with the discrete topology). Therefore, by condition (2),

$$\sigma(U\gamma(\mathbf{c})) = \bigvee_{\substack{\mathbf{a} \rightsquigarrow a \\ Uf(\mathbf{a})=\mathbf{b}, \\ f(a)=b}} \sigma(U\gamma(\mathbf{c})) \wedge \sigma(U\alpha(\mathbf{a})).$$

Since the functor  $U: \mathbf{Set} \rightarrow \mathbf{Set}$  weakly preserves pullbacks, for every  $\mathbf{a} \in UA$  with  $Uf(\mathbf{a}) = \mathbf{b}$ , there is some  $\mathfrak{w} \in UD$  with  $U\pi_1(\mathfrak{w}) = \mathbf{a}$  and  $U\pi_2(\mathfrak{w}) = \mathbf{c}$ . Therefore, applying also Lemma 3.8, we obtain

$$\bigvee_{\substack{\mathbf{a} \rightsquigarrow a \\ Uf(\mathbf{a})=\mathbf{b}, \\ f(a)=b}} \sigma(U\gamma(\mathbf{c})) \wedge \sigma(U\alpha(\mathbf{a})) = \bigvee_{\substack{\mathfrak{w} \in UD \\ U\pi_2(\mathfrak{w})=\mathbf{c}, \\ U\pi_1(\mathfrak{w}) \rightsquigarrow a, \\ f(a)=b}} \sigma(U\delta(\mathfrak{w})).$$

For every  $\mathfrak{w} \in UD$  with  $U\pi_2(\mathfrak{w}) = \mathbf{c}$ ,  $U\pi_1(\mathfrak{w}) \rightsquigarrow a$  and  $f(a) = b$ , we have that  $\mathfrak{w} \rightsquigarrow (a, c)$  in  $D$ . Since  $\delta: D \rightarrow X$  is continuous,  $U\delta(\mathfrak{w}) \rightsquigarrow \delta(a, c)$ , which is equivalent to  $\sigma(U\delta(\mathfrak{w})) \leq \delta(a, c)$ . Finally, by definition,

$$\delta(a, c) = \alpha(a) \wedge \gamma(c) \leq \gamma(c).$$

Summing up,  $\sigma(U\gamma(\mathbf{c})) \leq \gamma(c)$ , that is,  $U\gamma(\mathbf{c}) \rightsquigarrow \gamma(c)$ . We conclude that  $\gamma: C \rightarrow X$  is continuous. By Corollary 3.7,  $f$  is effective for descent in  $\mathbf{Top} \downarrow X$ .

Conversely, assume now that  $f: (A, \alpha) \rightarrow (B, \beta)$  is effective for descent in  $\mathbf{Top} \downarrow X$ . By Lemma 3.1,  $f: A \rightarrow B$  is effective for descent in  $\mathbf{Top}$ . To verify Condition (2), fix an ultrafilter convergence  $\mathbf{b} \rightsquigarrow b$  in  $B$  and  $w \in X$  with  $w \leq \sigma(U\beta(\mathbf{b}))$ . Consider the set  $C = B + \{\star\}$  (for simplicity of notation, we assume  $\star \notin B$  so that  $C = B \cup \{\star\}$ ) together with the topology where the only non-trivial ultrafilter convergence is  $\mathbf{b} \rightsquigarrow \star$ . Furthermore, we define the maps  $g: C \rightarrow B$  with  $g(\star) = b$  and  $g(c) = c$  for every  $c \in B$ , and

$$\gamma: C \longrightarrow X$$

$$c \longmapsto \begin{cases} \beta(c) \wedge w & \text{if } c \in B \\ \bigvee_{\substack{\mathbf{a} \rightsquigarrow a \\ Uf(\mathbf{a})=\mathbf{b}, \\ f(a)=b}} w \wedge \sigma(U\alpha(\mathbf{a})) & \text{if } c = \star \end{cases}$$

For the pullback (3.iv), we show now that  $\delta: D \rightarrow X$  is continuous. We can write the set  $D$  as

$$D = \underbrace{\{(a, c) \in D \mid c \in B\}}_{=D_1} + \underbrace{\{(a, c) \in D \mid c = \star\}}_{=D_2}.$$

Let  $\mathfrak{d} \rightsquigarrow (a, c)$  be an ultrafilter convergence in  $D$ . We put  $\mathbf{a} = U\pi_1(\mathfrak{d})$  and  $\mathbf{c} = U\pi_2(\mathfrak{d})$ .

- Assume that  $\mathfrak{d} \in UD_1$  and  $c \in B$ . Then

$$\sigma(U\delta(\mathfrak{d})) \leq \sigma(U\alpha(\mathbf{a})) \wedge \sigma(U\gamma(\mathbf{c})) \leq \sigma(U\alpha(\mathbf{a})) \wedge w \leq \alpha(a) \wedge w \leq \alpha(a) \wedge \beta(f(a)) \wedge w = \delta(a, c).$$

- Assume that  $\mathfrak{d} \in UD_2$  and  $c = \star$ . Then  $\mathbf{c} = \dot{\star}$  and

$$\sigma(U\delta(\mathfrak{d})) \leq \sigma(U\alpha(\mathbf{a})) \wedge \gamma(\star) \leq \alpha(a) \wedge \gamma(\star) = \delta(a, c).$$

- Assume that  $\mathfrak{d} \in UD_1$  and  $c = \star$ . Then  $\mathfrak{c} = \mathfrak{b}$  and therefore  $Uf(\mathfrak{a}) = \mathfrak{b}$ . Hence,  $\sigma(U\alpha(\mathfrak{a})) \wedge w = \gamma(\star)$ . Since also  $\sigma(U\alpha(\mathfrak{a})) \leq \alpha(a)$ , we obtain

$$\alpha(U\delta(\mathfrak{d})) \leq \sigma(U\alpha(\mathfrak{a})) \wedge w \leq \alpha(a) \wedge \gamma(\star) = \delta(a, \star).$$

- Finally,  $\mathfrak{d} \in UD_2$  and  $c \in B$  is not possible by the definition of the convergence in  $C$ .

Hence, we conclude that  $U\delta(\mathfrak{d}) \rightsquigarrow \delta(a, c)$  in  $X$ . This proves that  $\delta: D \rightarrow X$  is continuous. By Corollary 3.7,  $\gamma: C \rightarrow X$  is continuous as well. Therefore  $\sigma(U\gamma(\mathfrak{b})) \leq \gamma(\star)$ . Since also

$$\sigma(U\gamma(\mathfrak{b})) = \sigma(U\beta(\mathfrak{b})) \wedge w = w,$$

$w \leq \gamma(\star)$  and we have proven (2).  $\square$

**Lemma 3.10.** *Let  $f: (A, \alpha) \rightarrow (B, \beta)$  be a morphism in  $\mathbf{Top} \downarrow X$  satisfying Condition (2) of Theorem 3.9. Then  $f: (\alpha(a))_{a \in A} \rightarrow (\beta(b))_{b \in B}$  is a descent morphism in  $\mathbf{Fam}(X)$ .*

*Proof.* By [11, Lemma 2.1] (see also [38, 37]), the morphism  $f: (\alpha(a))_{a \in A} \rightarrow (\beta(b))_{b \in B}$  is a descent morphism in  $\mathbf{Fam}(X)$  if and only if, for all  $b \in B$  and  $w \leq \beta(b)$ ,

$$w = \bigvee_{a, f(a)=b} w \wedge \alpha(a).$$

To verify this condition, consider  $(\mathfrak{b} \rightsquigarrow b) = (\dot{b} \rightsquigarrow b)$  in Condition (2) of Theorem 3.9 and observe that  $\sigma(U\alpha(\mathfrak{a})) \leq \alpha(a)$  for every  $\mathfrak{a} \rightsquigarrow a$  in  $A$ .  $\square$

By [37, Corollary 4.6], every descent morphism in  $\mathbf{Fam}(X)$  is effective for descent provided that  $X$  is also a frame. Therefore we can state:

**Theorem 3.11.** *Assume that binary infima preserve smallest convergence points in the topological  $\wedge$ -semilattice  $X$  and, moreover, that the ordered set  $X$  is a frame. Then a morphism  $f: (A, \alpha) \rightarrow (B, \beta)$  in  $\mathbf{Top} \downarrow X$  is effective for descent if and only if*

- (1)  $f: A \rightarrow B$  is effective for descent in  $\mathbf{Top}$ ,
- (2) for all ultrafilter convergences  $\mathfrak{b} \rightsquigarrow b$  in  $B$ , we have

$$\sigma(U\beta(\mathfrak{b})) = \bigvee_{\substack{\mathfrak{a} \rightsquigarrow a \\ Uf(\mathfrak{a})=\mathfrak{b}, \\ f(a)=b}} \sigma(U\alpha(\mathfrak{a})).$$

There is a natural way of producing spaces  $X$  satisfying the hypotheses of Theorems 3.9 and 3.11, as follows.

**Proposition 3.12.** *If  $X$  is a continuous lattice equipped with the lower topology, then  $X$  is a topological  $\wedge$ -semilattice, with smallest convergence points preserved by binary infima.*



*Proof.* To prove this statement, we will explain how a continuous lattice equipped with the lower topology can be obtained as the dual space of an injective space. As we have already observed, if  $Y$  is an injective space, then  $Y$  is an op-continuous lattice with respect to the natural order and the topology of  $Y$  coincides with the lower Scott topology. Furthermore, every ultrafilter in  $Y$  has smallest convergence points; in fact, the left adjoint  $\sigma: UY \rightarrow Y$  of  $e_Y: Y \rightarrow UY$  is precisely the convergence of the Lawson topology of  $Y$  [16, Proposition VII-3.10]. Explicitly,

$$\sigma(\eta) = \bigwedge_{A \in \eta} \bigvee A,$$

for all  $\eta \in UY$ . The space  $Y$  is stably compact, hence we can define its dual space  $Y^{\text{op}}$ , whose topology is given by the convergence relation

$$\eta \rightsquigarrow y \iff y \leq \sigma(\eta);$$

this is the so-called co-compact topology. Hence the natural order of  $Y^{\text{op}}$  is the dual of the natural order of  $Y$  and also in  $Y^{\text{op}}$  every ultrafilter  $\mathfrak{x}$  has  $\sigma(\mathfrak{x})$  as smallest convergence point. Furthermore, by [16, Proposition VI-6.24], the co-compact topology of  $Y$  is the upper topology with respect to the natural order of  $Y$ , hence the lower topology with respect to the natural order of  $Y^{\text{op}}$ . By Lemma 1.9,  $Y^{\text{op}}$  is a topological  $\bigwedge$ -semilattice. Finally, since the map

$$\vee: Y \times Y \longrightarrow Y$$

is continuous and left adjoint to  $\Delta_Y: Y \rightarrow Y \times Y$ , it preserves smallest convergence points of ultrafilters. We conclude that binary infima in  $Y^{\text{op}}$

$$\wedge: Y^{\text{op}} \times Y^{\text{op}} \longrightarrow Y^{\text{op}}$$

preserve smallest convergence points of ultrafilters.

In conclusion, if  $X$  is a continuous lattice equipped with the lower topology, denoting by  $Y$  the topological space with the same underlying set and whose topology is the lower Scott topology with respect to the reverse of the natural order of  $X$ , then  $X = Y^{\text{op}}$  and the result follows.  $\square$

We note that, if  $X$  is a completely distributive lattice (see [39, 15] and [16, Corollary I-2.9]) equipped with the lower topology, in particular if  $X = \mathbb{S}$  and  $X = [0, 1]$  as in Examples 1.12, then  $X$  is both a continuous lattice and a frame. Hence:

**Corollary 3.13.** *If  $X$  is a completely distributive lattice equipped with the lower topology, then  $X$  satisfies the hypotheses of Theorem 3.11.*

**Example 3.14.** In  $\text{Top} \downarrow \mathbb{S}$ , by Theorem 3.11,  $f: (A, A_0) \rightarrow (B, B_0)$  is an effective descent morphism if and only if  $f: A \rightarrow B$  is an effective descent morphism in  $\text{Top}$  and  $f_0: A_0 \rightarrow B_0$  is a descent morphism in  $\text{Top}$ . In fact, Condition (2) of Theorem 3.11 specialises to: for all ultrafilter convergences  $\mathfrak{b} \rightsquigarrow b$  in  $B$  with  $B_0 \in \mathfrak{b}$  (and therefore  $b \in B_0$ ), there exists an ultrafilter convergence  $\mathfrak{a} \rightsquigarrow a$  in  $A$  with  $A_0 \in \mathfrak{a}$  (and therefore  $a \in A_0$ ) which is mapped to  $\mathfrak{b} \rightsquigarrow b$ , that is  $f_0$  is a descent morphism in  $\text{Top}$ .

**Example 3.15.** When we equip  $X = [0, 1]$  with the lower topology as in Example 1.12, for every ultrafilter  $\mathfrak{r}$  on  $[0, 1]$  (see [21], for example),

$$\sigma(\mathfrak{r}) = \sup\{u \in [0, 1] \mid \uparrow u \in \mathfrak{r}\}.$$

Hence, by Theorem 3.11, a morphism  $f: (A, (A_u)_{u \in [0, 1]}) \rightarrow (B, (B_u)_{u \in [0, 1]})$  in  $\mathbf{Top} \Downarrow [0, 1]$  is effective for descent if and only if  $f: A \rightarrow B$  is effective for descent in  $\mathbf{Top}$  and, for all  $u \in [0, 1]$  and all ultrafilter convergences  $\mathfrak{b} \rightsquigarrow b$  in  $B_u$ , for every  $v < u$  there exists an ultrafilter convergence  $\mathfrak{a} \rightsquigarrow a$  in  $A_v$  with  $Uf(\mathfrak{a}) = \mathfrak{b}$  and  $f(a) = b$ .

The interval  $[0, 1]$  is a completely distributive lattice, and its totally below relation  $\lll$  is given by  $<$ . The arguments used for  $[0, 1]$  generalise smoothly to completely distributive lattices, as stated in the following result.

**Theorem 3.16.** *Let  $X$  be a completely distributive lattice equipped with the lower topology. A morphism  $f: (A, \alpha) \rightarrow (B, \beta)$  is effective for descent in  $\mathbf{Top} \Downarrow X$  if and only if*

- (1)  $f: A \rightarrow B$  is effective for descent in  $\mathbf{Top}$ ;
- (2) for each  $u \in X$ , all ultrafilter convergences  $\mathfrak{b} \rightsquigarrow b$  in  $B_u$ , and each  $v \lll u$ , there exists an ultrafilter convergence  $\mathfrak{a} \rightsquigarrow a$  in  $A_v$  with  $Uf(\mathfrak{a}) = \mathfrak{b}$  and  $f(a) = b$ .

## References

- [1] Marco Abbadini, Vincenzo Marra, and Luca Spada. Stone-Gelfand duality for metrically complete lattice-ordered groups. *Advances in Mathematics*, 461:110067, 2025. [arXiv:2210.15341](https://arxiv.org/abs/2210.15341) [math.FA]. doi:10.1016/j.aim.2024.110067.
- [2] Jean Bénabou and Jacques Roubaud. Monades et descente. *Comptes rendus de l'Académie des Sciences Paris, Serie A*, 270:96–98, 1970.
- [3] Aurelio Carboni, Stephen Lack, and R. F. C. Walters. Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, 84(2):145–158, 1993. doi:10.1016/0022-4049(93)90035-r.
- [4] Maria Manuel Clementino and Dirk Hofmann. Triquotient maps via ultrafilter convergence. *Proceedings of the American Mathematical Society*, 130(11):3423–3431, 2002. eprint: [https://www.mat.uc.pt/preprints/eng\\_2000.html](https://www.mat.uc.pt/preprints/eng_2000.html). doi:10.1090/S0002-9939-02-06472-9.
- [5] Maria Manuel Clementino, Dirk Hofmann, and Walter Tholen. The convergence approach to exponentiable maps. *Portugaliae Mathematica. Nova Série*, 60(2):139–160, 2003. eprint: [https://www.mat.uc.pt/preprints/eng\\_2000.html](https://www.mat.uc.pt/preprints/eng_2000.html).
- [6] Maria Manuel Clementino and George Janelidze. A note on effective descent morphisms of topological spaces and relational algebras. *Topology and its Applications*, 158(17):2431–2436, 2011. doi:10.1016/j.topol.2011.04.014.
- [7] Maria Manuel Clementino and George Janelidze. Effective descent morphisms of filtered preorders. *Order*, 2024. doi:10.1007/s11083-024-09676-8.

- [8] Maria Manuel Clementino and Fernando Lucatelli Nunes. Lax comma categories of ordered sets. *Quaestiones Mathematicæ*, 46(sup1):145–159, 2023. [arXiv:2212.13541 \[math.CT\]](#). [doi:10.2989/16073606.2023.2247729](#).
- [9] Maria Manuel Clementino and Fernando Lucatelli Nunes. Lax comma 2-categories and admissible 2-functors. *Theory and Applications of Categories*, 40(6):180–226, 2024. [arXiv:2002.03132 \[math.CT\]](#).
- [10] Maria Manuel Clementino, Fernando Lucatelli Nunes, and Rui Prezado. Lax comma categories: cartesian closedness, extensivity, topologicity, and descent. *Theory and Applications of Categories*, 41(16):516–530, 2024. [arXiv:2405.03773 \[math.CT\]](#).
- [11] Maria Manuel Clementino and Rui Prezado. Effective descent morphisms of ordered families. *Quaestiones Mathematicæ*, pages 1–16, 2025. [arXiv:2407.08573 \[math.CT\]](#). [doi:10.2989/16073606.2025.2474091](#).
- [12] Brian J. Day and G. Max Kelly. On topologically quotient maps preserved by pullbacks or products. *Proceedings of the Cambridge Philosophical Society*, 67(03):553–558, 1970. [doi:10.1017/s0305004100045850](#).
- [13] Thomas Erker. Right Kan spaces and essentially complete T0-spaces. *Electronic Notes in Theoretical Computer Science*, 13:41–52, 1998. [doi:10.1016/s1571-0661\(05\)80671-1](#).
- [14] Martín Hötzel Escardó. Properly injective spaces and function spaces. *Topology and its Applications*, 89(1-2):75–120, 1998. [doi:10.1016/S0166-8641\(97\)00225-3](#).
- [15] Barry Fawcett and Richard J. Wood. Constructive complete distributivity. I. *Mathematical Proceedings of the Cambridge Philosophical Society*, 107(1):81–89, 1990. [doi:10.1017/S0305004100068377](#).
- [16] Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael W. Mislove, and Dana S. Scott. *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2003. [doi:10.1017/CB09780511542725](#).
- [17] Otomar Hájek. Notes on quotient maps. *Commentationes Mathematicæ Universitatis Carolinæ*, 7:319–323, 1966.
- [18] Otomar Hájek. Correction to: “Notes on quotient maps”. *Commentationes Mathematicæ Universitatis Carolinæ*, 8:171, 1967.
- [19] Melvin Hochster. Prime ideal structure in commutative rings. *Transactions of the American Mathematical Society*, 142:43–60, 1969. [doi:10.2307/1995344](#).
- [20] Rudolf-E. Hoffmann. Essentially complete T0-spaces. *Manuscripta Mathematica*, 27(4):401–432, 1979. [doi:10.1007/BF01507294](#).
- [21] Dirk Hofmann. Topological theories and closed objects. *Advances in Mathematics*, 215(2):789–824, 2007. [doi:10.1016/j.aim.2007.04.013](#).
- [22] Dirk Hofmann. The enriched Vietoris monad on representable spaces. *Journal of Pure and Applied Algebra*, 218(12):2274–2318, 2014. [arXiv:1212.5539 \[math.CT\]](#). [doi:10.1016/j.jpaa.2014.04.002](#).

- [23] Dirk Hofmann, Gavin J. Seal, and Walter Tholen, editors. *Monoidal Topology. A Categorical Approach to Order, Metric, and Topology*, volume 153 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2014. doi: [10.1017/cbo9781107517288](https://doi.org/10.1017/cbo9781107517288). Authors: Maria Manuel Clementino, Eva Colebunders, Dirk Hofmann, Robert Lowen, Rory Lucyshyn-Wright, Gavin J. Seal and Walter Tholen.
- [24] George Janelidze, Manuela Sobral, and Walter Tholen. Beyond Barr Exactness: Effective Descent Morphisms. In M. Cristina Pedicchio and Walter Tholen, editors, *Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory*, volume 97 of *Encyclopedia of Mathematics and its Applications*, pages 359–405. Cambridge University Press, Cambridge, 2004. doi: [10.1017/cbo9781107340985.011](https://doi.org/10.1017/cbo9781107340985.011).
- [25] George Janelidze and Walter Tholen. Facets of descent, I. *Applied Categorical Structures*, 2(3):245–281, 1994. doi: [10.1007/bf00878100](https://doi.org/10.1007/bf00878100).
- [26] Peter T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Reprint of the 1982 edition.
- [27] Fernando Lucatelli Nunes. Pseudo-Kan extensions and descent theory. *Theory and Applications of Categories*, 33(15):390–444., 2018. arXiv: [1606.04999](https://arxiv.org/abs/1606.04999) [math.CT].
- [28] Fernando Lucatelli Nunes. Semantic factorization and descent. *Applied Categorical Structures*, 30(6):1393–1433, 2022. arXiv: [1902.01225](https://arxiv.org/abs/1902.01225) [math.CT]. doi: [10.1007/s10485-022-09694-w](https://doi.org/10.1007/s10485-022-09694-w).
- [29] Fernando Lucatelli Nunes and Rui Prezado. Functors preserving effective descent morphisms. Technical report, 2024, arXiv: [2410.22876](https://arxiv.org/abs/2410.22876) [math.CT].
- [30] Fernando Lucatelli Nunes and Matthijs Vákár. CHAD for expressive total languages. *Mathematical Structures in Computer Science*, 33(4–5):311–426, 2023. arXiv: [2110.00446](https://arxiv.org/abs/2110.00446) [cs.PL]. doi: [10.1017/S096012952300018X](https://doi.org/10.1017/S096012952300018X).
- [31] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, New York, 2 edition, 1998. doi: [10.1007/978-1-4757-4721-8](https://doi.org/10.1007/978-1-4757-4721-8).
- [32] Ernest Michael. Topologies on spaces of subsets. *Transactions of the American Mathematical Society*, 71(1):152–182, 1951. doi: [10.1090/S0002-9947-1951-0042109-4](https://doi.org/10.1090/S0002-9947-1951-0042109-4).
- [33] Ernest Michael. Bi-quotient maps and Cartesian products of quotient maps. *Annales de l’institut Fourier*, 18(fasc. 2):287–302, 1968. doi: [10.5802/aif.301](https://doi.org/10.5802/aif.301).
- [34] Susan Niefield. Exponentiability in lax slices of Top. *Theory and Applications of Categories*, 16(11):218–235, 2006.
- [35] Susan B. Niefield. Cartesian spaces over  $T$  and locales over  $\Omega(T)$ . *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 23(3):257–267, 1982.
- [36] Susan B. Niefield. Cartesianness: topological spaces, uniform spaces, and affine schemes. *Journal of Pure and Applied Algebra*, 23(2):147–167, 1982. doi: [10.1016/0022-4049\(82\)90004-4](https://doi.org/10.1016/0022-4049(82)90004-4).
- [37] Rui Prezado. *Some aspects of descent theory and applications*. PhD thesis, University of Coimbra and University of Porto, 2023, arXiv: [2502.09337](https://arxiv.org/abs/2502.09337) [math.CT].

- [38] Rui Prezado. On effective descent  $\mathcal{V}$ -functors and familial descent morphisms. *Journal of Pure and Applied Algebra*, 228(5):107597, 2024. [arXiv:2305.04042 \[math.CT\]](#). doi: [10.1016/j.jpaa.2023.107597](#).
- [39] George N. Raney. Completely distributive complete lattices. *Proceedings of the American Mathematical Society*, 3(5):677–680, 1952. doi: [10.1090/s0002-9939-1952-0052392-3](#).
- [40] Jan Reiterman and Walter Tholen. Effective descent maps of topological spaces. *Topology and its Applications*, 57(1):53–69, 1994. doi: [10.1016/0166-8641\(94\)90033-7](#).
- [41] Günther Richter. An elementary approach to exponentiable maps. *Applied Categorical Structures*, 12(3):287–300, 2004. doi: [10.1023/b:apcs.0000031070.39281.21](#).
- [42] Günther Richter. Exponentiability for maps means fibrewise core-compactness. *Journal of Pure and Applied Algebra*, 187(1-3):295–303, 2004. doi: [10.1016/s0022-4049\(03\)00128-2](#).
- [43] Günther Richter and Walter Tholen. Perfect maps are exponentiable – categorically. *Theory and Applications of Categories*, 8(16):457–464, 2001.
- [44] Andrea Schalk. *Algebras for Generalized Power Constructions*. PhD thesis, Technische Hochschule Darmstadt, 1993.
- [45] Andrea Schalk. Domains arising as algebras for powerspace constructions. *Journal of Pure and Applied Algebra*, 89(3):305–328, 1993. doi: [10.1016/0022-4049\(93\)90059-3](#).
- [46] Dana Scott. Continuous lattices. In F. William Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136. Springer Berlin Heidelberg, 1972. doi: [10.1007/bfb0073967](#). Proceedings of the conference on Connections between Category Theory and Algebraic Geometry & Intuitionistic Logic, Dalhousie University, Halifax, Canada, January 16-19, 1971.
- [47] Leopold Vietoris. Bereiche zweiter Ordnung. *Monatshefte für Mathematik und Physik*, 32(1):258–280, 1922. doi: [10.1007/BF01696886](#).
- [48] Oswald Wyler. Dedekind complete posets and Scott topologies. In Bernhard Banaschewski and Rudolf-Eberhard Hoffmann, editors, *Continuous Lattices*, Lecture Notes in Mathematics, pages 384–389. Springer-Verlag, 1981. doi: [10.1007/BFb0089920](#). Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices (Workshop IV), University of Bremen, Germany, November 9–11, 1979.