ON THE FULLY NONLINEAR QUENCHING PROBLEM

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ABSTRACT. We study the fully nonlinear quenching problem and establish sharp $C^{1,\alpha}_{loc}-$estimates and optimal growth at the free boundary in two distinct scenarios: the uniformly parabolic case and the degenerate elliptic case with oscillatory singularities. For the former, we refine, in particular, the recent asymptotic results in [5].

1. INTRODUCTION

In this paper, we study local regularity properties of viscosity solutions to free boundary problems with singular absorption terms. The first model we consider is

$$\begin{cases} F(D^2u) - \partial_t u = \gamma u^{\gamma-1} & \text{in } \Omega_T \cap \{u>0\}, \\ u = \varphi & \text{on } \partial_p \Omega_T, \end{cases} \tag{1.1}$$

governed by a fully nonlinear uniformly parabolic operator F. The second model is

$$\begin{cases} |Du|^{\kappa(x)} F(D^2u) = \gamma(x) u^{\gamma(x)-1} & \text{in } \Omega \cap \{u>0\}, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

governed by a degenerate elliptic operator. Here, $\gamma, \gamma(x) \in (0,1)$ correspond to the singular absorption terms, while $\kappa(x) \geq 0$ in the second model represents the degeneracy associated with the gradient of the solution.

In the elliptic case and for $\kappa(x) = 0$, PDEs of this form arise as the Euler–Lagrange equations of the functional

$$\int \frac{1}{2}|Du|^2 + u^\gamma \, dx.$$

The case $\gamma = 0$ and $\gamma = 1$ correspond to the cavitation problem and the obstacle problem, respectively. The intermediate case $\gamma \in (0,1)$ is the quenching problem we will address.

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The cavitation problem, also known as the Alt–Caffarelli problem, was treated in the variational setting by Alt and Caffarelli in [1]. Later, Ricarte and Teixeira in [17] studied the fully nonlinear case. The obstacle problem was studied by Caffarelli in [10] and later extended to the fully nonlinear setting by Lee and Shahgholian in [14]. In the case of the quenching problem, also known as the Alt–Phillips problem, Alt and Phillips studied it in [2], and Araújo and Teixeira extended it to the fully nonlinear uniformly elliptic case in [7].

The quenching problem refers to a phenomenon for which a process or reaction abruptly stops or vanishes, often encountered in combustion theory, heat transfer, and chemical reaction models. The quenching problem has been extensively studied over the years, and numerous results are available in the literature. For the variational setting involving the Laplacian operator, we refer readers to [2, 16, 15, 20]. In the nonvariational setting with fully nonlinear uniformly elliptic operators, the authors in [7] obtained optimal regularity along the free boundary by investigating the fine oscillation decay of limiting solutions. For degenerate elliptic operators, Teixeira in [18] established optimal regularity of solutions along the free boundary. In the uniformly parabolic case, sharp regularity result was obtained in [5] by constructing proper barrier functions. We refer the reader to [4] and [6] for further related extensions.

In this paper, we first establish the existence of nonnegative viscosity solutions to (1.1) and (1.2) obtained as uniform limits of positive solutions to penalized problems. Subsequently, we derive sharp local regularity results by analyzing the regularity properties of the positive solutions to these penalized problems. For the uniformly parabolic case (1.1), we improve upon the result in [5], which provides regularity for exponents strictly less than the optimal value. By applying Jensen–Ishii's lemma twice, we obtain more refined estimates, ultimately enabling us to achieve the optimal exponent. In the degenerate elliptic case (1.2), we establish the result for the variable exponent case corresponding to oscillatory singularities under appropriate assumptions on $\kappa(x)$ and $\gamma(x)$. The proof relies upon the use of both Jensen–Ishii's lemma and Hopf's lemma.

The paper is organized as follows. In Section 2, we introduce notation, the basic assumptions and known results that will be used throughout. In Section 3, we establish the sharp local regularity and the optimal growth at the free boundary for uniformly parabolic operators by repeatedly applying Jensen–Ishii's lemma. In Section 4, we treat the case of a degenerate elliptic operator with oscillatory singularities.

2. Preliminaries

2.1. Notation and definitions. Let $\mathcal{S}^n$ denote the space of real $n\times n$ symmetric matrices. For parameters $0<\lambda\le\Lambda$, the Pucci extremal operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}:\mathcal{S}^n\to\mathbb{R}$ are defined as

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M)=\Lambda\sum_{e_i>0}e_i+\lambda\sum_{e_i<0}e_i \quad \text{and} \quad \mathcal{M}^{-}_{\lambda,\Lambda}(M)=\lambda\sum_{e_i>0}e_i+\Lambda\sum_{e_i<0}e_i,$$

where $e_i=e_i(M)$ are the eigenvalues of $M\in\mathcal{S}^n$. We denote with $\mathcal{A}_{\lambda,\Lambda}$ the set of symmetric matrices M such that $\lambda I\le M\ \le\Lambda I$. Note that

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M)=\sup_{A\in\mathcal{A}_{\lambda,\Lambda}}\operatorname{tr}(AM) \qquad \text{and} \qquad \mathcal{M}^{-}_{\lambda,\Lambda}(M)=\inf_{A\in\mathcal{A}_{\lambda,\Lambda}}\operatorname{tr}(AM).$$

For a bounded open domain $\Omega\subset\mathbb{R}^n$, with a smooth boundary, and $T>0$, let $\Omega_T=\Omega\times(-T,0]$. Denote by $\partial_p\Omega_T$ the parabolic boundary of Ω_T. For $(x_0,t_0)\in\Omega_T$ and $r>0$, we define the intrinsic parabolic cylinder

$$Q_r(x_0,t_0)=B_r(x_0)\times(t_0-r^2,t_0],$$

where $B_r(x_0)$ denotes an open ball in $\mathbb{R}^n$ centered at x_0 with radius r. For convenience, we donote $B_r=B_r(0)$ and $Q_r=Q_r(0,0)$.

Following [13], we introduce the definition of viscosity solution for the equation

$$F(D^2u)-\partial_t u=g(u,x,t)\quad\text{in }\Omega_T, \tag{2.1}$$

where $g\in C(\mathbb{R}\times\Omega_T)$. A similar definition applies to the elliptic case. We denote by $USC(\Omega_T)$, respectively $LSC(\Omega_T)$, the set of upper, respectively lower, semicontinuous functions on Ω_T.

Definition 2.1. *A function $u\in USC(\Omega_T)$ (resp., $u\in LSC(\Omega_T)$) is a viscosity subsolution (resp., supersolution) of (2.1) if, for every $(x_0,t_0)\in\Omega_T$ and $\phi\in C^{2,1}(\Omega_T)$ such that $u-\phi$ has a local maximum (resp. minimum) at (x_0,t_0), we have*

$$F(D^2\phi(x_0,t_0))-\partial_t\phi(x_0,t_0)\ge\ (\textit{resp.},\ \le)\ g(u(x_0,t_0),x_0,t_0).$$

We say that $u\in C(\Omega_T)$ is a viscosity solution if u is both a viscosity supersolution and a subsolution.

We next recall the concept of parabolic superjet/subjet introduced in [12, Section 8].

Definition 2.2. *Let $v : \Omega_T \to \mathbb{R}$ be an upper semicontinuous function. For every $(x,t) \in \Omega_T$, the parabolic superjet of v at (x,t) is the set*

$$\begin{aligned}\mathcal{P}^+(v)(x,t) =& \big\{(a,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mid \\ & v(y,s) \leqslant v(x,t) + a(s-t) + \langle p, y-x\rangle \\ & + \frac{1}{2}\langle X(y-x), y-x\rangle \\ & + o(|s-t| + |y-x|^2) \text{ as } (y,s) \to (x,t)\big\}.\end{aligned}$$

The corresponding limiting superjet of v at (x,t) is

$$\begin{aligned}\overline{\mathcal{P}}^+(v)(x,t) =& \big\{(a,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mid \\ & \exists (x_m, t_m, a_m, p_m, X_m) \text{ such that} \\ & (a_m, p_m, X_m) \in \mathcal{P}^+(v)(x_m, t_m), \text{ and} \\ & (x_m, t_m, v(x_m,t_m), a_m, p_m, X_m) \to (x,t,v(x,t),a,p,X) \\ & \text{as } m \to \infty\big\}.\end{aligned}$$

Subjets $\mathcal{P}^-$ and limiting subjets $\overline{\mathcal{P}}^-$ are defined analogously for lower semicontinuous functions, replacing $\leq$ with $\geq$ for the former and $\mathcal{P}^+$ with $\mathcal{P}^-$ for the latter. In the elliptic case, superjets and subjets are defined similarly, as described in [12, Section 2].

2.2. **Known results.** We recall here Jensen–Ishii's lemma (*cf.* [12]). We state it for the parabolic case, but a similar result also holds in the elliptic case.

Lemma 2.1 (Jensen–Ishii's lemma)**.** *Let $v \in C(Q_1)$ and suppose that*

$$\Phi(x,y,t) = v(x,t) - v(y,t) - L\phi(|x-y|) - K(|x|^2 + |y|^2 + (-t)^2)$$

has a local maximum at $(\overline{x}, \overline{y}, \overline{t}) \in Q_1$ with $\overline{x} \neq \overline{y}$ for $L, K > 0$. Then, for every sufficiently small $\iota > 0$, there exists $\tau \in \mathbb{R}, p \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}^n$ such that

$$(\tau + 2K\overline{t}, p + 2K\overline{x}, X) \in \overline{\mathcal{P}}^+(v)(\overline{x}, \overline{t}),$$
$$(\tau, p - 2K\overline{y}, Y) \in \overline{\mathcal{P}}^-(v)(\overline{y}, \overline{t}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq L \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + (2K + \iota)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{2.2}$$

where

$$|p| = L\phi'(|\overline{x} - \overline{y}|)$$

and

$$Z = \phi''(|\overline{x}-\overline{y}|)\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}\otimes\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}+\frac{\phi'(|\overline{x}-\overline{y}|)}{|\overline{x}-\overline{y}|}\left(I-\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}\otimes\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}\right).$$

Remark 2.1. *Note that applying the matrix inequality (2.2) to the vector (ξ,ξ), for $\xi\in\mathbb{R}^n$ arbitrary, implies that every eigenvalue of $Y-X$ is greater than or equal to $-(4K+2\iota)$. Similarly, applying (2.2) to the vector $\left(\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|},-\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}\right)$ shows that at least one eigenvalue of $Y-X$ is greater than or equal to $-4L\phi''(|\overline{x}-\overline{y}|)-(4K+2\iota)$.*

3. Optimal regularity in the uniformly parabolic case

In this section, we examine the fully nonlinear parabolic problem

$$\begin{cases} F(D^2u)-\partial_t u=\gamma u^{\gamma-1} & \text{in } \Omega_T\cap\{u>0\},\\ \qquad\qquad u=\varphi & \text{on } \partial_p\Omega_T,\end{cases}\tag{3.1}$$

and establish the existence and optimal regularity of a solution under the following assumptions.

(A1): F is (λ,Λ)-uniformly elliptic, *i.e.*,

$$\mathcal{M}^-_{\lambda,\Lambda}(M-N)\le F(M)-F(N)\le\mathcal{M}^+_{\lambda,\Lambda}(M-N),$$

for every $M,N\in\mathcal{S}^n$.

(A2): F is 1-homogeneous, *i.e.*,

$$F(tM)=tF(M),$$

for every $t\ge 0$ and $M\in\mathcal{S}^n$.

The first main result of this paper is the following.

Theorem 3.1. *Let $\gamma\in(0,1)$, $\varphi\in C(\partial_p\Omega_T)$, with $\varphi\ge 0$, and assume (A1) and (A2). There exists a nonnegative bounded viscosity solution u to (3.1) in the sense of Definition 2.1, and u is locally of class $C^{1,\alpha}$, for every*

$$\alpha\in\left(0,\frac{\gamma}{2-\gamma}\right]\cap(0,\alpha_F),$$

with the estimate

$$\sup_{(y,s)\in Q_r(x,t)}|u(y,s)-u(x,t)-Du(x,t)\cdot(y-x)|\le Cr^{1+\alpha},$$

for $Q_r(x,t)\Subset\Omega_T$, where $C=C(n,\lambda,\Lambda,\gamma,\alpha,\|u\|_{L^\infty})$. Moreover, for each free boundary point (x,t), u is of class $C^{1,\frac{\gamma}{2-\gamma}}$ at (x,t), with the estimate

$$\sup_{Q_r(x,t)}u\le Cr^{1+\frac{\gamma}{2-\gamma}},$$

for $Q_r(x,t) \Subset \Omega_T$, where $C = C(n, \lambda, \Lambda, \gamma, \|u\|_{L^\infty})$.

Remark 3.1. *The constant α_F in the statement of the theorem denotes the optimal exponent associated with the $C^{1+\mu,\frac{1+\mu}{2}}$–regularity theory for solutions of F-caloric functions, i.e., solutions of the equation $F(D^2h) - \partial_t h = 0$ (see [19]).*

We construct our solution to (3.1) as the limit of solutions to singularly penalized approximating problems. Let $\rho \in C^\infty(\mathbb{R})$ be a nonnegative smooth function with compact support in $[0,\, 1]$, such that $\int \rho = 1$. For each $\epsilon \in (0,1)$, define the real function

$$\beta_\epsilon(s) = \gamma \int_0^{\sigma(s)} \rho(\theta)\, d\theta,$$

where $\sigma(s) := s\epsilon^{\frac{2}{\gamma-2}} - \sigma_0$, for $\sigma_0 \in (0,1)$, which converges to $\gamma\chi_{\{s>0\}}$ as $\epsilon \to 0$. Here, χ_E denotes the characteristic function of a set E. We consider the penalized problem

$$\begin{cases} F(D^2u_\epsilon) - \partial_t u_\epsilon = \beta_\epsilon(u_\epsilon)u_\epsilon^{\gamma-1} & \text{in } \Omega_T, \\ \qquad\qquad\quad u_\epsilon = \varphi_\epsilon & \text{on } \partial_p\Omega_T, \end{cases} \tag{3.2}$$

where $\epsilon \in (0,1)$ and $\varphi_\epsilon = \varphi + \epsilon^{\frac{2}{2-\gamma}}$. The following result concerns the existence of a positive solution to (3.2) and is taken from [5, Section 3].

Proposition 3.1. *There exists a viscosity solution u_ϵ to (3.2). Moreover, u_ϵ satisfies*

$$0 < u_\epsilon \leq \|\varphi\|_{L^\infty(\partial_p\Omega_T)} + 1 \quad \text{in } \Omega_T.$$

For ease of notation, hereafter in this section, we will denote u_ϵ by u. Following the approach of [7], we will examine the regularity of the auxiliary function

$$v := u^{\frac{2-\gamma}{2}}.$$

By direct calculation, we have

$$Dv = \frac{2-\gamma}{2} u^{-\frac{\gamma}{2}} Du \tag{3.3}$$

and

$$D^2v = \frac{2-\gamma}{2} u^{-\frac{\gamma}{2}} D^2u - \frac{2-\gamma}{2}\frac{\gamma}{2} u^{-\frac{\gamma}{2}-1} Du \otimes Du. \tag{3.4}$$

Using (3.3), (3.4) and (A2), we rewrite the equation in (3.2) as (see [5])

$$F(D^2v + \delta v^{-1} Dv \otimes Dv) - \partial_t v = f(x,t)v^{-1} \quad \text{in } \Omega_T, \tag{3.5}$$

where

$$\delta = \frac{\gamma}{2-\gamma} \qquad \text{and} \qquad f(x,t) = \frac{2-\gamma}{2}\beta_\epsilon\left(v(x,t)^{\frac{2}{2-\gamma}}\right) \in [0,1).$$

The following Hölder regularity of v in the space variables has been obtained in [5, Thm. 1].

Proposition 3.2. *Let $v \in C(Q_1)$ be a positive viscosity solution to (3.5) in Q_1. Then, for each $\mu \in (0,1)$, there exists $C > 0$, depending only on $n, \lambda, \Lambda, \gamma, \mu$ and $\|v\|_{L^\infty(Q_1)}$, such that*

$$|v(x,t) - v(y,t)| \leq C|x-y|^\mu,$$

for every $(x,t), (y,t) \in Q_{\frac{1}{2}}$.

Now, we build upon Proposition 3.2 to obtain the Lipschitz regularity of v in the space variables, thus unlocking our optimal regularity result.

Theorem 3.2. *Let $v \in C(Q_1)$ be a positive viscosity solution to (3.5) in Q_1. Then, there exists $C > 0$, depending only on $n, \lambda, \Lambda, \gamma$ and $\|v\|_{L^\infty(Q_1)}$, such that*

$$|v(x,t) - v(y,t)| \leq C|x-y|, \tag{3.6}$$

for every $(x,t), (y,t) \in Q_{\frac{1}{2}}$.

Proof. Define in Q_1

$$\Phi(x,y,t) = v(x,t) - v(y,t) - Lw(|x-y|) - K(|x|^2 + |y|^2 + (-t)^2),$$

where, for a parameter $a \in (1,2)$ to be determined later,

$$w(t) = \begin{cases} t - \frac{1}{a}t^a & \text{if } 0 \leq t < 1, \\ 1 - \frac{1}{a} & \text{if } t \geq 1. \end{cases}$$

Then, for $0 < t < 1$, we have $w'(t) = 1 - t^{a-1}$ and $w''(t) = -(a-1)t^{a-2}$. Note also that

$$w(t) \geq \frac{t}{2}, \tag{3.7}$$

for sufficiently small t. We will prove that

$$\max_{\overline{B_{\frac{1}{2}}} \times \overline{B_{\frac{1}{2}}} \times [-\frac{1}{4}, 0]} \Phi \leq 0, \tag{3.8}$$

for sufficiently large L and K. Then (3.6) follows from the standard translation argument.

To prove (3.8), assume, by contradiction, that Φ attains its positive maximum at $(\overline{x}, \overline{y}, \overline{t}) \in \overline{B_{\frac{1}{2}}} \times \overline{B_{\frac{1}{2}}} \times [-1/4, 0]$. Then, we have $\overline{x} \neq \overline{y}$,

$$v(\overline{x}, \overline{t}) - v(\overline{y}, \overline{t}) \geq Lw(|\overline{x} - \overline{y}|) > 0, \tag{3.9}$$

and

$$v(\overline{x},\overline{t})-v(\overline{y},\overline{t})>K\left(|\overline{x}|^2+|\overline{y}|^2+(-\overline{t})^2\right).$$

By Proposition 3.2, we have

$$|v(\overline{x},\overline{t})-v(\overline{y},\overline{t})|\leq C|\overline{x}-\overline{y}|^{\mu},$$

where $\mu\in(0,1)$ is to be determined later. From

$$\frac{1}{3}(|\overline{x}|+|\overline{y}|+|\overline{t}|)^2\leq|\overline{x}|^2+|\overline{y}|^2+(-\overline{t})^2\leq\frac{v(\overline{x},\overline{t})-v(\overline{y},\overline{t})}{K}\leq\frac{C|\overline{x}-\overline{y}|^{\mu}}{K},$$

we get

$$|\overline{x}|+|\overline{y}|+|\overline{t}|\leq\left(\frac{3C}{K}\right)^{\frac{1}{2}}|\overline{x}-\overline{y}|^{\frac{\mu}{2}}. \tag{3.10}$$

From (3.10), by choosing K sufficiently large, we ensure that

$$(\overline{x},\overline{y},\overline{t})\in B_{\frac{1}{10}}\times B_{\frac{1}{10}}\times B_{\frac{1}{100}}.$$

Now, we can obtain $\tau\in\mathbb{R},p\in\mathbb{R}^n$ and $X,Y\in\mathcal{S}^n$, such that

$$(\tau+2K\overline{t},p_x,X)\in\overline{\mathcal{P}}^{+}(v)(\overline{x},\overline{t}), \tag{3.11}$$

$$(\tau,p_y,Y)\in\overline{\mathcal{P}}^{-}(v)(\overline{y},\overline{t}), \tag{3.12}$$

where $p_x=p+2K\overline{x}$ and $p_y=p-2K\overline{y}$, with the estimate given by Lemma 2.1. We can choose L sufficiently large so that

$$1\leq\frac{1}{2}|p|\leq|p_x|,|p_y|\leq\frac{3}{2}|p|. \tag{3.13}$$

Note that, by (3.10),

$$|p_y-p_x|=2K|\overline{x}+\overline{y}|\leq2(3CK)^{\frac{1}{2}}|\overline{x}-\overline{y}|^{\frac{\mu}{2}}. \tag{3.14}$$

By applying (3.11) and (3.12) to equation (3.5), we obtain the inequalities

$$F(X+\delta v(\overline{x},\overline{t})^{-1}p_x\otimes p_x)-(\tau+2K\overline{t})\geq f(\overline{x},\overline{t})v(\overline{x},\overline{t})^{-1},$$
$$F(Y+\delta v(\overline{y},\overline{t})^{-1}p_y\otimes p_y)-\tau\leq f(\overline{y},\overline{t})v(\overline{y},\overline{t})^{-1}.$$

Then, we get

$$\begin{aligned}F\big(Y+\delta v(\overline{y},\overline{t})^{-1}p_y\otimes p_y\big)-F\big(X+\delta v(\overline{x},\overline{t})^{-1}p_x\otimes p_x\big)\\ \leqslant K+f(\overline{y},\overline{t})v(\overline{y},\overline{t})^{-1}-f(\overline{x},\overline{t})v(\overline{x},\overline{t})^{-1}.\end{aligned} \tag{3.15}$$

On the other hand, for every $\eta>0$, there exists $M_\eta\in\mathcal{A}_{\lambda,\Lambda}$ such that

$$\begin{aligned}F\big(Y+\delta v(\overline{y},\overline{t})^{-1}p_y\otimes p_y\big)-F\big(X+\delta v(\overline{x},\overline{t})^{-1}p_x\otimes p_x\big)\\ \geqslant \operatorname{tr}\Big(M_\eta\Big(Y+\delta v(\overline{y},\overline{t})^{-1}p_y\otimes p_y\\ -X-\delta v(\overline{x},\overline{t})^{-1}p_x\otimes p_x\Big)\Big)-\eta.\end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$K+\eta \geqslant \operatorname{tr}\big(M_\eta(Y-X)\big) + v(\overline{y},\overline{t})^{-1}\big(\delta \operatorname{tr}(M_\eta p_y\otimes p_y) - f(\overline{y},\overline{t})\big) - v(\overline{x},\overline{t})^{-1}\big(\delta \operatorname{tr}(M_\eta p_x\otimes p_x) - f(\overline{x},\overline{t})\big). \tag{3.17}$$

From Lemma 2.1,

$$\begin{aligned}\operatorname{tr}(M_\eta(Y-X)) &\geq \lambda(-4Lw''(|\overline{x}-\overline{y}|)-(4K+2\iota))-\Lambda(n-1)(4K+2\iota)\\ &\geq -4\lambda Lw''(|\overline{x}-\overline{y}|)-(\lambda+\Lambda(n-1))(4K+2\iota)\\ &\geq -3\lambda Lw''(|\overline{x}-\overline{y}|),\end{aligned} \tag{3.18}$$

for sufficiently large L. From (3.13), we know

$$\delta \operatorname{tr}(M_\eta p_y\otimes p_y)-f(\overline{y},\overline{t}) \geq \delta\lambda|p_y|^2-\|f\|_{L^\infty} \geq \frac{\delta\lambda}{4}|p|^2-1>0, \tag{3.19}$$

for sufficiently large L. Using (3.9), (3.18) and (3.19), from (3.17), we obtain

$$\begin{aligned}K+\eta &\geqslant \operatorname{tr}\big(M_\eta(Y-X)\big)\\ &\quad + v(\overline{x},\overline{t})^{-1}\big(\delta \operatorname{tr}(M_\eta p_y\otimes p_y)-f(\overline{y},\overline{t})\big)\\ &\quad - v(\overline{x},\overline{t})^{-1}\big(\delta \operatorname{tr}(M_\eta p_x\otimes p_x)-f(\overline{x},\overline{t})\big)\\ &\geqslant -3\lambda Lw''(|\overline{x}-\overline{y}|)\\ &\quad - v(\overline{x},\overline{t})^{-1}\big(2\delta n\Lambda|p_y||p_y-p_x|+\delta\Lambda|p_y-p_x|^2\\ &\quad - f(\overline{x},\overline{t})+f(\overline{y},\overline{t})\big).\end{aligned} \tag{3.20}$$

Note that

$$f(\overline{x},\overline{t})-f(\overline{y},\overline{t}) = \frac{2-\gamma}{2}(\beta_\epsilon(v(\overline{x},\overline{t})^{\frac{2}{2-\gamma}})-\beta_\epsilon(v(\overline{y},\overline{t})^{\frac{2}{2-\gamma}}))\geq 0, \tag{3.21}$$

which follows from $v(\overline{x},\overline{t})>v(\overline{y},\overline{t})$, and β_ϵ being a nondecreasing function. Denote $\Delta=|\overline{x}-\overline{y}|$. Using (3.7), (3.9), (3.13), (3.14) and (3.21), from (3.20), we have

$$\begin{aligned}K+\eta &\geq -3\lambda Lw''(\Delta)-L^{-1}w(\Delta)^{-1}\Big(6(3C)^{\frac12}\delta n\Lambda K^{\frac12}\Delta^{\frac{\mu}{2}}Lw'(\Delta)\\ &\quad +12\delta CK\Lambda\Delta^{\mu}\Big)\\ &\geq L\Delta^{a-2}\Big(3(a-1)\lambda-12(3C)^{\frac12}\delta n\Lambda K^{\frac12}L^{-1}\Delta^{1-a+\frac{\mu}{2}}\\ &\quad -24\delta CK\Lambda L^{-2}\Delta^{1-a+\mu}\Big)\\ &\geq \frac{3(a-1)\lambda}{2}L,\end{aligned}$$

for sufficiently large L, provided

$$1-a+\frac{\mu}{2}\geq 0 \qquad \text{and} \qquad 1-a+\mu\geq 0.$$

Note that we used $\Delta \leq 1$. Now, choose $a = \dfrac{5}{4}$ and $\mu = \dfrac{3}{4}$, and take the limit as $\eta \to 0$ and L sufficiently large, to obtain a contradiction. □

The next result is an improvement of [5, Thm. 2].

Theorem 3.3. *Let $v \in C(Q_1)$ be a positive viscosity solution to (3.5) in Q_1. Then, there exist r_0, $C > 0$, depending only on $n, \lambda, \Lambda, \gamma$ and $\|v\|_{L^\infty(Q_1)}$, such that*

$$|v(x,t) - v(x,s)| \leq C|t-s|^{\frac{1}{2}},$$

for every $x \in B_{\frac{1}{2}}$ and $t, s \in (-r_0, 0]$.

Proof. Upon construction of a proper barrier function and application of the comparison principle, the conclusion follows from Theorem 3.2 and [5, Lemma 2]. □

Proof of Theorem 3.1. Once Theorem 3.2 and Theorem 3.3 are proven, the remaining parts of the proof are similar to those in [5], where v was shown to exhibit $C^{1-,1/2-}$–regularity. Here, we have established $C^{1,1/2}$–regularity, an improvement enabling us to achieve the optimal regularity. Since only minor modifications are required, we omit further details.

Remark 3.2. *If we consider a variable coefficient operator $F = F(M, x, t)$, we can obtain the same result under the assumption that, for some $\mu > 0$, there exists a μ–Hölder modulus of continuity $\tilde{\omega}$ such that*

$$|F(M,x,t) - F(M,y,t)| \leq \tilde{\omega}(|x-y|)\|M\|.$$

4. Degenerate elliptic case with oscillatory singularities

In this section, we examine the degenerate elliptic problem

$$\begin{cases} |Du|^{\kappa(x)} F(D^2 u) = \gamma(x) u^{\gamma(x)-1} & \text{in } \Omega \cap \{u > 0\}, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

for degeneracy, and oscillatory singularity, exponents $\kappa(\cdot)$ and $\gamma(\cdot)$, respectively. We assume the following extra hypotheses in addition to (A1) and (A2).

(A3): The functions $\kappa, \gamma : \Omega \to \mathbb{R}$ are continuous and there exist constants κ_1, γ_0 and γ_1 such that, for all $x \in \Omega$,

$$0 \leq \kappa(x) \leq \kappa_1 \qquad \text{and} \qquad 0 < \gamma_0 \leq \gamma(x) \leq \gamma_1 < 1.$$

Moreover, there exists a modulus of continuity $\overline{\omega}$ such that

$$\limsup_{t \to 0+} \overline{\omega}(t) \log\left(\frac{1}{t}\right) \leq C,$$

for a constant $C > 0$, and

$$|\kappa(x) - \kappa(y)| \le \overline{\omega}(|x - y|),$$
$$|\gamma(x) - \gamma(y)| \le \overline{\omega}(|x - y|),$$

for all $x, y \in \Omega$.

(A4): There exists a modulus of continuity ω such that, for all $\alpha > 0$, $x, y \in \Omega$, and $X, Y \in \mathcal{S}^n$, we have

$$|\alpha(x-y)|^{\kappa(x)} F(X) - |\alpha(x-y)|^{\kappa(y)} F(Y) \le \omega(\alpha|x-y|^2),$$

provided

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

The second main result of this paper is the following.

Theorem 4.1. *Let $\varphi \in C(\partial\Omega)$, with $\varphi \ge 0$, and assume (A1)–(A4). There exists a nonnegative bounded viscosity solution u to (4.1), and u is locally of class $C^{1,\alpha}$, for every*

$$\alpha \in \left(0, \frac{\gamma(x)}{\kappa(x) + 2 - \gamma(x)}\right] \cap (0, \alpha_F),$$

with the estimate

$$\sup_{y \in B_r(x)} |u(y) - u(x) - Du(x) \cdot (y - x)| \le C r^{1+\alpha}, \tag{4.2}$$

for $B_r(x) \Subset \Omega$, where $C = C(n, \lambda, \Lambda, \gamma, \kappa, \alpha, \|u\|_{L^\infty})$.

Moreover, for each free boundary point x, u is of class $C^{1,\frac{\gamma(x)}{\kappa(x)+2-\gamma(x)}}$ at x, with the estimate

$$\sup_{B_r(x)} u \le C r^{1+\frac{\gamma(x)}{\kappa(x)+2-\gamma(x)}}, \tag{4.3}$$

for $B_r(x) \Subset \Omega$, where $C = C(n, \lambda, \Lambda, \gamma, \kappa, \|u\|_{L^\infty})$.

Remark 4.1. *The constant α_F in the statement of the theorem denotes the optimal exponent associated with the $C^{1+\mu}-$regularity theory for solutions of F-harmonic functions, i.e., solutions of the equation $F(D^2 h) = 0$ (see [11]).*

We will start by establishing the existence of a solution to (4.1). Similarly to Section 3, we first define, for $\epsilon \in (0, 1)$,

$$\beta_\epsilon(s) = \int_0^{\frac{s}{\epsilon^{1+\alpha}} - \sigma_0} \rho(\theta)\, d\theta,$$

where $\rho \in C^\infty(\mathbb{R})$ is a nonnegative smooth function with compact support in $[0,1]$, satisfying $\int \rho = 1$, $\sigma_0 \in (0,1)$ and

$$\alpha = \frac{\gamma_0}{\kappa_1 + 2 - \gamma_0}.$$

Note that $\beta_\epsilon(s) \to \chi_{\{s>0\}}$ as $\epsilon \to 0$.

Now, we analyze the penalized equation

$$\begin{cases} |Du_\epsilon|^{\kappa(x)} F(D^2 u_\epsilon) - \epsilon u_\epsilon + \dfrac{\sigma_0 \epsilon^{2+\alpha}}{2} = \gamma(x)\beta_\epsilon(u_\epsilon) u_\epsilon^{\gamma(x)-1} & \text{in } \Omega, \\ \qquad\qquad u_\epsilon = \varphi_\epsilon & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

where $\varphi_\epsilon = \varphi + \epsilon^{1+\alpha}$.

Proposition 4.1. *For each $\epsilon \in (0,1)$, there exists a viscosity solution u_ϵ to (4.4). Moreover, u_ϵ satisfies*

$$0 < u_\epsilon \le \|\varphi\|_{L^\infty(\partial\Omega)} + 1 \quad \textit{in } \Omega. \tag{4.5}$$

Proof. Let $\overline{u}$ be a viscosity solution to

$$\begin{cases} |Du|^{\kappa(x)} F(D^2 u) - \epsilon u + \dfrac{\sigma_0 \epsilon^{2+\alpha}}{2} = 0 & \text{in } \Omega, \\ \qquad\qquad u = \varphi_\epsilon & \text{on } \partial\Omega, \end{cases}$$

and $\underline{u}$ be a viscosity solution to

$$\begin{cases} |Du|^{\kappa(x)} F(D^2 u) - \epsilon u + \dfrac{\sigma_0 \epsilon^{2+\alpha}}{2} = \gamma_1(\sigma_0 \epsilon^{1+\alpha})^{\gamma_0 - 1} & \text{in } \Omega, \\ \qquad\qquad u = \varphi_\epsilon & \text{on } \partial\Omega. \end{cases}$$

Note that

$$G(M,p,r,x) = |p|^{\kappa(x)} F(M) - \epsilon r + \frac{\sigma_0 \epsilon^{2+\alpha}}{2}$$

is degenerate elliptic and strictly decreasing in r. The existence of $\overline{u}$ and $\underline{u}$ follows from Perron's method ([12, Theorem 4.1]), together with the comparison principle ([8, Lemma 6.3]) and [8, Lemma 6.4]. Since

$$0 \le \gamma(x)\beta_\epsilon(u) u^{\gamma(x)-1} \le \gamma_1(\sigma_0 \epsilon^{1+\alpha})^{\gamma_0-1} \quad \text{in } \Omega,$$

it follows that $\overline{u}$ and $\underline{u}$ are a viscosity supersolution and a viscosity subsolution to (4.4), respectively. By the comparison principle, we know that

$$\underline{u} \le \overline{u} \quad \text{in } \Omega.$$

Then, by [3, Theorem 2.1], there exists a viscosity solution u_ϵ to (4.4) such that $\underline{u} \le u_\epsilon \le \overline{u}$.

Now, we prove (4.5). First, we claim that

$$u_\epsilon \ge \frac{\sigma_0}{2}\epsilon^{1+\alpha}.$$

Assume, for the sake of contradiction, the set

$$\mathcal{A} := \{x \in \Omega \mid u_\epsilon(x) < \frac{\sigma_0}{2}\epsilon^{1+\alpha}\},$$

is nonempty. Since

$$u_\epsilon = \varphi_\epsilon \geq \epsilon^{1+\alpha} > \frac{\sigma_0}{2}\epsilon^{1+\alpha} \quad \text{on } \partial\Omega,$$

we have

$$u_\epsilon \geq \frac{\sigma_0}{2}\epsilon^{1+\alpha} \quad \text{on } \partial\mathcal{A}.$$

By the definitions of β_ϵ and $\mathcal{A}$, u_ϵ is a viscosity solution to

$$|Du|^{\kappa(x)}F(D^2u) - \epsilon u + \frac{\sigma_0\epsilon^{2+\alpha}}{2} = 0 \quad \text{in } \mathcal{A}. \tag{4.6}$$

Note that the constant function $\frac{\sigma_0}{2}\epsilon^{1+\alpha}$ is a solution to (4.6). Then applying the comparison principle, we deduce $u_\epsilon \geq \frac{\sigma_0}{2}\epsilon^{1+\alpha}$ in $\mathcal{A}$ which contradicts the definition of $\mathcal{A}$.

In addition, the constant function $\|\varphi_\epsilon\|_{L^\infty(\partial\Omega)}$ is a viscosity supersolution to

$$|Du|^{\kappa(x)}F(D^2u) - \epsilon u + \frac{\sigma_0\epsilon^{2+\alpha}}{2} = 0 \quad \text{in } \Omega,$$

since $\varphi_\epsilon \geq \epsilon^{1+\alpha}$ in Ω. By the comparison principle, we have

$$u_\epsilon \leq \overline{u} \leq \|\varphi_\epsilon\|_{L^\infty(\partial\Omega)} \leq \|\varphi\|_{L^\infty(\partial\Omega)} + 1 \quad \text{in } \Omega,$$

and (4.5) follows. □

As before, for simplicity of notation, we omit the subscript ϵ in u_ϵ from now on. We now consider the equation

$$|Du|^{\kappa(x)}F(D^2u) - h_1(x)u + h_2(x) = h_3(x)\beta_\epsilon(u)u^{\gamma(x)-1} \quad \text{in } \Omega, \tag{4.7}$$

where h_1, h_2 and h_3 are nonnegative functions, all uniformly bounded by the universal constant $\overline{C}$.

Remark 4.2. *We will examine the scaling invariance of solutions to* (4.7)*. Let u be a positive viscosity solution to* (4.7) *in $B_R(x_0) \Subset \Omega$. Then, for parameters $R > 0$ and $A > 0$, the rescaled function*

$$\tilde{u}(x) = \frac{u(x_0 + Rx)}{A}$$

satisfies

$$|D\tilde{u}|^{\tilde{\kappa}(x)}F(D^2\tilde{u}) - \tilde{h}_1(x)\tilde{u} + \tilde{h}_2(x) = \tilde{h}_3(x)\beta_{\tilde{\epsilon}}(\tilde{u})\tilde{u}^{\tilde{\gamma}(x)-1} \quad in\ B_1,$$

in the viscosity sense, where

$$\begin{aligned}
\tilde{\epsilon} &= \epsilon A^{-\frac{1}{1+\alpha}};\\
\tilde{\kappa}(x) &= \kappa(x_0+Rx);\\
\tilde{\gamma}(x) &= \gamma(x_0+Rx);\\
\tilde{h}_1(x) &= \frac{R^{\kappa(x_0+Rx)+2}}{A^{\kappa(x_0+Rx)}} h_1(x_0+Rx);\\
\tilde{h}_2(x) &= \frac{R^{\kappa(x_0+Rx)+2}}{A^{\kappa(x_0+Rx)+1}} h_2(x_0+Rx);\\
\tilde{h}_3(x) &= \frac{R^{\kappa(x_0+Rx)+2}}{A^{\kappa(x_0+Rx)+2-\gamma(x_0+Rx)}} h_3(x_0+Rx).
\end{aligned}$$

Note that, as a consequence of (A3), $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$ are nonnegative functions, uniformly bounded by a constant that depends only on the universal constants, A and R.

Let us now denote

$$\alpha(x) := \frac{\gamma(x)}{\kappa(x)+2-\gamma(x)}.$$

To obtain the optimal growth of the solution, we first prove the following estimate.

Theorem 4.2. *Let $u \in C(B_1)$ be a positive viscosity solution to (4.7) in B_1. Then, for each $\mu \in (0,1)$, there exists $C > 0$, depending only on $n, \lambda, \Lambda, \gamma, \kappa, \mu$ and $\|u\|_{L^\infty(B_1)}$, such that*

$$\left| u(x)^{\frac{1}{1+\inf\limits_{B_1}\alpha}} - u(y)^{\frac{1}{1+\inf\limits_{B_1}\alpha}} \right| \le C|x-y|^\mu, \tag{4.8}$$

for every $x, y \in B_{\frac{1}{2}}$.

Proof. Denote

$$\alpha_0 := \inf_{B_1} \alpha \;\in\; \left[\tfrac{\gamma_0}{\kappa_1+2-\gamma_0}, \tfrac{\gamma_1}{2-\gamma_1}\right],$$

and let $v = u^{\frac{1}{1+\alpha_0}}$. Applying (A2), we rewrite (4.7) as

$$\begin{aligned}
|Dv|^{\kappa(x)} F(D^2 v + \alpha_0 v^{-1} Dv \otimes Dv) - \overline{h}_1(x) v^{1-\alpha_0\kappa(x)} + \overline{h}_2 v^{-\alpha_0(\kappa(x)+1)}&\\
= \overline{h}_3(x)\beta_\epsilon(v^{1+\alpha_0}) v^{\tilde{\alpha}(x)} \quad \text{in } B_1,&
\end{aligned} \tag{4.9}$$

where

$$\tilde{\alpha}(x) = (1+\alpha_0)(\gamma(x)-\kappa(x)-2)+\kappa(x)+1$$

and

$$\overline{h}_i(x) = \left(\frac{1}{1+\alpha_0}\right)^{\kappa(x)+1} h_i(x),$$

for $i = 1, 2, 3$. Note that

$$-1 \leq \tilde{\alpha}(x) \leq 0 \quad \text{in } B_1, \tag{4.10}$$

and $\overline{h}_1, \overline{h}_2$ and $\overline{h}_3$ are nonnegative functions, uniformly bounded by $\overline{C}$. Denote also

$$f(x) := \overline{h}_3(x)\beta_\epsilon\left(v(x)^{1+\alpha_0}\right),$$

which is a nonnegative bounded function.

Defining

$$\Phi(x,y) = v(x) - v(y) - L|x-y|^\mu - K(|x|^2 + |y|^2) \quad \text{in } B_1,$$

we will prove that

$$\max_{\overline{B_{\frac{1}{2}}} \times \overline{B_{\frac{1}{2}}}} \Phi \leq 0, \tag{4.11}$$

for sufficiently large L and K, thus obtaining (4.8). To obtain (4.11), assume that Φ attains its positive maximum at $(\overline{x}, \overline{y}) \in \overline{B_{\frac{1}{2}}} \times \overline{B_{\frac{1}{2}}}$. This implies that

$$\overline{x} \neq \overline{y}, \qquad v(\overline{x}) > v(\overline{y}) + L|\overline{x} - \overline{y}|^\mu,$$

and

$$L|\overline{x} - \overline{y}|^\mu + K(|\overline{x}|^2 + |\overline{y}|^2) \leq 2\|v\|_\infty. \tag{4.12}$$

From (4.12), by choosing K sufficiently large, we ensure that

$$(\overline{x}, \overline{y}) \in B_{\frac{1}{4}} \times B_{\frac{1}{4}}.$$

Now, we can obtain $p \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}^n$, such that

$$(p_x, X) \in \overline{\mathcal{P}}^+(v)(\overline{x}), \tag{4.13}$$

$$(p_y, Y) \in \overline{\mathcal{P}}^-(v)(\overline{y}), \tag{4.14}$$

where $p_x = p + 2K\overline{x}$ and $p_y = p - 2K\overline{y}$, with the estimate given by Lemma 2.1. We can choose L sufficiently large so that

$$1 \leq \frac{1}{2}\mu L \leq \frac{1}{2}\mu L|\overline{x} - \overline{y}|^{\mu-1} \leq |p_x|, |p_y| \leq \frac{3}{2}\mu L|\overline{x} - \overline{y}|^{\mu-1}. \tag{4.15}$$

By applying (4.13) and (4.14) to equation (4.9), we obtain the inequalities

$$\begin{aligned} |p_x|^{\kappa(\overline{x})} F\big(X + \alpha_0 v(\overline{x})^{-1} p_x \otimes p_x\big) - \overline{h}_1(\overline{x}) v(\overline{x})^{1-\alpha_0\kappa(\overline{x})} \\ + \overline{h}_2(\overline{x}) v(\overline{x})^{-\alpha_0(\kappa(\overline{x})+1)} \geq f(\overline{x}) v(\overline{x})^{\tilde{\alpha}(\overline{x})}, \end{aligned}$$

and

$$|p_y|^{\kappa(\overline{y})}F\big(Y+\alpha_0 v(\overline{y})^{-1}p_y\otimes p_y\big)-\overline{h}_1(\overline{y})v(\overline{y})^{1-\alpha_0\kappa(\overline{y})}$$
$$+\overline{h}_2(\overline{y})v(\overline{y})^{-\alpha_0(\kappa(\overline{y})+1)}\leq f(\overline{y})v(\overline{y})^{\tilde{\alpha}(\overline{y})}.$$

Then, we get

$$\begin{aligned}
&F\big(Y+\alpha_0 v(\overline{y})^{-1}p_y\otimes p_y\big)-F\big(X+\alpha_0 v(\overline{x})^{-1}p_x\otimes p_x\big)\\
&\leq |p_y|^{-\kappa(\overline{y})}\big(f(\overline{y})v(\overline{y})^{\tilde{\alpha}(\overline{y})}+\overline{h}_1(\overline{y})v(\overline{y})^{1-\alpha_0\kappa(\overline{y})}-\overline{h}_2(\overline{y})v(\overline{y})^{-\alpha_0(\kappa(\overline{y})+1)}\big)\\
&\quad-|p_x|^{-\kappa(\overline{x})}\big(f(\overline{x})v(\overline{x})^{\tilde{\alpha}(\overline{x})}+\overline{h}_1(\overline{x})v(\overline{x})^{1-\alpha_0\kappa(\overline{x})}-\overline{h}_2(\overline{x})v(\overline{x})^{-\alpha_0(\kappa(\overline{x})+1)}\big).
\end{aligned}\tag{4.16}$$

On the other hand, for every $\eta>0$, there exists $M_\eta\in\mathcal{A}_{\lambda,\Lambda}$ such that

$$\begin{aligned}
&F(Y+\alpha_0 v(\overline{y})^{-1}p_y\otimes p_y)-F(X+\alpha_0 v(\overline{x})^{-1}p_x\otimes p_x)\\
&\geq \operatorname{tr}(M_\eta(Y+\alpha_0 v(\overline{y})^{-1}p_y\otimes p_y-X-\alpha_0 v(\overline{x})^{-1}p_x\otimes p_x))-\eta.
\end{aligned}\tag{4.17}$$

From (4.16) and (4.17), we obtain

$$\begin{aligned}
\eta &\geq \operatorname{tr}(M_\eta(Y-X))\\
&\quad+v(\overline{y})^{-1}\Big(\alpha_0\operatorname{tr}(M_\eta p_y\otimes p_y)-|p_y|^{-\kappa(\overline{y})}\big(f(\overline{y})v(\overline{y})^{\tilde{\alpha}(\overline{y})+1}\\
&\quad+\overline{h}_1(\overline{y})v(\overline{y})^{2-\alpha_0\kappa(\overline{y})}-\overline{h}_2(\overline{y})v(\overline{y})^{1-\alpha_0(\kappa(\overline{y})+1)}\big)\Big)\\
&\quad-v(\overline{x})^{-1}\Big(\alpha_0\operatorname{tr}(M_\eta p_x\otimes p_x)-|p_x|^{-\kappa(\overline{x})}\big(f(\overline{x})v(\overline{x})^{\tilde{\alpha}(\overline{x})+1}\\
&\quad+\overline{h}_1(\overline{x})v(\overline{x})^{2-\alpha_0\kappa(\overline{x})}-\overline{h}_2(\overline{x})v(\overline{x})^{1-\alpha_0(\kappa(\overline{x})+1)}\big)\Big),
\end{aligned}\tag{4.18}$$

and, from Lemma 2.1,

$$\begin{aligned}
\operatorname{tr}(M_\eta(Y-X)) &\geq \lambda\Big(4\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2}-(4K+2\iota)\Big)\\
&\quad-\Lambda(n-1)(4K+2\iota)\\
&\geq 4\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2}\\
&\quad-(\lambda+\Lambda(n-1))(4K+2\iota)\\
&\geq 3\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2},
\end{aligned}\tag{4.19}$$

for sufficiently large L. From (4.10) and (4.15), we get

$$\begin{aligned}
&\alpha_0 \operatorname{tr}(M_\eta p_y \otimes p_y) - |p_y|^{-\kappa(\overline{y})}\Big(f(\overline{y})v(\overline{y})^{\tilde{\alpha}(\overline{y})+1} + \overline{h}_1(\overline{y})v(\overline{y})^{2-\alpha_0\kappa(\overline{y})} \\
&\quad - \overline{h}_2(\overline{y})v(\overline{y})^{1-\alpha_0(\kappa(\overline{y})+1)}\Big) \\
&\quad \geq \frac{\gamma_0}{\kappa_1 + 2 - \gamma_0}\lambda|p_y|^2 - \overline{C}\max\left(1, \|v\|_{L^\infty}\right) - \overline{C}\max\left(1, \|v\|_{L^\infty}^2\right) \\
&\quad \geq \frac{\gamma_0}{\kappa_1 + 2 - \gamma_0}\lambda\left(\frac{1}{2}\mu L\right)^2 - \overline{C}\max\left(1, \|v\|_{L^\infty}\right) - \overline{C}\max\left(1, \|v\|_{L^\infty}^2\right) \\
&\quad > 0,
\end{aligned} \tag{4.20}$$

for sufficiently large L. Using that $v(\overline{y})^{-1} > v(\overline{x})^{-1}$, from (4.15), (4.18), (4.19) and (4.20), we obtain

$$\begin{aligned}
\eta &\geq \operatorname{tr}(M_\eta(Y - X)) \\
&\quad + v(\overline{x})^{-1}\Big(\alpha_0 \operatorname{tr}(M_\eta p_y \otimes p_y) - |p_y|^{-\kappa(\overline{y})}\big(f(\overline{y})v(\overline{y})^{\tilde{\alpha}(\overline{y})+1} \\
&\quad + \overline{h}_1(\overline{y})v(\overline{y})^{2-\alpha_0\kappa(\overline{y})} - \overline{h}_2(\overline{y})v(\overline{y})^{1-\alpha_0(\kappa(\overline{y})+1)}\big)\Big) \\
&\quad - v(\overline{x})^{-1}\Big(\alpha_0 \operatorname{tr}(M_\eta p_x \otimes p_x) - |p_x|^{-\kappa(\overline{x})}\big(f(\overline{x})v(\overline{x})^{\tilde{\alpha}(\overline{x})+1} \\
&\quad + \overline{h}_1(\overline{x})v(\overline{x})^{2-\alpha_0\kappa(\overline{x})} - \overline{h}_2(\overline{x})v(\overline{x})^{1-\alpha_0(\kappa(\overline{x})+1)}\big)\Big) \\
&\geq 3\lambda\mu(1-\mu)L|\overline{x} - \overline{y}|^{\mu-2} \\
&\quad - v(\overline{x})^{-1}\Big\{2\alpha_0 n\Lambda|p_y||p_y - p_x| + \alpha_0\Lambda|p_y - p_x|^2 + |p_y|^{-\kappa(\overline{y})}\big(f(\overline{y}) \\
&\quad v(\overline{y})^{\tilde{\alpha}(\overline{y})+1} + \overline{h}_1(\overline{y})v(\overline{y})^{2-\alpha_0\kappa(\overline{y})}\big) + |p_x|^{-\kappa(\overline{x})}\overline{h}_2(\overline{x})v(\overline{x})^{1-\alpha_0(\kappa(\overline{x})+1)}\Big\} \\
&\geq 3\lambda\mu(1-\mu)L|\overline{x} - \overline{y}|^{\mu-2} \\
&\quad - v(\overline{x})^{-1}\Big\{3\alpha_0 n\Lambda K\mu L|\overline{x} - \overline{y}|^{\mu-1} + \alpha_0\Lambda K^2 + 2\overline{C}\max(1, \|v\|_{L^\infty}) \\
&\quad + \overline{C}\max\left(1, \|v\|_{L^\infty}^2\right)\Big\}.
\end{aligned}$$

Note that we used the fact that

$$0 \leq \alpha_0(\kappa(\overline{x}) + 1) \leq \gamma(\overline{x})\frac{\kappa(\overline{x}) + 1}{\kappa(\overline{x}) + 2 - \gamma(\overline{x})} \leq 1.$$

Finally, since $v(\overline{x})^{-1} < L^{-1}|\overline{x}-\overline{y}|^{-\mu}$ and $|\overline{x}-\overline{y}| \le 1$, we get

$$\begin{aligned}
\eta &\ge 3\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2} \\
&\quad - L^{-1}|\overline{x}-\overline{y}|^{-\mu}\Big(3\alpha_0 n\Lambda K\mu L|\overline{x}-\overline{y}|^{\mu-1} + \alpha_0\Lambda K^2 + C\Big) \\
&\ge L|\overline{x}-\overline{y}|^{\mu-2}\Big\{3\lambda\mu(1-\mu) - 3\alpha_0 n\Lambda K\mu L^{-1}|\overline{x}-\overline{y}|^{1-\mu} \\
&\quad - (\alpha_0\Lambda K^2 + C)L^{-2}|\overline{x}-\overline{y}|^{2-2\mu}\Big\} \\
&\ge \lambda\mu(1-\mu)L,
\end{aligned}$$

for sufficiently large L. By taking the limit as $\eta \to 0$, we obtain a contradiction and conclude the proof. $\square$

Remark 4.3. *Let $R \le 1$ and $u \in C(B_R(x_0))$ be a viscosity solution to*

$$|Du|^{\kappa(x)}F(D^2u) - h_1(x)u + h_2(x) = h_3(x)\beta_\epsilon(u)u^{\gamma(x)-1} \quad \text{in } B_R(x_0).$$

Then, for $\mu \in (0,1)$, it follows from Theorem 4.2 that

$$\left| u(x)^{\frac{1}{1+\inf\limits_{B_R(x_0)}\alpha}} - u(y)^{\frac{1}{1+\inf\limits_{B_R(x_0)}\alpha}} \right| \le C|x-y|^\mu, \tag{4.21}$$

for every $x, y \in B_{\frac{R}{2}}(x_0)$. Indeed, denote $\overline{\alpha} := \inf\limits_{B_R(x_0)} \alpha$ and let

$$\tilde{u}(x) = \frac{u(x_0+Rx)}{R^{1+\overline{\alpha}}} \quad \text{in } B_1.$$

By Remark 4.2, $\tilde{u}$ satisfies

$$|D\tilde{u}|^{\tilde{\kappa}(x)}F(D^2\tilde{u}) - \tilde{h}_1(x)\tilde{u} + \tilde{h}_2(x) = \tilde{h}_3(x)\beta_{\tilde{\epsilon}}(\tilde{u})\tilde{u}^{\tilde{\gamma}(x)-1} \quad \text{in } B_1,$$

in the viscosity sense, where

$$\begin{aligned}
\tilde{\epsilon} &= \epsilon R^{-\frac{1+\overline{\alpha}}{1+\alpha}}; \\
\tilde{\kappa}(x) &= \kappa(x_0+Rx); \\
\tilde{\gamma}(x) &= \gamma(x_0+Rx); \\
\tilde{h}_1(x) &= R^{2-\overline{\alpha}\kappa(x_0+Rx)}h_1(x_0+Rx); \\
\tilde{h}_2(x) &= R^{1-\overline{\alpha}(\kappa(x_0+Rx)+1)}h_2(x_0+Rx); \\
\tilde{h}_3(x) &= R^{\gamma(x_0+Rx)-\overline{\alpha}(\kappa(x_0+Rx)+2-\gamma(x_0+Rx))}h_3(x_0+Rx),
\end{aligned}$$

for $x \in B_1$. For $0 \le \beta \le 1+\alpha(x_0)$, defining in B_1

$$\begin{aligned}
h_{1,\beta}(x) &= R^{\kappa(x_0+Rx)+2-\beta\kappa(x_0+Rx)}h_1(x_0+Rx), \\
h_{2,\beta}(x) &= R^{\kappa(x_0+Rx)+2-\beta(\kappa(x_0+Rx)+1)}h_2(x_0+Rx), \\
h_{3,\beta}(x) &= R^{\kappa(x_0+Rx)+2-\beta(\kappa(x_0+Rx)+2-\gamma(x_0+Rx))}h_3(x_0+Rx),
\end{aligned}$$

we have that $h_{1,\beta}, h_{2,\beta}$ and $h_{3,\beta}$ are uniformly bounded. Indeed, from

$$\gamma(x_0) - \alpha(x_0)(\kappa(x_0) + 2 - \gamma(x_0)) = 0$$

and (A3), we have

$$\begin{aligned}
&R^{\kappa(x_0+Rx)+2-\beta(\kappa(x_0+Rx)+2-\gamma(x_0+Rx))} \\
&\leq R^{\gamma(x_0+Rx)-\alpha(x_0)(\kappa(x_0+Rx)+2-\gamma(x_0+Rx))} \\
&= R^{\gamma(x_0+Rx)-\gamma(x_0)+\alpha(x_0)(\kappa(x_0)-\kappa(x_0+Rx)+\gamma(x_0+Rx)-\gamma(x_0))} \\
&\leq R^{-3\overline{\omega}(R)} \\
&\leq C,
\end{aligned}$$

in B_1. By similar calculations, we can see that

$$R^{\kappa(x_0+Rx)+2-\beta\kappa(x_0+Rx)} \qquad \text{and} \qquad R^{\kappa(x_0+Rx)+2-\beta(\kappa(x_0+Rx)+1)}$$

are universally bounded. The functions $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$ correspond to the case $\beta = 1 + \overline{\alpha}$. Then, applying Theorem 4.2 to $\tilde{u}$, we obtain

$$\left| \tilde{u}(x)^{\frac{1}{1+\inf\limits_{B_1} \tilde{\alpha}}} - \tilde{u}(y)^{\frac{1}{1+\inf\limits_{B_1} \tilde{\alpha}}} \right| \leq C|x-y|^{\mu}, \tag{4.22}$$

for every $x, y \in B_{\frac{1}{2}}$, where

$$\tilde{\alpha}(x) = \frac{\tilde{\gamma}(x)}{\tilde{\kappa}(x) + 2 - \tilde{\gamma}(x)} = \alpha(x_0 + Rx).$$

Note that $\inf\limits_{B_1} \tilde{\alpha} = \overline{\alpha}$. Hence, (4.22) implies (4.21).

We can now prove the optimal growth of the solution using Theorem 4.2.

Theorem 4.3. *Let u be a positive viscosity solution to (4.7) in B_1. There exists a universal constant $C > 0$, depending only on n, λ, Λ, γ, κ and $\|u\|_{L^\infty(B_1)}$, but not depending on ϵ, such that*

$$\sup_{B_r} u \leq C\left(u(0) + r^{1+\alpha(0)}\right), \tag{4.23}$$

for every $r \leq \dfrac{1}{2}$.

Proof. Let us first observe that to establish (4.23), it suffices to show

$$\sup_{B_r} u \leq C\left(u(0) + r^{1+\inf\limits_{B_{2r}} \alpha}\right), \tag{4.24}$$

for every $r \leq 1/2$. Indeed, by the Mean Value Theorem, together with (A3), we obtain, for $x, y \in B_1$,

$$
\begin{aligned}
|\alpha(x) - \alpha(y)| &\leq \left| \tfrac{\gamma(x)}{\kappa(x)+2-\gamma(x)} - \tfrac{\gamma(x)}{\kappa(y)+2-\gamma(x)} \right| + \left| \tfrac{\gamma(x)}{\kappa(y)+2-\gamma(x)} - \tfrac{\gamma(y)}{\kappa(y)+2-\gamma(y)} \right| \\
&\leq \sup_{t \geq 0} \tfrac{\gamma(x)}{(t+2-\gamma(x))^2} |\kappa(x) - \kappa(y)| + \sup_{0 \leq s \leq 1} \tfrac{\kappa(y)+2}{(\kappa(y)+2-s)^2} |\gamma(x) - \gamma(y)| \\
&\leq |\kappa(x) - \kappa(y)| + 2|\gamma(x) - \gamma(y)| \\
&\leq 3\overline{\omega}(|x-y|). \qquad (4.25)
\end{aligned}
$$

Using (4.25) and again (A3), we conclude

$$
\begin{aligned}
r^{1+\inf\limits_{B_{2r}} \alpha} &\leq r^{1+\alpha(0)} r^{-3\overline{\omega}(2r)} \\
&\leq C r^{1+\alpha(0)}.
\end{aligned}
$$

To prove (4.24), we assume, by contradiction, that for every integer l, there exist F_l, u_l, κ_l, γ_l, $h_{1,l}$, $h_{2,l}$, $h_{3,l}$, ϵ_l and $r_l \leq 1/2$ such that

$$
|Du_l|^{\kappa_l(x)} F_l(D^2 u_l) - h_{1,l}(x) u_l + h_{2,l}(x) = h_{3,l}(x) \beta_{\epsilon_l}(u_l) u_l^{\gamma_l(x)-1} \quad \text{in } B_1,
$$

in the viscosity sense,

$$
0 \leq h_{1,l}, h_{2,l}, h_{3,l} \leq \overline{C} \quad \text{in } B_1
$$

but

$$
s_l := \sup_{B_{r_l}} u_l > l \left(u_l(0) + r_l^{1+\inf\limits_{B_{2r_l}} \alpha_l} \right). \qquad (4.26)
$$

Denote $\overline{\alpha_l} = \inf\limits_{B_{2r_l}} \alpha_l$ and define w_l

$$
w_l(x) := \frac{u_l(r_l x)}{s_l}.
$$

Then, by Remark 4.2, w_l satisfies

$$
|Dw_l|^{\tilde{\kappa}_l(x)} F_l(D^2 w_l) - \tilde{h}_{1,l}(x) w_l + \tilde{h}_{2,l}(x) = \tilde{h}_{3,l}(x) \beta_{\tilde{\epsilon}_l}(w_l) w_l^{\tilde{\gamma}_l(x)-1} \quad \text{in } B_1, \qquad (4.27)
$$

in the viscosity sense, where, for $x \in B_1$,

$$
\begin{aligned}
\tilde{\epsilon_l} &= \epsilon_l s_l^{-\frac{1}{1+\alpha}};\\
\tilde{\kappa}_l(x) &= \kappa_l(r_l x);\\
\tilde{\gamma}_l(x) &= \gamma_l(r_l x);\\
\tilde{h}_{1,l}(x) &= \frac{r_l^{\kappa_l(r_l x)+2}}{s_l^{\kappa_l(r_l x)}} h_{1,l}(r_l x);\\
\tilde{h}_{2,l}(x) &= \frac{r_l^{\kappa_l(r_l x)+2}}{s_l^{\kappa_l(r_l x)+1}} h_{2,l}(r_l x);\\
\tilde{h}_{3,l}(x) &= \frac{r_l^{\kappa_l(r_l x)+2}}{s_l^{\kappa_l(r_l x)+2-\gamma_l(r_l x)}} h_{3,l}(r_l x).
\end{aligned}
$$

Also, from (4.25) and (4.26), we have

$$
w_l(0) = o(1), \qquad \sup_{B_1} w_l = 1, \tag{4.28}
$$

and

$$
\frac{r_l^{1+\alpha_l(r_l x)}}{s_l} \le \frac{r_l^{1+\overline{\alpha_l}}}{s_l} \le \frac{1}{l} \to 0 \quad \text{as} \quad l \to \infty, \tag{4.29}
$$

in B_1. Using (4.29) and

$$
\kappa_l(r_l x) + 2 - \gamma_l(r_l x) \ge 1 \quad \text{in } B_1,
$$

we obtain

$$
\frac{r_l^{\kappa_l(r_l x)+2}}{s_l^{\kappa_l(r_l x)+2-\gamma_l(r_l x)}} = \left(\frac{r_l^{1+\alpha_l(r_l x)}}{s_l} \right)^{\kappa_l(r_l x)+2-\gamma_l(r_l x)} \le \frac{1}{l} \to 0 \quad \text{as} \quad l \to \infty, \tag{4.30}
$$

in B_1. Note that, from (4.30),

$$
\tilde{h}_{i,l} \to 0 \quad \text{as} \quad l \to \infty, \tag{4.31}
$$

for $i = 1, 2, 3$. By Theorem 4.2 and Remark 4.3, along with (4.28) and (4.31), for $0 < \mu < 1$, we have that $\{w_l\}_l$ is equicontinuous and

$$
\begin{aligned}
\sup_{B_{\frac{R}{2}}(x_0)} w_l &\le \left(w_l(x_0)^{\frac{1}{1+\inf\limits_{B_R(x_0)} \tilde{\alpha}_l}} + C R^{\mu} \right)^{1+\inf\limits_{B_R(x_0)} \tilde{\alpha}_l}\\
&\le C \left(w_l(x_0)^{\frac{2-\gamma_1}{2}} + R^{\mu} \right)^{1+\frac{\gamma_0}{\kappa_1+2-\gamma_0}},
\end{aligned} \tag{4.32}
$$

for $B_R(x_0) \subset B_1$. Note that we used
$$\frac{\gamma_0}{\kappa_1 + 2 - \gamma_0} \le \tilde{\alpha}_l \le \frac{\gamma_1}{2-\gamma_1}.$$
From
$$\begin{aligned} |\tilde{\kappa}_l(x) - \tilde{\kappa}_l(y)| &= |\kappa_l(r_l x) - \kappa_l(r_l y)| \\ &\le \overline{\omega}(|r_l(x-y)|) \\ &\le \overline{\omega}(|x-y|), \end{aligned}$$
and
$$\begin{aligned} |\tilde{\gamma}_l(x) - \tilde{\gamma}_l(y)| &= |\gamma_l(r_l x) - \gamma_l(r_l y)| \\ &\le \overline{\omega}(|r_l(x-y)|) \\ &\le \overline{\omega}(|x-y|), \end{aligned}$$
for $x, y \in B_1$, we know that also $\{\tilde{\kappa}_l\}_l$, $\{\tilde{\gamma}_l\}_l$ are equicontinuous. Then, by the Arzelà–Ascoli Theorem, the equicontinuity of $\{\tilde{\kappa}_l\}_l$, $\{\tilde{\gamma}_l\}_l$ and $\{w_l\}_l$, combined with (A3) and (4.28), implies the existence of $\tilde{\kappa}_0, \tilde{\gamma}_0, w_0 \in C(B_1)$ such that, up to a subsequence,
$$\tilde{\kappa}_l \to \tilde{\kappa}_0, \qquad \tilde{\gamma}_l \to \tilde{\gamma}_0, \qquad w_l \to w_0,$$
locally uniformly in B_1. Furthermore, by (A1), possibly after passing to a subsequence, F_l converges locally uniformly to F_0, which satisfies (A1). We can rewrite (4.27) as
$$\begin{aligned} w_l^{1-\tilde{\gamma}_l(x)} |Dw_l|^{\tilde{\kappa}_l(x)} F_l(D^2 w_l) - \tilde{h}_{1,l}(x) w_l^{2-\tilde{\gamma}_l(x)} + \tilde{h}_{2,l}(x) w_l^{1-\tilde{\gamma}_l(x)} \\ = \tilde{h}_{3,l}(x) \beta_{\tilde{\epsilon}_l}(w_l) \quad \text{in } B_1, \end{aligned}$$
in the viscosity sense. From stability of viscosity solutions and (4.31), w_0 satisfies
$$w_0^{1-\tilde{\gamma}_0(x)} |Dw_0|^{\tilde{\kappa}_0(x)} F_0(D^2 w_0) = 0 \quad \text{in } B_1.$$
Note that, by the cutting lemma ([13, Lemma 6]), we have
$$F_0(D^2 w_0) = 0 \quad \text{in } \{w_0 > 0\} \cap B_1, \tag{4.33}$$
in the viscosity sense. By (4.32), we get
$$\sup_{B_{\frac{R}{2}}(x_0)} w_0 \le C \left(w_0(x_0)^{\frac{2-\gamma_1}{2}} + R^{\mu} \right)^{1 + \frac{\gamma_0}{\kappa_1 + 2 - \gamma_0}}, \tag{4.34}$$
for $0 < \mu < 1$ and $B_R(x_0) \subset B_1$. Furthermore, by (4.28), we obtain
$$w_0 \ge 0 \quad \text{in } B_1, \quad w_0(0) = 0 \quad \text{and} \quad \sup_{B_1} w_0 = 1. \tag{4.35}$$

From (4.35), there exist $z_+ \in \{w_0 > 0\} \cap B_1$ and $z_0 \in \{w_0 = 0\} \cap B_1$ such that

$$\operatorname{dist}(z_+, \{w_0 = 0\}) = |z_+ - z_0|.$$

By Hopf's lemma with (4.33), we obtain

$$\liminf_{h \to 0+} \frac{w_0(z_0 + h(z_+ - z_0)) - w_0(z_0)}{h} > 0. \tag{4.36}$$

On the other hand, by applying (4.34) with $x_0 = z_0$, we obtain

$$\sup_{B_{\frac{R}{2}}(z_0)} w_0 \le CR^{\mu\left(1 + \frac{\gamma_0}{\kappa_1 + 2 - \gamma_0}\right)},$$

for $0 < \mu < 1$ and sufficiently small $R > 0$. Then, choosing μ satisfying

$$\mu\left(1 + \tfrac{\gamma_0}{\kappa_1 + 2 - \gamma_0}\right) > 1,$$

we get

$$\begin{aligned} \limsup_{h \to 0+} \frac{w_0(z_0 + h(z_+ - z_0)) - w_0(z_0)}{h} &= \limsup_{h \to 0+} \frac{w_0(z_0 + h(z_+ - z_0))}{h} \\ &\le \limsup_{h \to 0+} Ch^{\mu\left(1 + \frac{\gamma_0}{\kappa_1 + 2 - \gamma_0}\right) - 1} \\ &= 0, \end{aligned}$$

which contradicts (4.36). ☐

Now, we will prove the Lipschitz continuity of the solution.

Proposition 4.2. *Let u be a positive viscosity solution to (4.7) in B_1. There exists a universal constant $C > 0$, depending only on $n, \lambda, \Lambda, \gamma, \kappa$ and $\|u\|_{L^\infty(B_1)}$, but not depending on ϵ, such that*

$$\sup_{B_r} |u(x) - u(0)| \le Cr,$$

for every $r \le \frac{1}{2}$.

Proof. Let $r \le 1/2$. We will consider two cases based on the range of r in terms of $u(0) =: \theta$. By Remark 4.2, we may assume $\theta \le 1/2$.

Case 1) $r \ge \theta$: by Theorem 4.3, we obtain

$$\sup_{B_r} u \le C(\theta + r^{1+\alpha(0)}) \le C(\theta + r) \le 2Cr,$$

so we have

$$\sup_{B_r} |u(x) - u(0)| \le \sup_{B_r} u + u(0) \le 2Cr + \theta \le (2C + 1)r. \tag{4.37}$$

Case 2) $0 < r < \theta$: define $w(x) = \dfrac{u(\theta x)}{\theta}$ in B_1. Then, by Remark 4.2, w satisfies

$$|Dw|^{\tilde{\kappa}(x)} F(D^2 w) - \tilde{h}_1(x) w + \tilde{h}_2(x) = \tilde{h}_3(x) \beta_{\tilde{\epsilon}}(w) w^{\tilde{\gamma}(x)-1} \quad \text{in } B_1, \tag{4.38}$$

in the viscosity sense, where, for $x \in B_1$,

$$\begin{aligned} \tilde{\epsilon} &= \epsilon \theta^{-\frac{1}{1+\alpha}}; \\ \tilde{\kappa}(x) &= \kappa(\theta x); \\ \tilde{\gamma}(x) &= \gamma(\theta x); \\ \tilde{h}_1(x) &= \theta^2 h_1(\theta x); \\ \tilde{h}_2(x) &= \theta h_2(\theta x); \\ \tilde{h}_3(x) &= \theta^{\gamma(\theta x)} h_3(\theta x). \end{aligned}$$

Note that

$$w(x) = \frac{u(\theta x)}{\theta} \le \frac{\sup_{B_\theta} u}{\theta} \le \frac{C(u(0) + \theta)}{\theta} = 2C \quad \text{in } B_1, \tag{4.39}$$

which follows from Theorem 4.3. We also have

$$\frac{\gamma_1}{2 - \gamma_1} \ge \inf_{B_1} \tilde{\alpha} = \inf_{B_\theta} \alpha =: \alpha_\theta,$$

where

$$\tilde{\alpha}(x) = \frac{\tilde{\gamma}(x)}{\tilde{\kappa}(x) + 2 - \tilde{\gamma}(x)}.$$

Applying Theorem 4.2 to w, with $\mu = \dfrac{1}{2}$, we obtain

$$|w(x)^{\frac{1}{1+\alpha_\theta}} - w(0)^{\frac{1}{1+\alpha_\theta}}| = |w(x)^{\frac{1}{1+\alpha_\theta}} - 1| \le C|x|^{\frac{1}{2}} \quad \text{in } B_{\frac{1}{2}},$$

which in turn implies

$$w(x) \ge \left(1 - C|x|^{\frac{1}{2}}\right)^{1+\alpha_\theta} \ge \left(1 - C|x|^{\frac{1}{2}}\right)^{\frac{2}{2-\gamma_1}} \quad \text{in } B_{\frac{1}{2}}.$$

Then,

$$w \ge \frac{1}{2} \quad \text{in} \quad B_{\delta_0}, \tag{4.40}$$

for a universal constant $\delta_0 > 0$. We can rewrite (4.38) as

$$|Dw|^{\tilde{\kappa}(x)} F(D^2 w) = \tilde{h}_1(x) w - \tilde{h}_2(x) + \tilde{h}_3(x) \beta_{\tilde{\epsilon}}(w) w^{\tilde{\gamma}(x)-1} \quad \text{in } B_1, \tag{4.41}$$

and, by (4.39) and (4.40), the right-hand side of (4.41) is bounded by

$$\overline{C} \left(2C + 1 + \left(\frac{1}{2}\right)^{\gamma_0 - 1} \right) \quad \text{in } B_{\delta_0}.$$

The regularity result from [9] implies that

$$\sup_{B_r} |w(x) - w(0)| \leq Cr,$$

for $0 < r \leq \frac{\delta_0}{2}$. By scaling back, we obtain

$$\sup_{B_r} |u(x) - u(0)| \leq Cr,$$

for $0 < r \leq \frac{\theta\delta_0}{2}$.

For $\frac{\theta\delta_0}{2} < r < \theta$, using (4.37), we get

$$\begin{aligned} \sup_{B_r} |u(x) - u(0)| &\leq \sup_{B_\theta} |u(x) - u(0)| \\ &\leq (2C+1)\theta \\ &\leq \frac{2}{\delta_0}(2C+1)r. \end{aligned}$$

□

As a consequence of Proposition 4.2, we derive a gradient bound for the solution, which is instrumental in establishing its sharp local regularity.

Proposition 4.3. *Let u be a positive viscosity solution to (4.7) in B_1. There exists a universal constant $C > 0$, depending only on $n, \lambda, \Lambda, \gamma, \kappa$ and $\|u\|_{L^\infty(B_1)}$, but not depending on ϵ, such that*

$$|Du(x)| \leq Cu(x)^{\frac{\gamma(x)}{\kappa(x)+2}} \quad \textit{in } B_{\frac{1}{2}}.$$

Proof. Let $x_0 \in B_{\frac{1}{2}}$ and

$$r_0 = \left(\frac{u(x_0)}{M}\right)^{\frac{1}{1+\alpha(x_0)}},$$

where M is a constant chosen such that $r_0 \leq \frac{1}{4}$. Define

$$w(x) := \frac{u(x_0 + r_0 x)}{r_0^{1+\alpha(x_0)}} \quad \text{in } B_1.$$

Then, by Remark 4.2, w satisfies

$$|Dw|^{\tilde{\kappa}(x)} F(D^2 w) - \tilde{h}_1(x) w + \tilde{h}_2(x) = \tilde{h}_3(x)\beta_{\tilde{\epsilon}}(w) w^{\tilde{\gamma}(x)-1} \quad \text{in } B_1,$$

in the viscosity sense, where, for $x \in B_1$,

$$
\begin{aligned}
\tilde{\epsilon} &= \epsilon r_0^{-\frac{1+\alpha(x_0)}{1+\alpha}};\\
\tilde{\kappa}(x) &= \kappa(x_0 + r_0 x);\\
\tilde{\gamma}(x) &= \gamma(x_0 + r_0 x);\\
\tilde{h}_1(x) &= r_0^{2-\alpha(x_0)\kappa(x_0+r_0x)} h_1(x_0 + r_0 x);\\
\tilde{h}_2(x) &= r_0^{1-\alpha(x_0)(\kappa(x_0+r_0x)+1)} h_2(x_0 + r_0 x);\\
\tilde{h}_3(x) &= r_0^{\gamma(x_0+r_0x)-\alpha(x_0)(\kappa(x_0+r_0x)+2-\gamma(x_0+r_0x))} h_3(x_0 + r_0 x).
\end{aligned}
$$

Recall that, by Remark 4.3, $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$ are uniformly bounded.
Using Theorem 4.3, we get

$$
\sup_{B_1} w = \sup_{B_1} \frac{u(x_0 + r_0 x)}{r_0^{1+\alpha(x_0)}} \leq \frac{C(u(x_0) + r_0^{1+\alpha(x_0)})}{r_0^{1+\alpha(x_0)}} = C(M+1).
$$

Therefore, applying Proposition 4.2, we conclude

$$
\begin{aligned}
|Du(x_0)| &= r_0^{\alpha(x_0)} |Dw(0)|\\
&\leq C r_0^{\alpha(x_0)}\\
&= C\left(\frac{u(x_0)}{M}\right)^{\frac{\alpha(x_0)}{1+\alpha(x_0)}}\\
&= \tilde{C} u(x_0)^{\frac{\gamma(x_0)}{\kappa(x_0)+2}}.
\end{aligned}
$$

□

Using the optimal growth and gradient bound for the solutions, we obtain the following sharp local estimates, uniform in ϵ.

Theorem 4.4. *Let u be a positive viscosity solution to (4.7) in B_1. For $x_0 \in B_{\frac{1}{2}}$ and $\beta \in (0, \alpha(x_0)] \cap (0, \alpha_F)$, there exists a universal constant $C > 0$, depending only on $n, \lambda, \Lambda, \gamma, \kappa, \beta$ and $\|u\|_{L^\infty(B_1)}$, and independent of ϵ, such that*

$$
\sup_{x \in B_r(x_0)} |u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)| \leq C r^{1+\beta}, \tag{4.42}
$$

for every $r \leq \frac{1}{4}$.

Proof. Fix $x_0 \in B_{\frac{1}{2}}$, $\beta \in (0, \alpha(x_0)] \cap (0, \alpha_F)$ and $r \leq \frac{1}{4}$. Let

$$
r_0 = \left(\frac{u(x_0)}{M}\right)^{\frac{1}{1+\beta}},
$$

where $M > 1$ is a constant chosen in such a way that $r_0 \le \frac{1}{4}$. We will consider two cases based on the range of r in terms of r_0.

Case 1) $r \ge r_0$: by the definition of r_0, $u(x_0) = Mr_0^{1+\beta} \le Mr^{1+\beta}$; then, applying Theorem 4.3 and Proposition 4.3, we obtain

$$\begin{aligned}
\sup_{x\in B_r(x_0)} &\big|u(x) - u(x_0) - Du(x_0)\cdot(x-x_0)\big| \\
&\le C\big(u(x_0) + r^{1+\alpha(x_0)}\big) + u(x_0) + Cru(x_0)^{\frac{\gamma(x_0)}{\kappa(x_0)+2}} \\
&\le C\Big\{Mr^{1+\beta} + r^{1+\alpha(x_0)} + r(Mr^{1+\beta})^{\frac{\gamma(x_0)}{\kappa(x_0)+2}}\Big\} \\
&\le Cr^{1+\beta}. \qquad (4.43)
\end{aligned}$$

Note that we used that

$$\begin{aligned}
1 + (1+\beta)\frac{\gamma(x_0)}{\kappa(x_0)+2} - (1+\beta) &= 1 - (1+\beta)\frac{\kappa(x_0)+2-\gamma(x_0)}{\kappa(x_0)+2} \\
&= 1 - (1+\beta)\frac{1}{1+\alpha(x_0)} \\
&\ge 0.
\end{aligned}$$

Case 2) $0 < r < r_0$: define

$$w(x) := \frac{u(x_0 + r_0 x)}{r_0^{1+\beta}} \quad \text{in } B_1.$$

Then, by Remark 4.2, w satisfies

$$|Dw|^{\tilde\kappa(x)}F(D^2w) - \tilde h_1(x)w + \tilde h_2(x) = \tilde h_3(x)\beta_{\tilde\epsilon}(w)w^{\tilde\gamma(x)-1} \quad \text{in } B_1, \qquad (4.44)$$

in the viscosity sense, where, for $x \in B_1$,

$$\begin{aligned}
\tilde\epsilon &= \epsilon r_0^{-\frac{1+\beta}{1+\alpha}}; \\
\tilde\kappa(x) &= \kappa(x_0 + r_0 x); \\
\tilde\gamma(x) &= \gamma(x_0 + r_0 x); \\
\tilde h_1(x) &= r_0^{2-\beta\kappa(x_0+r_0x)}h_1(x_0 + r_0 x); \\
\tilde h_2(x) &= r_0^{1-\beta(\kappa(x_0+r_0x)+1)}h_2(x_0 + r_0 x); \\
\tilde h_3(x) &= r_0^{\gamma(x_0+r_0x)-\beta(\kappa(x_0+r_0x)+2-\gamma(x_0+r_0x))}h_3(x_0 + r_0 x).
\end{aligned}$$

Recall that, by Remark 4.3, $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$ are uniformly bounded. Applying Theorem 4.3, we get

$$\begin{aligned}\sup_{B_1} w = \sup_{B_1} \frac{u(x_0 + r_0 x)}{r_0^{1+\beta}} &\leq \frac{C(u(x_0) + r_0^{1+\alpha(x_0)})}{r_0^{1+\beta}} \\ &\leq C(M + r_0^{\alpha(x_0)-\beta}) \\ &\leq C(M+1). \end{aligned} \tag{4.45}$$

The definition of r_0 implies that $w(0) = M > 1$, thus, using Proposition 4.2, we obtain

$$w \geq \frac{1}{2} \text{ in } B_{\delta_0}, \tag{4.46}$$

for universal a constant $\delta_0 > 0$. We can now rewrite (4.44) as

$$|Dw|^{\tilde{\kappa}(x)} F(D^2 w) = \tilde{h}_1(x) w - \tilde{h}_2(x) + \tilde{h}_3(x) \beta_{\tilde{\epsilon}}(w) w^{\tilde{\gamma}(x)-1} \quad \text{in } B_1, \tag{4.47}$$

and by (4.45) and (4.46), the right-hand side of (4.47) is universally bounded in B_{δ_0}. Since

$$\beta \leq \alpha(x_0) = \frac{\gamma(x_0)}{\kappa(x_0) + 2 - \gamma(x_0)} = \frac{\tilde{\gamma}(0)}{\tilde{\kappa}(0) + 2 - \tilde{\gamma}(0)} < \frac{1}{\tilde{\kappa}(0) + 1},$$

the regularity result from [9] implies that there exists a universal constant $C > 0$ such that

$$\sup_{B_r} |w(x) - w(0) - Dw(0) \cdot x| \leq C r^{1+\beta},$$

for $0 < r \leq \dfrac{\delta_0}{2}$. By scaling back, we have

$$\sup_{x \in B_r(x_0)} |u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)| \leq C r^{1+\beta},$$

for $0 < r \leq \dfrac{r_0 \delta_0}{2}$.

For $\dfrac{r_0\delta_0}{2} < r < r_0$, from (4.43), we obtain

$$\begin{aligned} &\sup_{x \in B_r(x_0)} \big| u(x) - u(x_0) - Du(x_0) \cdot (x - x_0) \big| \\ &\qquad \leq \sup_{x \in B_{r_0}(x_0)} \big| u(x) - u(x_0) - Du(x_0) \cdot (x - x_0) \big| \\ &\qquad \leq C r_0^{1+\beta} \\ &\qquad \leq C \left(\frac{2}{\delta_0} \right)^{1+\beta} r^{1+\beta}. \end{aligned}$$

□

Combining the previous results through the limiting process, we finally establish the existence and sharp local regularity of viscosity solutions to equation (1.2).

Proof of Theorem 4.1. By Proposition 4.1, the sequence $\{u_\epsilon\}_\epsilon$ is uniformly bounded, and by Proposition 4.2, it is equicontinuous. Therefore, by the Arzelà–Ascoli Theorem, there exists a continuous function u such that, up to a subsequence, u_ϵ converges locally uniformly to u. By the properties of u_ϵ, the limit function u is nonnegative and bounded. Now, we show that u is a viscosity solution to (1.2). For $x \in \{u > 0\} \cap \Omega$, the continuity of u implies that $u > u(x)/2$ in $B_\delta(x)$, for some $\delta > 0$. Then by the uniform convergence of u_ϵ to u, we obtain $u_\epsilon > u(x)/4 > (\sigma_0 + 1)\epsilon^{1+\alpha}$ in $B_\delta(x)$, for sufficiently small ϵ. By the definition of β_ϵ, we know that u_ϵ satisfies

$$|Du_\epsilon|^{\kappa(x)} F(D^2 u_\epsilon) - \epsilon u_\epsilon + \frac{\sigma_0 \epsilon^{2+\alpha}}{2} = \gamma(x) u_\epsilon^{\gamma(x)-1} \text{ in } B_\delta(x),$$

in the viscosity sense. Taking the limit as $\epsilon \to 0$ and using the stability of viscosity solutions, we conclude that u is a viscosity solution to (1.2). The regularity along the free boundary with the estimate (4.3) follows from (4.23) and the limiting process. Similarly, the local regularity result with estimate (4.2) follows from (4.42) and the limiting process.

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