MEASURE AND INTEGRATION ON σ-SUBLOCALES

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ABSTRACT. This paper deals with aspects of measure and integration in the context of point-free topology. It establishes the Lebesgue integral for general functions on σ-locales, extending the author's study for simple functions in [5]. The integral is described with respect to a measure defined on the coframe of all σ-sublocales. This makes it possible to define the notion of integrable function for any function and to compute the integral over any σ-sublocale of L. In particular, it is shown how the new point-free integral generalises the classical Lebesgue integral.

1. INTRODUCTION

In 2012, by replacing "subsets" with "σ-sublocales", Simpson ([10]) provided a method to extend a measure on a σ-locale L to a measure on the coframe $\mathcal{S}(L)$ of all σ-sublocales of L. This turned out to be a remarkable alternative to overcome some classical paradoxes, such as the ones of Vitali [11] and Banach-Tarski [1]. Moreover, since a measure on $\mathcal{S}(L)$ assigns a value to every σ-sublocale of L, this suggests that one can drop any formal notion of measurability.

Inspired by this viewpoint, we have tried to develop an approach to measure theory in the framework of point-free topology. We aim to describe the integral with respect to measures defined on coframes and to extend the notion of an integrable function (usually reserved for measurable functions) to general functions. The novelty of this work lies in the fact that it goes beyond the constraints of Boolean algebras and a specified notion of "measurability".

Our underlying idea is to think of σ-locales as "generalised measurable spaces". Let L be a *σ-frame* (=*σ-locale*), that is, a lattice with joins of all countable subsets $A \subseteq L$, satisfying the distributive law

$$\Big(\bigvee_{a\in A} a\Big) \wedge b = \bigvee_{a\in A} (a \wedge b)$$

for every countable $A \subseteq L$ and $b \in L$. A map between σ-frames that preserves finite meets and countable joins is called a *σ-frame homomorphism.*

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Let $\sigma\mathsf{Frm}$ be the category of σ-frames and σ-frame homomorphisms. The opposite category $\sigma\mathsf{Loc} = \sigma\mathsf{Frm}^{op}$ is the category of *σ-locales* and *σ-localic maps*.

In stark contrast to subobjects in the category of locales and localic maps (called *sublocales* [9]), in general, a σ-sublocale does not have a concrete description as a subset of L. A subobject S of an object L in $\sigma\mathsf{Loc}$, known as a *σ-sublocale*, is described by a σ-frame quotient L/θ_S given by a *σ-frame congruence* θ_S on L, that is, an equivalence relation on L satisfying the congruence properties

(C1) $(x,y),(x',y') \in \theta_S \;\Rightarrow\; (x \wedge x', y \wedge y') \in \theta_S$,
(C2) $(x_a, y_a) \in \theta_S$ $(a \in A, A = \text{countable}) \;\Rightarrow\; \big(\bigvee_{a\in A} x_a, \bigvee_{a\in A} y_a\big) \in \theta_S$.

The set $\mathcal{C}(L)$ of all congruences on a σ-frame L ordered by inclusion is a frame [8]. Hence, the dual lattice $\mathcal{S}(L) = \mathcal{C}(L)^{op}$ of all σ-sublocales of L is a coframe.

In particular, the *open* and *closed* σ-sublocales associated with an element $a \in L$ are represented, respectively, by the open and closed congruences

$$\Delta_a \coloneqq \{(x,y) \in L \times L \mid x \wedge a = y \wedge a\},$$

$$\nabla_a \coloneqq \{(x,y) \in L \times L \mid x \vee a = y \vee a\}.$$

The σ-sublocales of the form $\mathfrak{o}(a) \coloneqq L/\Delta_a$ and $\mathfrak{c}(a) \coloneqq L/\nabla_a$ play a relevant role in this work as $\nabla(a \mapsto \nabla_a)\colon L \to \nabla[L]$ embeds L in $\mathcal{C}(L)$ and $\mathfrak{o}(a \mapsto \mathfrak{o}(a))\colon L \to \mathfrak{o}[L]$ embeds L in $\mathcal{S}(L)$.

With this framework in mind, we initiated this research program with [3], where we studied measurable functions in the point-free setting. A *measurable real function* on a σ-locale L (previously mentioned in [2] as a σ-continuous map) is a σ-frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$ from the frame of reals into L. Similarly, a *measurable extended real function* on L is a σ-frame homomorphism $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ from the frame of extended reals into L. Let $\mathsf{M}(L)$ and $\overline{\mathsf{M}}(L)$ be the sets of all measurable real functions and all measurable extended real functions on L, respectively. We denote by

$$\mathsf{F}(L) \coloneqq \mathsf{M}(\mathcal{C}(L)) = \sigma\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{C}(L))$$
$$\text{and} \quad \overline{\mathsf{F}}(L) \coloneqq \overline{\mathsf{M}}(\mathcal{C}(L)) = \sigma\mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{C}(L))$$

the sets of all arbitrary real functions and arbitrary extended real functions on a σ-frame L. Identifying each $f \in \overline{\mathsf{M}}(L)$ with $\nabla \circ f \in \overline{\mathsf{F}}(L)$, we have $\overline{\mathsf{M}}(L) \subseteq \overline{\mathsf{F}}(L)$. Moreover, an $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ is measurable on L if and only if

$$f(p,\text{—}),\; f(\text{—},q) \in \nabla[L] \quad \text{for all } p,q \in \mathbb{Q}.$$

In [5], we set out the details of the ring of simple functions and the integral of simple functions. Following [5], in this paper we extend the integral to more general functions, and we show that the integral established in this framework generalises the classical Lebesgue integral.

The paper is structured as follows. In Section 2, we review some of the major points studied in [5] and extend the definition of the integral on σ-locales from simple functions to general functions. In Section 3, we illustrate how this theory can be applied with the standard Lebesgue measure. Then, in Section 4, we study some elementary properties of the integral of general functions. In Section 5, we prove that the indefinite integral of a nonnegative function is a measure on $\mathcal{S}(L)$. In Section 6, we focus on proving a point-free counterpart of the Monotone Converge Theorem. Using it, we deduce a point-free counterpart of Fatou's Lemma in Section 7, and we analyse sufficient conditions for the integral to be additive in Section 8. Finally, in Section 9, we prove that our localic integral is an extension of the classical Lebesgue integral.

It should be emphasized that the extension of the integral from simple functions to general functions addressed in this paper is far from straightforward. Notably, since a coframe is not inherently complemented, the fact that the indefinite integral of a nonnegative function is a measure on $\mathcal{S}(L)$ came as a pleasant surprise. Nevertheless, the formulation of a point-free counterpart of the Monotone Convergence Theorem or Fatou's Lemma, as well as the study of the additivity of the integral, were difficult tasks. Additional conditions are needed, not only to guarantee the existence of the limits or inferior limits but also because the extension of Proposition 4.4 to more general σ-sublocales has been a roundabout pursuit. If we restrict ourselves to functions measurable on L, the results are clean and straightforward. However, the formulations become significantly more intricate when we try to establish these results for a broader class of functions. For that reason, to avoid too many technicalities, in this paper, we focus on the integral over complemented σ-sublocales. The details about the general case can be consulted in [6].

2. The general setting

Our general reference for point-free topology and lattice theory is Picado-Pultr [9]. For σ-frames (σ-locales) and congruences on σ-frames, we use Madden [8] and Frith-Schauerte [7]. We follow our previous paper [5] for the integral of localic simple functions. Many results in it will be used throughout this paper. In this section, we briefly grasp some of its major points, weaving the path to establish the point-free integral for the general case.

For each $\theta_S \in \mathcal{C}_B(L)$ (where $\mathcal{C}_B(L)$ denotes the lattice of complemented elements of $\mathcal{C}(L)$), the *characteristic function associated with* θ_S is the function

$\chi_{\theta_S}\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ given by

$$\chi_{\theta_S}(p,\text{—}) = \begin{cases} 1 & \text{if } p < 0 \\ \theta_S & \text{if } 0 \leq p < 1 \\ 0 & \text{if } p \geq 1 \end{cases} \quad \text{and} \quad \chi_{\theta_S}(\text{—},q) = \begin{cases} 0 & \text{if } q \leq 0 \\ \theta_S^c & \text{if } 0 < q \leq 1 \\ 1 & \text{if } q > 1. \end{cases}$$

We say that a function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ is *simple* if it is a finite linear combination of characteristic functions. In other words, an $f \in \mathsf{F}(L)$ is simple if there exist $n \in \mathbb{N}$, $r_1, \ldots, r_n \in \mathbb{Q}$ and $\theta_{S_1}, \ldots, \theta_{S_n} \in \mathcal{C}_B(L)$, such that

$$f = \sum_{i=1}^{n} r_i \cdot \chi_{\theta^c_{S_i}}.$$

Whenever $r_1 < r_2 < \cdots < r_n$ and $\theta^c_{S_1}, \ldots, \theta^c_{S_n} \in \mathcal{C}_B(L) \smallsetminus \{0\}$ are pairwise disjoint with $\bigvee_{i=1}^{n} \theta^c_{S_i} = 1$, we say that f is written in its *canonical form*. Every simple function has one and only one canonical form.

A simple function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ is *measurable* on L if and only if each θ_{S_i} in its canonical form is a clopen congruence. The sets of all simple functions and all simple functions that are measurable on L are denoted by $\mathsf{SM}(\mathcal{C}(L))$ and $\mathsf{SM}(L)$, respectively, and we have

$$\begin{tikzcd}
& \mathsf{M}(L) \arrow[dr, "\subseteq"] & \\
\mathsf{SM}(L) \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \mathsf{F}(L). \\
& \mathsf{SM}(\mathcal{C}(L)) \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

With the operations induced by $\mathsf{F}(L)$, $\mathsf{SM}(\mathcal{C}(L))$ is a sublattice and a subring of $\mathsf{F}(L)$. Combining [5, Theorem 6.2] and [5, Lemma 6.1], we state an important result about simple functions that will be crucial later on:

Proposition 2.1. *Let $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ be a nonnegative function. If any countable join in $\{f(p,\text{—}) \mid p \in \mathbb{Q}\}$ is complemented in $\mathcal{C}(L)$, then we can write*

$$f = \lim_{k\to+\infty} f_k = \sup_{k\in\mathbb{N}} f_k,$$

where $(f_k)_{k\in\mathbb{N}}$ is an increasing sequence in $\mathsf{SM}(\mathcal{C}(L))$ such that

$$\begin{cases} \forall p \in \mathbb{Q}, \exists\, r > p \,(r \in \mathbb{Q}) : f_k(p,\text{—}) = f(r,\text{—}); \\ f(t,\text{—}) = \bigvee_{k\in\mathbb{N}} f_k(t,\text{—}). \end{cases}$$

Moreover, $(f_k)_{k\in\mathbb{N}}$ is a sequence in $\mathsf{SM}(L)$ if $f(p,\text{—}) \in \nabla[L] \cap \Delta[L]$ for all $p \in \mathbb{Q}$.

Let μ be a measure on $\mathcal{S}(L)$. Recall that a *measure* on a join-σ-complete lattice M ([10]) is a map $\mu\colon M \to [0,+\infty]$ satisfying

(M1) $\mu(0_M) = 0$;

(M2) $\forall x, y \in M,\ x \leq y \Rightarrow \mu(x) \leq \mu(y)$;
(M3) $\forall x, y \in M,\ \mu(x) + \mu(y) = \mu(x \vee y) + \mu(x \wedge y)$;
(M4) $\forall (x_i)_{i\in\mathbb{N}} \subseteq M,\ \forall i \in \mathbb{N},\ x_i \leq x_{i+1} \Rightarrow \mu(\bigvee_{i\in\mathbb{N}} x_i) = \sup_{i\in\mathbb{N}} \mu(x_i)$.

Given a nonnegative simple function $g \in \mathsf{SM}(\mathcal{C}(L))$ with canonical representation

$$g = \sum_{i=1}^{n} r_i \cdot \chi_{\theta^c_{S_i}},$$

we established in [5] the *μ-integral of g over a σ-sublocale S* as the value

$$\int_S g\, d\mu := \sum_{i=1}^{n} r_i \mu(S_i \wedge S).$$

The integral of g does not take an indeterminate form because each $r_i \geq 0$ and, by convention, we consider $0 \cdot \infty := 0$. In addition, we proved that for any representation $g = \sum_{i=1}^{m} s_i \cdot \chi_{\theta^c_{T_i}}$ with $\theta^c_{T_1}, \dots, \theta^c_{T_m}$ pairwise disjoint,

$$\int_S g\, d\mu = \sum_{i=1}^{n} r_i \mu(S_i \wedge S) = \sum_{i=1}^{m} s_i \mu(T_i \wedge S).$$

Now we will extend this integral to more general functions. This will be done in two steps. First, let us focus on the nonnegative case. A nonnegative function $f \in \overline{\mathsf{F}}(L)$ can be approximated via simple functions by Proposition 2.1. Thus, we are encouraged to approximate the integral of f by the integral of simple functions.

Definition 2.2 (**Integral of a nonnegative function).** Given a nonnegative $f \in \overline{\mathsf{F}}(L)$, the *$\mu$-integral of f over an $S \in \mathcal{S}(L)$* is defined by

$$\int_S f\, d\mu := \sup\Big\{ \int_S g\, d\mu \mid \mathbf{0} \leq g \leq f,\ g \in \mathsf{SM}(\mathcal{C}(L)) \Big\}.$$

The μ-integral of f over $L = 1_{\mathcal{S}(L)}$ is called the *μ-integral of f* and we write it as $\int f\, d\mu$.

This definition generalises the definition presented in [5] for nonnegative simple functions. In fact, if f is a nonnegative simple function, it is clear that

$$\int_S^{\text{Def. } 2.2} f\, d\mu \geq \int_S f\, d\mu.$$

Conversely, since the integral of a nonnegative simple function preserves the inequality of the functions ([5, Proposition 8.9]), we have that for each simple function g satisfying $\mathbf{0} \leq g \leq f$,

$$\int_S g\, d\mu \leq \int_S f\, d\mu.$$

We further point out that for any nonnegative function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ and $S \in \mathcal{S}(L)$, we have

$$\int_S f\, d\mu \geq 0$$

because the integral of a nonnegative simple function is always nonnegative ([5, Section 7]).

In the second and final step, we extend the integral to more general functions as follows. Each $f \in \overline{\mathsf{F}}(L)$ can be written as

$$f = f^+ - f^-,$$

with $f^+ := f \vee \mathbf{0}$ and $f^- := (-f) \vee \mathbf{0}$ both nonnegative functions on L (see [5]). This suggests that we can compute the integral of f via the integrals of f^+ and f^-.

Definition 2.3 (Integral of a general function). A function $f \in \overline{\mathsf{F}}(L)$ is *μ-integrable over $S \in \mathcal{S}(L)$* if

$$\int_S f^+\, d\mu < \infty \qquad \text{or} \qquad \int_S f^-\, d\mu < \infty,$$

and its *μ-integral over S* is given by

$$\int_S f\, d\mu := \int_S f^+\, d\mu - \int_S f^-\, d\mu.$$

The μ-integral of f over $L = 1_{\mathcal{S}(L)}$ is called the *μ-integral of f*. (When there is no ambiguity, we drop the prefix μ.)

A nonnegative function f is always integrable over any $S \in \mathcal{S}(L)$ because $f^- = \mathbf{0}$. Moreover, Definition 2.3 is a generalisation of 2.2 in the sense that

$$\int_S^{\text{Def. } 2.3} f\, d\mu = \int_S f^+\, d\mu - \int_S f^-\, d\mu = \int_S f\, d\mu - 0 = \int_S^{\text{Def. } 2.2} f\, d\mu.$$

That being so, it also generalises the integral of a general simple function established in [5, Definition 8.1].

It is important to note that this definition allows the integral to be $+\infty$ or $-\infty$. An integrable function whose integral is finite is called summable. More precisely, an $f \in \overline{\mathsf{F}}(L)$ is *summable over $S \in \mathcal{S}(L)$* if

$$\int_S f^+\, d\mu < \infty \qquad \text{and} \qquad \int_S f^-\, d\mu < \infty.$$

We say that f is *summable* if it is summable over $L = 1_{\mathcal{S}(L)}$.

3. Illustrative example: Lebesgue measure

Before going into the details and results of the point-free integral, we will try to demonstrate the power of this approach. We illustrate it with an example of how to approach the problem of applying this theory.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\Omega(\mathbb{R}^n)$ its lattice of open sets. We denote the Lebesgue measure on $\mathbb{R}^n$ and the class of Lebesgue measurable subsets of $\mathbb{R}^n$ by $\mathfrak{L}^n$ and $\mathfrak{L}(\mathbb{R}^n)$, respectively.

Take the measure space $(\mathbb{R}^n, \mathfrak{L}(\mathbb{R}^n), \mathfrak{L}^n)$. The map $\mathfrak{L}^n\colon \mathfrak{L}(\mathbb{R}^n) \to [0,+\infty]$ is a measure in the classical sense (i.e., it is a σ-additive map with $\mathfrak{L}^n(\emptyset) = 0$). As a consequence of $\mathfrak{L}(\mathbb{R}^n)$ being a σ-algebra, $\mathfrak{L}^n$ is also a measure in the sense of a measure on a join-σ-complete lattice L for $L = \mathfrak{L}(\mathbb{R}^n)$.

In addition, recall that given a σ-frame L, the open neighbourhood filter of $S \in \mathcal{S}(L)$ is given by

$$\mathcal{N}(S) := \{a \in L \mid S \subseteq \mathfrak{o}(a)\},$$

and we say that the σ-frame L is fit if for every σ-sublocale $S \in \mathcal{S}(L)$,

$$S = \bigwedge_{a\in\mathcal{N}(S)} \mathfrak{o}(a).$$

Every element of $\mathfrak{L}(\mathbb{R}^n)$ is complemented. Consequently, $\mathfrak{L}(\mathbb{R}^n)$ is regular and therefore fit (see [10]). Thus, applying Simpson's [10, Theorem 1], the outer measure

$$\begin{aligned}(\mathfrak{L}^n)^\diamond\colon \mathcal{S}(\mathfrak{L}(\mathbb{R}^n)) &\longrightarrow [0,+\infty]\\ S &\longmapsto (\mathfrak{L}^n)^\diamond(S) := \inf_{A\in\mathcal{N}(S)} \mathfrak{L}^n(A)\end{aligned}$$

associated with $\mathfrak{L}^n$ is a measure on the coframe $\mathcal{S}(\mathfrak{L}(\mathbb{R}^n))$. In particular, for all $A \in \mathfrak{L}(\mathbb{R}^n)$, we have

$$(\mathfrak{L}^n)^\diamond(\mathfrak{o}(A)) = \mathfrak{L}^n(A).$$

Hence, we can regard $(\mathfrak{L}^n)^\diamond$ as an extension of $\mathfrak{L}^n$ to $\mathcal{S}(\mathfrak{L}(\mathbb{R}^n))$ through the isomorphism $\mathfrak{L}(\mathbb{R}^n) \cong \mathfrak{o}[\mathfrak{L}(\mathbb{R}^n)]$, where $\mathfrak{o}[\mathfrak{L}(\mathbb{R}^n)]$ is the set of open σ-sublocales of $\mathfrak{L}(\mathbb{R}^n)$.

Now, for each subset $Y \subseteq \mathbb{R}^n$, take the *congruence θ_Y induced by Y on* $\mathfrak{L}(\mathbb{R}^n)$, which is defined by

$$\theta_Y := \{(U,V) \in \mathfrak{L}(\mathbb{R}^n) \times \mathfrak{L}(\mathbb{R}^n) \mid U \cap Y = V \cap Y\} \in \mathcal{C}(\mathfrak{L}(\mathbb{R}^n)).$$

The correspondence $Y \mapsto \theta_Y$ is injective if and only if $(\mathbb{R}^n, \mathfrak{L}(\mathbb{R}^n))$ is a T_D σ-space, that is, if for each $x \in \mathbb{R}^n$, there exists $U_x \in \mathfrak{L}(\mathbb{R}^n)$ containing x such that $U_x \setminus \{x\} \in \mathfrak{L}(\mathbb{R}^n)$ [4]. But as $\emptyset, \{x\} \in \mathfrak{L}^n(\mathbb{R})$ (for any $x \in \mathbb{R}^n$), it is clear that $(\mathbb{R}^n, \mathfrak{L}(\mathbb{R}^n))$ is T_D. (More generally, any Hausdorff σ-space is a

T_D σ-space.) As a result, there is a bijection between the subsets of $\mathbb{R}^n$ and the induced congruences on $\mathfrak{L}(\mathbb{R}^n)$, which yields the inclusion

$$\mathcal{P}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathfrak{L}(\mathbb{R}^n)) = \mathcal{C}(\mathfrak{L}(\mathbb{R}^n))^{op}$$

through the mapping $Y \mapsto S_Y$, where $S_Y := L/\theta_Y$. This mapping preserves arbitrary joins but not necessarily finite meets. Setting

$$(\mathfrak{L}^n)^\diamond(Y) := (\mathfrak{L}^n)^\diamond(S_Y),$$

the measure $(\mathfrak{L}^n)^\diamond$ assigns a value to each subset $Y \in \mathbb{R}^n$ (and we can also prove that it extends the classic Lebesgue outer measure).

In summary, applying Simpson's idea, we were able to extend the standard Lebesgue measure $\mathfrak{L}^n$ to a measure $(\mathfrak{L}^n)^\diamond$ on $\mathcal{S}(\mathfrak{L}(\mathbb{R}^n))$. Then, using the theory presented in the next sections, we will be able to apply the measure $(\mathfrak{L}^n)^\diamond$ to compute a point-free $(\mathfrak{L}^n)^\diamond$-integral extending the classic $\mathfrak{L}^n$-integral in two different senses.

First, since we get a wider class of "integrable" functions in the point-free setting, we can regard the class of $\mathfrak{L}^n$-integrable functions as a subset of the class of $(\mathfrak{L}^n)^\diamond$-integrable functions (see Section 9 for more details). Let us denote such inclusion by i. By the end of this paper, we will see that for any $\mathfrak{L}^n$-integrable $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}$ and $A \in \mathfrak{L}(\mathbb{R}^n)$,

$$\int_A \tilde{f}\, d\mathfrak{L}^n = \int_{\mathfrak{o}(A)} i(\tilde{f})\, d(\mathfrak{L}^n)^\diamond,$$

where the member on the left side represents the standard $\mathfrak{L}^n$-integral and the member on the right denotes the point-free $(\mathfrak{L}^n)^\diamond$- integral.

Finally, while in the classical theory, we can only define the $\mathfrak{L}^n$-integral over an $A \in \mathfrak{L}^n(\mathbb{R})$, in the point-free framework we can define the $(\mathfrak{L}^n)^\diamond$-integral over any $S \in \mathcal{S}(\mathfrak{L}^n(\mathbb{R}))$. And for a nonnegative measurable function $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}$, the standard indefinite integral

$$\begin{aligned} \mathfrak{L}(\mathbb{R}^n) &\to [0, +\infty] \\ A &\mapsto \int_A \tilde{f}\, d\mathfrak{L}^n \end{aligned}$$

is extended to a measure

$$\begin{aligned} \mathcal{S}(\mathfrak{L}(\mathbb{R}^n)) &\to [0, +\infty] \\ S &\mapsto \int_S i(\tilde{f})\, d(\mathfrak{L}^n)^\diamond, \end{aligned}$$

regarding $\mathfrak{L}^n(\mathbb{R}) \subset \mathcal{P}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathfrak{L}^n(\mathbb{R}))$ through the correspondence $Y \mapsto S_Y$.

Remark 3.1. In the classical theory, the famous paradoxes found by Vitali [11] and Banach and Tarski [1] show that it is impossible to define a nonzero measure on all subsets of $\mathbb{R}^n$, for $n = 1$ and $n = 3$ (respectively), invariant under the Euclidean isometries. Nonetheless, the measure $(\mathfrak{L}^n)^\diamond$ not only assigns a value to each subset of $\mathbb{R}^n$ but we can also prove that it is invariant under Euclidean transformations (following an idea similar to the one used in [10, Example 4.7]).

Remark 3.2. In [10], Simpson proposed a point-free version of the Lebesgue measure extending the restriction of $\mathfrak{L}^n$ to $\Omega(\mathbb{R}^n)$ to a measure λ_n^* on the coframe of all sublocales of $\Omega(\mathbb{R}^n)$. However, although λ_n^* has the advantage of dealing only with the theory of frames and locales (because, as $\Omega(\mathbb{R}^n)$ is a strongly Lindelöf σ-frame, we can drop the prefix σ), for each characteristic function χ_Y where Y is Lebesgue measurable but not open, χ_Y does not have a point-free counterpart as a localic measurable function on $\Omega(\mathbb{R}^n)$. The measure λ_n^* is then suitable to generalise the $\mathfrak{L}^n_{|\Omega(\mathbb{R}^n)}$-integral but not the $\mathfrak{L}^n$-integral. For that reason, we sought a different approach to get a point-free version of the Lebesgue measure.

4. Basic properties

From now on, we will work on an arbitrary σ-frame L and with a measure μ on $\mathcal{S}(L)$. We start by proving some basic properties of the integral of a general function $f \in \overline{\mathsf{F}}(L)$. In most cases, the proofs follow a set pattern:

(1) First, we focus on showing the result for nonnegative simple functions (when applicable, see [5] for the proofs);
(2) Then, we prove it for nonnegative functions;
(3) Lastly, we extend it to the general case.

Proposition 4.1. *Let $f \in \overline{\mathsf{F}}(L)$ and $\lambda \in \mathbb{Q}$. If f is integrable over $S \in \mathcal{S}(L)$, then $\lambda \cdot f$ is also integrable over S and*

$$\int_S \lambda \cdot f \, d\mu = \lambda \int_S f \, d\mu.$$

Proof. If $\lambda = 0$, then $\lambda \cdot f = \mathbf{0}$ and the result holds trivially. Suppose that $\lambda > 0$. Let f be a nonnegative function. Since the claim holds for nonnegative simple functions by [5, Proposition 8.7], we get

$$\lambda \int_S f \, d\mu = \sup\Big\{ \int_S \lambda \cdot g \, d\mu \mid \mathbf{0} \le g \le f,\ g \in \mathsf{SM}(\mathcal{C}(L)) \Big\} \le \int_S \lambda \cdot f \, d\mu.$$

Thus,

$$\tfrac{1}{\lambda} \int_S \lambda \cdot f \, d\mu \le \int_S \tfrac{1}{\lambda} \cdot (\lambda \cdot f) \, d\mu = \int_S f \, d\mu,$$

which implies that

$$\lambda \int_S f \, d\mu \le \int_S \lambda \cdot f \, d\mu = \lambda \cdot \tfrac{1}{\lambda} \int_S \lambda \cdot f \, d\mu \le \lambda \int_S f \, d\mu.$$

Now, take an $f \in \overline{\mathsf{F}}(L)$ integrable over S. Since $(\lambda \cdot f)^+ = \lambda \cdot f^+$, $(\lambda \cdot f)^- = \lambda \cdot f^-$ and f^+, f^- are nonnegative, we have

$$\int_S (\lambda \cdot f)^+ \, d\mu = \lambda \int_S f^+ \, d\mu \qquad \text{and} \qquad \int_S (\lambda \cdot f)^- \, d\mu = \lambda \int_S f^- \, d\mu.$$

Therefore, $\lambda \cdot f$ is also integrable over S and

$$\int_S \lambda \cdot f \, d\mu = \lambda \int_S f^+ \, d\mu - \lambda \int_S f^- \, d\mu = \lambda \int_S f \, d\mu.$$

Finally, suppose that $\lambda < 0$. The statement is a consequence of the case for a positive scalar because $\lambda \cdot f = (-\lambda) \cdot (-f)$, where $-\lambda > 0$ is nonnegative, $(\lambda \cdot f)^+ = (-\lambda) \cdot f^-$ and $(\lambda \cdot f)^- = (-\lambda) \cdot f^+$. □

The following result will often be helpful in determining an integral over a complemented σ-sublocale.

Proposition 4.2. *If $f \in \overline{\mathsf{F}}(L)$ is integrable over a complemented $S \in \mathcal{S}(L)$, we have*

$$\int_S f \, d\mu = \int f \cdot \chi_{\theta_S^c} \, d\mu.$$

Proof. Let f be a nonnegative function. As the claim holds for a nonnegative simple function ([5, Proposition 8.5]),

$$\int_S f \, d\mu = \sup\Big\{ \int g \cdot \chi_{\theta_S^c} \, d\mu \mid \mathbf{0} \le g \le f,\ g \in \mathsf{SM}(\mathcal{C}(L)) \Big\} \le \int f \cdot \chi_{\theta_S^c} \, d\mu.$$

Conversely, for any $h \in \mathsf{F}(L)$, $\mathbf{0} \le h \le f \cdot \chi_{\theta_S^c}$ if and only if $h(\text{—}, 0) = 0$ and $(f \cdot \chi_{\theta_S^c} - h)(\text{—}, 0) = 0$. Moreover, $(f \cdot \chi_{\theta_S^c} - h)(\text{—}, 0) = 0$ if and only if $\theta_S^c \wedge (f - h)(\text{—}, 0) = 0$ and $\theta_S \wedge h(0, \text{—}) = 0$, with the latter implying that $h \cdot \chi_{\theta_S} = \mathbf{0}$. Hence, for every simple function $h \in \mathsf{SM}(\mathcal{C}(L))$ satisfying $\mathbf{0} \le h \le f \cdot \chi_{\theta_S^c}$, we have $h = h \cdot \mathbf{1} = h \cdot \chi_{\theta_S \vee \theta_S^c} = h \cdot (\chi_{\theta_S} + \chi_{\theta_S^c})$ and

$$\begin{aligned} \int h \, d\mu &= \int h \cdot (\chi_{\theta_S} + \chi_{\theta_S^c}) \, d\mu \\ &= \int h \cdot \chi_{\theta_S} \, d\mu + \int h \cdot \chi_{\theta_S^c} \, d\mu \\ &= \int h \cdot \chi_{\theta_S^c} \, d\mu \\ &= \int_S h \, d\mu, \end{aligned}$$

where the second equality follows from the fact that $h, h\cdot\chi_{\theta_S}$ and $h\cdot\chi_{\theta_S^c}$ are all nonnegative simple functions ([5, Proposition 8.8]). As a consequence,

$$\begin{aligned}\int f\cdot\chi_{\theta_S^c}\,d\mu &= \sup\Big\{\int h\,d\mu \mid \mathbf{0}\le h\le f\cdot\chi_{\theta_S^c},\ h\in\mathsf{SM}(\mathcal{C}(L))\Big\}\\ &= \sup\Big\{\int_S h\,d\mu \mid \mathbf{0}\le h\le f\cdot\chi_{\theta_S^c},\ h\in\mathsf{SM}(\mathcal{C}(L))\Big\}\\ &\le \sup\Big\{\int_S h\,d\mu \mid \mathbf{0}\le h\le f,\ h\in\mathsf{SM}(\mathcal{C}(L))\Big\}\\ &= \int_S f\,d\mu.\end{aligned}$$

If $f\in\overline{\mathsf{F}}(L)$ is a general function integrable over S, since f^+ and f^- are nonnegative,

$$\int_S f^+\,d\mu = \int f^+\cdot\chi_{\theta_S^c}\,d\mu \qquad\text{and}\qquad \int_S f^-\,d\mu = \int f^-\cdot\chi_{\theta_S^c}\,d\mu.$$

The claim follows from the fact that $f^+\cdot\chi_{\theta_S^c} = (f\cdot\chi_{\theta_S^c})^+$ and $f^-\cdot\chi_{\theta_S^c} = (f\cdot\chi_{\theta_S^c})^-$. $\square$

The integral of a general function preserves the inequality of the functions:

Proposition 4.3. *Let $f,g\in\overline{\mathsf{F}}(L)$ be integrable over $S\in\mathcal{S}(L)$. If $f\le g$, then*

$$\int_S f\,d\mu \le \int_S g\,d\mu.$$

Proof. If f and g are nonnegative, then from Definition 2.2 and $f\le g$, it is straightforward that

$$\int_S f\,d\mu \le \int_S g\,d\mu.$$

If f,g are general functions integrable over S, it follows from $f\le g$ that $f^+\le g^+$ and $g^-\le f^-$. Thus, applying the nonnegative case, we get that

$$\int_S f\,d\mu = \int_S f^+\,d\mu - \int_S f^-\,d\mu \le \int_S g^+\,d\mu - \int_S g^-\,d\mu = \int_S g\,d\mu. \qquad\square$$

Nevertheless, although $f\le g$ is a sufficient condition, it might not be necessary. In fact, for any complemented σ-sublocale S, if $\theta_S^c\wedge(g-f)(\text{—},0) = 0$ the inequality still holds.

Proposition 4.4. *Let $f,g\in\overline{\mathsf{F}}(L)$ be integrable over a complemented $S\in\mathcal{S}(L)$. Suppose that g and $-f$ are sum-compatible. If $\theta_S^c\wedge(g-f)(\text{—},0)=0$, then*

$$\int_S f\,d\mu \le \int_S g\,d\mu.$$

Proof. The case where f, g are simple and nonnegative was already proved in [5, Proposition 8.10]. Suppose that f, g are nonnegative functions. Note that

$$0 = \theta_S^c \wedge (g-f)(\text{—},0) = ((g-f)\cdot \chi_{\theta_S^c})(\text{—},0) = (g\cdot\chi_{\theta_S^c} - f\cdot\chi_{\theta_S^c})(\text{—},0),$$

which means that $f\cdot\chi_{\theta_S^c} \leq g\cdot\chi_{\theta_S^c}$. Hence, by Proposition 4.2,

$$\int_S f\,d\mu = \int f\cdot\chi_{\theta_S^c}\,d\mu \leq \int g\cdot\chi_{\theta_S^c}\,d\mu = \int_S g\,d\mu.$$

Finally, take $f, g \in \overline{\mathsf{F}}(L)$ integrable over S. If g and $-f$ are sum-compatible, then g^+ and $-f^+$ are sum-compatible, as well as f^- and $-g^-$. Moreover, $\theta_S^c \wedge (g-f)(\text{—},0) = 0$ implies that $\theta_S^c \wedge (g^+ - f^+)(\text{—},0) = 0$ and $\theta_S^c \wedge (f^- - g^-)(\text{—},0) = 0$ because

$$\theta_S^c \wedge (g^+ - f^+)(\text{—},0) \vee \theta_S^c \wedge (f^- - g^-)(\text{—},0) \leq \theta_S^c \wedge (g-f)(\text{—},0) = 0.$$

As a consequence, it follows from the nonnegative case that

$$\int_S f^+\,d\mu \leq \int_S g^+\,d\mu \qquad \text{and} \qquad \int_S g^-\,d\mu \leq \int_S f^-\,d\mu.$$

Therefore,

$$\int_S f\,d\mu = \int_S f^+\,d\mu - \int_S f^-\,d\mu \leq \int_S g^+\,d\mu - \int_S g^-\,d\mu = \int_S g\,d\mu. \qquad \square$$

Similarly, a function need not be nonnegative for its integral over S to be nonnegative. Setting $f = \mathbf{0}$ in the previous proposition gives the following corollary:

Corollary 4.5. *Let $f \in \overline{\mathsf{F}}(L)$ be integrable over a complemented $S \in \mathcal{S}(L)$. If $\theta_S^c \wedge f(\text{—},0) = 0$, then*

$$\int_S f\,d\mu \geq 0.$$

Remark 4.6. In case S is not complemented, it is still possible to establish some conditions under which the inequality

$$\int_S f\,d\mu \leq \int_S g\,d\mu$$

holds without requiring that $f \leq g$. The details can be found in [6].

5. The indefinite integral

Given an $f \in \overline{\mathsf{F}}(L)$ integrable over any σ-sublocale $S \in \mathcal{S}(L)$, the *indefinite integral* of f is the map $\eta\colon \mathcal{S}(L) \to [-\infty, +\infty]$ defined by

$$\eta(S) := \int_S f\,d\mu.$$

By Corollary 4.5, the codomain of η can be restricted to $[0,+\infty]$ whenever f is nonnegative. In this section, we want to show that the indefinite integral of any nonnegative function is a measure on $\mathcal{S}(L)$.

Lemma 5.1. *If $f \in \overline{\mathsf{F}}(L)$ is nonnegative and $S,T \in \mathcal{S}(L)$ are such that $S \leq T$, then*

$$\int_S f\, d\mu \leq \int_T f\, d\mu.$$

Proof. By [5, Proposition 7.5], the claim holds for any nonnegative simple function. If f is a (general) nonnegative function, not only it is integrable over S and T, but it also satisfies

$$\begin{aligned}\int_S f\, d\mu &= \sup\Big\{\int_S g\, d\mu \mid \mathbf{0} \leq g \leq f,\ g \in \mathsf{SM}(\mathcal{C}(L))\Big\}\\ &\leq \sup\Big\{\int_T g\, d\mu \mid \mathbf{0} \leq g \leq f,\ g \in \mathsf{SM}(\mathcal{C}(L))\Big\} = \int_T f\, d\mu. \qquad \square\end{aligned}$$

Consequently, an $f \in \overline{\mathsf{F}}(L)$ is integrable over any σ-sublocale $S \in \mathcal{S}(L)$ whenever f is integrable over L. In fact, as f^+ and f^- are nonnegative functions and $L = 1_{\mathcal{S}(L)}$, we have

$$\int_S f^+\, d\mu \leq \int_L f^+\, d\mu \qquad \text{and} \qquad \int_S f^-\, d\mu \leq \int_L f^-\, d\mu$$

for all $S \in \mathcal{S}(L)$. To simplify the statements, the next lemmas will be formulated for functions that are integrable over L.

Lemma 5.2. *For any $f \in \overline{\mathsf{F}}(L)$ integrable over L and any $S,T \in \mathcal{S}(L)$,*

$$\int_S f\, d\mu + \int_T f\, d\mu = \int_{S\vee T} f\, d\mu + \int_{S\wedge T} f\, d\mu.$$

Proof. Suppose that f is nonnegative and set

$$\mathfrak{S}_f \coloneqq \{g \in \mathsf{SM}(\mathcal{C}(L)) \mid \mathbf{0} \leq g \leq f\}.$$

Then

$$\begin{aligned}\int_S f\, d\mu + \int_T f\, d\mu &= \sup\Big\{\int_S g\, d\mu + \int_T h\, d\mu \mid g,h \in \mathfrak{S}_f\Big\} \text{ and}\\ \int_{S\vee T} f\, d\mu + \int_{S\wedge T} f\, d\mu &= \sup\Big\{\int_{S\vee T} g\, d\mu + \int_{S\wedge T} h\, d\mu \mid g,h \in \mathfrak{S}_f\Big\}.\end{aligned}$$

Since the claim holds for simple functions ([5, Lemma 9.1]) and $\mathsf{SM}(\mathcal{C}(L))$ is a sublattice of $\mathsf{F}(L)$ (so $g, h \in \mathfrak{S}_f$ implies that $g \vee f \in \mathfrak{S}_f$), we have

$$\begin{aligned}\int_S f\,d\mu + \int_T f\,d\mu &\leq \sup\Big\{\int_S g\vee h\,d\mu + \int_T g\vee h\,d\mu \mid g,h\in\mathfrak{S}_f\Big\}\\ &= \sup\Big\{\int_{S\vee T} g\vee h\,d\mu + \int_{S\wedge T} g\vee h\,d\mu \mid g,h\in\mathfrak{S}_f\Big\}\\ &\leq \int_{S\vee T} f\,d\mu + \int_{S\wedge T} f\,d\mu.\end{aligned}$$

Conversely, similar arguments yield the opposite inequality:

$$\begin{aligned}\int_{S\vee T} f\,d\mu + \int_{S\wedge T} f\,d\mu &\leq \sup\Big\{\int_{S\vee T} g\vee h\,d\mu + \int_{S\wedge T} g\vee h\,d\mu \mid g,h\in\mathfrak{S}_f\Big\}\\ &= \sup\Big\{\int_S g\vee h\,d\mu + \int_T g\vee h\,d\mu \mid g,h\in\mathfrak{S}_f\Big\}\\ &\leq \int_S f\,d\mu + \int_T f\,d\mu.\end{aligned}$$

Therefore, the claim also holds for nonnegative functions and whenever f is a general function integrable over L,

$$\begin{aligned}\int_S f\,d\mu + \int_T f\,d\mu &= \int_S f^+\,d\mu - \int_S f^-\,d\mu + \int_T f^+\,d\mu - \int_T f^-\,d\mu\\ &= \int_{S\vee T} f^+\,d\mu + \int_{S\wedge T} f^+\,d\mu - \int_{S\vee T} f^-\,d\mu - \int_{S\wedge T} f^-\,d\mu\\ &= \int_{S\vee T} f\,d\mu + \int_{S\wedge T} f\,d\mu. \qquad \square\end{aligned}$$

Lemma 5.3. *Let $f \in \overline{\mathsf{F}}(L)$ be integrable over L. If $(B_k)_{k\in\mathbb{N}}$ is an increasing sequence in $\mathcal{S}(L)$ with $B = \bigvee_{k\in\mathbb{N}} B_k$, then*

$$\int_B f\,d\mu = \lim_{k\to+\infty} \int_{B_k} f\,d\mu.$$

In particular, if f is nonnegative,

$$\int_B f\,d\mu = \sup_{k\in\mathbb{N}} \int_{B_k} f\,d\mu.$$

Proof. We have already proved that the claim holds whenever f is a simple function ([5, Lemma 9.2]). Now, suppose that f is a nonnegative function. For each $S \in \mathcal{S}(L)$, f is integrable over S and we can write

$$\int_S f\,d\mu = \sup\Big\{\int_S g\,d\mu \mid \mathbf{0}\leq g\leq f,\ g\in\mathsf{SM}(\mathcal{C}(L))\Big\} = \sup_{g\in\mathfrak{S}_f}\int_S g\,d\mu,$$

where $\mathfrak{S}_f := \{g \in \mathsf{SM}(\mathcal{C}(L)) \mid \mathbf{0} \leq g \leq f\}$. Hence,

$$\lim_{k\to+\infty} \int_{B_k} f\, d\mu = \lim_{k\to+\infty} \sup_{g\in\mathfrak{S}_f} \int_{B_k} g\, d\mu = \sup_{k\in\mathbb{N}} \sup_{g\in\mathfrak{S}_f} \int_{B_k} g\, d\mu$$

because $(B_k)_{k\in\mathbb{N}}$ being increasing implies that $\left(\int_{B_k} g\, d\mu\right)_{k\in\mathbb{N}}$ is increasing, so $\left(\sup_{g\in\mathfrak{S}_f} \int_{B_k} g\, d\mu\right)_{k\in\mathbb{N}}$ is also increasing. Moreover,

$$\begin{aligned}\lim_{k\to+\infty} \int_{B_k} f\, d\mu = \sup_{k\in\mathbb{N}} \sup_{g\in\mathfrak{S}_f} \int_{B_k} g\, d\mu = \sup_{g\in\mathfrak{S}_f} \sup_{k\in\mathbb{N}} \int_{B_k} g\, d\mu = \sup_{g\in\mathfrak{S}_f} \int_{B} g\, d\mu \\ = \int_B f\, d\mu,\end{aligned}$$

where the second to last equality follows from the fact that the claim holds for simple functions.

Finally, if f is a general function integrable over L,

$$\begin{aligned}\lim_{k\to+\infty} \int_{B_k} f\, d\mu &= \lim_{k\to+\infty} \left[\int_{B_k} f^+\, d\mu - \int_{B_k} f^-\, d\mu\right] \\ &= \lim_{k\to+\infty} \int_{B_k} f^+\, d\mu - \lim_{k\to+\infty} \int_{B_k} f^-\, d\mu \\ &= \int_B f^+\, d\mu - \int_B f^-\, d\mu \\ &= \int_B f\, d\mu.\end{aligned}$$

□

Theorem 5.4. *The indefinite integral of any nonnegative $f \in \overline{\mathsf{F}}(L)$ is a measure on $\mathcal{S}(L)$.*

Proof. Let f be a nonnegative function. By Corollary 4.5, $\eta(S) \geq 0$ for every $S \in \mathcal{S}(L)$, and we can consider the indefinite integral η as a map with codomain $[0, +\infty]$. We need to verify whether η satisfies (M1)-(M4). Since (M1) holds for any nonnegative simple function ([5, Theorem 9.3]), (M1) also holds for any nonnegative function by Definition 2.2. (M2), (M3) and (M4) follow from Lemma 5.1, Lemma 5.2 and Lemma 5.3, respectively. □

6. A localic version of the Monotone Convergence Theorem

In this section, we establish a point-free version of the Monotone Convergence Theorem. Our goal is to relate the integral of the limit with the limit of the integrals. First, we will take an increasing sequence of nonnegative functions, and then we will focus on an increasing sequence of nonnegative functions that are measurable on L.

In [5] we have studied sufficient conditions to ensure the existence of the limit of an increasing sequence. The following lemma summarises these conditions.

Lemma 6.1. *Let $(f_n : \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L))_{n\in\mathbb{N}}$ be an increasing sequence in $\overline{\mathsf{F}}(L)$. If $\bigvee_{n\in\mathbb{N}} f_n(p, —)$ is complemented for every $p \in \mathbb{Q}$, then $\lim_{n\to+\infty} f_n$ exists in $\overline{\mathsf{F}}(L)$, and*

$$\big(\lim_{n\to+\infty} f_n\big)(p, —) = (\sup_{n\in\mathbb{N}} f_n)(p, —) = \bigvee_{n\in\mathbb{N}} f_n(p, —),$$
$$\big(\lim_{n\to+\infty} f_n\big)(—, q) = (\sup_{n\in\mathbb{N}} f_n)(—, q) = \bigvee_{r<q} \big(\bigvee_{n\in\mathbb{N}} f_n(r, —)\big)^c.$$

Proceeding by induction and using Lemma 5.2, we can show that the indefinite integral of any integrable function is finitely additive.

Lemma 6.2. *Let $f \in \overline{\mathsf{F}}(L)$ be integrable over L and take $S_1, \ldots, S_n \in \mathcal{S}(L)$ $(n \in \mathbb{N})$. If $S_1, \ldots, S_n$ are pairwise disjoint in $\mathcal{S}(L)$, then*

$$\sum_{j=1}^{n} \int_{S_j} f\, d\mu = \int_{\bigvee_{j=1}^{n} S_j} f\, d\mu.$$

Proof. The case $n = 1$ is trivial. Assume that the statement holds for some $n - 1 \geq 1$. Then

$$\sum_{j=1}^{n} \int_{S_j} f\, d\mu = \sum_{j=1}^{n-1} \int_{S_j} f\, d\mu + \int_{S_n} f\, d\mu = \int_{\bigvee_{j=1}^{n-1} S_j} f\, d\mu + \int_{S_n} f\, d\mu$$
$$= \int_{\bigvee_{j=1}^{n} S_j} f\, d\mu,$$

where the last equality follows from Proposition 5.2. □

Theorem 6.3 (Generalised version of the Monotone Convergence Theorem). *Let $(f_n : \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L))_{n\in\mathbb{N}}$ be an increasing sequence of non-negative functions in $\overline{\mathsf{F}}(L)$. Suppose that $\lim_{n\to+\infty} f_n$ exists in $\overline{\mathsf{F}}(L)$ with*

$$\big(\lim_{n\to+\infty} f_n\big)(p, —) = \bigvee_{n\in\mathbb{N}} f_n(p, —) \text{ for all } p \in \mathbb{Q}.$$

If any countable join in $\{f_n(p, —) \mid p \in \mathbb{Q}\}$ is complemented in $\mathcal{C}(L)$ (for each $n \in \mathbb{N}$), then

$$\lim_{n\to+\infty} \int f_n\, d\mu = \int \lim_{n\to+\infty} f_n\, d\mu.$$

Proof. Let us write

$$f := \lim_{n\to+\infty} f_n = \sup_{n\in\mathbb{N}} f_n \in \overline{\mathsf{F}}(L).$$

Since $f_n \leq f$ for any $n \in \mathbb{N}$, by Proposition 4.3, we have that

$$\lim_{n\to+\infty} \int f_n\, d\mu = \sup_{n\in\mathbb{N}} \int f_n\, d\mu \leq \int f\, d\mu.$$

Conversely, as $f_n \geq \mathbf{0}$ for each $n \in \mathbb{N}$, it is clear that $f \geq \mathbf{0}$. Thus,

$$\int f\, d\mu = \sup\Big\{\int g\, d\mu \mid \mathbf{0} \leq g \leq f,\ g \in \mathsf{SM}(\mathcal{C}(L))\Big\}.$$

Take a $g \in \mathsf{SM}(\mathcal{C}(L))$ such that $\mathbf{0} \leq g \leq f$. Write $g = \sum_{i=1}^{n} r_i \cdot \chi_{\theta^c_{S_i}}$, where $r_1 < \ldots < r_n$, $r_i \neq 0$ for $i = 1, \ldots, n$, and $\theta^c_{S_1}, \ldots, \theta^c_{S_n}$ are pairwise disjoint. Fix $0 < s < 1$ ($s \in \mathbb{Q}$).

For any $h \in \overline{\mathsf{F}}(L)$, h and $-s \cdot \boldsymbol{r_j}$ are sum-compatible (because $-s \cdot \boldsymbol{r_j}$ is finite) and

$$\begin{aligned}
(h - s \cdot \boldsymbol{r_j})(0,—) &= \bigvee_{r \in \mathbb{Q}} [h(r,—) \wedge s \cdot \boldsymbol{r_j}(—, r)] \\
&= \bigvee_{\substack{r \in \mathbb{Q} \\ r \leq sr_j}} [h(r,—) \wedge s \cdot \boldsymbol{r_j}(—, r)] \vee \bigvee_{\substack{r \in \mathbb{Q} \\ r > sr_j}} [h(r,—) \wedge s \cdot \boldsymbol{r_j}(—, r)] \\
&= \bigvee_{\substack{r \in \mathbb{Q} \\ r \leq sr_j}} [h(r,—) \wedge 0] \vee \bigvee_{\substack{r \in \mathbb{Q} \\ r > sr_j}} [h(r,—) \wedge 1] \\
&= \bigvee_{\substack{r \in \mathbb{Q} \\ r > sr_j}} h(r,—).
\end{aligned}$$

As a result, for each $k \in \mathbb{N}$,

$$\begin{aligned}
(f_k - s \cdot \boldsymbol{r_j})(0,—) &= \bigvee_{\substack{r \in \mathbb{Q} \\ r > sr_j}} f_k(r,—) \text{ and} \\
(f - s \cdot \boldsymbol{r_j})(0,—) &= \bigvee_{\substack{r \in \mathbb{Q} \\ r > sr_j}} f(r,—) = \bigvee_{\substack{r \in \mathbb{Q} \\ r > sr_j}} \bigvee_{k \in \mathbb{N}} f_k(r,—) = \bigvee_{k \in \mathbb{N}} (f_k - s \cdot \boldsymbol{r_j})(0,—).
\end{aligned}$$

Moreover, for each $k \in \mathbb{N}$, $(f_k - s \cdot \boldsymbol{r_j})(0,—) \in \mathcal{C}_B(L)$ in consequence of any countable join in $\{f_k(p,—) \mid p \in \mathbb{Q}\}$ being complemented in $\mathcal{C}(L)$.

Set

$$\theta_{B_{jk}} := \theta_{S_j} \vee (f_k - s \cdot \boldsymbol{r_j})(0,—)^c.$$

We have

$$\begin{aligned}
\bigwedge_{k \in \mathbb{N}} \theta_{B_{jk}} &= \bigwedge_{k \in \mathbb{N}} [\theta_{S_j} \vee (f_k - s \cdot \boldsymbol{r_j})(0,—)^c] \\
&= \bigwedge_{k \in \mathbb{N}} [\theta^c_{S_j} \wedge (f_k - s \cdot \boldsymbol{r_j})(0,—)]^c \\
&= \Big(\bigvee_{k \in \mathbb{N}} [\theta^c_{S_j} \wedge (f_k - s \cdot \boldsymbol{r_j})(0,—)]\Big)^c \\
&= \Big(\theta^c_{S_j} \wedge \bigvee_{k \in \mathbb{N}} (f_k - s \cdot \boldsymbol{r_j})(0,—)\Big)^c \\
&= \big(\theta^c_{S_j} \wedge (f - s \cdot \boldsymbol{r_j})(0,—)\big)^c \\
&= \theta_{S_j}
\end{aligned}$$

because $\theta^c_{S_j}$ is complemented and

$$\theta^c_{S_j} = \theta^c_{S_j} \wedge (f - s\cdot \boldsymbol{r_j})(0,—).$$

In fact, from $g\cdot \chi_{\theta^c_{S_j}} = \left(\sum_{i=1}^n r_i \cdot \chi_{\theta^c_{S_i}}\right)\cdot \chi_{\theta^c_{S_j}} = r_j \cdot \chi_{\theta^c_{S_j}}$, it follows that

$$\begin{aligned}
\theta^c_{S_j} \wedge (f - s\cdot \boldsymbol{r_j})(0,—) &= ((f - s\cdot \boldsymbol{r_j})\cdot \chi_{\theta^c_{S_j}})(0,—)\\
&\geq ((g - s\cdot \boldsymbol{r_j})\cdot \chi_{\theta^c_{S_j}})(0,—)\\
&= ((\boldsymbol{r_j} - s\cdot \boldsymbol{r_j})\cdot \chi_{\theta^c_{S_j}})(0,—)\\
&= (\boldsymbol{r_j} - s\cdot \boldsymbol{r_j})(0,—) \wedge \theta^c_{S_j}\\
&= \Big(\bigvee_{r\in\mathbb{Q}} \boldsymbol{r_j}(r,—) \wedge s\cdot \boldsymbol{r_j}(—,r)\Big) \wedge \theta^c_{S_j}\\
&= 1 \wedge \theta^c_{S_j}\\
&= \theta^c_{S_j},
\end{aligned}$$

where the second to last equality is a consequence of existing $t \in \mathbb{Q}$ such that $sr_j < t < r_j$.

In addition, each $\theta_{B_{jk}}$ is complemented in $\mathcal{C}(L)$ and

$$\theta^c_{B_{jk}} \wedge (f_k - s\cdot \boldsymbol{r_j})(—,0) = \theta^c_{S_j} \wedge (f_k - s\cdot \boldsymbol{r_j})(0,—) \wedge (f_k - s\cdot \boldsymbol{r_j})(—,0) = 0.$$

Hence, taking note that $S_j \geq B_{jk}$, from Lemma 5.1, Lemma 6.2 and Proposition 4.4, we have

$$\int f_k\, d\mu \geq \int_{\bigvee_{j=1}^n S_j} f_k\, d\mu = \sum_{j=1}^n \int_{S_j} f_k\, d\mu \geq \sum_{j=1}^n \int_{B_{jk}} f_k\, d\mu \geq \sum_{j=1}^n \int_{B_{jk}} s\cdot \boldsymbol{r_j}\, d\mu.$$

As $1_{\mathcal{C}(L)} = \theta_{0_{\mathcal{S}(L)}} = \theta^c_{1_{\mathcal{S}(L)}}$ and $(B_{jk})_{k\in\mathbb{N}}$ is an increasing sequence,

$$\begin{aligned}
\sum_{j=1}^n \int_{B_{jk}} s\cdot \boldsymbol{r_j}\, d\mu &= s\sum_{j=1}^n \int_{B_{jk}} r_j \cdot \chi_{1_{\mathcal{C}(L)}}\, d\mu\\
&= s\sum_{j=1}^n r_j \mu(1_{\mathcal{S}(L)} \wedge B_{jk})\\
&= s\sum_{j=1}^n r_j \mu(B_{jk})\\
&\geq s\sum_{j=1}^n r_j \mu(B_{jl}) \qquad \text{(for any } l \leq k\text{)}.
\end{aligned}$$

Thus,

$$\inf_{k\geq l} \int f_k\, d\mu \geq \inf_{k\geq l} \Big(s\sum_{j=1}^n r_j\mu(B_{jk})\Big) = s\sum_{j=1}^n r_j\mu(B_{jl}),$$

and since μ is a measure and $\bigvee_{l\in\mathbb{N}} B_{jl} = S_j$, we get

$$\begin{aligned}\lim_{k\to+\infty}\int f_k\,d\mu = \sup_{l\in\mathbb{N}}\inf_{k\geq l}\int f_k\,d\mu &\geq \sup_{l\in\mathbb{N}}\big(s\sum_{j=1}^{n} r_j\mu(B_{jl})\big)\\ &= s\sum_{j=1}^{n} r_j \sup_{l\in\mathbb{N}}\mu(B_{jl})\\ &= s\sum_{j=1}^{n} r_j\mu(\bigvee_{l\in\mathbb{N}} B_{jl})\\ &= s\sum_{j=1}^{n} r_j\mu(S_j)\\ &= s\int g\,d\mu.\end{aligned}$$

Therefore,

$$\lim_{k\to+\infty}\int f_k\,d\mu = \sup_{s\in]0,1[\cap\mathbb{Q}}\Big[\lim_{k\to+\infty}\int f_k\,d\mu\Big] \geq \sup_{s\in]0,1[\cap\mathbb{Q}}\Big[s\int g\,d\mu\Big] = \int g\,d\mu$$

for any simple function g on L satisfying $\mathbf{0}\leq g\leq f$, which means that

$$\lim_{k\to+\infty}\int f_k\,d\mu \geq \int f\,d\mu. \qquad \square$$

If $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence of nonnegative functions that are measurable on L, we get what we will call the point-free version of the Monotone Convergence Theorem.

Corollary 6.4 (**Point-free Monotone Convergence Theorem**). *Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of nonnegative functions in $\overline{\mathsf{M}}(L)$. Then $\lim_n f_n$ exists in $\overline{\mathsf{F}}(L)$ and*

$$\lim_{n\to+\infty}\int f_n\,d\mu = \int \lim_{n\to+\infty} f_n\,d\mu.$$

Combining the previous corollary with Proposition 4.2, the interchangeability of the limit and the integral over L can be extended to the integral over a complemented σ-sublocale.

Corollary 6.5. *Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of nonnegative functions in $\overline{\mathsf{M}}(L)$ and take a complemented $S\in\mathcal{S}(L)$. Then $\lim_n f_n$ exists in $\overline{\mathsf{F}}(L)$ and*

$$\lim_{n\to+\infty}\int_S f_n\,d\mu = \int_S \lim_{n\to+\infty} f_n\,d\mu.$$

In [6], we have an extension of Corollary 6.5 to more general σ-sublocales.

7. A point-free version of the Fatou's Lemma

Now, we seek to formulate a point-free version of Fatou's Lemma. Given a sequence of nonnegative functions on L (not necessarily increasing), we will relate the integral of the inferior limit to the inferior limit of the integral.

Theorem 7.1 (**Generalised Fatou's Lemma**). *Let* $(f_n\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L))_{n\in\mathbb{N}}$ *be a sequence of nonnegative functions in* $\overline{\mathsf{F}}(L)$ *such that* $\lim_n \inf f_n$ *exists in* $\overline{\mathsf{F}}(L)$ *and*

$$\big(\liminf_n f_n\big)(p,—) = \bigvee_{n\in\mathbb{N}} (\inf_{k\geq n} f_k)(p,—).$$

Set $g_n := \inf_{k\geq n} f_k$. *If any countable join in* $\{g_n(p,—) \mid p \in \mathbb{Q}\}$ *is complemented in* $\mathcal{C}(L)$ *(for each* $n \in \mathbb{N}$*), then*

$$\int \lim_{n\to+\infty} \inf f_n \, d\mu \leq \lim_{n\to+\infty} \inf \int f_n \, d\mu.$$

Proof. The sequence $(g_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\overline{\mathsf{F}}(L)$ of nonnegative functions, and by the definition of inferior limit,

$$\lim_{n\to+\infty} \inf f_n = \sup_{n\in\mathbb{N}} \inf_{k\geq n} f_k = \lim_{n\to+\infty} g_n.$$

Thus, applying Theorem 6.3 to the sequence $(g_n)_{n\in\mathbb{N}}$, we get

$$\int \lim_{n\to+\infty} \inf f_n \, d\mu = \int \lim_{n\to+\infty} g_n \, d\mu = \lim_{n\to+\infty} \int g_n \, d\mu = \sup_{n\in\mathbb{N}} \int g_n \, d\mu.$$

Since $g_n \leq f_m$ for any $m \geq n$, $\displaystyle\int g_n \, d\mu \leq \int f_m \, d\mu$ for all $m \geq n$. Hence, for each $n \in \mathbb{N}$

$$\int g_n \, d\mu \leq \inf_{m\geq n} \int f_m \, d\mu,$$

which implies that

$$\int \lim_{n\to+\infty} \inf f_n \, d\mu = \sup_{n\in\mathbb{N}} \int g_n \, d\mu \leq \sup_{n\in\mathbb{N}} \inf_{m\geq n} \int f_m \, d\mu = \lim_{n\to+\infty} \inf \int f_n \, d\mu.$$

□

We point out that applying the generalised Fatou's Lemma to an increasing sequence of nonnegative functions yields the generalised version of the Monotone Convergence Theorem (Theorem 6.3): if the sequence $(f_n)_{n\in\mathbb{N}}$ is increasing, then

$$\inf_{k\geq n} f_k = f_n \qquad \text{and} \qquad \liminf_n f_n = \sup_n f_n = \lim_n f_n.$$

The generalised Fatou's Lemma implies that

$$\int \lim_{n\to+\infty} f_n \, d\mu \leq \lim_{n\to+\infty} \int f_n \, d\mu,$$

and the equality follows from the fact that $(f_n)_{n\in\mathbb{N}}$ is increasing.

Whenever L is σ-complete (i.e., it has countable joins and countable meets) and $(f_n\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L))_{n\in\mathbb{N}}$ is a sequence of nonnegative functions that are measurable on L, that is,

$$f_n(p,\text{—}), f_n(\text{—},q) \in \nabla[L] \text{ for all } p,q \in \mathbb{Q}, n \in \mathbb{N},$$

we can ensure the existence of $\lim_n \inf f_n$. Indeed, in this case, $\nabla[L]$ is closed under countable joins and countable meets. Hence, $\bigvee_{k\geq n} f_k(\text{—},q) \in \nabla[L]$ is complemented for all $q \in \mathbb{Q}$, and $g_n := \inf_{k\geq n} f_k$ not only exists in $\overline{\mathsf{F}}(L)$ ([5]) but it is also measurable:

$$\begin{aligned} g_n(\text{—},q) &= \inf_{k\geq n} f_k(\text{—},q) = \bigvee_{k\geq n} f_k(\text{—},q) \in \nabla[L], \\ g_n(p,\text{—}) &= \inf_{k\geq n} f_k(p,\text{—}) = \bigvee_{r>p} \big(\bigvee_{k\geq n} f_k(\text{—},r)\big)^c = \bigvee_{r>p} \bigwedge_{k\geq n} f_k(r,\text{—}) \in \nabla[L]. \end{aligned}$$

Since $\bigvee_{n\in\mathbb{N}} g_n(p,\text{—})$ is complemented for each $p \in \mathbb{Q}$, $\lim_n \inf f_n = \sup_{n\in\mathbb{N}} g_n$ exists in $\overline{\mathsf{F}}(L)$. The other condition in the Generalised Fatou's Lemma is also satisfied, and therefore we have:

Corollary 7.2 (Point-free Fatou's Lemma). *Let L be a σ-frame with countable meets. For any sequence $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathsf{M}}(L)$, $\lim_n \inf f_n$ exists in $\overline{\mathsf{F}}(L)$ and*

$$\int \lim_{n\to+\infty} \inf f_n \, d\mu \leq \lim_{n\to+\infty} \inf \int f_n \, d\mu.$$

8. Sum of integrals

We want now to discuss the integral of the sum and to identify classes of functions in which the integral will be linear. In [5], we have shown that the integral of simple functions is linear on the class of summable simple functions as well as on the class of nonnegative simple functions:

Lemma 8.1. *For any $r,s \in \mathbb{Q}$ (resp. $r,s \in \mathbb{Q}_0^+$) and any $g,h \in \mathsf{SM}(\mathcal{C}(L))$ that are summable over $S \in \mathcal{S}(L)$ (resp. any $g,h \in \mathsf{SM}(\mathcal{C}(L))$ with $g \wedge h \geq \mathbf{0}$),*

$$\int_S (r\cdot g + s\cdot h)\, d\mu = r \int_S g \, d\mu \;+\; s \int_S h \, d\mu.$$

We aim to find a counterpart of this result for more general functions. Let us first focus on nonnegative functions.

Theorem 8.2. *Let $f,g \in \overline{\mathsf{F}}(L)$ be nonnegative functions on L. If any countable join in $\{f(p,\text{—}) \mid p \in \mathbb{Q}\}$ and in $\{g(p,\text{—}) \mid p \in \mathbb{Q}\}$ is complemented in $\mathcal{C}(L)$, then*

$$\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

Proof. By Proposition 2.1, there are increasing sequences $(f_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$ of nonnegative simple functions in $\mathsf{SM}(\mathcal{C}(L))$ such that,

$$f = \lim_{k\to+\infty} f_k, \quad \text{with} \quad f(p,—) = \bigvee_{k\in\mathbb{N}} f_k(p,—) \text{ for } p \in \mathbb{Q},$$
$$g = \lim_{k\to+\infty} g_k, \quad \text{with} \quad g(p,—) = \bigvee_{k\in\mathbb{N}} g_k(p,—) \text{ for } p \in \mathbb{Q}.$$

Define

$$h_k \coloneqq f_k + g_k.$$

The sequence $(h_k)_{k\in\mathbb{N}}$ is increasing, and for each $k \in \mathbb{N}$, h_k is simple and nonnegative. Since f_k and g_k are simple functions, $\{f_k(p,—) \mid p \in \mathbb{Q}\}$ and $\{g_k(p,—) \mid p \in \mathbb{Q}\}$ are finite sets of complemented elements ([5, Corollary 5.5]). As a result, $\{f_k(r,—) \wedge g_k(s,—) \mid r, s \in \mathbb{Q}\}$ is also a finite set of complemented elements in $\mathcal{C}(L)$,

$$\bigvee_{k\in\mathbb{N}} h_k(p,—) = \bigvee_{k\in\mathbb{N}} \bigvee_{t\in\mathbb{Q}} \big(f_k(t,—) \wedge g_k(p-t,—)\big)$$

is complemented in $\mathcal{C}(L)$ for each $p \in \mathbb{Q}$, and $\sup_{k\in\mathbb{N}} h_k$ exists in $\overline{\mathsf{F}}(L)$ with

$$\big(\sup_{k\in\mathbb{N}} h_k\big)(p,—) = \bigvee_{k\in\mathbb{N}} h_k(p,—).$$

In addition,

$$\lim_{k\to+\infty} h_k = \sup_{k\in\mathbb{N}} h_k = \sup_{k\in\mathbb{N}}(f_k + g_k) = \sup_{k\in\mathbb{N}} f_k + \sup_{k\in\mathbb{N}} g_k = f + g,$$

where the third equality follows from the fact that $(f_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$ are increasing. Indeed, on the one hand,

$$\sup_{k\in\mathbb{N}} f_k + \sup_{k\in\mathbb{N}} g_k = \sup\{f_n + g_m \mid n, m \in \mathbb{N}\} \geq \sup_{k\in\mathbb{N}}(f_k + g_k).$$

On the other hand,

$$\sup_{n\in\mathbb{N}} f_n + \sup_{m\in\mathbb{N}} g_m = \sup_{n\in\mathbb{N}} \big(f_n + \sup_{m\in\mathbb{N}} g_m\big) = \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} (f_n + g_m),$$

and

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} (f_n + g_m) &= \sup_{n\in\mathbb{N}} \big(\sup_{m\leq n}(f_n + g_m) \vee \sup_{m>n}(f_n + g_m)\big) \\
&\leq \sup_{n\in\mathbb{N}} \big((f_n + g_n) \vee \sup_{m>n}(f_m + g_m)\big) \\
&= \sup_{n\in\mathbb{N}} \sup_{m\geq n}(f_m + g_m) \\
&= \sup_{m\in\mathbb{N}}(f_m + g_m) \\
&= \sup_{m\in\mathbb{N}} h_m.
\end{aligned}$$

For each $k \in \mathbb{N}$, any countable join in $\{h_k(p,—) \mid p \in \mathbb{Q}\}$ is also complemented because h_k is a simple function. Hence, by Theorem 6.3,

$$\int f + g \, d\mu = \int \lim_{k\to+\infty} h_k \, d\mu = \lim_{k\to+\infty} \int h_k \, d\mu = \lim_{k\to+\infty} \int f_k + g_k \, d\mu.$$

Applying Lemma 8.1 and once again Theorem 6.3, we get

$$\begin{aligned}\int f + g \, d\mu &= \lim_{k\to+\infty} \Big(\int f_k \, d\mu + \int g_k \, d\mu \Big) \\ &= \lim_{k\to+\infty} \int f_k \, d\mu + \lim_{k\to+\infty} \int g_k \, d\mu \\ &= \int f \, d\mu + \int g \, d\mu. \end{aligned}$$

□

Reformulating the proof of Theorem 8.2, we can easily obtain the corresponding result for a finite sum with more than two terms:

Theorem 8.3. *Let $f_1, \ldots, f_n \in \overline{\mathsf{F}}(L)$ be nonnegative functions on L. If any countable join in $\{f_i(p,—) \mid p \in \mathbb{Q}\}$, for each $i = 1, \ldots, n$, is complemented in $\mathcal{C}(L)$, then*

$$\int f_1 + \ldots + f_n \, d\mu = \int f_1 \, d\mu + \ldots + \int f_n \, d\mu.$$

Provided that $f_1, \ldots, f_n$ are nonnegative measurable functions on L, the conditions in the previous theorem are trivially satisfied in consequence of L having a σ-frame structure. Thus, the integral is linear for nonnegative functions in $\overline{\mathsf{M}}(L)$.

Corollary 8.4. *For any $r, s \in \mathbb{Q}_0^+$ and any nonnegative $f, g \in \overline{\mathsf{M}}(L)$,*

$$\int (r \cdot f + s \cdot g) \, d\mu = r \int f \, d\mu + s \int g \, d\mu.$$

Combining Corollary 6.4 and Corollary 8.4, we can describe the integral of a countable sum of nonnegative measurable functions on L.

Theorem 8.5. *If $(f_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative functions in $\overline{\mathsf{M}}(L)$, then*

$$\int \sum_{n\in\mathbb{N}} f_n \, d\mu = \sum_{n\in\mathbb{N}} \int f_n \, d\mu.$$

Proof. For each $n \in \mathbb{N}$, the functions $f_1, \ldots, f_n$ are sum-compatible because they are nonnegative. Set $g_n := \sum_{i=1}^n f_i$. The sequence $(g_n)_{n\in\mathbb{N}}$ is an increasing sequence of nonnegative functions in $\overline{\mathsf{M}}(L)$. By Corollary 6.4, $\sum_{n\in\mathbb{N}} f_n := \lim_n g_n$ exists and

$$\int \sum_{n\in\mathbb{N}} f_n \, d\mu = \int \lim_{n\to+\infty} g_n \, d\mu = \lim_{n\to+\infty} \int g_n \, d\mu = \lim_{n\to+\infty} \int \sum_{i=1}^n f_i \, d\mu.$$

Finally, by Corollary 8.4, we get

$$\int \sum_{n\in\mathbb{N}} f_n \, d\mu = \lim_{n\to+\infty} \int \sum_{i=1}^{n} f_i \, d\mu = \lim_{n\to+\infty} \sum_{i=1}^{n} \int f_i \, d\mu = \sum_{i=1}^{\infty} \int f_i \, d\mu. \qquad \square$$

Consider now that $f, g \in \overline{\mathsf{F}}(L)$ are general integrable functions on L. Since any $h \in \overline{\mathsf{F}}(L)$ can be written as $h = h^+ - h^-$, where h^+ and h^- are non-negative functions, it is possible to study the linearity of the integral for general integrable functions using Theorem 8.2. Given the complexity of the conditions that would need to be imposed, in this paper we cover only the class of summable functions that are measurable on L. An extension to a broader class of functions is presented in detail in [6].

Theorem 8.6. *If $f, g \in \overline{\mathsf{M}}(L)$ are summable functions, then $f + g$ is also summable and*

$$\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$$

whenever f and g are sum-compatible.

Proof. Set $h := f + g$. Note that $h^+ \leq f^+ + g^+$ and $h^- \leq f^- + g^-$. Thus, by Proposition 4.3 and Corollary 8.4,

$$\int h^+ \, d\mu \leq \int f^+ + g^+ \, d\mu = \int f^+ \, d\mu + \int g^+ \, d\mu < \infty$$

$$\text{and} \quad \int h^- \, d\mu \leq \int f^- + g^- \, d\mu = \int f^- \, d\mu + \int g^- \, d\mu < \infty.$$

Moreover, since $f = f^+ - f^-$, $g = g^+ - g^-$ and $h = h^+ - h^-$, we have that

$$h^+ + f^- + g^- = f^+ + g^+ + h^-.$$

Therefore, once again by Corollary 8.4, we get

$$\begin{aligned}
&\int h^+ + f^- + g^- \, d\mu = \int f^+ + g^+ + h^- \, d\mu \\
\Leftrightarrow &\int h^+ \, d\mu + \int f^- \, d\mu + \int g^- \, d\mu = \int f^+ \, d\mu + \int g^+ \, d\mu + \int h^- \, d\mu \\
\Leftrightarrow &\int h^+ \, d\mu - \int h^- \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu + \int g^+ \, d\mu - \int g^- \, d\mu \\
\Leftrightarrow &\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu. \qquad \square
\end{aligned}$$

Corollary 8.7. *For any $r, s \in \mathbb{Q}$ and any summable functions $f, g \in \overline{\mathsf{M}}(L)$,*

$$\int (r \cdot f + s \cdot g) \, d\mu = r \int f \, d\mu + s \int g \, d\mu$$

whenever $r \cdot f$ and $s \cdot g$ are sum-compatible.

So far we have only covered the integral over L. For an extension of the previous results to the integral over an arbitrary σ-sublocale $S \in \mathcal{S}(L)$ see [6].

9. Point-free setting versus classic setting

In this final section, our main goal is to show that the point-free integral established in Definition 2.3 extends the standard Lebesgue integral.

Let $(X, \mathcal{A})$ be a measurable space, that is, a pair where X is a set and $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra on X. Take the space of extended real numbers equipped with the Borel σ-algebra, and set

$$\mathsf{Meas}((X, \mathcal{A}), \overline{\mathbb{R}}) := \{\tilde{f} \colon X \to \overline{\mathbb{R}} \mid \tilde{f} \text{ is measurable}\}.$$

Given a measurable extended real function $\tilde{f} \colon X \to \overline{\mathbb{R}}$, the localic counterpart of $\tilde{f}$ is the σ-frame homomorphism $\Phi(\tilde{f}) \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{A} \in \overline{\mathsf{M}}(\mathcal{A})$ determined by

$$\Phi(\tilde{f})(p, \text{—}) = \tilde{f}^{-1}(]p, +\infty]) \qquad \text{and} \qquad \Phi(\tilde{f})(\text{—}, q) = \tilde{f}^{-1}([-\infty, q[)$$

for all $p, q \in \mathbb{Q}$. The correspondence Φ yields a bijection

$$\Phi \colon \mathsf{Meas}((X, \mathcal{A}), \overline{\mathbb{R}}) \longrightarrow \sigma\mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{A}) = \overline{\mathsf{M}}(\mathcal{A})$$

that preserves the order and the ring operations in $\mathsf{Meas}((X, \mathcal{A}), \overline{\mathbb{R}})$. Furthermore, through the isomorphism $\nabla(E \mapsto \nabla(E)) \colon \mathcal{A} \to \nabla[\mathcal{A}]$ that embeds $\mathcal{A}$ in $\mathcal{C}(\mathcal{A})$, we can identify each $\Phi(\tilde{f})$ with

$$\nabla \circ \Phi(\tilde{f}) \in \mathsf{F}(\mathcal{A})$$

and view $\Phi(\tilde{f})$ as an element of $\mathsf{F}(\mathcal{A})$.

Let $\lambda \colon \mathcal{A} \to [0, +\infty]$ be a measure on the measurable space $(X, \mathcal{A})$. By definition, λ is a σ-additive map such that $\lambda(\emptyset) = 0$. Thus, since $\mathcal{A}$ is a σ-algebra, λ is also a measure in the sense of a measure on a join-σ-complete lattice L for $L = \mathcal{A}$ [10].

Through the isomorphism $\mathfrak{o}(E \mapsto \mathfrak{o}(E)) \colon \mathcal{A} \mapsto \mathfrak{o}[\mathcal{A}]$, we can regard λ as a measure $\lambda \colon \mathfrak{o}[\mathcal{A}] \to [0, +\infty]$. Hence, for any measure λ on $\mathcal{A} \cong \mathfrak{o}[\mathcal{A}] \subseteq \mathcal{S}(\mathcal{A})$, there exists a measure $\lambda^\diamond \colon \mathcal{S}(\mathcal{A}) \to [0, +\infty]$ extending λ. (Indeed, we can get a map $\lambda^\diamond$ satisfying the required conditions by applying Theorem 1 of [10] because λ is a measure on $\mathcal{A}$ and every element of $\mathcal{A}$ is complemented.)

Theorem 9.1. *Given a measure space $(X, \mathcal{A}, \lambda)$, let $\lambda^\diamond \colon \mathcal{S}(\mathcal{A}) \to [0, +\infty]$ be a measure on $\mathcal{S}(\mathcal{A})$ extending λ in the sense that $\lambda^\diamond(\mathfrak{o}(E)) = \lambda(E)$ for each $E \in \mathcal{A}$. A measurable function $\tilde{f} \colon X \to \overline{\mathbb{R}}$ is λ-integrable if and only if $\nabla \circ \Phi(\tilde{f})$ is $\lambda^\diamond$-integrable, and for any $E \in \mathcal{A}$,*

$$\int_{\mathfrak{o}(E)} \nabla \circ \Phi(\tilde{f})\, d\lambda^\diamond = \int_E \tilde{f}\, d\lambda.$$

Proof. **Case 1: $\mathbb{Q}$-Simple functions.** Consider a simple function $\tilde{f}\colon X \to \mathbb{R}$ with codomain in $\mathbb{Q}$ (briefly called a $\mathbb{Q}$-simple function through this proof), that is, a function of the form

$$\tilde{f} = \sum_{i=1}^{n} r_i \mathbb{1}_{E_i}$$

for some $n \in \mathbb{N}$ and $r_1, \ldots, r_n \in \mathbb{Q}$, where $\mathbb{1}_{E_i}\colon X \to \{0,1\}$ is the indicator (characteristic) function of E_i and $E_1, \ldots, E_n \in \mathcal{A}$ are pairwise disjoint subsets of X such that $\bigcup_{i=1}^{n} E_i = X$. We proved in [5, Section 10] that the class of $\mathbb{Q}$-simple functions on X is in bijection with $\mathsf{SM}(\mathcal{A}) \subseteq \mathsf{SM}(\mathcal{C}(\mathcal{A}))$, $\nabla \circ \Phi(\tilde{f}) \in \mathsf{SM}(\mathcal{A})$ and

$$\int_{\mathcal{A}} \nabla \circ \Phi(\tilde{f})\, d\lambda^{\diamond} = \int_X \tilde{f}\, d\lambda.$$

Case 2: Nonnegative measurable functions. If $\tilde{f} \in \mathsf{Meas}((X,\mathcal{A}), \overline{\mathbb{R}})$ is nonnegative, the integral of $\tilde{f}$ is defined as

$$\int_X \tilde{f}\, d\lambda := \sup\left\{ \int_X \tilde{g}\, d\lambda \mid 0 \leq \tilde{g} \leq \tilde{f},\ \tilde{g}\colon X \to \overline{\mathbb{R}} \text{ is simple}\right\}.$$

However, as a consequence of the fact that $f = \lim_n \sum_{i=1}^{n} \frac{1}{i}\mathbb{1}_{E_i}$ for a sequence $(E_i)_{i\in\mathbb{N}}$ in $\mathcal{A}$ and the Monotone Convergence Theorem, we can write

$$\int_X \tilde{f}\, d\lambda = \sup\left\{ \int_X \tilde{g}\, d\lambda \mid 0 \leq \tilde{g} \leq \tilde{f},\ \tilde{g}\colon X \to \overline{\mathbb{R}} \text{ is } \mathbb{Q}\text{-simple}\right\}.$$

Since Φ preserves the partial order and $\Phi(0) = \mathbf{0}$, we have that for any $\mathbb{Q}$-simple function $\tilde{g}\colon X \to \overline{\mathbb{R}}$, $0 \leq \tilde{g} \leq \tilde{f}$ if and only if $\mathbf{0} \leq \nabla\circ\Phi(\tilde{g}) \leq \nabla\circ\Phi(\tilde{f})$. Thus, from case 1:

$$\begin{aligned}
\int_X \tilde{f}\, d\lambda &= \sup\left\{ \int_X \tilde{g}\, d\lambda \mid 0 \leq \tilde{g} \leq \tilde{f},\ \tilde{g}\colon X \to \overline{\mathbb{R}} \text{ is } \mathbb{Q}\text{-simple}\right\} \\
&= \sup\left\{ \int_{\mathcal{A}} \nabla\circ\Phi(\tilde{g})\, d\lambda^{\diamond} \mid 0 \leq \tilde{g} \leq \tilde{f},\ \tilde{g}\colon X \to \overline{\mathbb{R}} \text{ is } \mathbb{Q}\text{-simple}\right\} \\
&\leq \sup\left\{ \int_{\mathcal{A}} h\, d\lambda^{\diamond} \mid \mathbf{0} \leq h \leq \nabla\circ\Phi(\tilde{f}),\ h \in \mathsf{SM}(\mathcal{C}(\mathcal{A}))\right\} \\
&= \int_{\mathcal{A}} \nabla\circ\Phi(\tilde{f})\, d\lambda^{\diamond}.
\end{aligned}$$

Conversely, since $\nabla \circ \Phi(\tilde{f})$ is nonnegative, measurable on $\mathcal{A}$ and $\mathcal{A}$ is Boolean, there is, by Proposition 2.1, an increasing sequence $(g_n\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{A})_{n\in\mathbb{N}}$ of nonnegative functions in $\mathsf{SM}(\mathcal{A})$ such that

$$\Phi(\tilde{f}) = \lim_{n\to+\infty} g_n = \sup_{n\in\mathbb{N}} g_n.$$

Note that $\mathbf{0} \leq g_n \leq \Phi(\tilde{f})$ for each $n \in \mathbb{N}$. Hence $(\Phi^{-1}(g_n))_{n\in\mathbb{N}}$ is an increasing sequence of $\mathbb{Q}$-simple functions in $\mathsf{Meas}((X,\mathcal{A}),\overline{\mathbb{R}})$ with $0 \leq \Phi^{-1}(g_n) \leq \tilde{f}$. As a result, we have

$$\nabla \circ \Phi(\tilde{f}) = \nabla \circ \lim_{n\to+\infty} g_n = \lim_{n\to+\infty} (\nabla \circ g_n) = \sup_{n\in\mathbb{N}} (\nabla \circ g_n),$$

and by Corollary 6.4,

$$\begin{aligned}
\int_{\mathcal{A}} \nabla \circ \Phi(\tilde{f})\, d\lambda^{\diamond} &= \lim_{n\to+\infty} \int_{\mathcal{A}} \nabla \circ g_n \, d\lambda^{\diamond} \\
&= \sup \Big\{ \int_{\mathcal{A}} \nabla \circ g_n \, d\lambda^{\diamond} \mid n \in \mathbb{N} \Big\} \\
&= \sup \Big\{ \int_{X} \Phi^{-1}(g_n) \, d\lambda \mid n \in \mathbb{N} \Big\} \\
&\leq \sup \Big\{ \int_{X} \tilde{g} \, d\lambda \mid 0 \leq \tilde{g} \leq \tilde{f},\ \tilde{g} \colon X \to \overline{\mathbb{R}} \text{ is } \mathbb{Q}\text{-simple} \Big\} \\
&= \int_X \tilde{f} \, d\lambda.
\end{aligned}$$

Case 3: Measurable functions. Given a measurable function $\tilde{f} \colon X \to \overline{\mathbb{R}}$ on $(X,\mathcal{A})$, we say that $\tilde{f}$ is λ-integrable if

$$\int_X \tilde{f}^+ \, d\lambda < \infty \qquad \text{or} \qquad \int_X \tilde{f}^- \, d\lambda < \infty,$$

and its integral is the value

$$\int_X \tilde{f} \, d\lambda := \int_X \tilde{f}^+ \, d\lambda - \int_X \tilde{f}^- \, d\lambda.$$

Take an $\tilde{f} \in \mathsf{Meas}((X,\mathcal{A}),\overline{\mathbb{R}})$. The functions f^+ and f^- are nonnegative, measurable on $(X,\mathcal{A})$, $\Phi(\tilde{f}^+) = \Phi(\tilde{f})^+$ and $\Phi(\tilde{f}^-) = \Phi(\tilde{f})^-$. Moreover, as $(\nabla \circ \Phi(\tilde{f}))^+ = \nabla \circ \Phi(\tilde{f})^+$ and $(\nabla \circ \Phi(\tilde{f}))^- = \nabla \circ \Phi(\tilde{f})^-$, we have

$$\begin{aligned}
\int_X \tilde{f}^+ \, d\lambda &= \int_{\mathcal{A}} \nabla \circ \Phi(\tilde{f})^+ \, d\lambda^{\diamond} = \int_{\mathcal{A}} (\nabla \circ \Phi(\tilde{f}))^+ \, d\lambda^{\diamond}, \\
\int_X \tilde{f}^- \, d\lambda &= \int_{\mathcal{A}} \nabla \circ \Phi(\tilde{f})^- \, d\lambda^{\diamond} = \int_{\mathcal{A}} (\nabla \circ \Phi(\tilde{f}))^- \, d\lambda^{\diamond}.
\end{aligned}$$

Therefore, not only $\tilde{f}$ is λ-integrable if and only if $\nabla \circ \Phi(\tilde{f})$ is $\lambda^{\diamond}$-integrable, but we also get that

$$\int_X \tilde{f} \, d\lambda = \int_{\mathcal{A}} \nabla \circ \Phi(\tilde{f}) \, d\lambda^{\diamond}$$

whenever $\tilde{f}$ is λ-integrable.

For any $E \in \mathcal{A}$, the integral of $\tilde{f}$ over E is given by

$$\int_E \tilde{f}\, d\lambda := \int_X \tilde{f} \cdot \mathbb{1}_E\, d\lambda = \int_{\mathcal{A}} \nabla \circ \Phi(\tilde{f} \cdot \mathbb{1}_E)\, d\lambda^\diamond.$$

But as $\Phi(\tilde{f} \cdot \mathbb{1}_E) = \Phi(\tilde{f}) \cdot \Phi(\mathbb{1}_E) = \Phi(\tilde{f}) \cdot \chi_E \in \overline{\mathsf{M}}(\mathcal{A})$, we get

$$\int_E \tilde{f}\, d\lambda = \int_{\mathcal{A}} (\nabla \circ \Phi(\tilde{f})) \cdot \chi_{\nabla_E}\, d\lambda^\diamond = \int_{\mathfrak{o}(E)} \nabla \circ \Phi(\tilde{f})\, d\lambda^\diamond. \qquad \square$$

Summing up, as $\mathsf{Meas}((X, \mathcal{A}), \overline{\mathbb{R}}) \cong \overline{\mathsf{M}}(\mathcal{A})$ and $\overline{\mathsf{M}}(\mathcal{A}) \subseteq \overline{\mathsf{F}}(\mathcal{A})$, regarding $\mathsf{Meas}((X, \mathcal{A}), \overline{\mathbb{R}})$ as a subset of $\overline{\mathsf{F}}(\mathcal{A})$ and setting

$$\int_{\mathcal{A}} \tilde{f}\, d\lambda^\diamond := \int_{\mathcal{A}} \nabla \circ \Phi(\tilde{f})\, d\lambda^\diamond$$

for any λ-integrable $\tilde{f} \in \mathsf{Meas}((X, \mathcal{A}), \overline{\mathbb{R}})$, our point-free integral can be seen as an extension of the Lebesgue integral from

$$\{\tilde{f} \in \mathsf{Meas}((X, \mathcal{A}), \overline{\mathbb{R}}) \mid \tilde{f} \text{ is } \lambda\text{-integrable}\}$$

to

$$\{f \in \mathsf{F}(\mathcal{A}) \mid f \text{ is } \lambda^\diamond\text{-integrable}\}.$$

In addition, fixing a λ-integrable $\tilde{f} \in \mathsf{Meas}((X, \mathcal{A}), \overline{\mathbb{R}})$ and recalling that $\mathcal{A} \cong \mathfrak{o}[\mathcal{A}] \subseteq \mathcal{S}(\mathcal{A})$, our point-free integral allows us to regard the indefinite integral of $\nabla \circ \Phi(\tilde{f})$,

$$\begin{aligned} \eta \colon \mathcal{S}(\mathcal{A}) &\to [-\infty, +\infty] \\ S &\mapsto \int_S \nabla \circ \Phi(\tilde{f})\, d\lambda^\diamond, \end{aligned}$$

as an extension of the indefinite integral of $\tilde{f}$, which is, by definition, a map

$$\begin{aligned} \tilde{\eta} \colon \mathcal{A} &\to [-\infty, +\infty] \\ E &\mapsto \int_E \tilde{f}\, d\lambda. \end{aligned}$$

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References

1. S. Banach and A. Tarski, *Sur la décomposition des ensembles de points en parties respectivement congruentes*, Fund. Math. **6** (1924), 244–277.
2. B. Banaschewski and C.R.A. Gilmour, *Rings of continuous functions on σ-frames*, Topology Appl. **158** (2011), 861–868.
3. R. Bernardes, *Measurable functions on σ-frames*, Topology Appl. **336** (2023), Paper No. 108609, 26. MR 4601090
4. ______, *Induced congruences in σ-frames*, presentation, http://www.mat.uc.pt/~mat1413/TACL24_presentation.pdf, 7 2024, Accessed: 2025-03-28.
5. ______, *Lebesgue integration on σ-locales: simple functions*, submitted manuscript, https://arxiv.org/pdf/2408.13911, 2024.
6. ______, *Lebesgue integration, measure and σ-sublocales*, PhD thesis, University of Coimbra, in preparation.
7. J. Frith and A. Schauerte, *The congruence frame and the Madden quotient for partial frames*, Algebra Universalis **79** (2018), no. 3, Paper No. 73, 27.
8. J. Madden, *k-Frames*, J. Pure Appl. Algebra **70** (1991), 107–127.
9. J. Picado and A. Pultr, *Frames and locales: topology without points*, Frontiers in Mathematics, vol. 28, Springer, Basel, 2012.
10. A. Simpson, *Measure, randomness and sublocales*, Ann. Pure Appl. Logic **163** (2012), 1642–1659.
11. G. Vitali, *Sul problema della misura dei gruppi di punti di una retta*, Bologna, 1905.

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