REGULARITY OF CONVEX ENVELOPES (A GEOMETRIC APPROACH)

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ABSTRACT. By leveraging an inherently geometric argument, we establish new properties of convex functions Γ defined on a convex domain Ω. Suppose the graph of (convex) Γ contains a line segment $[Y^1, Y^2]$, where $Y^j = (y^j, \Gamma(y^j))$, with $y^j \in \Omega$, and that $Y^* = (y^*, \Gamma(y^*))$ lies on this segment. Given a second-order polynomial P, whose graph touches (locally) the graph of Γ from below at Y^*, then a horizontal/vertical translation of $P/3$ touches the convex graph of Γ at least at one of the points Y^j. This, in light of the viscosity approach in PDEs, has interesting consequences for the regularity of convex envelopes of supersolutions to a large class of partial differential equations. These include degenerate fully nonlinear models and quasi-linear problems that have not been treated in the literature. Our methods are versatile, and we expect them to find applications in a broader class of models.

1. Introduction

1.1. **Background.** We establish new geometric properties of the graphs of convex functions and explore their consequences for the regularity of convex envelopes of supersolutions to a wide range of partial differential equations (PDEs).

For clarity, we introduce a few notations that will remain consistent throughout the paper. Our domains $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ are always bounded and convex, with the required smoothness specified when needed. Convex functions over $\overline{\Omega}$ are always denoted by Γ, and they are required to be continuous. For a given continuous function u defined on $\overline{\Omega}$, or (merely) on the boundary $\partial\Omega$, we denote its convex envelope by Γ_u, defined as

$$\Gamma_u(x) := \sup \left\{ \ell(x) \mid \ell \text{ is convex in } \Omega, \text{ and } \ell \leq u \text{ on } \overline{\Omega} \right\}. \tag{1}$$

Although the graph of Γ (or Γ_u) will be denoted by $X := (x, \Gamma(x))$ for $x \in \overline{\Omega}$, we shall, when convenient for readability, make no distinction between the function and its graph. We exclusively use x, y, z for points in $\overline{\Omega}$, while capital letters X, Y, Z are exclusively reserved for points on the graph of the function under discussion.

We shall also use the concept of touching polynomials, borrowed from the viscosity approach in the theory of PDEs. More precisely, we say that a (second-order) polynomial P *touches* the function Γ from below at a point $y \in \Omega$ if $P(x) \leq \Gamma(x)$ in a small neighborhood $B_r(y) \subset \Omega$ of y and $P(y) = \Gamma(y)$; see Definition 1. We also emphasize that, unless otherwise stated, all touching

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polynomials P in this paper are of second-order

$$P(x) := c + \sum_{j=1}^{d} \left(a_j x_j^2 + b_j x_j \right), \qquad a_j, b_j, c \in \mathbb{R}.$$

To set the scene, suppose a polynomial P touches Γ from below at $y^* \in \Omega$. Suppose further that $Y^* = (y^*, \Gamma(y^*)) = \lambda_1 Y^1 + \lambda_2 Y^2$, with $Y^j = (y^j, \Gamma(y^j))$, and $y^j \in \Omega$. We prove that, for at least one $j \in \{1, 2\}$

$$P^j(x) := \frac{1}{3} P(x + y^* - y^j) + \Gamma(y^j) - \frac{1}{3} P(y^*)$$

touches Γ from below at y^j.

This simple observation has spillovers on the regularity of convex envelopes for supersolutions to a wide class of PDEs, when defined in terms of viscosity, i.e. touching polynomials. Indeed, it is intuitive to think that the smoothness of Γ_u is dictated by the smoothness of the contact set $\{u = \Gamma_u\}$.[1]

Now, if u satisfies a PDE inequality at contact points, meaning that

$$Lu \leq f \quad \text{ in } \quad \{u = \Gamma_u\},$$

where L denotes an elliptic PDE, then our results state that this property is propagated along interior line segments of Γ_u, whose end points are in the contact set $\{u = \Gamma_u\}$. This, in turn, translates to $L\Gamma_u \leq 3f$ on this (interior) line segment. What remains to be worked out are those line segments that have at least one of its endpoints on the boundary. This is taken care of by a simple idea of using the cone generated by the interior end point and the boundary as a barrier that touches the actual point from above, and is smooth due to the smoothness of the boundary. In either case, we obtain bounds on the PDE from the above.

At the same time, the convexity of Γ_u (formally) yields $D_{ee}\Gamma_u \geq 0$. This, in particular, shows that all we need from the given PDE is that the pure second derivatives can be estimated from above by the PDE itself, and hence by f. This is naturally the (only) technical aspect specific to each PDE under consideration. For example, when L is the Laplace operator, one easily concludes that pure second derivatives of Γ_u are bounded at contact sets. However, for more general PDEs, estimating pure second derivatives from below is more involved and is encoded in the structure of the equation, see (3).

1.2. **Main results.** Our first result, which forms the pillar of this paper, concerns the touching properties of convex graphs with second-order polynomials. Specifically, we aim to prove that if a polynomial touches the graph of a convex function on a flat (polygonal) piece of dimension N, then a translated and scaled version of this polynomial must touch the graph at some vertex of such a piece.

[1]Recall that the convex hull of a C^2-domain is $C^{1,1}$ (see, for instance, [14]), which is a consequence of the smoothness of the boundary of the convex hull intersected with the boundary of the C^2-domain.

Theorem 1 (Touching property for convex graphs). *Let $\Omega \subset \mathbb{R}^d$ be a convex domain, not necessarily smooth, and $\Gamma : \overline{\Omega} \to \mathbb{R}$ be a continuous convex function. Suppose the graph of Γ contains a polygonal piece $\mathcal{P}$ of dimension N, generated by a set of points $\{Y^j\}_1^{N+1}$, where $1 < N \leq d$, $Y^j = (y^j, \Gamma(y^j))$, with $y^j \in \Omega$. For $y^* \in \Omega$ suppose the graph of P touches $\mathcal{P}$ from below at $Y^* = (y^*, \Gamma(y^*))$, in $B_r(y^*) \cap \Omega$. Then, for at least one j, the polynomial*

$$P^j(x) = P^j_{y^*}(x) := \frac{1}{3N} P(x + y^* - y^j) + \Gamma(y^j) - \frac{1}{3N} P(y^*)$$

touches Γ from below at y^j within $B_{r/2}(y^j) \cap \Omega$.

We now highlight the consequences of Theorem 1 regarding PDEs. Notice that the geometric properties in Theorem 1 are expressed in terms of touching paraboloids. Hence, they naturally relate to supersolutions in the viscosity sense, which we define now.

Definition 1 (Viscosity supersolution). *Let $F : S(d) \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$. Suppose $f \in C(\Omega)$. We say that $u \in C(\Omega)$ is a viscosity supersolution to*

$$F(D^2u, Du, u, x) = f \quad \text{in} \quad \Omega$$

if, for every second-order polynomial P and $x_0 \in \Omega$ such that $u(x_0) = P(x_0)$ with $u(x) \geq P(x)$ in a neighborhood of x_0, we have

$$F(D^2P(x_0), DP(x_0), u(x_0), x_0) \leq f(x_0). \tag{2}$$

Suppose that Γ is a continuous convex function, and a viscosity supersolution to a fully nonlinear PDE F, i.e.,

$$F(D^2\Gamma, D\Gamma, \Gamma, x) \leq C$$

for some $C > 0$, in the viscosity sense. Additionally, assume that F satisfies the following lower bound

$$F(M, \xi, t, x) \geq (c_1|\xi|^{\gamma_1} + c_2|t|^{\gamma_2} + c_3)\, \lambda_i, \qquad \forall\, i = 1, \cdots, d, \tag{3}$$

where $M = M_{d\times d}$ is a positive definite matrix, $\xi \in \mathbb{R}^d$, $t \in \mathbb{R}$, $x \in \Omega$, $c_1, c_2, c_3, \gamma_1, \gamma_2$ are positive constants and λ_i denotes the eigenvalues of M. Observe that the right-hand side is smaller than the negative Pucci operator in this particular case, and many uniform and degenerate/singular operators satisfy this property.

Indeed, if $F = F(M)$ is a (λ, Λ)-uniformly elliptic operator, then (3) holds with $c_1 = c_2 = 0$ and $c_3 = \lambda$. For degenerate operators, various models satisfy (3). In particular, the fully nonlinear counterpart of the p-Laplace operator is given by

$$F(M, \xi) = |\xi|^p G(M),$$

where $p > 1$ and G is a (λ, Λ)-elliptic operator. This operator satisfies (3) with $c_1 = 1$, $\gamma_1 = p$, $c_2 = 0$, and $c_3 = \lambda$.

We notice the p-Laplacian also satisfies (3). Indeed, the non-variational formulation of Δ_p can be written as

$$\Delta_p u = \operatorname{Tr}\left(A(Du, x)D^2u\right),$$

where $\xi^T A(Du, x)\xi \sim |Du|^{p-2}|\xi|^2$, for $1 < p < \infty$.

The role of (3) in the regularity of Γ stems from the supersolution property in the viscosity sense. Obviously, we have that Γ is universally Lipschitz, where the norm depends on $\partial\Omega$; see [10, Theorem 1, Sec. 6.3]. Therefore, if $c_3 > 0$, it follows immediately that λ_i is universally bounded above, with the bound depending in part on c_3. Since the matrix is positive definite, we also have $\lambda_i > 0$. In particular, because Γ is convex, its pure second derivatives – and consequently, all of its second derivatives – are bounded.

When $c_2 = c_3 = 0$ and $c_1 > 0$, the previous argument no longer applies, allowing us to investigate the possibility of universal Hölder continuity for $D\Gamma$. Establishing such regularity requires a more refined analysis of the PDE's structure.

Nevertheless, the supersolution property of Γ, combined with its convexity, imposes bounds on the PDE. In elliptic theory, such bounds often lead to some degree of smoothness in the solutions. One expects that the convexity of Γ should further enhance the regularity, potentially yielding optimal smoothness.

We now formalize this discussion in the following theorem, whose proof will be presented in detail.

Let $F = F(M, p, r, x)$ be so that

$$F(M, p, r, x) - F(M, p, s, y) \leq \omega_1(|r - s|) + \omega_2(|x - y|), \tag{4}$$

for locally uniformly bounded moduli of continuity $\omega_i : \mathbb{R}^+ \to \mathbb{R}^+$. For a convex function Γ, we define $\mathcal{E}_\Gamma$ to be the set of all points $y \in \Omega$ such that $(y, \Gamma(y))$ is an extremal point of Γ, i.e., $(y, \Gamma(y))$ is not a strict convex combination of any two other points on Γ.

Theorem 2. *Let Ω be a C^2-regular domain. Suppose $u \in C(\overline{\Omega})$ satisfies*

$$F(D^2u, Du, u, x) \leq C, \qquad \text{on } \mathcal{E}_{\Gamma_u},$$

in the viscosity sense, and agrees with $g \in C^2(\partial\Omega)$ on $\partial\Omega$. Suppose F satisfies (4). Then, for every $\Omega' \Subset \Omega$, there exists a positive constant $C_1 = C_1(C, d, \partial\Omega, \Omega', g, F)$ such that

$$F(D^2\Gamma_u, D\Gamma_u, \Gamma_u, x) \leq C_1, \qquad \text{in } \Omega',$$

in the viscosity sense.

Observe that the supersolution property for u is only required on $\mathcal{E}_{\Gamma_u}$. The above theorem has interesting implications for the smoothness of convex envelopes, dictated by the ellipticity of the PDE.

Remark 1. *In Theorem 2, if u is much larger than the boundary value, so that $\Gamma_u = \Gamma_g$, the convex envelope of the boundary value, then our proof yields a simple version of existing results, but only in the interior. Compare Theorem 4, for the case of less smooth boundary values.*

We use Theorem 2 to prove new results on the regularity of the convex envelope for supersolutions to nonlinear degenerate equations. Namely, the p-Poisson equation and its fully nonlinear counterpart. The case of the latter follows from Theorem 2 directly. The case of the p-Laplacian combines Theorem 2 with intrinsic characteristics of the operator, exploring different regularity regimes depending on the magnitude of the gradient. The regularity of the convex envelope for the supersolutions of these model-problems is the subject of the next theorem.

Theorem 3 (Regularity of the convex envelope)**.** *Let $u \in C(\Omega)$ be a viscosity solution to*

$$F(D^2u, Du, u, x) \leq C \qquad \text{on} \quad \{u = \Gamma_u\},$$

and suppose Ω is a C^2-regular domain. Suppose further that u agrees with g on $\partial\Omega$, for some $g \in C^2(\partial\Omega)$. Then,

i. if $F(M, \xi) = |\xi|^p G(M)$, where $p > 1$ and G is a convex (λ, Λ)-elliptic operator, then $D\Gamma_u \in C^{\frac{1}{1+p}}_{\mathrm{loc}}(\Omega)$;

ii. if F is the p-Laplacian operator, then $D\Gamma_u \in C^{\alpha}_{\mathrm{loc}}(\Omega)$, where

$$\alpha = \min\left(1, \frac{1}{p-1}\right).$$

In addition, these hold uniformly in every $\Omega' \Subset \Omega$.

Remark 2. (Optimality of Theorem 3) *The optimal regularity in Theorem 3 holds only in the interior of the domain, and we are currently unable to establish its uniform validity up to the boundary, even in cases where such behavior is expected. In fact, even in the simple case where Ω is the unit ball and $g \equiv 0$, we have neither been able to prove optimal boundary regularity nor to construct a counterexample. Although it is tempting to conjecture one or another way, we leave it as an open question without suggesting which one is potentially viable.*

Our reasoning so far has relied on Theorem 1 and on geometric information stemming from the C^2-regularity assumptions on the boundary — both on $\partial\Omega$ and on the boundary data — as stated in Lemma 1 below. The observation in Lemma 1, within the context of $C^{1,\gamma}$-regularity assumptions on the boundary, yields an additional insight.

Indeed, if $g \in C^{1,\beta}(\partial\Omega)$ and $\partial\Omega$ is locally of class $C^{1,\alpha}$, our argument builds upon previous results (e.g., [8, Theorem 1.2]) to obtain local $C^{1,\gamma}$-regularity estimates for the convex envelope of g. This is the content of the next theorem.

Theorem 4 (Lower regularity of the data)**.** *Let Ω be a $C^{1,\alpha}$-regular domain, whereas $g \in C^{1,\beta}(\partial\Omega)$, for some $0 < \alpha, \beta < 1$. Then $\Gamma_g \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$, where $\gamma := \min\{\alpha, \beta\}$. In addition, for every $\Omega' \Subset \Omega$,* *there exists $C>0$, depending on g, Ω', the geometry of $\partial\Omega$, and the space dimension such that*

$$\|\Gamma_g\|_{C^{1,\gamma}(\Omega')} \leq C.$$

1.3. **Existing results.** Let $v \in C(\Omega)$ be a supersolution to

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 v) = f \quad \text{in} \quad \Omega,$$

where $\mathcal{M}^-_{\lambda,\Lambda}$ is the smallest (λ,Λ)-Pucci extremal operator and $f \in L^\infty(\Omega)$. Then one concludes that $\Gamma_v \in C^{1,1}_{\rm loc}(\Omega)$, [5, Lemma 2]; see also [6, Lemma 3.3]. For recent generalizations in the context of L^d-viscosity solutions, see [4].

A different perspective on the connection of PDE and the convex envelope of a given function is pursued in [16]. In that paper, the author considers a function $g : \mathbb{R}^d \to \mathbb{R}$ and shows that its convex envelope Γ_g is a viscosity solution to

$$\max\left\{v(x) - g(x), -\lambda_1(D^2 v(x))\right\} = 0 \quad \text{in} \quad \mathbb{R}^d,$$

where $\lambda_1(M)$ stands for the smallest eigenvalue of the matrix M. Here, the interest is not in the effects of a PDE structure of the convex envelope of the solutions but, conversely, in deriving an equation characterizing the envelope. A PDE characterizing the convex envelope of g in a bounded domain is the subject of [17].

The analysis of regularity properties of the convex envelope in a general setting is the topic of [11, 3, 13], to name just a few. In [13], the authors consider an extended function $v : \mathbb{R}^d \to \mathbb{R}\cup\{+\infty\}$; if $v \in C^{1,\alpha}_{\rm loc}(\{v < +\infty\})$, then $\Gamma_v \in C^{1,\alpha}_{\rm loc}(\{\Gamma_v < +\infty\})$, for any $\alpha \in [0,1]$. They require $v(x) \to +\infty$ as $|x| \to \infty$. The results in [13], as well as the growth condition imposed on v, relate to the developments in [11, 3].

In [8], the authors study the geometry of the convex envelope Γ_v by exploring the regularity of v and the geometry of the convex domain Ω. Indeed, if $v \in C^{1,\alpha}(\Omega)$ and the domain is of class $C^{1,\beta}$, they prove that Γ_v is of class $C^{1,\gamma}$, where $\gamma := \min\{\alpha,\beta\}$. When it comes to ensuring $C^{1,1}$-regularity for Γ_v, the authors find that both v and the boundary of Ω must be of class $C^{3,1}$ for this result to hold in general. Compare the latter with the findings in [7].

The remainder of the paper is organized as follows. Section 2 gathers two auxiliary lemmas used in the paper, whereas Section 3.1 details the proof of Theorem 1. The proof of Theorem 2 is the subject of Section 3.2, and Section 3.3 presents the proof of Theorem 3. Section 3.4 accounts for the proof of Theorem 4 and closes the paper.

2. Two technical Lemmas

Let $g \in C^2(\partial\Omega)$ and denote with $\mathcal{G}(\partial\Omega)$ the set

$$\mathcal{G}(\partial\Omega) = \mathcal{G}_g(\partial\Omega) := \left\{(x, g(x)) \in \mathbb{R}^{d+1} \mid x \in \partial\Omega\right\}.$$

For $y \in \Omega$, we define the cone generated by the graph of g on $\partial\Omega$ and y as

$$C_g(y,\partial\Omega) := \left\{(x,\tau) \in \mathbb{R}^{d+1} \,|\, (x,\tau) = (z + \lambda(y-z), \lambda),\ \text{for } z \in \mathcal{G}(\partial\Omega) \text{ and } \lambda \in \mathbb{R}\right\}.$$

We are interested in the curvature of $C_g(y,\partial\Omega)$ for certain values of λ. For $I\subset(-\infty,1]$, denote with $C_g(y,\partial\Omega,I)$ the subset of $C_g(y,\partial\Omega)$ defined as

$$C_g(y,\partial\Omega,I):=\left\{(x,\tau)\in\mathbb{R}^{d+1}\,|\,(x,\tau)=(z+\lambda(y-z),\lambda),\ \text{for } z\in\mathcal{G}(\partial\Omega) \text{ and } \lambda\in I\right\}.$$

For $x\in C_g(y,\partial\Omega,I)$, denote by $\kappa(x)$ the curvature of the cone at x. Next, we study the curvature of $C_g(y,\partial\Omega,[-1/2,1/2])$.

Lemma 1 (Uniform bounds for the cone curvature)**.** *Let $\Omega\subset\mathbb{R}^d$ be a bounded, convex domain of class C^2. Let further $g\in C^2(\partial\Omega)$. For $y\in\Omega$, consider the cone generated by y and the graph of g on $\partial\Omega$. Then there exists $C>0$, depending on the dimension, the curvature of $\partial\Omega$, and the smoothness of g, such that $\kappa(x)<C$ for every $x\in C_g(y,\partial\Omega,[-1/2,1/2])$.*

Proof. Because the radial curvature of the cone is zero, we examine the tangential curvature at $x\in C_g(y,\partial\Omega,[-1/2,1/2])$. Let $y^j\in\partial\Omega$ and $s=(s_1,\ldots,s_{d-1})$ be local coordinates near y^j. Denote with $x(s)$ a local parametrization of $\partial\Omega$. The cone generated by $\{(x,g(x)\mid x\in\partial\Omega\}$ and $(y^*,-T)$ is parametrized near y^j as

$$C(s,\lambda)=\lambda(y^*,T)+(1-\lambda)\left(x(s),g(x(s))\right),$$

where $y^*\in\Omega$ and $T>0$ are fixed, though arbitrary. We compute the derivatives of $C(s,\lambda)$ with respect to s. Indeed,

$$\frac{\partial C}{\partial s_i}=(1-\lambda)\left[\sum_{j=1}^{d}\frac{\partial x_j}{\partial s_i}e_j+\sum_{j=1}^{d}\left(\frac{\partial g}{\partial x_j}\frac{\partial x_j}{\partial s_i}\right)e_{d+1}\right]$$

and

$$\frac{\partial^2 C}{\partial s_\ell\partial s_i}=(1-\lambda)\left[\sum_{j=1}^{d}\frac{\partial^2 x_j}{\partial s_\ell\partial s_i}e_j+\sum_{k,j=1}^{d}\left(\frac{\partial^2 g}{\partial x_k\partial x_j}\frac{\partial x_j}{\partial s_i}+\frac{\partial g}{\partial x_j}\frac{\partial^2 x_j}{\partial s_\ell\partial s_i}\right)e_{d+1}\right].$$

Because $\lambda\in[-1/2,1/2]$ confines the analysis to a strip away from (y^*,T), we get

$$|\kappa(x)|\leq C_1\sum_{\ell,i=1}^{d}\frac{\partial^2 C(x,\lambda)}{\partial s_\ell\partial s_i}\leq C,$$

where $C>0$ depends on the geometry of $\partial\Omega$ and the C^2-norm of the function g. □

We emphasize we use Γ for a convex function as well as for its graph, whenever it improves readability.

Lemma 2 (Smoothness from the boundary)**.** *Let $\Omega'\Subset\Omega$ be such that* $\operatorname{dist}(\Omega',\partial\Omega)>\tau$*, for some $\tau>0$ fixed, though arbitrary. Let $g\in C^2(\partial\Omega)$ and Γ be the convex envelope of a function agreeing with g on $\partial\Omega$. Let also Y^* be a point on the graph of Γ. Suppose Y^* is generated by $Y^1,Y^2\in\Gamma$, with y^1 or y^2, or both, lying on $\partial\Omega$. Suppose a polynomial P touches Γ from below at Y^*. Then there exists $C>0$ such that*

$$D^2P(y^*)\leq CI.$$

The constant $C>0$ depends only on the dimension, the geometry of $\partial\Omega$, the boundary value g, and τ.

Proof. Consider the segment $[Y^1,Y^2]$ and suppose first both y^1 and y^2 are on $\partial\Omega$. Suppose without loss of generality that $y^*=\lambda y^1+(1-\lambda)y^2$, for some $\lambda\in(0,1/2)$. Notice λ is uniformly bounded from below by a constant depending only on τ and the diameter of Ω.

Let $\tilde{y}^i:=(y^*+y^i)/2$, for $i=1,2$. Clearly, $\operatorname{dist}(\tilde{y}^i,\partial\Omega)>\tau/2$. Now, Theorem 1 ensures that a shift of $P/3$ touches Γ at $\tilde{Y}^1$ or $\tilde{Y}^2$. However, Lemma 1 ensures the curvature of the cone generated by Y^* and $\mathcal{G}(\partial\Omega)$ is uniformly bounded at the points $(\tilde{y}^i,\tilde{w}^i)\in C(Y^*,\partial\Omega)$. Since the opening of P at $\tilde{y}^i$ is controlled by the curvature of the cone, there exists a universal constant $C>0$ such that $D^2\tilde{P}(\tilde{y}^i)\leq\tilde{C}I$. By taking $C:=\tilde{C}/3d$, one completes the proof in that case.

If only one point, say y^1, lies on the boundary $\partial\Omega$, extend the segment $[Y^1,Y^2]$ until it intersects $\partial\Omega$. Denote the intersection point with y^2, after relabeling and arguing as before, one finishes the argument. □

It might be insightful to compare the claim in Lemma 2 with the following example. Let $\Omega:=B_1$ and $\Gamma=(1+x_1)^{2-\epsilon}$. Notice $\Gamma\in C^{3,1-2\epsilon}(\partial\Omega)$, yet Γ is not uniformly $C^{1,1}$ in the interior of Ω; see [8, Section 2].

The main geometric reason for the failure of smoothness in this example (in light of Lemma 2) is that the line segments $[Y^1,Y^2]$ (or flat pieces) on $\Gamma=(1+x_1)^{2-\epsilon}$ are orthogonal to the x_1-axis. As we approach the point $z:=(-1,0')$, these segments become closer and closer to the tangent line to the domain at z. In the words of the lemma, $\tau\to 0$.

Remark 3. *It is noteworthy, that above discussion, about the segment $[Y^1,Y^2]$ becoming eventually tangential in the limit as we approach the boundary point z, is the main technical part in many existing proofs for optimal smoothness up to the boundary. This is also the main trouble point in our study, forcing us to stay inside the domain, and not get uniform estimates up to the boundary.*

3. Proof of Theorems

3.1. **Proof of Theorem 1.** Suppose first $y^1,\ldots,y^{N+1}$ are in Ω. Consider now the case when Y^* is generated by two points Y^1,Y^2, so that $N=1$ and

$$y^*=\lambda_1y^1+\lambda_2y^2.$$

We prove that at least one of P^j will touch Γ at the point y^j.

Now, towards a contradiction, suppose that neither P^1 nor P^2 touch Γ at y^1 or y^2, respectively. Next we subtract the linear function $(x-y^*)\cdot\nabla P(y^*)$ (which represents a supporting plane) from Γ, and translate such that $y^1=\mathbf{0}=(0',0)$ and $y^2=(0',x_d^*)$, and assume $y^*=(0',s)$ for some $s\geq x_d^*/2$. If $s<x_d^*/2$, then we may change the role of y^1,y^2. We keep the same notation for this P. Furthermore, since $\partial_dP\equiv 0$, we need not to translate the polynomial anymore, as it is "cylindrical" in x_d-direction.

Without loss of generality, we may assume that $\partial_j P(x) = 0$ for all directions x_j where $\partial_j P(x) \leq 0$ (including the x_d-direction). Moreover, P is non-negative ($P \geq 0$) and homogeneous of order two.

These reductions imply that, to prove the theorem, it suffices to show that the x_d-independent polynomial P, which touches Γ at y^*, is such that $P/3$ touches Γ from below at one of the points y^1 or y^2.

If this does not happen, then we have a point $X^* = (x^*, x^*_{d+1})$ on the graph of Γ such that

$$x^*_{d+1} = \Gamma(x^*) < \frac{1}{3}P(x^*). \tag{5}$$

Define now L^t $(0 \leq t \leq 1)$ to be the line segment connecting the origin to X^*, so that

$$L^t := t(X^*) = t(x^*, x^*_{d+1}), \qquad L^0 = \mathbf{0}, \quad L^1 = X^*.$$

The convexity of Γ ensures that L^t stays above the graph of Γ, i.e. it belongs to the convex set $\{x_{d+1} \geq \Gamma(x)\}$.

It follows by inspection that L^t has to cut the graph of the polynomial P at some point. Hence, for some t_0 we have

$$t_0 x^*_{d+1} = P(t_0 x^*) = t_0^2 P(x^*),$$

so that $x^*_{d+1} = t_0 P(x^*)$. By virtue of (5) and the fact that $P \geq 0$, we obtain $t_0 < 1/3$. This means that at the point X^*, where L^t intersects the graph of P, the x_d-coordinate satisfies $t_0 x^*_d < x^*_d/3$. This contradicts the assumption that $y^* = (0', s)$, with $s \geq x^*_d/2$, is a touching point for P, which implies that $P \leq \Gamma$ in a neighborhood of this point.

Now, suppose Y^* is generated by several (or all) of the points $\{Y^j\}_{j=1}^N$. In this case, starting from Y^* and following line segments on $\mathcal{P}$, one reaches either a vertex or a point on a lower-dimensional edge of the polygon $\mathcal{P}$, with the touching property holding for at least one of them. In the first case, the proof is completed. In the second case, we apply a dimension reduction argument or simply continue along the segments, repeating the process until we reach a vertex with the touching property, which must happen within at most N steps. The proof of Theorem 1 is completed.

3.2. **Proof of Theorem 2.** Let $Y^* = [y^*, \Gamma_u(y^*)]$ be a point in Γ_u, with $y^* \in \Omega' \Subset \Omega$, and P be a paraboloid touching Γ_u from below at y^*. We prove there exists a universal constant $C_1 > 0$ such that

$$F(D^2P(y^*), DP(y^*), \Gamma_u(y^*), y^*) \leq C_1. \tag{6}$$

Indeed, if $y^* \in \{\Gamma_u = u\}$, the supersolution property for u ensures (6) holds with $C_1 = C$. Suppose otherwise that $y^* \in \Omega \setminus \{\Gamma_u = u\}$.

Then there exists a (smallest) polygonal piece $\mathcal{P}^1 \subset \Gamma_u$ of dimension $N \leq d$, generated by points $\{Y^j\}_{j=1}^{N+1}$, such that $Y^* \in \mathcal{P}^1$ and $y^j \in \{x \in \overline{\Omega} : \ \Gamma_u = u\}$ for every $j = 1, \ldots, N+1$. With *smallest* we mean it is impossible to generate Y^* with M points, for $M < N$. Notice that some, or even all, points y^j may belong to $\partial\Omega$. We start the analysis with the case $N = 1$ and examine the general scenario using a reduction argument. We split the proof into three steps for clarity.

Step 1 - Suppose $N = 1$. In that case, $Y^* = \lambda Y^1 + (1 - \lambda)Y^2$. Three possibilities arise: $(i.)$ y^1 and y^2 are in the interior of Ω, $(ii.)$ y^1 is on $\partial\Omega$ and y^2 is in Ω, or $(iii.)$ both points are on $\partial\Omega$.

In Case $(i.)$, Theorem 1 ensures that there exists at least one $j \in \{1, 2\}$ such that P^j touches y^j; for definiteness, suppose $j = 1$. Hence,

$$DP(y^*) = 3DP^1(y^1) \qquad \text{and} \qquad D^2P(y^*) = 3D^2P^1(y^1).$$

Calculating the PDE and using the properties of F, see (4), we have

$$\begin{aligned} F(D^2P(y^*), DP(y^*), P(y^*), y^*) &= F(3D^2P^1(y^1), 3DP^1(y^1), \Gamma(y^*), y^*) \\ &\leq C + \omega_1(|\Gamma(y^*) - \Gamma(y^1)|) + \omega_2(|y^* - y^1|) \\ &\leq C_1, \end{aligned}$$

where $C_1 > 0$ depends on C, the dimension d, the domain Ω and F, through ω_i and its homogeneity degree.

In Case $(ii.)$ we may assume P^2 does not touch Γ_u at y^2 since then, by the supersolution property, we would be done. Denote by $\tilde{y}^1$ the point

$$\tilde{y}^1 := \frac{1}{2}\left(y^1 + y^*\right).$$

An application of Lemma 2 ensures the existence of a universal constant $C > 0$ such that

$$D^2P(y^*) \leq CI.$$

Finally, if Case $(iii.)$ holds, Lemma 2 once again ensures the former upper bound for $D^2P(y^*)$ is available, perhaps with a different constant, still universal. Therefore, one finds $C_1 > 0$ such that

$$F(D^2P(y^*), DP(y^*), P(y^*), y^*) \leq C_1.$$

In the sequel, we treat the case $1 < N \leq d$.

Step 2 - Let $1 < N \leq d$. As before, if $y^1, \ldots, y^{N+1} \in \Omega$, the result follows from Theorem 1 combined with the supersolution property for u. We continue by supposing there exists $j \in \{1, \ldots, N + 1\}$ such that $y^j \in \partial\Omega$. For definiteness, let $y^1 \in \partial\Omega$. Consider the segment $\left[Y^1, Y^*\right]$ and extend it until it intersects the boundary of $\mathcal{P}^1$. Denote the intersection point with Z^1, which lies in a polygon $\mathcal{P}^2$ with $\dim \mathcal{P}^2 = N - 1$; indeed, were $\dim \mathcal{P}^2 < N - 1$, Y^* would be generated by less than $N + 1$ points, and $\mathcal{P}^1$ would not be the smallest polygon generating Y^*. In particular, $z^1 \in \Omega$.

Step 3 - At the point z^1 either one of the following holds:

I - No shifted version of P touches Γ_u at Z^1;

II - A shifted version of P touches Γ_u at Z^1.

In Case I, we set $\tilde{y}^1 := \frac{1}{2}\left(y^1 + y^*\right) \in \Omega$, and invoke Lemma 2, which yields $D^2P(y^*) \leq CI$, for some universal constant $C > 0$, leading to

$$F(D^2P(y^*), DP(y^*), P(y^*), y^*) \leq C_1.$$

In Case II, we assume $z^1 \in \Omega \setminus \{\Gamma_u = u.\}$, since otherwise, the supersolution property yields the required estimate.

Now, letting z^1 play the role of y^*, we may iterate the argument above and reach to either of the cases, again. Repeating in finite steps (maximum N steps), we arrive at the case when a final shift of P touches $\{\Gamma_u = u\}$, or alternatively it does not, and we are in Case I. In either case, we conclude again the bound of the PDE for Γ_u. Theorem 2 is complete.

3.3. **Proof of Theorem 3.** We divide the argument into two cases, corresponding to the different operators appearing in the statement. For each operator in the theorem, we employ a distinct approach.

Case 1) $|Du|^pG(D^2u)$: By Theorem 3 we have

$$\left||D\Gamma_u|^pG(D^2\Gamma_u)\right| \leq C \quad \text{in} \;\; B_{3/4},$$

for some universal $C > 0$. Hence, [12, Theorem 1] implies $D\Gamma_u \in C^{\alpha^*}_{\text{loc}}(B_{3/4})$, with local uniform estimates, where α^* is the minimum between the exponent associated with the Krylov-Safonov regularity theory and $1/(p+1)$. The convexity of G ensures the former is equal to 1, and the argument is complete in this case, as was pointed out in [2, Corollary 3.2].

Case 2) $\Delta_p u$: We recall that Theorem 2 implies $\Delta_p\Gamma_u \in L^\infty(B_{99/100})$. Consequently, we conclude there exists $\alpha \in (0,1)$ such that $\Gamma_u \in C^{1,\alpha}_{\text{loc}}(B_{99/100})$, with estimates (see, for instance, [9]; see also [1] and references therein). To obtain the optimal regularity of the convex hull, we refine the analysis by considering two regimes.

First, we fix $x_0 \in B_{99/100}$ and set $\beta \in (0,1)$ as

$$\beta := \frac{1}{p-1}. \tag{7}$$

Suppose first that

$$|D\Gamma_u(x_0)| \leq r^\beta, \tag{8}$$

and that we can find $C > 0$ such that

$$\sup_{x \in B_r(x_0)} |\Gamma_u(x) - \Gamma_u(x_0)| \leq Cr^{1+\beta}. \tag{9}$$

The triangle inequality then yields the desired result

$$\sup_{x\in B_r(x_0)} |\Gamma_u(x)-\Gamma_u(x_0)-D\Gamma_u(x_0)\cdot(x-y)|\leq Cr^{1+\beta}.$$

We thus need to show inequality (9). Consider the scaled function

$$\overline{\Gamma}_u(x):=\frac{\Gamma_u(rx+x_0)-\Gamma_u(x_0)}{r^{1+\beta}}.$$

We have

$$\Delta_p\overline{\Gamma}_u=r^{1-\beta(p-1)}\Delta_p\Gamma_u;$$

the choice of β in (7) builds upon Theorem 2 to yield $\Delta_p\overline{\Gamma}_u\in L^\infty(B_{99/100})$. Therefore there exists $C>0$ such that

$$0\leq\Delta_p\overline{\Gamma}_u\leq C,\qquad \text{in } B_{99/100}.$$

Now, by the assumption (8) we have $|D\Gamma_u(x_0)|\leq r^\beta$, implying

$$\overline{\Gamma}_u(x)+1\geq\overline{\Gamma}_u(x)-D\overline{\Gamma}_u(0)\cdot x=\frac{\Gamma_u(rx+x_0)-\Gamma_u(x_0)}{r^{1+\beta}}-D\Gamma_u(x_0)\cdot x\geq 0,$$

for every $x\in B_1$, where the last inequality is due to the convexity of Γ_u.

Using the weak Harnack inequality for non-negative super-solutions [15, Theorem 3.13], we conclude that

$$\|\overline{\Gamma}_u(x)+C\|_{L^s(B_{3/4})}\leq C_1(\inf\overline{\Gamma}_u(x)+C)=C_1C,$$

since $\overline{\Gamma}_u(0)=0$.

Next, we apply the maximum principle (cf. [15, Corollary 3.10]) to the p-subharmonic function $\overline{\Gamma}_u(x)$ to obtain

$$\sup_{B_{1/2}}\overline{\Gamma}_u(x)\leq\|\overline{\Gamma}_u(x)+C\|_{L^s(B_{3/4})}\leq C.$$

This gives us (9) and leads to

$$\sup_{x\in B_r(x_0)}|\Gamma_u(x)-\Gamma_u(x_0)-D\Gamma_u(x_0)\cdot(x-y)|\leq Cr^{p/(p-1)}.$$

Now we consider the case $|D\Gamma_u(x_0)|\geq r^\beta$, for $0<r<1/4$. Here we set $r_{x_0}^{1/(p-1)}:=|D\Gamma_u(x_0)|$ and define $\tilde{\Gamma}_u:B_1\to\mathbb{R}$ as

$$\tilde{\Gamma}_u(x):=\frac{\Gamma_u(r_{x_0}x+x_0)-\Gamma_u(x_0)}{r_{x_0}^{p/(p-1)}}.$$

By Case 1, $|\tilde{\Gamma}_u(x)|\leq 1$ on $B_{99/100}$. Moreover $0\leq\Delta_p\tilde{\Gamma}_u\leq C$, and $|D\tilde{\Gamma}_u(0)|=1$. Therefore, by continuity of $D\tilde{\Gamma}_u$, there exists $0<r^*<1$ such that

$$\left|D\tilde{\Gamma}_u(x)\right|>\frac{1}{2}\tag{10}$$

in B_{r^*}. We conclude $\tilde{\Gamma}_u$ satisfies

$$0\leq\operatorname{div}\left(A(x)D\tilde{\Gamma}_u\right)\leq C\quad\text{in}\quad B_{99/100},$$

for some $A \in C^{\alpha}(B_{99/100}, \mathbb{R}^d)$. Because of (10), we notice A is uniformly elliptic. As a consequence, we obtain $\Gamma_u \in C^{1,\alpha}_{\mathrm{loc}}(B_{r^*/2})$ for every $\alpha \in (0,1)$, with estimates. The previous steps combine to complete the proof.

3.4. **Proof of Theorem 4.** Fix $\Omega' \Subset \Omega$ and take $y \in \Omega'$. Let $y^1 \in \partial\Omega$ be the closest boundary point to y. Let $z \in \Omega$ be such that $y = \lambda y^1 + (1-\lambda)z$, for some $\lambda \in (0, 1/2)$.

Consider the cone generated by z and $g(\partial\Omega)$. Away from the vertex, the cone is of class $C^{1,\gamma}$, where $\gamma := \min(\alpha, \beta)$. This norm is also uniform, if $\lambda \leq 1/2$, as we have assumed; compare with Lemma 1. As a consequence, the convex graph Γ_g is touched from above by a $C^{1,\gamma}$-graph.

The above, in light of [8, Definition 1.1], means that Γ_g is $(1+\gamma)$-semi-concave. Hence, by Theorem 1.2 in [8], the result follows.

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