IMPROVED REGULARITY FOR A NONLOCAL DEAD-CORE PROBLEM

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ABSTRACT. We obtain improved regularity results for solutions to a nonlocal dead-core problem at branching points. Our approach, which does not rely on the maximum principle, introduces a new strategy for analyzing two-phase problems within the local framework, an area that remains largely unexplored.

1. INTRODUCTION

In this work, we obtain improved regularity properties for solutions of the *two-phase* nonlocal problem

$$-(-\Delta)^s u = u_+^\gamma - u_-^\gamma \quad \text{in } B_1, \tag{1.1}$$

where $(-\Delta)^s$ is the fractional Laplacian and $\gamma \in (0, 1/3)$. Local versions of the *one-phase* problem have attracted increasing attention in recent years (see, for example, [2, 3, 14, 19]) due to their wide range of applications, for instance, in optimizing resources in catalysis processes. In such reaction-diffusion models, the presence of a catalyst accelerates the rate of chemical reactions, leading to the formation of regions where the reactant concentration drops to zero and no reaction occurs. These regions are commonly known as *dead-cores* in the literature, and placing catalysts there would be ineffective and result in resource wastage. The mathematical study of these phenomena has a long-standing history, dating back to the seminal work on the classical Alt-Phillips problem (cf. [1, 4, 15, 16, 21, 22]).

The dead-core problem (1.1) is nonlocal in nature, and the techniques used to treat the corresponding local models cannot be applied in our setting, requiring an alternative approach. For example, we cannot rely on the maximum principle to derive sharp regularity estimates as done in the one-phase local model of [19] corresponding to $s = 1$ and $u \geq 0$. The reason for

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this is two-fold. First, for the maximum principle to hold in the nonlocal setting, solutions would need to have a sign in the whole complement of the unit ball (see, for example, [20, Theorem 6.1]). Second, while in one-phase models, points on the free boundary $\partial\{u > 0\}$ correspond to local minima of the function, this property does not extend to the two-phase scenario.

Expanding on ideas from [18], our strategy relies instead on the study of the growth of solutions according to the natural scaling of the equation, combined with an auxiliary Liouville-type result. This leads to improved sharp regularity results at branching points, where the function vanishes with some of its derivatives, typically up to order two. The approach comes with the caveat of restricting $\gamma \in (0, 1/3)$, whereas in the *one-phase* local case, we can take $\gamma \in (0, 1)$. This is induced by the nonlocal nature of the problem as the new approach requires a certain integrability condition (see the proof of Theorem 4.1 below) only valid in this range. However, unlike the local result in [19], this new scheme offers two key advantages. First, it extends to the two-phase case, regardless of whether the left-hand side of (1.1) has a sign. In fact, the equation

$$(-\Delta)^s u = u_+^\gamma$$

has no direct local analogue as, for $s = 1$, no dead-core phenomenon occurs, *i.e.*, no non-negative solutions vanish in an interior region without being identically zero. The second significant advantage builds on the first, introducing a new strategy for analyzing the two-phase problem in the *local* setting. Since our estimates remain uniform, we can establish sharp regularity results for solutions of the local two-phase problem by passing to the limit as $s \nearrow 1^-$ in (1.1) (see Theorem 5.1). The one-phase local problem, studied in [19] (see also [3, 14]), relies heavily on the maximum principle, which is why the corresponding two-phase problem remained unresolved. Our approach circumvents this difficulty, as it does not rely on the maximum principle or the sign of the right-hand side.

To remain at least in the $C^{1,\sigma}-$regularity regime, we restrict our analysis to the range $s \in (1/2, 1)$. Observe that as long as the right-hand side is bounded, we have, *a priori*, that solutions of (1.1) are locally of class $C^{1,\sigma}$, for any $0 < \sigma < 2s - 1$ (see Theorem 2.3 below). Since, in general, u_+ and u_- are at most Lipschitz and the right-hand side in (1.1) is then locally C^γ, the best regularity one can hope for solutions of (1.1), using Schauder theory, is $C^{2s+\gamma}_{\text{loc}}$. However, our main result reveals that solutions are indeed $C^{\frac{2s}{1-\gamma}}$ at branching points. Note that

$$1 + \frac{2s - 1 + \gamma}{1 - \gamma} = \boxed{\frac{2s}{1-\gamma} > 2s + \gamma} = 1 + \frac{2s - 1 + \gamma}{1},$$

for any $\gamma \in (0,1)$ and $s \in (1/2,1)$, *i.e.*, we obtain higher regularity than one could hope for relying on Schauder theory alone. More precisely, assuming x_0 is a branching point for u, *i.e.*, $u(x_0) = |D^{\nu-1}u(x_0)| = |D^\nu u(x_0)| = 0$, we show that around it one has precisely the growth given by

$$|u(x)| \le C|x-x_0|^{\frac{2s}{1-\gamma}}. \tag{1.2}$$

Here ν is a constant defined by

$$\nu = \begin{cases} 1, & s < 1-\gamma, \\ 2, & s > 1-\frac{\gamma}{2}, \end{cases}$$

The idea of the proof is to study the growth of the scaled function

$$v_r(x) := \frac{u(rx)}{r^{\frac{2s}{1-\gamma}}},$$

as $r \to 0$. Obtaining a Liouville-type theorem, we ensure that v_r grows like a polynomial of degree ν, which implies (1.2) by contradiction.

In the nonlocal setting, a similar problem was studied in [22]. However, in addition to the fact that the model considered therein does not generate dead-cores, the result heavily depends on the celebrated Caffarelli-Silvestre extension argument (*cf.* [9]), which limits its flexibility and prevents generalizations to a broader class of operators. On the contrary, our approach is flexible enough to be extended to fully nonlinear equations (*cf.* Section 6).

The paper is organized as follows. In Section 2, we introduce the necessary notation and auxiliary results. The primary technical tool, a Liouville-type theorem, is presented in Section 3, while our main result on improved regularity at branching points is proved in Section 4. In Section 5, we explore some consequences of the main result, including its local implications and a Liouville-type property. In the final Section 6, we address the extension to the fully nonlinear case.

2. Notation and auxiliary results

This section gathers some notation, the notion of solution, some remarks about existence and a few auxiliary well-known results.

2.1. **Notation.** We denote with $B_r(x_0)$ the open ball of radius r centered at x_0, and write $B_r := B_r(0)$. For a multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, as usual, we put $|\beta| := \beta_1 + \beta_2 + \ldots + \beta_n$. For $\alpha \in (0,1)$, the Hölder semi-norm is defined as

$$[u]_{C^\alpha} := \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$

$$[u]_{C^{1+\alpha}} := \max_{|\beta|=1} \left[D^\beta u\right]_{C^\alpha},$$

and

$$[u]_{C^{2+\alpha}} := \max_{|\beta|=2} \left[D^\beta u \right]_{C^\alpha},$$

where $D^\beta u := \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n} u$.

The fractional Laplacian is the nonlocal operator defined by

$$(-\Delta)^s u(x) := c_{n,s}\, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy,$$

where $s \in (0, 1)$ and $c_{n,s}$ is a normalization constant, depending only on n and s. We use $u_+ := \max(u, 0)$ and $u_- := -u_+$. We also use the norm

$$\|u\|_{L^1_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+2s}} dy.$$

2.2. **Existence of solutions.** In the spirit of [6, 7] (see also [8, 10, 12]), and with the goal of extending our results to a broader class of fully nonlinear operators (see Section 6), solutions to (1.1) are understood in the viscosity sense according to the following definition.

Definition 2.1. *A function $u : \mathbb{R}^n \to \mathbb{R}$ is called a viscosity subsolution (supersolution) of (1.1), and we write*

$$-(-\Delta)^s u \geq (\leq)\, u_+^\gamma - u_-^\gamma,$$

if u is upper (lower) semi-continuous in $\overline{B_1}$ and, whenever $x_0 \in B_1$, $B_r(x_0) \subset B_1$, for some r, and $\varphi \in C^2(\overline{B_r(x_0)})$ satisfies

$$\varphi(x_0) = u(x_0) \quad \text{and} \quad \varphi(y) > (<)\, u(y),\ \forall y \in B_r(x_0) \setminus \{x_0\},$$

then, if we let

$$v := \begin{cases} \varphi & in \;\; B_r(x_0) \\ \\ u & in \;\; \mathbb{R}^n \setminus B_r(x_0), \end{cases}$$

we have $-(-\Delta)^s v(x_0) \geq (\leq)\, v_+^\gamma(x_0) - v_-^\gamma(x_0)$.

A function is called a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

For the existence of viscosity solutions, we refer the reader to [6, Theorem 1.2]. The rough idea is that, since the comparison principle holds for the fractional Laplacian (see [10, Theorem 5.2], [12, Theorem 2.5] or [20, Corollary 6.1]), the classical Perron's method then leads to the existence of the solution to the Dirichlet problem (see [13], for example). The comparison principle for

$$Lu := -(-\Delta)^s u - u_+^\gamma + u_-^\gamma$$

follows from [7, Theorem 3]. Its proof relies on the nonlocal Jensen-Ishii lemma, established in [7, Lemma 1], and makes use of the technique of jets with ideas from [13]. For the sake of completeness, we state it below.

Theorem 2.1. *If $u_1, u_2 \in C(\mathbb{R}^n)$, $Lu_1 \le 0 \le Lu_2$ in B_1, and $u_1 \ge u_2$ in $\mathbb{R}^n \setminus B_1$, then $u_1 \ge u_2$ in $\mathbb{R}^n$.*

We conclude this section with regularity results for solutions of the homogeneous and the non-homogeneous fractional Laplace equation. For the proof of the following theorem, we refer the reader to [11, Theorem 27] (see also [10, Theorem 13.1]).

Theorem 2.2. *Let $s \ge s_0 > 0$ and $u \in C(\overline{B}_1)$ be such that $\|u\|_{L^1_s(\mathbb{R}^n)} < \infty$. If $(-\Delta)^s u = 0$ in B_1, then there exists $\sigma > 0$, depending only on n and s_0, such that $u \in C^{1+\sigma}(B_{1/2})$. Moreover,*

$$\|u\|_{C^{1+\sigma}(B_{1/2})} \le C\left(\|u\|_{L^\infty(B_1)} + \|u\|_{L^1_s(\mathbb{R}^n)}\right),$$

where $C > 0$ is a constant, depending only on n and s_0.

The proof of the following theorem can be found in [11, Theorem 61] (see also [11, Theorem 52]).

Theorem 2.3. *If $s \in (1/2, 1)$, and u is a bounded solution of*

$$-(-\Delta)^s u = f \quad \text{in } B_1,$$

where $f \in L^\infty(B_1)$, then $u \in C^{1+\sigma}(B_{1/2})$, for any $\sigma < 2s - 1$. Moreover,

$$\|u\|_{C^{1+\sigma}(B_{1/2})} \le C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1)}\right),$$

for a constant $C > 0$, depending only on n.

Finally, we recall Schauder-type estimates for the fractional Laplacian (see, for example, [20, Remark 9.1]).

Theorem 2.4. *If $s \in (0, 1)$ and $(-\Delta)^s u \in C^\sigma(B_1) \cap C(\overline{B}_1)$, for some $\sigma > 0$, then $u \in C^{\sigma+2s}_{\text{loc}}(B_1)$.*

3. Liouville-type results

This section and the next form the core of the paper. We prove Liouville-type results in the spirit of [18, 23] that will be used later to derive the sharp regularity for the dead-core problem.

Theorem 3.1. *Let $s \in (1/2, 1)$ and*

$$(-\Delta)^s \left(u(x+h) - u(x)\right) = 0, \quad x \in B_1, \tag{3.1}$$

for all $h \in \mathbb{R}^n$. If, for $0 \le \alpha < \beta < 1$, there holds

$$[u]_{C^{\nu+\alpha}(B_R)} \le C R^{\beta-\alpha}, \quad \forall R \ge 1, \tag{3.2}$$

where $\nu = 1, 2$, and $C > 0$ is a constant, depending only on α and β, then

$$u(x) = u(0) + Du(0) \cdot x + \frac{\nu - 1}{2} x^T \cdot D^2 u(0) x, \quad x \in \mathbb{R}^n.$$

Proof. We divide the proof into two steps, depending on the value of ν.

STEP 1. Suppose $\nu = 1$, and let $e \in \mathbb{R}^n$, with $|e| = 1$. Set

$$w(x) := u(x) - u(0) - Du(0) \cdot x,$$

and note that $w(0) = |Dw(0)| = 0$, and w satisfies (3.1)-(3.2). Define $w_e := Dw \cdot e$. Since $w_e(0) = 0$, (3.2) yields

$$\begin{aligned} |w_e(x)| = |w_e(x) - w_e(0)| &\leq [w]_{C^{1,\alpha}(B_{|x|})}|x|^\alpha \\ &\leq C|x|^{\beta-\alpha}|x|^\alpha = C|x|^\beta, \end{aligned} \tag{3.3}$$

for all $x \notin B_1$. Hence, since $\beta < 1 < 2s$,

$$\int_{B_1^c} \frac{|w_e(y)|}{|y|^{n+2s}}\, dy \leq C \int_{B_1^c} \frac{|y|^\beta}{|y|^{n+2s}}\, dy = C \int_{B_1^c} \frac{1}{|y|^{n+2s-\beta}}\, dy < \infty.$$

On the other hand, (3.2) with $R = 1$ gives

$$|w_e(x)| \leq C, \quad x \in B_1. \tag{3.4}$$

By stability (see [11, Lemma 5], and also [10, Lemma 4.5 and Corollary 4.7]), (3.1) then implies

$$(-\Delta)^s w_e = 0 \quad \text{in } B_1.$$

By Theorem 2.2, there exists a constant $\sigma > 0$, depending only on n, such that

$$\|w_e\|_{C^{1,\sigma}(B_{1/2})} < C_1,$$

for a constant $C_1 > 0$, depending only on n and β.

Now, for $\rho > 0$, set

$$v(x) := \rho^{-\beta} w_e(\rho x)$$

and, recalling (3.3), note that

$$|v(x)| = \rho^{-\beta}|w_e(\rho x)| \leq C|x|^\beta, \quad \text{for all } \ x \notin B_{1/\rho}.$$

Also, using (3.4), in $B_{1/\rho}$ we have

$$|v(x)| = \rho^{-\beta}|w_e(\rho x)| \leq C\rho^{-\beta} < C,$$

for a $\rho > 0$ large enough. Thus,

$$\|v\|_{L^\infty(B_1)} < C.$$

Hence, again by Theorem 2.2, there exists $\sigma_* > 0$ such that

$$\|v\|_{C^{1,\sigma_*}(B_{1/2})} \leq C_2 \left(\|v\|_{L^\infty(B_1)} + \|v\|_{L^1_s(\mathbb{R}^n)} \right),$$

where $C_2 > 0$ is a constant, depending only on n. Since

$$\|Dv\|_{L^\infty(B_{1/2})} = \rho^{1-\beta}\|Dw_e\|_{L^\infty(B_{1/2\rho})},$$

if follows that

$$\|Dw_e\|_{L^\infty(B_{1/2\rho})} \leq \rho^{\beta-1}C_3, \tag{3.5}$$

where $C_3 > 0$ is a constant, depending only on n, β and $C > 0$ from (3.2). Letting $\rho \to \infty$ in (3.5), one gets $Dw_e(0) = 0$, and since the fractional Laplacian is translation invariant, we deduce that $Dw_e \equiv 0$, for any $e \in \mathbb{R}^n$, $|e| = 1$. Hence, w_e is a constant, and as $w_e(0) = 0$, w_e must be identically zero, implying that w is a constant. But $w(0) = 0$, therefore $w \equiv 0$, and hence,

$$u(x) = u(0) + Du(0) \cdot x.$$

STEP 2. If $\nu = 2$, set

$$w(x) := u(x) - u(0) - Du(0) \cdot x - \frac{1}{2}x^T \cdot D^2u(0)x,$$

and observe that it satisfies (3.1)-(3.2), while $w(0) = |w_e(0)| = |w_{ee}(0)| = 0$, where $e \in \mathbb{R}^n$ is a unit vector. Arguing as above, we arrive at $Dw_{ee} = 0$ for any unit vector $e \in \mathbb{R}^n$, meaning that w_{ee} is a constant, *i.e.*, $w_{ee} \equiv 0$. The latter yields $w_e = 0$, which then gives $w \equiv 0$. Thus,

$$u(x) = u(0) + Du(0) \cdot x + \frac{1}{2}x^T \cdot D^2u(0)x.$$

$\square$

The following result is a simple consequence of the mean value theorem.

Lemma 3.1. *Let $u \in C^{\nu,\alpha}(\mathbb{R}^n)$, where $\nu = 1, 2$ and $\alpha \in (0, 1)$. If*

$$u(0) = |D^{\nu-1}u(0)| = |D^\nu u(0)| = 0$$

and

$$\sup_{0<r<1/2} r^{\alpha-\beta}[u]_{C^{\nu+\alpha}(B_r)} \leq A,$$

for $\beta > \alpha$ and a constant $A > 0$, then

$$|u(x)| \leq A|x|^{\nu+\beta}, \quad x \in B_{1/2}.$$

Proof. As $u(0) = 0$, by the mean value theorem, for $x \in B_{1/2}$, one has

$$|u(x)| = |u(x) - u(0)| \leq |Du(\xi_1)||x|, \tag{3.6}$$

for some $\xi_1 \in B_{|x|}$.

For $\nu = 1$, since $Du(0) = 0$ and $Du \in C^\alpha\left(B_{1/2}\right)$, we have

$$\frac{|Du(\xi_1)||x|}{|x|^{1+\beta}} = \frac{|Du(\xi_1) - Du(0)||x|}{|x|^{1+\beta}} \leq |x|^{\alpha-\beta}\,[u]_{C^{1+\alpha}\left(B_{|x|}\right)} \leq A,$$

which combined with (3.6) concludes the proof in this case.

Similarly, if $\nu = 2$, since $|Du(0)| = |D^2u(0)| = 0$, employing the mean value theorem once more, for some $\xi_2 \in B_{|\xi_1|}$, we estimate

$$\begin{aligned}\frac{|Du(\xi_1)||x|}{|x|^{2+\beta}} &= \frac{|Du(\xi_1) - Du(0)||x|}{|x|^{2+\beta}}\\ &\leq \frac{|D^2u(\xi_2) - D^2u(0)||x|^2}{|x|^{2+\beta}}\\ &\leq |x|^{\alpha-\beta}\,[u]_{C^{2+\alpha}\left(B_{|x|}\right)}\\ &\leq A,\end{aligned}$$

which concludes the proof also for $\nu = 2$, thanks to (3.6). □

4. Improved regularity at branching points

In this section, we prove the main result of this paper. As observed earlier, unlike the local one-phase problem treated in [19], we cannot rely on the maximum principle to derive sharp regularity estimates. We start with the precise definition of a branching point.

Definition 4.1. *A point $x_0 \in B_1$ is called a branching point for the function $u : B_1 \to \mathbb{R}$ if*

$$u(x_0) = |Du(x_0)| = |D^2u(x_0)| = 0.$$

Theorem 4.1. *Let $\gamma \in (0, 1/3)$ and $s > 1 - \frac{\gamma}{2}$. If $u \in L^\infty(\mathbb{R}^n)$ is a viscosity solution of (1.1) and $x_0 \in B_{1/2}$ is its branching point, then there exists a constant $C > 0$, depending only on n and γ, such that*

$$|u(x)| \leq C\|u\|_{L^\infty(\mathbb{R}^n)}|x - x_0|^{\frac{2s}{1-\gamma}}, \quad \forall x \in B_{1/2}(x_0). \tag{4.1}$$

As a consequence, $u \in C^{\frac{2s}{1-\gamma}}$ at branching points.

Proof. By Theorem 2.3 and Theorem 2.4, we have $u \in C^{2s+\gamma}(B_{1/2})$. Moreover,

$$\|u\|_{C^{2s+\gamma}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|u\|^\gamma_{L^\infty(B_1)}\right),$$

for a constant $C > 0$, depending only on n and s. With no loss of generality, we may assume $x_0 = 0$ and $\|u\|_{L^\infty(\mathbb{R}^n)} = 1$, and we need to show that

$$|u(x)| \leq C|x|^{\frac{2s}{1-\gamma}}. \tag{4.2}$$

We argue by contradiction and assume that (4.2) fails. Then, there exist sequences u_k of solutions of (1.1) and points x_k, such that 0 is a branching point of u_k, and

$$[u_k]_{C^{2s+\gamma}(B_{1/2})} \leq 2C,$$

with

$$|u_k(x_k)| > k|x_k|^{\frac{2s}{1-\gamma}}. \tag{4.3}$$

Set

$$\theta_k(r') := \sup_{r'<r<1/2} r^{2s+\gamma-\frac{2s}{1-\gamma}} \left[u_k\right]_{C^{2s+\gamma}(B_r)}. \tag{4.4}$$

Observe that $2s+\gamma-\frac{2s}{1-\gamma}<0$ and

$$\lim_{r'\to 0}\theta_k(r') = \sup_{0<r<1/2} r^{2s+\gamma-\frac{2s}{1-\gamma}} \left[u_k\right]_{C^{2s+\gamma}(B_r)}.$$

Then, (4.3) and Lemma 3.1 yield

$$\lim_{r'\to 0}\theta_k(r') > k.$$

Thus, there exists a sequence $r_k > \frac{1}{k}$, such that

$$r_k^{2s+\gamma-\frac{2s}{1-\gamma}} \left[u_k\right]_{C^{2s+\gamma}(B_{r_k})} \geq \frac{1}{2}\theta_k(1/k) \geq \frac{1}{2}\theta_k(r_k). \tag{4.5}$$

From the first inequality in (4.5), we conclude that

$$r_k \longrightarrow 0, \quad \text{as} \quad k\to\infty.$$

Set now

$$v_k(x_k) := \frac{u_k(r_k x_k)}{\theta_k(r_k) r_k^{\frac{2s}{1-\gamma}}} \tag{4.6}$$

and note that, for $1\leq R\leq \frac{1}{2r_k}$, one has

$$\begin{aligned}
\left[v_k\right]_{C^{2s+\gamma}(B_R)} &= \frac{1}{\theta_k(r_k) r_k^{\frac{2s}{1-\gamma}}} \left[u_k\right]_{C^{2s+\gamma}(B_{r_k R})} r_k^{2s+\gamma} \\
&= \frac{(r_k R)^{2s+\gamma-\frac{2s}{1-\gamma}}}{\theta_k(r_k)} \left[u_k\right]_{C^{2s+\gamma}(B_{r_k R})} R^{\frac{2s}{1-\gamma}-2s-\gamma}.
\end{aligned} \tag{4.7}$$

Since

$$(r_k R)^{2s+\gamma-\frac{2s}{1-\gamma}} \left[u_k\right]_{C^{2s+\gamma}(B_{r_k R})} \leq \theta_k(r_k R) \leq \theta_k(r_k),$$

employing (4.7), we get

$$\left[D^2 v_k\right]_{C^{2s+\gamma-2}(B_R)} = \left[v_k\right]_{C^{2s+\gamma}(B_R)} \leq R^{\frac{2s}{1-\gamma}-2s-\gamma}, \quad \forall R\geq 1. \tag{4.8}$$

Since 0 is a branching point for v_k, using the mean value theorem and (4.8), one has

$$\begin{aligned}
|v_k(x)| &\leq |Dv_k(\xi)||x| \leq |D^2 v_k(\xi')||x|^2 \\
&\leq \left[D^2 v_k\right]_{C^{2s+\gamma-2}(B_R)} |x|^{2s+\gamma} \\
&\leq R^{\frac{2s}{1-\gamma}-2s-\gamma}|x|^{2s+\gamma},
\end{aligned}$$

where ξ is a point on the line segment connecting the origin to x, and ξ' is a point on the line segment connecting the origin to ξ. Now, if η is a smooth function such that $\eta \equiv 1$ in $B_{1/2}$ and $\eta = 0$ outside B_1, then for any unit vector e, one has

$$\int_{B_1} \eta \cdot D_{ee} v_k \, dx = \int_{B_1} D_{ee}\eta \cdot v_k \, dx \leq C_n,$$

where $C_n > 0$ is a constant, depending only on n. Therefore, there exists $z \in B_1$ such that $|D^2 v_k(z)| \leq C_n$, and (4.8) implies

$$|D_{ee} v_k(x) - D_{ee} v_k(z)| \leq R^{\frac{2s}{1-\gamma} - 2s - \gamma} |x - z|^{2s+\gamma-2}.$$

Hence, for $x \in B_R$ and $1 \leq R \leq \frac{1}{2r_k}$, we get

$$|D_{ee} v_k(x)| \leq C_n + R^{\frac{2s}{1-\gamma} - 2s - \gamma} |x - z|^{2s+\gamma-2} \leq C R^{\frac{2s}{1-\gamma} - 2},$$

for a constant $C > 0$, depending only on n. Thus, up to a subsequence, v_k converges to some v_0 in $C^{2s+\gamma}_{\text{loc}}(B_R)$, as $k \to \infty$, and thanks to (4.8),

$$[v_0]_{C^{2s+\gamma}(B_R)} \leq R^{\frac{2s}{1-\gamma} - 2s - \gamma}, \quad \forall R \geq 1. \tag{4.9}$$

Additionally, from the second inequality in (4.5), we deduce that

$$[v_0]_{C^{2s+\gamma}(B_1)} \geq \frac{1}{2}. \tag{4.10}$$

Observe that, for a fixed $h \in \mathbb{R}^n$, using the mean value theorem and (4.8), for $|x| \geq 1$, we have

$$\begin{aligned}
|v_k(x+h) - v_k(x)| &\leq |Dv_k(\eta)||h| \\
&\leq |D^2 v_k(\eta')|(|x| + |h|)|h| \\
&\leq [v_k]_{C^{2s+\gamma}(B_{|x|+|h|})}(|x| + |h|)^{2s+\gamma-1}|h| \\
&\leq (|x| + |h|)^{\frac{2s}{1-\gamma} - 1}|h| \\
&\leq C_h |x|^{\frac{2s}{1-\gamma} - 1},
\end{aligned} \tag{4.11}$$

where $C_h > 0$ is a constant, depending only on h and $\frac{2s}{1-\gamma}$. Here, η is a point on the line segment connecting x to $x + h$, and η' is a point on the line segment connecting the origin to h. Since $\gamma \in (0, 1/3)$, then

$$\gamma < \frac{1}{1+2s},$$

which guarantees that the right-hand side of (4.11) is in $L^1_s(\mathbb{R}^n)$. Furthermore, as u_k solves (1.1), a direct calculation reveals that

$$\begin{aligned}
&- (-\Delta)^s \left(v_k(x+h) - v_k(x) \right) \\
&= \frac{1}{\theta_k(r_k) r_k^{\frac{2s\gamma}{1-\gamma}}} \left[(u_k(x+h))_+^\gamma - (u_k(x))_+^\gamma - (u_k(x+h))_-^\gamma + (u_k(x))_-^\gamma \right].
\end{aligned} \tag{4.12}$$

We can then pass to the limit, as $k \to \infty$, in (4.12) (see [11, Lemma 5], and also [10, Lemma 4.5 and Corollary 4.7]) and arrive at

$$(-\Delta)^s (v_0(x+h) - v_0(x)) = 0 \quad \text{in} \quad \mathbb{R}^n.$$

Note that since 0 is a branching point for u_k, (4.6) implies that it is a branching point for v_0. This fact, combined with Theorem 3.1 applied to v_0, with

$$\nu = 2, \quad \beta := \frac{2s}{1-\gamma} - 2 < 1 \quad \text{and} \quad \alpha := 2s + \gamma - 2 < 1,$$

implies $v_0 \equiv 0$, which contradicts (4.10). □

Corollary 4.1. *Under the conditions of Theorem 4.1, there exists a constant $C > 0$, depending only on n and γ, such that for any $r < 1/2$ one has*

$$\sup_{B_r(x_0)} |u| \leq C \|u\|_{L^\infty(\mathbb{R}^n)} r^{\frac{2s}{1-\gamma}}.$$

Remark 4.1. *Note that if $s \in (1/2, 1-\gamma)$, with $\gamma \in (0, 1/3)$, and $u \in L^\infty(\mathbb{R}^n)$ is a viscosity solution of (1.1), then the conclusion of Theorem 4.1 still holds, provided $x_0 \in B_{1/2}$ is such that $u(x_0) = |Du(x_0)| = 0$. The proof is similar to that of Theorem 4.1; at the end, to get a contradiction, one needs to apply Theorem 3.1 to v_0 with*

$$\nu = 1, \quad \beta := \frac{2s}{1-\gamma} - 1 < 1 \quad \text{and} \quad \alpha := 2s + \gamma - 1 < 1.$$

5. Consequences and beyond

In this section, we prove three consequences of our main result. We start by observing that the approach to prove Theorem 4.1 introduces a new strategy for studying the regularity of solutions for two-phase problems in the *local* framework.

Theorem 5.1. *Let u be a viscosity solution of*

$$\Delta u = u_+^\gamma - u_-^\gamma \quad \text{in } B_1, \tag{5.1}$$

where $\gamma \in (0, 1/3)$. If $x_0 \in B_{1/2}$ is a branching point for u, then there exists a constant $C > 0$, depending only on n and γ, such that

$$|u(x)| \leq C \|u\|_{L^\infty(\mathbb{R}^n)} |x - x_0|^{\frac{2}{1-\gamma}}, \quad \forall x \in B_{1/2}(x_0).$$

As a consequence, $u \in C^{\frac{2}{1-\gamma}}$ at branching points.

Proof. Since

$$\lim_{s\nearrow 1^-}(-\Delta)^s u = -\Delta u$$

(see, for example, [20, Lemma 4.1]), using Theorem 4.1 and passing to the limit in (4.1) as $s \nearrow 1^-$, we conclude the desired result. □

Remark 5.1. *Since all points in* $\{x_n = 0\}$ *are branching points for*

$$u(x) := \begin{cases} x_n^{\frac{2}{1-\gamma}} & , \quad x_n \geq 0 \\ -(-x_n)^{\frac{2}{1-\gamma}} & , \quad x_n < 0, \end{cases}$$

the improved regularity result of Theorem 5.1 is optimal.

Proposition 5.1. *If* u *is a viscosity solution of*

$$\Delta u = u_+^\gamma \quad \text{in } B_1, \tag{5.2}$$

then all the points on $\partial\{u > 0\}$ *are branching points for* u.

Proof. Indeed, if $x_0 \in \partial\{u > 0\}$, then $u(x_0) = |Du(x_0)| = 0$, where the last equality follows from the fact that u takes its minimum at x_0. For the same reason, $D^2u(x_0)$ is non-negative definite, *i.e.*, all the eigenvalues of $D^2u(x_0)$ are non-negative. On the other hand, if $\lambda_i \geq 0$ are the eigenvalues of $D^2u(x_0)$, one has

$$0 = u_+^\gamma(x_0) = \Delta u(x_0) = \operatorname{trace}(D^2u(x_0)) = \sum_{i=1}^n \lambda_i,$$

therefore, λ_i=0, for all $i = 1, 2, \ldots, n$. Hence, $D^2u(x_0) = 0$. □

Remark 5.2. *Solutions of* (5.2) *are* $C^{\frac{2}{1-\gamma}}$ *at the free boundary* $\partial\{u > 0\}$ *(see* [19, Theorem 2]*). In view of Proposition 5.1, Theorem 5.1 generalizes this result to the two-phase case. Note, however, that while for* (5.2) *the range of* γ *is* $(0, 1)$*, for* (5.1) γ *ranges in* $(0, 1/3)$.

We now prove the second consequence of our main result.

Theorem 5.2. *Under the conditions of Theorem 4.1, there exists a constant* $C > 0$*, depending only on* n*,* γ *and* $\|u\|_{L^\infty(\mathbb{R}^n)}$*, such that*

$$\sup_{B_r(x_0)} |Du| \leq C r^{\frac{2s}{1-\gamma}-1}, \tag{5.3}$$

for $r < 1/2$*, where* $x_0 \in B_{1/2}$ *is a branching point.*

Proof. With no loss of generality, let $x_0 = 0$. We argue by contradiction and assume the conclusion fails. Then, for all $k \in \mathbb{N}$ and for a constant $C_* > 0$ to be chosen later,

$$\mu_{k+1} \geq C_* 2^{-(k+1)(\frac{2s}{1-\gamma}-1)}, \tag{5.4}$$

where

$$\mu_k := \sup_{B_{2^{-k}}} |Du|.$$

Observe that

$$\mu_{k+1} \geq 2^{-(\frac{2s}{1-\gamma}-1)}\mu_k, \tag{5.5}$$

since otherwise, iterating one would have

$$\mu_{k+1} \leq 2^{-(\frac{2s}{1-\gamma}-1)}\mu_k \leq 2^{-2(\frac{2s}{1-\gamma}-1)}\mu_{k-1} \leq \ldots \leq 2^{-k(\frac{2s}{1-\gamma}-1)}\mu_1.$$

The latter, recalling Theorem 2.3, implies

$$\mu_{k+1} \leq 2^{-k(\frac{2s}{1-\gamma}-1)}\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|u\|^\gamma_{L^\infty(B_1)}\right),$$

which leads to (5.3), as any $r \in (0,1/2)$ can be trapped in an interval $(2^{-(k+1)}, 2^{-k})$, for some $k \in \mathbb{N}$. As (5.3) is assumed to fail, then (5.5) must hold. On the other hand, Corollary 4.1 yields the existence of a constant $C > 0$, depending only on n, s and γ, such that

$$\sup_{B_1} |u(2^{-k}x)| = \sup_{B_{2^{-k}}} |u(x)| \leq C(2^{-k})^{\frac{2s}{1-\gamma}}. \tag{5.6}$$

Set

$$v_k(x) := \frac{u(2^{-k}x)}{2^{-k}\mu_{k+1}}, \quad x \in B_1.$$

Employing (5.6) and (5.4), we estimate

$$|v_k(x)| = \frac{|u(2^{-k}x)|}{2^{-k}\mu_{k+1}} \leq \frac{C}{\mu_{k+1}}(2^{-k})^{\frac{2s}{1-\gamma}-1} \leq \frac{C}{C_*}2^{\frac{2s}{1-\gamma}-1}.$$

In the last inequality, choosing

$$C_* \geq Ck2^{\frac{2s}{1-\gamma}-1}, \tag{5.7}$$

we obtain

$$|v_k(x)| \leq \frac{1}{k}, \quad x \in B_1. \tag{5.8}$$

Note that

$$\sup_{B_{1/2}} |Dv_k| = 1. \tag{5.9}$$

Furthermore, from (5.4), (5.7) and (5.8), we get

$$\begin{aligned} |(-\Delta)^s v_k(x)| &= \mu_{k+1}^{-1} 2^{(1+2s)k} |(-\Delta)^s u(2^{-k}x)| \\ &= \mu_{k+1}^{-1} 2^{(1+2s)k} u_+^\gamma(2^{-k}x) \\ &= \mu_{k+1}^{\gamma-1} 2^{(1+2s-\gamma)k} v_+^\gamma(x) \\ &\leq C_*^{\gamma-1} 2^{(1+2s-\gamma)} v_+^\gamma(x) \\ &\leq \frac{\tilde{C}}{k^\gamma}, \end{aligned} \tag{5.10}$$

for a universal constant $\tilde{C} > 0$, depending only on γ and s. Additionally, for $0 < \alpha < \sigma < 2s - 1$ and $2^{k-1} \geq R \geq 1$, recalling (5.5), one has

$$\begin{aligned}[v_k]_{C^{1+\alpha}(B_R)} &= \frac{2^{-k\alpha}}{\mu_{k+1}}[u]_{C^{1+\alpha}(B_{2^{-k}R})} \\ &\leq \frac{2^{-k\alpha}}{\mu_{k+1}}(2^{-k+1}R)^{\sigma-\alpha}[u]_{C^{1+\sigma}(B_{2^{-k}R})} \\ &\leq \frac{2^{k(\frac{2s}{1-\gamma}-1-\sigma)+\sigma-\alpha}}{\mu_1}R^{\sigma-\alpha}[u]_{C^{1+\sigma}(B_{2^{-k}R})} \\ &\leq CR^{\sigma-\alpha}, \end{aligned} \tag{5.11}$$

for a constant $C > 0$, depending only on s and γ. Here, the last inequality follows by choosing $\sigma > 0$ sufficiently close to $2s - 1$. Thus, up to a subsequence, v_k converges to some v_0 in $C^{1,\alpha}(B_R)$, as $k \to \infty$. Moreover, thanks to (5.10) and (5.11),

$$(-\Delta)^s v_0 = 0$$

and

$$[v_0]_{C^{1+\alpha}(B_R)} \leq CR^{\sigma-\alpha}.$$

Theorem 3.1 applies to v_0, providing $v_0 \equiv 0$, which is a contradiction, since from (5.9), one has

$$\sup_{B_{1/2}} |Dv_0| = 1.$$

$\square$

We finish this section by proving a Liouville-type result for solutions of (1.1) with a certain growth at infinity.

Theorem 5.3. *Let $u \in L^\infty(\mathbb{R}^n)$ be a viscosity solution of*

$$-(-\Delta)^s u = u_+^\gamma - u_-^\gamma \quad \text{in } \mathbb{R}^n,$$

with $\gamma \in (0, 1/3)$ and $s \in (1/2, 1)$. If $x_0 \in \partial\{u > 0\} \cap \{|Du| = 0\}$, and

$$u(x) = o\left(|x - x_0|^{\frac{2s}{1-\gamma}}\right), \quad \text{as} \quad |x| \to \infty \tag{5.12}$$

then $u \equiv 0$.

Proof. Without loss of generality, we may assume $x_0 = 0$. With $R > 1$, set

$$v_R(x) := \frac{u(Rx)}{R^{\frac{2s}{1-\gamma}}}.$$

Since $v_R(0) = 0$, and v_R is a viscosity solution of (1.1), Theorem 4.1 gives

$$|v_R(x)| \leq C\|v_R\|_{L^\infty(\mathbb{R}^n)}|x|^{\frac{2s}{1-\gamma}}. \tag{5.13}$$

Observe that if $|Rx|$ is bounded, then $u(Rx)$ is bounded. Therefore,

$$v_R \longrightarrow 0, \quad \text{as} \quad R \to \infty. \tag{5.14}$$

In fact, (5.14) remains true also when $|Rx| \to \infty$, as $R \to \infty$. Indeed, using (5.12), for any fixed $x \neq 0$, one gets

$$v_R(x) = \frac{u(Rx)}{|Rx|^{\frac{2s}{1-\gamma}}} |x|^{\frac{2s}{1-\gamma}} \longrightarrow 0, \quad \text{as } R \to \infty.$$

We aim to show that $u \equiv 0$. Suppose this is not the case, and there is a point $y \in \mathbb{R}^n$ such that $|u(y)| > 0$. By choosing $R > 0$ large enough so that $y \in B_R$, and using (5.13) and (5.14), we estimate

$$\frac{|u(y)|}{|y|^{\frac{2s}{1-\gamma}}} \leq \sup_{B_R} \frac{|u(x)|}{|x|^{\frac{2s}{1-\gamma}}} = \sup_{B_1} \frac{|v_R(x)|}{|x|^{\frac{2s}{1-\gamma}}} < \frac{|u(y)|}{2|y|^{\frac{2s}{1-\gamma}}},$$

reaching a contradiction. □

6. Fully nonlinear case

As observed earlier, our main result can be generalized to the fully non-linear setting. More precisely, let

$$F(D^{2s}u) = u_+^\gamma - u_-^\gamma \quad \text{in } B_1, \tag{6.1}$$

where $D^{2s}u(x)$ is the matrix with $(i,j)-$entry

$$\int_{\mathbb{R}^n} \delta u(x,y) \frac{\langle e_i, y\rangle \langle e_j, y\rangle}{|y|^{n+2s+2}}\, dy,$$

where $\delta u(x,y) := u(x+y) + u(x-y) - 2u(x)$ is the symmetric difference, and $\{e_i\}$ is the standard orthonormal basis of $\mathbb{R}^n$. In (6.1), the operator $F : \mathrm{Sym}(n) \to \mathbb{R}$ is assumed to be a uniformly elliptic operator that vanishes at the origin, *i.e.*, $F(0) = 0$, and

$$\lambda \|N\| \leq F(M+N) - F(M) \leq \Lambda \|N\|, \tag{6.2}$$

for some constants $0 < \lambda \leq \Lambda$, and for any $M, N \in \mathrm{Sym}(n)$ with $N \geq 0$. Furthermore, we suppose that $F(M)$ is differentiable with respect to M, and

$$\|DF(M) - DF(N)\| \leq \omega(\|M - N\|), \tag{6.3}$$

for a modulus of continuity ω, and for all $M, N \in \mathrm{Sym}(n)$. The following result is from [23, Proposition 2.2] (see also [5, 17]). It unlocks the proof of the corresponding improved regularity result for (6.1).

Proposition 6.1. *If $F_k : \mathrm{Sym}(n) \to \mathbb{R}$ is a sequence of operators vanishing at the origin and satisfying (6.2)-(6.3) with the same ellipticity constants, then, up to a subsequence,*

$$\rho^{-1} F_k(\delta M) \longrightarrow DF_0(0)M,$$

locally uniformly, as $\rho \to 0$, where F_0 is an operator satisfying the same conditions.

The following result generalizes Theorem 4.1, and its proof is based on similar arguments. We sketch it here for the reader's convenience.

Theorem 6.1. *Let F satisfy (6.2)-(6.3) and $F(0) = 0$. If $u \in L^\infty(\mathbb{R}^n)$ is a viscosity solution of (6.1), for $\gamma \in (0, 1/3)$ and $s > 1 - \frac{\gamma}{2}$, and $x_0 \in B_{1/2}$ is a branching point of u, then there exists a constant $C > 0$, depending only on n, γ, λ and Λ, such that*

$$|u(x)| \leq C\|u\|_{L^\infty(\mathbb{R}^n)}|x - x_0|^{\frac{2s}{1-\gamma}}, \quad \forall x \in B_{1/2}(x_0).$$

Proof. Without loss of generality, we assume $x_0 = 0$. We make use of Proposition 6.1 and argue as in the proof of Theorem 4.1. More precisely, if the conclusion fails, then there exist sequences u_k, x_k, and F_k such that 0 is a branching point of u_k,

$$F_k(D^{2s}u_k) = (u_k)_+^\gamma - (u_k)_-^\gamma,$$

$$[u_k]_{C^{2s+\gamma}(B_{1/2})} \leq 2C,$$

but

$$|u_k(x_k)| > k|x_k|^{\frac{2s}{1-\gamma}}.$$

Let now θ_k be as in (4.4). As observed in the proof of Theorem 4.1, there exists $r_k \to 0$, such that (4.5) holds. A direct calculation shows that

$$\begin{aligned}
&\tilde{\delta}_k^{-1}F_k(\tilde{\delta}_k D^{2s}(v_k(x+h) - v_k(x)) + D^{2s}u_k(r_k x)) \\
&= \frac{1}{\theta_k^{1-\gamma}(r_k)}\left[(v_k(x+h))_+^\gamma - (v_k(x+h))_-^\gamma\right],
\end{aligned} \tag{6.4}$$

where v_k is defined by (4.6), and $\tilde{\delta}_k := \theta_k(r_k)r_k^{\frac{\gamma}{1-\gamma}}$. Furthermore, as noted in the proof of Theorem 4.1, up to a subsequence, v_k converges, as $k \to \infty$, to some v_0 in $C^{2s+\gamma}_{\text{loc}}(B_R)$, as $k \to \infty$, for any $R \geq 1$, which satisfies (4.10). Since

$$\theta_k(r_k)r_k^{\frac{\gamma}{1-\gamma}} \longrightarrow 0,$$

as $k \to \infty$, using Proposition 6.1, and passing to the limit in (6.4), we get

$$DF_0(0)D^{2s}v_0 = 0. \tag{6.5}$$

On the other hand, since $DF_0(0)D^{2s}$ is a constant coefficient linear operator, then up to an affine change of variables, from (6.5), we conclude

$$(-\Delta)^s v_0 = 0 \quad \text{in } \mathbb{R}^n.$$

Therefore, as in the proof of Theorem 4.1, Theorem 3.1 implies $v_0 \equiv 0$, which contradicts (4.10). $\square$

Remark 6.1. *By passing to the limit as $s \nearrow 1^-$, we extend the regularity result of* [19] *to the two-phase setting, also covering the case of fully nonlinear elliptic operators.*

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References

[1] H.W. Alt and D. Phillips, *A free boundary problem for semilinear elliptic equations*, J. Reine Angew. Math. 368 (1986), 63–107.
[2] D.J. Araújo, R. Leitão and E.V. Teixeira, *Infinity Laplacian equation with strong absorptions*, J. Funct. Anal. 270 (2016), 2249–2267.
[3] D.J. Araújo and R. Teymurazyan, *Fully nonlinear dead-core systems*, J. Funct. Anal. 287 (2024), Paper No. 110586.
[4] D.J. Araújo, R. Teymurazyan and J.M. Urbano, *Geometric properties of free boundaries in degenerate quenching problems*, submitted.
[5] S.N. Armstrong, L. Silvestre and C.K. Smart, *Partial regularity of solutions of fully nonlinear, uniformly elliptic equations*, Comm. Pure Appl. Math. 65 (2012), 1169–1184.
[6] G. Barles, E. Chasseigne and C. Imbert, *On the Dirichlet problem for second-order elliptic integro-differential equations*, Indiana Univ. Math. J. 57 (2008), 213–246.
[7] G. Barles and C. Imbert, *Second-order elliptic integro-differential equations: viscosity solutions' theory revisited*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 25 (2008), 567–585.
[8] L. Caffarelli, R. Leitão and J.M. Urbano, *Regularity for anisotropic fully nonlinear integro-differential equations*, Math. Ann. 360 (2014), 681–714.
[9] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32 (2007), 1245–1260.
[10] L. Caffarelli and L. Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. 62 (2009), 597–638.
[11] L. Caffarelli and L. Silvestre, *Regularity results for nonlocal equations by approximation*, Arch. Ration. Mech. Anal. 200 (2011), 59–88.
[12] L. Caffarelli, R. Teymurazyan and J.M. Urbano, *Fully nonlinear integro-differential equations with deforming kernels*, Comm. Partial Differential Equations 45 (2020), 847–871.
[13] M.G. Crandall, H. Ishii and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) 27 (1992), 1–67.
[14] N.M.L. Diehl and R. Teymurazyan, *Reaction-diffusion equations for the infinity Laplacian*, Nonlinear Anal. 199 (2020), 111956, 12pp.
[15] D. Phillips, *A minimization problem and the regularity of solutions in the presence of a free boundary*, Indiana Univ. Math. J. 32 (1983), 1–17.
[16] D. Phillips, *Hausdorff measure estimates of a free boundary for a minimum problem*, Comm. Partial Differential Equations 8 (1983), 1409–1454.

[17] D. dos Prazeres and E.V. Teixeira, *Asymptotics and regularity of flat solutions to fully nonlinear elliptic problems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 15 (2016), 485–500.

[18] J. Serra, $C^{\sigma+\alpha}$ *regularity for concave nonlocal fully nonlinear elliptic equations with rough kernels*, Calc. Var. Partial Differential Equations 54 (2015), 3571–3601.

[19] E.V. Teixeira, *Regularity for the fully nonlinear dead-core problem*, Math. Ann. 364 (2016), 1121–1134.

[20] R. Teymurazyan, *The fractional Laplacian: a primer*, Coimbra Mathematical Texts, Springer Cham, to appear.

[21] Y. Wu and H. Yu, *On the fully nonlinear Alt-Phillips equation*, Int. Math. Res. Not. IMRN (2022), 8540–8570.

[22] R. Yang, *Optimal regularity and nondegeneracy of a free boundary problem related to the fractional Laplacian*, Arch. Ration. Mech. Anal. 208 (2013), 693–723.

[23] H. Yu, *Small perturbation solutions for nonlocal elliptic equations*, Comm. Partial Differential Equations 42 (2017), 142–158.

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