Numerical methods for fully nonlinear degenerate diffusions

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Abstract

We propose finite difference methods for degenerate fully nonlinear elliptic equations and prove the convergence of the schemes. Our focus is on the pure equation and a related free boundary problem of transmission type. The cornerstone of our argument is a regularisation procedure. It decouples the degeneracy term from the elliptic operator driving the diffusion process. In the free boundary setting, the absence of degenerate ellipticity entails new, genuine difficulties. To bypass them, we resort to the intrinsic properties of the regularised problem. We present numerical experiments supporting our theoretical results. Our methods are flexible, and our approach can be extended to a broader class of non-variational problems.

Keywords: Degenerate fully nonlinear equations; free boundary problems; viscosity solutions; finite difference methods; convergent numerical scheme.

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1 Introduction

We propose numerical methods for fully nonlinear degenerate equations of the form

$$\begin{cases} |Du|^\theta F(D^2u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $F : S(d) \to \mathbb{R}$ is a fully nonlinear elliptic operator, $f \in C(\Omega) \cap L^\infty(\Omega)$, and $g \in C(\partial\Omega)$. The exponent $\theta > 1$ is a degeneracy rate, governing how the vanishing of the gradient affects the ellipticity of the problem.

After designing a finite difference scheme for (1), we embed this equation into a free boundary problem. Namely, we consider the free transmission model

$$\begin{cases} |Du|^{\theta_1}F(D^2u) = f & \text{in } \ \Omega \cap \{u > 0\} \\ |Du|^{\theta_2}F(D^2u) = f & \text{in } \ \Omega \cap \{u < 0\} \\ u = g & \text{on } \ \partial\Omega, \end{cases} \tag{2}$$

where $1 \leq \theta_1 < \theta_2$. In both settings, the unknown u is understood in the sense of viscosity solutions [19, 17, 18, 13].

The PDE in (1) accounts for a fully nonlinear counterpart of the p-laplacian. As in the latter, the ellipticity in (1) is no longer uniform, since it depends on a power of the gradient. This problem has been largely studied in the literature [5, 6, 7, 8, 9, 20]. Regularity results for the solutions to (1) are the subject of [25, 10, 1, 27, 26, 11]. The transmission problem in (2) was introduced in [24]; see also the survey [4] and the references therein.

Numerical methods for nonlinear partial differential equations (PDE) of Hamilton-Jacobi type have appeared in [2, 15, 16, 21, 31], to name just a few. In [3], the authors examine a strategy for the approximation of viscosity solutions to fully nonlinear equations. They also devise a set of conditions under which a family of numerical approximations converge to the (unique) viscosity solution of the underlying PDE. The authors prove that solutions to a monotone, consistent and stable scheme converge to the unique viscosity solution of the approximated problem. In [28], the author verifies that degenerate ellipticity, properness and Lipschitz continuity of the numerical method ensure monotonicity and stability and lead to convergence. We also mention [22, 29, 23, 14].

Solving a nonlinear approximation scheme designed for a stationary PDE benefits from a solution operator in the spirit of the Euler method. In [28], the author resorts to this method and derives a non-linear Courant-Friedrichs-Lewy (CFL) condition ensuring the solution operator is a contraction on a Banach space. Such a condition depends only on the Lipschitz constant of the approximation scheme.

Problems (1) and (2) introduce genuine difficulties concerning numerical approximations. The first one regards monotonicity. It is well-known that there exists a monotone approximation for $F(D^2u)$, provided F satisfies a diagonal-dominance condition (see [3, Theorem 3.4]). On the other hand, an upwind monotone discretisation of Du yields a monotone approximation for $|Du|^\theta$. However, the *product* of monotone operators are not necessarily monotone.

The second main difficulty concerns the (two-phase) free boundary problem (2). Unlike the obstacle problem, (2) cannot be immediately written in terms of a single equation in the entire domain. Compare with the [28, Section 3]. Therefore, an approximation scheme in Ω is not available. In addition, the dependence of the degeneracy rates on the sign of the solution affects the maximum and the comparison principles.

We start our analysis with the pure equation (1). For $0 < \varepsilon \ll 1$, we propose a regularisation of the form

$$\begin{cases} \left(\varepsilon + |Du^\varepsilon|^2\right)^{\frac{\theta}{2}} \left(\varepsilon u^\varepsilon + F(D^2 u^\varepsilon)\right) = f & \text{in } \Omega \\ u^\varepsilon = g & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Arguing along the same lines as in [24], we prove the existence of a (unique) viscosity solution to (3). We also verify that $u^\varepsilon \to u$, locally uniformly, where $u \in C(\Omega)$ is a viscosity solution to (1). This is summarised in our first result.

Theorem 1 (Existence of solutions)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain satisfying a uniform exterior sphere condition. Suppose Assumptions A1 and A3, to be detailed further, hold. Then there exists a unique viscosity solution to* (3)*, denoted with $u^\varepsilon \in C(\Omega)$. Moreover, $u^\varepsilon \to u$ locally uniformly in Ω, where $u \in C(\Omega)$ is a viscosity solution to* (1)*.*

Thus, our goal is to produce a numerical approximation to (3). To that end, we fix $0 < h \ll 1$ and consider a discrete approximation of Ω, which we denote with Ω_h. Then we propose an approximation of the form

$$\begin{cases} \varepsilon u_h^\varepsilon(x) + F(D_h^2 u_h^\varepsilon(x)) - \frac{f(x)}{\left(\varepsilon + |D_h u_h^\varepsilon(x)|^2\right)^{\frac{\theta}{2}}} = 0 & \text{in } \Omega_h \\ u_h^\varepsilon(x) = g(x) & \text{on } \partial\Omega_h. \end{cases} \tag{4}$$

We prove the method in (4) is monotone, consistent with (3), and stable. Therefore, an off-the-shelf application of the Barles-Souganidis theory implies that $u_h^\varepsilon \to u^\varepsilon$ locally uniformly [3]. Finally, by sending $\varepsilon \to 0$, one concludes that u_h^ε provides an approximation to the solutions to (1).

We emphasise the regularisation in (3) decouples the product in (1), unlocking monotonicity. In addition, the proof of the existence of viscosity solutions to (3) relies on global barriers. We show that a discrete version of those barriers amounts to sub- and super-solutions to the numerical method in (4). Such discrete barriers are independent of the mesh size h. Together with the degenerate ellipticity of the method, it provides us with the stability for u_h^ε. To prove the consistency, we rely on standard arguments. Our first main result is the following.

Theorem 2 (Convergence of the numerical method I)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain satisfying a uniform exterior sphere condition. Suppose Assumptions A1-A3, to be detailed further, hold true. Then the method* (4) *is monotone, consistent and stable. Therefore, the family $(u_h^\varepsilon)_{0<h,\varepsilon<1}$ converges locally uniformly to u as $\varepsilon, h \to 0$.*

Theorem 2 verifies that the method in (4) is monotone and consistent. After proving that global barriers available for (3) provide us with global barriers for (4), uniformly in h, we prove the method is stable. Hence, u_h^ε converges locally uniformly to the viscosity solution u^ε of (3).

Consequently, the method (4) provides a numerical approximation for (1). It remains to notice that the existence of solutions to (4) follows from the Euler method, under an appropriate CFL condition.

Once the numerical approximation for the pure equation is understood, we address the free transmission problem (2). Once again, we resort to a regularisation strategy. However, to accommodate a free boundary problem whose degeneracy law depends on the solutions, we start with an auxiliary function $\theta_\varepsilon(u)$. Indeed, for $0 < \varepsilon \ll 1$, consider

$$\Theta_\varepsilon(t) := \begin{cases} \theta_1 \mathbb{1}_{\{t<-\varepsilon\}} + \theta_2 \mathbb{1}_{\{t>\varepsilon\}} & \text{if} \quad (x,t) \in \mathbb{R} \setminus [-\varepsilon, \varepsilon] \\ \frac{\theta_2 - \theta_1}{2\varepsilon} t + \frac{\theta_1 + \theta_2}{2} & \text{if} \quad (x,t) \in (-\varepsilon, \varepsilon). \end{cases}$$

Now, define $\theta_\varepsilon := \Theta_\varepsilon * \eta_\varepsilon$. Notice that $\theta_\varepsilon \in [\theta_1, \theta_2]$. Also, if $t > \varepsilon$ we get $\theta_\varepsilon(t) = \theta_2$; if $t < -\varepsilon$, we get $\theta_\varepsilon(t) = \theta_1$. In the interval $(-\varepsilon, \varepsilon)$, the exponent $\theta_\varepsilon(t)$ is non-decreasing and smooth. We consider

$$\begin{cases} \left(\varepsilon + |Du^\varepsilon|^2\right)^{\frac{\theta_\varepsilon(u^\varepsilon)}{2}} \left(\varepsilon u^\varepsilon + F(D^2 u^\varepsilon)\right) = f & \text{in} \quad \Omega \\ u^\varepsilon = g & \text{on} \quad \partial\Omega. \end{cases} \tag{5}$$

A viscosity solution u^ε to (5) converges locally uniformly to a viscosity solution of (2). We observe, however, that (5) introduces an important limitation to our method.

Indeed, our approach to the numerical approximation of (1) and (2) stems from the regularisation introduced in [30, 24]. In those papers, the authors address the free transmission problem through regularisations that depend on two parameters. The first one is $0 < \varepsilon \ll 1$, whereas the second one is a function $v \in C(\overline{\Omega})$. To pass from the regularised problem to (2), it is critical to produce a fixed-point argument for the functional parameter v. Consequently, a numerical approximation of (2) through the (ε, v)-regularisation would rely on a similar strategy. The introduction of θ_ε, as above, aims at bypassing this constraint.

We propose a discrete approximation for (5) of the form

$$\begin{cases} \varepsilon u_h^\varepsilon(x) + F(D_h^2 u_h^\varepsilon(x)) - \frac{f(x)}{\left(\varepsilon + |D_h u_h^\varepsilon(x)|^2\right)^{\frac{\theta_\varepsilon(u_h^\varepsilon(x))}{2}}} = 0 & \text{in } \Omega_h \\ u_h^\varepsilon(x) = g(x) & \text{on } \partial\Omega_h. \end{cases} \tag{6}$$

Under general assumptions, this method is monotone and consistent with the regularisation in (5). Nevertheless, the dependence of θ_ε on the solution introduces a further difficulty. It jeopardises the comparison principle, in the sense of [28].

To overcome this difficulty, we construct new barriers $\underline{w} < \overline{w}$ and compare them with solutions u_h. By construction, the former are also comparable to the threshold parameters $\pm\varepsilon$. As a consequence, we enforce $\theta_\varepsilon(\underline{w}) = \theta_1$ and $\theta_\varepsilon(\overline{w}) = \theta_2$. Our second main result reads as follows.

Theorem 3 (Convergence of the numerical method II)**.** *Suppose Assumptions A2-A4, to be detailed further, are in force. Then the method in* (6) *is monotone, consistent with* (5)*, and stable. Suppose there exists a unique viscosity solution to* (5)*. Then $u_h^\varepsilon \to u$ locally uniformly, where u is a viscosity solution to* (2)*.*

The proof of Theorem 3 follows along the same arguments as in the proof of Theorem 2. The main difference is due to the lack of a discrete comparison principle. We find a way around this difficulty by exploring the monotonicity of θ_ε and using specific barriers.

The remainder of this paper is organised as follows. Section 2 gathers preliminary material, whereas Section 3 examines the existence of solutions to (3). The proof of Theorem 2 is the subject of Section 4. In Section 5 we detail the proof of Theorem 3. A final section presents a few numerical experiments, as an attempt to illustrate our strategy and validate our method.

2 Main assumptions and preliminary material

We denote with $S(d)$ the space of symmetric matrices of order d, and notice $S(d) \sim \mathbb{R}^{\frac{d(d+1)}{2}}$. The norm of $M \in S(d)$ is given by.

$$\|M\| := \sup_{e \in \mathbb{S}^{d-1}} e^T M e.$$

For $\xi \in \mathbb{R}^d$, we define $\|\xi\| := \sqrt{\xi \cdot \xi}$. The matrix norm $\|N\|$ is defined as

$$\|N\| := \max_{i=1,\dots,d} |e_i|\,,$$

where $\{e_1, e_2, \dots, e_d\}$ are the eigenvalues of the matrix N. We also notice that the matrix inequality $M \geq N$ means that $M - N$ is positive semidefinite.

Our first main assumption concerns the uniform ellipticity of the fully nonlinear operator driving (1) and (2).

A 1 (Uniform ellipticity)**.** *Let $0 < \lambda \leq \Lambda$ be fixed, though arbitrary. We suppose F is uniformly elliptic, or (λ, Λ)-elliptic. Namely, for every $M, N \in S(d)$, we have*

$$\lambda \|N\| \leq F(M) - F(M+N) \leq \Lambda \|N\|\,,$$

provided $N \geq 0$.

A well-known consequence of uniform ellipticity is the Lipschitz continuity of the operator F concerning the Hessian entry. Also, a uniformly elliptic operator can be written as an Isaacs-type operator [12, Remark 1.5]. Building on that observation, we introduce our second assumption.

A 2 (Diagonal dominance)**.** *We suppose the operator $F : S(d) \to \mathbb{R}$ is of Isaacs-type. That is,*

$$F(M) = \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \left(-\mathrm{Tr}\left(A_{\alpha,\beta} M\right)\right),$$

where the matrices

$$A_{\alpha,\beta} = \left(a_{i,j}^{\alpha,\beta}\right)_{i,j=1}^{d} \in S(d)$$

are (λ, Λ)*-elliptic and the sets* $\mathcal{A}$ *and* $\mathcal{B}$ *are compact. Also,* $A_{\alpha,\beta}$ *satisfies a diagonal dominance condition of the form*

$$a_{i,i}^{\alpha,\beta} \geq \sum_{\substack{i,j=1\\ i\neq j}}^{d} \left| a_{i,j}^{\alpha,\beta} \right|,$$

for every $i = 1, \ldots, d$*,* $\alpha \in \mathcal{A}$ *and* $\beta \in \mathcal{B}$*.*

The diagonal dominance condition in Assumption A2 ensures that there exists a monotone approximation scheme for the operator F; see [3, Theorem 3.4].

A 3 (Regularity of the data)**.** *We suppose* $f \in C(\Omega) \cap L^\infty(\Omega)$*. We also suppose* $g \in C(\partial\Omega)$*.*

When working under Assumption A3, one may consider $f \equiv 1$, for simplicity. We also impose conditions on the exponents θ_1 and θ_2.

A 4 (Degeneracy rates)**.** *We suppose* $1 \leq \theta_1 < \theta_2$*.*

We continue with the definition of degenerate ellipticity.

Definition 1 (Degenerate elliptic operators)**.** *We say the operator* $F : S(d) \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ *is degenerate elliptic if*

$$F(N, p, r, x) \leq F(M, p, r, x),$$

for every $p \in \mathbb{R}^d$*,* $r \in \mathbb{R}$ *and* $x \in \Omega$*, whenever* $M \leq N$*.*

A (λ, Λ)-elliptic operator is degenerate elliptic. Of particular interest for us is the operator $F_\varepsilon = F_\varepsilon(M, p, r, x)$ defined as

$$(\varepsilon + |p|)^{\theta_\varepsilon(r)} \left(\varepsilon r + F(M)\right).$$

Notice F_ε is degenerate elliptic provided F is degenerate elliptic.

Let $G : S(d) \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a (λ, Λ)-elliptic operator and consider the Dirichlet problem

$$\begin{cases} G(D^2u, Du, u, x) = f & \text{in} \quad \Omega \\ u = g & \text{in} \quad \partial\Omega, \end{cases} \tag{7}$$

where $f \in L^p(\Omega) \cap C(\Omega)$ and $g \in C(\partial\Omega)$. We define the (perhaps discontinuous) operator $F : S(d) \times \mathbb{R}^d \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ as

$$F(M, p, r, x) := \begin{cases} G(M, p, r, x) - f(x) & \text{if} \quad x \in \Omega \\ r - g(x) & \text{if} \quad x \in \partial\Omega. \end{cases}$$

The operator F simultaneously accounts for the PDE and the boundary condition. To define a viscosity solution to (7), one requires the notion of semicontinuous envelopes, which is the subject of the next definition.

Definition 2 (Semicontinuous envelopes)**.** *Let $F : S(d) \times \mathbb{R}^d \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$. We define the upper semicontinuous envelope $F^* : S(d) \times \mathbb{R}^d \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ as*

$$F^*(M, p, r, x_0) := \limsup_{x \to x_0} F(M, p, r, x).$$

We also define the lower semicontinuous envelope $F_* : S(d) \times \mathbb{R}^d \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ as

$$F_*(M, p, r, x_0) := \liminf_{x \to x_0} F(M, p, r, x).$$

We proceed with the definition of viscosity solution to (7) in terms of the semicontinuous envelopes for F.

Definition 3 (Viscosity solution)**.** *An upper semicontinuous function $u : \overline{\Omega} \to \mathbb{R}$ is a viscosity subsolution to* (7) *if for every $\varphi \in C^2(\overline{\Omega})$ such that $u - \varphi$ attains a maximum at $x_0 \in \overline{\Omega}$, we have*

$$F_*(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq 0.$$

A lower semicontinuous function $u : \overline{\Omega} \to \mathbb{R}$ is a viscosity supersolution to (7) *if for every $\varphi \in C^2(\overline{\Omega})$ such that $u - \varphi$ attains a minimum at $x_0 \in \overline{\Omega}$, we have*

$$F^*(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \geq 0.$$

A continuous function that is both a subsolution and a supersolution to (7) *is a viscosity solution to* (7)*.*

The previous definition resorts to C^2-regular test functions, setting the framework in the context of C-viscosity solutions [18].

Let $0 < h_0 \ll 1$ be fixed and set $\mathcal{H}_0 := (0, h_0)$. Given a domain $\Omega \subset \mathbb{R}^d$ and $h \in \mathcal{H}_0$, we design a structured grid $\overline{\Omega}_h$ providing a discrete approximation of $\overline{\Omega}$.

For $i = 1, \dots, d$, let Ω_i be a family of hyperplanes, orthogonal to the canonical unit vector e_i; each hyperplane in Ω_i is a copy of $\mathbb{R}^{d-1}$. Suppose the hyperplanes in Ω_i are h-apart. Denote with $\Omega_{1,\dots,d}$ the collection of all the points determined by the intersection of d hyperplanes. We define Ω_h as

$$\Omega_h := \Omega \cap \Omega_{1,\dots,d}.$$

Clearly, Ω_h is a discrete approximation of Ω, comprised of points that are h-apart. To complete the discretisation of $\overline{\Omega}$, consider the intersection

$$\partial\Omega_h := \partial\Omega \cap (\Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_d)\,.$$

Finally,

$$\overline{\Omega_h} := \Omega_h \cup \partial\Omega_h.$$

For $0 < h \ll 1$ fixed, the cardinality of $\overline{\Omega^h}$ is finite, and denoted with N_h. For $0 < h < h_0$, we use discrete approximations of the Hessian and the gradient of u, respectively.

Under Assumption A2, we define $D_h^2 u_h(x) = \left(\partial^2 x_i, x_j u_h(x)\right)_{i,j=1}^2$ by setting

$$\partial^2_{x_i,x_i} u_h(x) := \frac{u_h(x+he_i) + u_h(x-he_i) - 2u_h(x)}{h^2}$$

and

$$\begin{aligned}\partial^2_{x_i,x_i} u_h(x) := &\frac{-2u_h(x) + u_h(x+e_ih-e_jh) + u_h(x-e_ih+e_jh)}{2h^2}\\ &+ \frac{u_h(x+e_ih) + u_h(x-e_ih) + u_h(x+e_jh) + u_h(x-e_jh)}{2h^2}.\end{aligned}$$

As concerns $|D_h u_h|^2$, we consider

$$|D_h u_h(x)|^2 := \frac{1}{h^2}\sum_{i=1}^{d} \max\left(\left(u_h(x) - u(x+he_i)\right), \left(u_h(x) - u(x-he_i)\right), 0\right)^2.$$

For $0 < h < h_0$ fixed, though arbitrary, we denote with $\mathcal{F}_h$ the set of functions defined on $\overline{\Omega}_h$. A *numerical scheme* (or numerical method, or approximation scheme) is a family $(G_h)_{h\in(0,h_0)}$ of rules $G_h : \mathcal{F}_h \times \overline{\Omega}_h \to \mathbb{R}$. We abuse terminology and sometimes refer to a single rule G_h as a numerical scheme. A solution to the numerical scheme is a function $u_h \in \mathcal{F}_h$ such that $G_h(u_h(x), x) = 0$ for every $x \in \overline{\Omega}_h$. Typically, G_h depends on u_h through

$D_h^2 u_h$, $D_h u_h$ and u_h and we are interested in

$$G_h(u_h(x), x) = G_h(D_h^2 u_h(x), D_h u_h(x), u_h(x), x).$$

Combining the former notation with the discretisations introduced above, we notice $G_h(u_h(x), x)$ also depends on u_h in neighbouring points $y \in \overline{\Omega}_h$. That is,

$$G_h = G_h\left(u_h(x), u_h(y)\Big|_{y \in N(x)}, x\right),$$

where $N(x)$ stands for the neighbouring points of x used in the required discretisations. We use this notation only in the definition of degenerate elliptic schemes. Elsewhere in the paper, we adhere to $G_h = G_h(u_h(x), x)$ and leave the dependence on the neighbouring points implicit. We continue with the definition of degenerate elliptic methods.

Definition 4 (Degenerate elliptic)**.** *We say the numerical method* $(G_h)_{0<h<h_0}$ *is degenerate elliptic if*

$$G_h(v_h(x), x) = G_h\left(v_h(x), v_h(y)\Big|_{y \in N(x)}, x\right)$$

is non-decreasing with respect to $v_h(x)$ *and non-increasing with respect to* $v_h(y)$*, for every* $y \in N(x)$*.*

Notice that both $D_h^2 u_h$ and $D_h u_h$ are degenerate elliptic discretisations. Next, we recall the definitions of monotonicity, consistency and stability for an approximation scheme.

Definition 5 (Monotone scheme)**.** *We say the numerical method* $(G_h)_{0<h<h_0}$ *is monotone if, for every* $0 < h < h_0$*, and every* $u_h, v_h : \overline{\Omega}_h \to \mathbb{R}$ *with* $u_h(x) = v_h(x)$ *and* $u_h \leq v_h$*, we have*

$$G_h(v_h(x), x) \leq G_h(u_h(x), x).$$

The next definition concerns the consistency of the numerical scheme.

Definition 6 (Consistent scheme)**.** *We say the numerical method* $(G_h)_{0<h<h_0}$ *is consistent with* (7) *if*

$$\limsup_{\substack{h \to 0 \\ y \to x \\ \xi \to 0}} G_h(\varphi(y) + \xi, y) \leq F^*(D^2\varphi(x), D\varphi(x), \varphi(x), x)$$

and

$$\liminf_{\substack{h\to 0\\ y\to x\\ \xi\to 0}} G_h(\varphi(y)+\xi,y) \geq F_*(D^2\varphi(x), D\varphi(x), \varphi(x), x),$$

for every $\varphi \in C^\infty(\overline{\Omega})$ *and* $x \in \overline{\Omega}$.

We proceed with defining stability. As usual, a function $u_h : \overline{\Omega}_h \to \mathbb{R}$ denotes a solution to $G_h = 0$.

Definition 7 (Stability)**.** *We say the numerical method* $(G_h)_{0<h<h_0}$ *is stable if there exists* $C > 0$ *such that*

$$\sup_{h\in(0,h_0)} \max_{x\in\overline{\Omega}_h} |u_h(x)| \leq C.$$

Definitions 5-7 are the main ingredients in the criterion for convergence of the numerical scheme introduced in [3]. We recall it in the sequel, in the form of a proposition.

Proposition 1 (Convergence of the numerical method)**.** *Suppose the numerical method* $(G_h)_{0<h<h_0}$ *is monotone, stable, and consistent with* (7)*. Then* $(u_h)_{h\in(0,h_0)}$ *converges, as* $h \to 0$ *to the unique viscosity solution of* (7)*.*

We close this section with a strategy to solve $G_h = 0$. Namely, we consider the solution operator

$$S_\rho[u(x)] := u(x) - \rho G_h(u(x), x),$$

where $0 < \rho \ll 1$ is a parameter chosen to ensure, among other things, that S_ρ has a fixed point; see [28]. The choice of ρ depends on h through a non-linear CFL condition. We also refer to S_ρ as *Euler operator*. We resort to this strategy in the proof of Theorem 2 as well as in our numerical tests. Our last preliminary result regards the conditions ensuring that the solution operator is a contraction. See [28, Theorem 7].

Proposition 2 (The solution operator is a contraction)**.** *Let* $h \in (0, h_0)$ *be fixed, though arbitrary. Suppose* G_h *is degenerate elliptic and Lipschitz-continuous, with Lipschitz constant* $C = C(d, \lambda, \Lambda, \varepsilon, h, \theta)$*. If* $0 < \rho < C$*, the operator* S_ρ *is a strict contraction.*

3 A detour on the existence of solutions

For completeness, we detail the proof of Theorem 1. We argue as usual in the theory of viscosity solutions. Namely, for $\varepsilon > 0$ fixed, we prove a comparison principle for (3) and construct global sub- and super-solutions. Then Perron's method yields the existence of a viscosity solution u_ε to that problem. At this point, regularity estimates for (3) allow us to consider the limit $\varepsilon \to 0$. The stability of viscosity solutions yields the existence of a function $u \in C(\Omega)$ solving (1). We proceed with the comparison principle.

Proposition 3 (Comparison principle)**.** *Suppose Assumptions A1 and A3 are in force. Let $u_\varepsilon \in \mathrm{USC}(\overline{\Omega})$ be a viscosity sub-solution to the PDE in* (3) *and $w_\varepsilon \in \mathrm{LSC}(\overline{\Omega})$ be a viscosity super-solution to the PDE in* (3)*. Suppose further that $u_\varepsilon \leq w_\varepsilon$ on $\partial\Omega$. Then $u_\varepsilon \leq w_\varepsilon$ in $\overline{\Omega}$.*

Proof. The argument follows along the same lines as in the proof of [24, Proposition 4] and is omitted. □

We proceed by constructing global sub- and super-solutions.

Proposition 4 (Existence of global sub and super-solutions)**.** *Suppose Assumptions A1 and A3 are in force. Then, for every $\varepsilon \in (0,1)$, there exists a viscosity sub-solution $\underline{w} \in C(\overline{\Omega})$ to* (3) *and a viscosity super-solution $\overline{w} \in C(\overline{\Omega})$ to* (3)*. In addition, $\underline{w} = \overline{w} = g$ on $\partial\Omega$. Finally, $\underline{w}$ and $\overline{w}$ do not depend on ε.*

Proof. The argument follows along the same lines as in the proof of [24, Lemma 2] and is omitted. □

For every $0 < \varepsilon < 1$, Perron's method ensures the existence of a viscosity solution $u_\varepsilon \in C(\Omega)$ to (3), agreeing with g on the boundary of Ω. In addition, $\underline{w} \leq u_\varepsilon \leq \overline{w}$, for every $\varepsilon \in (0,1)$. We continue with an observation on the Hölder-regularity of the family $(u_\varepsilon)_{\varepsilon\in(0,1)}$.

Lemma 1 (Uniform compactness)**.** *Suppose Assumptions A1 and A3 are in force. Then there exists a unique viscosity solution $u_\varepsilon \in C(\Omega)$ to* (3)*. Moreover, we have $\underline{w} \leq u \leq \overline{w}$. Finally, there exists $\alpha \in (0,1)$ such that $u_\varepsilon \in C^\alpha_{\mathrm{loc}}(\Omega)$ and, for every $\Omega' \Subset \Omega$, we have*

$$\|u_\varepsilon\|_{C^\alpha(\Omega')} \leq C,$$

where $C = C(d, \lambda, \Lambda, \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega)}, \mathrm{dist}(\Omega', \partial\Omega))$.

Proof. Notice that u_ε satisfies

$$-\|f\|_{L^\infty(\Omega)} - \|u_\varepsilon\|_{L^\infty(\Omega)} \leq F(D^2 u_\varepsilon) \leq \|f\|_{L^\infty(\Omega)} + \|u_\varepsilon\|_{L^\infty(\Omega)}$$

in $\Omega \cap \{|Du| > 1\}$. Arguing as in the proof of [24, Corollary 2], one obtains the result. $\square$

Proof of Theorem 1. The family $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is uniformly bounded in some Hölder space. Therefore, there exists a convergent subsequence, still denoted with $(u_\varepsilon)_{\varepsilon\in(0,1)}$, satisfying $u_\varepsilon \longrightarrow u$, locally uniformly in Ω, where $u \in C(\Omega)$. Standard results on the stability of viscosity solutions ensure that u solves (1) in the viscosity sense. $\square$

4 Degenerate fully nonlinear equations

We now introduce a discrete approximation of (3) given by

$$G_h(u_h, x) := \begin{cases} \varepsilon u_h(x) + F(D_h^2 u_h(x)) - \frac{f(x)}{(\varepsilon + |D_h u_h(x)|^2)^{\frac{\theta}{2}}} & \text{if}\, x \in \Omega_h \\ u_h(x) = g(x) & \text{if}\, x \in \partial\Omega_h, \end{cases} \tag{8}$$

We proceed with the monotonicity of $G_h(u_h, x)$.

Proposition 5 (Monotonicity of G_h)**.** *Let G_h be defined as in (8). Suppose Assumptions A2 and A3 are in force. Suppose $u_h, v_h : \overline{\Omega}_h \to \mathbb{R}$ are such that $u_h(x) = v_h(x)$ for some $x \in \Omega_h$, with $u_h \leq v_h$ in Ω_h. Then $G_h(v_h, x) \leq G_h(u_h, x)$.*

Proof. If $x \in \partial\Omega_h$, we conclude $G_h(u_h, x) = g(x) = G_h(v_h, x)$. Suppose otherwise that $x \in \Omega_h$. Under Assumption A2, we have $F(D_h^2 v_h(x)) \leq F(D_h^2 u_h(x))$. Also, $|D_h v_h(x)|^2 \leq |D_h u_h(x)|^2$. Hence,

$$\begin{aligned} G_h(u_h, x) &= \varepsilon v_h(x) + F(D_h^2 u_h(x)) - \frac{f(x)}{(\varepsilon + |D_h u_h(x)|^2)^{\frac{\theta}{2}}} \\ &\geq \varepsilon v_h(x) + F(D_h^2 v_h(x)) - \frac{f(x)}{(\varepsilon + |D_h v_h(x)|^2)^{\frac{\theta}{2}}} \\ &= G_h(v_h, x), \end{aligned}$$

which completes the proof. $\square$

In what follows, we verify that $G_h(u_h, x)$ is Lipschitz-continuous.

Proposition 6 (Lipschitz continuity of G_h)**.** *Let G_h be defined as in* (8)*. Suppose Assumptions A2 and A3 hold true. Then*

$$\max_{x\in\overline{\Omega}_h} |G_h(u_h,x) - G_h(v_h,x)| \leq C \max_{x\in\overline{\Omega}_h} |u_h(x) - v_h(x)| ,$$

for every $u_h, v_h : \overline{\Omega}_h \to \mathbb{R}$. Moreover, the constant $C > 0$ depends on the dimension d, the ellipticity constants $0 < \lambda \leq \Lambda$, $0 < \varepsilon \ll 1$, $0 < h \ll 1$, and $\|f\|_{L^\infty(\overline{\Omega}_h)}$.

Proof. If $x \in \partial\Omega_h$, $G_h(u_h,x) - G_h(v_h,x) = 0$ and there is nothing to prove. Suppose otherwise and notice

$$\begin{aligned}|G_h(u_h,x) - G_h(v_h,x)| &\leq \left(\varepsilon + \frac{C(\lambda,\Lambda,d)}{h^2}\right)|u_h(x) - v_h(x)| \\ &\quad + \left|\frac{f(x)}{(\varepsilon + |D_h u_h(x)|^2)^{\frac{\theta}{2}}} - \frac{f(x)}{(\varepsilon + |D_h v_h(x)|^2)^{\frac{\theta}{2}}}\right| \\ &=: \left(\varepsilon + \frac{C(\lambda,\Lambda,d)}{h^2}\right)|u_h(x) - v_h(x)| + I.\end{aligned} \tag{9}$$

We proceed by examining the term I. Indeed,

$$I \leq \|f\|_{L^\infty(\Omega)} \left| \frac{\left(\varepsilon + |D_h u_h(x)|^2\right)^{\frac{\theta}{2}} - \left(\varepsilon + |D_h v_h(x)|^2\right)^{\frac{\theta}{2}}}{(\varepsilon + |D_h u_h(x)|^2)^{\frac{\theta}{2}}\,(\varepsilon + |D_h v_h(x)^2)^{\frac{\theta}{2}}} \right| \tag{10}$$

and

$$\begin{aligned}&\left| \frac{\left(\varepsilon + |D_h u_h(x)|^2\right)^{\frac{\theta}{2}} - \left(\varepsilon + |D_h v_h(x)|^2\right)^{\frac{\theta}{2}}}{(\varepsilon + |D_h u_h(x)|^2)^{\frac{\theta}{2}}\,(\varepsilon + |D_h v_h(x)^2)^{\frac{\theta}{2}}} \right| \\ &\leq \frac{\max\left(\left(\varepsilon + |D_h u_h(x)|^2\right), \left(\varepsilon + |D_h v_h(x)|^2\right)\right)^{\frac{\theta}{2}-1}}{(\varepsilon + |D_h u_h(x)|^2)^{\frac{\theta}{2}}\,(\varepsilon + |D_h v_h(x)|^2)^{\frac{\theta}{2}}} \left| |D_h u_h(x)|^2 - |D_h v_h(x)|^2 \right| \\ &\leq \frac{C(d,\varepsilon)}{h} \max_{x\in\overline{\Omega}_h} |u_h(x) - v_h(x)| ,\end{aligned} \tag{11}$$

By combining (9) and (10), and (11), we conclude

$$\max_{x\in\overline{\Omega}_h} |G_h(u_h(x),x) - G_h(v_h(x),x)| \leq C \max_{x\in\overline{\Omega}_h} |u(x) - v_h(x)| ,$$

where

$$C := \left(\varepsilon + \frac{C(\lambda,\Lambda,d)}{h^2} + \frac{C(d,\varepsilon)}{h}\right) \sim \frac{1}{h^2}. \tag{12}$$

$\square$

Remark 1 (Courant-Friedrichs-Lewy condition)**.** *In the sequel, we resort to an Euler scheme with parameter $0 < \rho \ll 1$ to prove the existence of solutions to $G_h = 0$. In that context, the constant in* (12) *is pivotal. Indeed, we impose a Courant-Friedrichs-Lewy condition of the form*

$$\rho < \left(\varepsilon + \frac{C(\lambda,\Lambda,d)}{h^2} + \frac{C(d,\varepsilon)}{h}\right)^{-1}.$$

Having in mind that our goal is to examine the case $0 < \varepsilon, h \ll 1$, we can simplify the former inequality and work under

$$\rho \ll \frac{1}{h^2}.$$

We continue with the consistency of G_h. To that end, write

$$G(D^2u, Du, u, x) := \begin{cases} \varepsilon u(x) + F(D^2u(x)) - \frac{f(x)}{(\varepsilon+|Du(x)|^2)^{\frac{\theta}{2}}} & \text{in } \Omega \\ u(x) - g(x) & \text{on } \partial\Omega. \end{cases} \tag{13}$$

A viscosity solution to $G = 0$ solves (3) in the viscosity sense. We prove that G_h is consistent with G.

Proposition 7 (Consistency)**.** *Suppose Assumptions A2-A4 hold. Then G_h is consistent with G.*

Proof. For $\varphi \in C^\infty(\overline{\Omega})$ and $x \in \overline{\Omega}$, we prove that

$$\limsup_{\substack{h\to 0\\ y\to x\\ \xi\to 0}} G_h(\varphi(y)+\xi, y) \le G^*\left(D^2\varphi(x), D\varphi(x), \varphi(x), x\right). \tag{14}$$

We split the proof into three steps, depending on the point $x \in \overline{\Omega}$. We start by considering $x \in \Omega$.

Step 1 - If $x \in \Omega$, we can suppose the points y approaching it are also interior points. In this case,

$$G_h(\varphi(y)+\xi,y) = \varepsilon(\varphi(y)+\xi) + F(D_h^2(\varphi(y)+\xi)) - \frac{f(y)}{(\varepsilon+|D_h(\varphi(y)+\xi)|^2)^{\frac{\theta}{2}}}$$

The regularity of φ implies

$$D_h^2(\varphi(y)+\xi) = D^2\varphi(x) + O(h^2) \quad \text{and} \quad D_h(\varphi(y)+\xi) = D\varphi(x) + O(h).$$

Now, the continuity of F and θ_ε builds upon this information to ensure

$$\begin{aligned}\limsup_{\substack{h\to 0\\ y\to x\\ \xi\to 0}} G_h(\varphi(y)+\xi,y) &= \varepsilon\varphi(x) + F(D^2\varphi(x)) - \frac{f(x)}{(\varepsilon+|D\varphi(x)|^2)^{\frac{\theta}{2}}}\\ &\leq G^*\left(D^2\varphi(x), D\varphi(x), \varphi(x), x\right).\end{aligned}$$

Step 2 - Consider next $x \in \partial\Omega$. In this case, one can approach x by points $y \in \Omega_h$ or $y \in \partial\Omega_h$, or both. Since we work with limit superiors, we consider only $y \in \Omega_h$ or $y \in \partial\Omega_h$. In the latter case, we have

$$G_h(\varphi(y)+\xi,y) = \varphi(y)+\xi-g(y) \longrightarrow \varphi(x)-g(x) = G(D^2\varphi, D\varphi, \varphi, x)$$

as $h \to 0$, $y \to x$ and $\xi \to 0$, and (14) follows. Suppose now $y \in \Omega_h$. Then, arguing as before,

$$\begin{aligned}&\limsup_{\substack{h\to 0\\ y\to x\\ \xi\to 0}} G_h(\varphi(y)+\xi,y)\\ &\leq \limsup_{\substack{h\to 0\\ y\to x\\ \xi\to 0}} \left(\varepsilon(\varphi(y)+\xi) + F(D_h^2(\varphi(y)+\xi)) - \frac{f(y)}{(\varepsilon+|D_h(\varphi(y)+\xi)|^2)^{\frac{\theta}{2}}}\right)\\ &\leq G^*\left(D^2\varphi(x), D\varphi(x), \varphi(x), x\right).\end{aligned}$$

□

The consistency of the method is fundamental for convergence. However, it also plays a role in building global barriers for G_h. Indeed, by modifying the sub and super-solutions in Proposition 4, one can find $\underline{w}$ and $\overline{w}$ such that $\underline{w} = \overline{w} = g$ on $\partial\Omega_h$ and $G_h(\underline{w}(x),x) \leq 0 \leq G_h(\overline{w}(x),x)$ in Ω_h for every

$0 < h \ll 1$ small enough. We formalise this heuristic in the next proposition.

Proposition 8 (Discrete global barriers)**.** *Suppose Assumption A2-A4 hold. There exists* $0 < h_0 \ll 1$ *such that, for* $0 < h < h_0$*, one can find* $\underline{w}, \overline{w} : \overline{\Omega}_h \to \mathbb{R}$ *with* $\underline{w} \leq g \leq \overline{w}$ *on* $\partial\Omega_h$*, satisfying*

$$G_h(\underline{w}(x), x) \leq 0 \leq G_h(\overline{w}(x), x),$$

for every $x \in \overline{\Omega}_h$*.*

Proof. For constants $C_1, C_2 > 0$, define

$$\overline{w}(x) \coloneqq C_2 - \frac{C_1}{2\lambda d} \|x - x_0\|^2 ,$$

where $x_0 \in \mathbb{R}^d$ is such that $|x_i - x_{0,i}| > \sqrt{\lambda}$ for every $i = 1, \ldots, d$, and C_2 is such that $\overline{w}(x) \geq \|g\|_{L^\infty(\partial\Omega)}$. Note

$$D_h^2 \overline{w}(x) = -C_1 I,$$

whereas

$$|D_h u_h(x)|^2 = \frac{C_1^2}{4(\lambda d)^2} \sum_{i=1}^{d} \left(\max\left(2|x_i - x_{0,i}| - h, 0\right)\right)^2 .$$

Now, let $x \in \partial\Omega_h$. Then

$$G_h(\overline{w}(x), x) = \overline{w}(x) - g(x) \geq 0,$$

because of the choice of C_1. If $x \in \Omega_h$, we have

$$G_h(\overline{w}(x), x) \geq \varepsilon\overline{w}(x) + C_1 - \frac{C_1}{(\varepsilon C_1^2)^{\frac{\theta}{2}}} \geq 0.$$

Hence, $G_h(\overline{w}(x), x) \geq 0$ for every $x \in \overline{\Omega}_h$. The construction of $\underline{w}$ is entirely analogous. □

An important aspect of Proposition 8 concerns uniform bounds on $\underline{w}$ and $\overline{w}$. Because these functions are obtained as (uniform) variants of the global barriers in Proposition 4, we conclude there exists $C > 0$, depending only on the dimension d, ellipticity $0 < \lambda \leq \Lambda$, and the norms $\|f\|_{L^\infty(\Omega)}$ and

$\|g\|_{L^\infty(\partial\Omega)}$ such that

$$-C \leq \underline{w} \leq \overline{w} \leq C.$$

It is critical to notice that C does not depend on h.

Proposition 9 (Stability)**.** *Let $h \in (0, h_0)$ be fixed. Let $u_h : \overline{\Omega}_h \to \mathbb{R}$ be a solution to $G_h(u_h(x), x) = 0$ in $\overline{\Omega}_h$. Suppose assumptions A2-A4 are in force. Then there exists $C > 0$ such that*

$$\sup_{h \in (0,h_0)} \max_{x \in \overline{\Omega}_h} |u_h(x)| \leq C.$$

In addition, C depends on the dimension d, the ellipticity $0 < \lambda \leq \Lambda$, and the norms $\|f\|_{L^\infty(\Omega)}$ and $\|g\|_{L^\infty(\partial\Omega)}$, but does not depend on h.

Proof. Let $\overline{w}$, be the barrier function from Proposition 8. We claim that $u_h \leq \overline{w}$ in $\overline{\Omega}_h$. Suppose otherwise; if this is the case, there exists $\overline{x} \in \overline{\Omega}_h$ such that $u_h(\overline{x}) > \overline{w}(\overline{x})$. Clearly, such a point has to be in Ω_h. Also, $u_h(\overline{x}) \geq u_h(x)$ for every $x \in \Omega_h$. Hence,

$$u_h(\overline{x}) - u_h(x) \geq \overline{w}(\overline{x}) - \overline{w}(x),$$

for every $x \in \Omega_h$. It follows that

$$\begin{aligned} G_h(u_h(\overline{x}), \overline{x}) &= \varepsilon u_h(\overline{x}) - F(D_h^2 u_h(x)) - \frac{f(\overline{x})}{(\varepsilon + |D_h u_h(\overline{x})|^2)^{\frac{\theta}{2}}} \\ &> \varepsilon \overline{w}(\overline{x}) - F(D_h^2 \overline{w}(x)) - \frac{f(\overline{x})}{(\varepsilon + |D_h \overline{w}(\overline{x})|^2)^{\frac{\theta}{2}}}. \end{aligned} \tag{15}$$

Therefore,

$$G_h(\overline{w}(\overline{x}), \overline{x}) < G_h(u(\overline{x}), \overline{x}) = 0,$$

which contradicts Proposition 8. Therefore, $u_h \leq \overline{w}$. Arguing as before, with $\underline{w}$ instead of $\overline{w}$, one concludes

$$u_h \geq -\overline{w}$$

in $\overline{\Omega}_h$ and completes the proof. $\square$

We have established that the method in (8) is monotone, consistent with (3) and stable. Therefore, we are in a position to prove Theorem 2.

Proof of Theorem 2. The method G_h is monotone, stable and consistent with (3). Hence, Proposition 1 ensures that the family $(u_h^\varepsilon)_{h\in(0,h_0)}$ converges to the unique viscosity solution u_ε to (3). In addition, because G_h is degenerate elliptic and Lipschitz continuous, with a Lipschitz constant of the order h^{-2}, the solution operator S_ρ is a strict contraction, provided we choose $\rho \ll h^{-2}$. Therefore, seen as a functional on ℓ_∞, S_ρ admits a unique fixed point. To complete the proof, we recall u_ε converges locally uniformly to a viscosity solution to (1). $\square$

5 The fully nonlinear free boundary problem

In this section, we detail the proof of Theorem 3. For $0 < \varepsilon \ll 1$, we introduce

$$\Theta_\varepsilon(t) := \begin{cases} \theta_1 \mathbb{1}_{\{t<-\varepsilon\}} + \theta_2 \mathbb{1}_{\{t>\varepsilon\}} & \text{if} \quad (x,t) \in \mathbb{R}\setminus[-\varepsilon,\varepsilon] \\ \frac{\theta_2-\theta_1}{2\varepsilon} t + \frac{\theta_1+\theta_2}{2} & \text{if} \quad (x,t) \in (-\varepsilon,\varepsilon). \end{cases}$$

and define $\theta_\varepsilon := \Theta_\varepsilon * \eta_\varepsilon$. It is paramount to emphasise that $\theta_\varepsilon \in [\theta_1,\theta_2]$ satisfies $\theta_\varepsilon(t) = \theta_2$ if $t > \varepsilon$ and $\theta_\varepsilon(t) = \theta_1$ if $t < -\varepsilon$. Also, the exponent $\theta_\varepsilon(t)$ is non-decreasing and smooth in $[-\varepsilon,\varepsilon]$. We propose the method $G_h(u_h(x),x)$ defined as

$$G_h(u_h(x),x) := \begin{cases} \varepsilon u_h + F(D_h^2 u_h) - \frac{f(x)}{(\varepsilon+|D_h u_h|^2)^{\frac{\theta_\varepsilon(u_h)}{2}}} = 0 & \text{in} \quad \Omega_h \\ u_h(x) = g(x) & \text{on} \quad \partial\Omega_h. \end{cases} \tag{16}$$

Now, we verify that G_h is monotone in the sense of Definition 5.

Proposition 10 (Monotonicity of G_h)**.** *Let G_h be defined as in* (16)*. Suppose Assumptions A2 and A3 are in force. Suppose $u_h, v_h : \overline{\Omega}_h \to \mathbb{R}$ are such that $u_h(x) = v_h(x)$ for some $x \in \Omega_h$, with $u_h \le v_h$ in Ω_h. Then $G_h(v_h,x) \le G_h(u_h,x)$.*

Proof. The proof follows along the same lines as in Proposition 5, noticing that $\theta_\varepsilon(u_h(x)) = \theta_\varepsilon(v_h(x))$. $\square$

As before, we verify that G_h is consistent.

Proposition 11 (Consistency)**.** *Suppose Assumptions A2-A4 are in force. Then G_h is consistent with* (5)*.*

Proof. The argument is the same as in the proof of Proposition 7, once we notice $\theta_\varepsilon \in C(\mathbb{R})$. $\square$

We proceed with the stability of the method. To that end, we build sub- and super-solutions to $G_h = 0$. Once those functions are available, we compare them with solutions u_h at maximum points and take advantage of the asymptotic behaviour of θ_ε.

Proposition 12 (Stability)**.** *Let $h \in (0, h_0)$ be fixed. Let $u_h : \overline{\Omega}_h \to \mathbb{R}$ be a solution to $G_h(u_h(x), x) = 0$ in $\overline{\Omega}_h$. Suppose assumptions A2-A4 are in force. Then*

$$|u_h(x)| \leq \frac{1}{\varepsilon}\left(\|g\|_{L^\infty(\partial\Omega)} + 1 + \frac{\|f\|_{L^\infty(\Omega)}}{\varepsilon^{\frac{\theta_2}{2}}}\right).$$

Proof. For ease of presentation, we split the proof into three steps. Set $\overline{w} : \overline{\Omega}_h \to \mathbb{R}$ as

$$\overline{w}(x) := \frac{1}{\varepsilon}\left(\|g\|_{L^\infty(\partial\Omega)} + 1 + \frac{\|f\|_{L^\infty(\Omega)}}{\varepsilon^{\frac{\theta_2}{2}}}\right).$$

Step 1 - We start by verifying that

$$G_h(\overline{w}(x), x) \geq 0, \tag{17}$$

for every $x \in \overline{\Omega}_h$. If $x \in \partial\Omega_h$,

$$G_h(\overline{w}(x), x) = \overline{w}(x) - g(x) \geq \frac{1}{\varepsilon}\left(\|g\|_{L^\infty(\partial\Omega)} + 1 + \frac{\|f\|_{L^\infty(\Omega)}}{\varepsilon^{\theta+1}}\right) - \|g\|_{L^\infty(\partial\Omega_h)} \geq 0.$$

Now, suppose $x \in \Omega_h$. In this case,

$$G_h(\overline{w}(x), x) = \|g\|_{L^\infty(\partial\Omega)} + 1 + \frac{\|f\|_{L^\infty(\Omega)}}{\varepsilon^{\frac{\theta_2}{2}}} - \frac{\|f\|_{L^\infty(\Omega)}}{\varepsilon^{\frac{\theta_\varepsilon(\overline{w}(x))}{2}}} \geq 0.$$

Step 2 - We claim that $u_h \leq \overline{w}$ in $\overline{\Omega}_h$. Suppose otherwise; if this is the case, there exists $\overline{x} \in \overline{\Omega}_h$ such that $u_h(\overline{x}) > \overline{w}(\overline{x})$. Such a point has to be in Ω_h. Also, $u_h(\overline{x}) \geq u_h(x)$ for every $x \in \Omega_h$. Hence,

$$u_h(\overline{x}) - u_h(x) \geq \overline{w}(\overline{x}) - \overline{w}(x),$$

for every $x \in \Omega_h$. It follows that

$$\begin{aligned} G_h(u_h(\overline{x}), \overline{x}) &= \varepsilon u_h(\overline{x}) - F(D_h^2 u_h(x)) - \frac{f(\overline{x})}{(\varepsilon + |D_h u_h(\overline{x})|^2)^{\frac{\theta_\varepsilon(\overline{x}, u_h(\overline{x}))}{2}}} \\ &> \varepsilon \overline{w}(\overline{x}) - F(D_h^2 \overline{w}(x)) - \frac{f(\overline{x})}{(\varepsilon + |D_h \overline{w}(\overline{x})|^2)^{\frac{\theta_\varepsilon(\overline{x}, u_h(\overline{x}))}{2}}} \end{aligned} \tag{18}$$

By construction, $u_h(\overline{x}) > \overline{w} > \varepsilon$. Therefore, the definition of θ_ε yields

$$\theta_\varepsilon(\overline{x}, u_h(\overline{x})) = \theta_\varepsilon(\overline{x}, \overline{w}(\overline{x})) = \theta_2.$$

The former observation builds upon (18) and the fact that u_h is a solution to ensure

$$G_h(\overline{w}(\overline{x}), \overline{x}) < G_h(u(\overline{x}), \overline{x}) = 0,$$

which contradicts (17). Therefore, $u_h \leq \overline{w}$.

Step 3 - Consider now $\underline{w} := -\overline{w}$. Arguing as before, one concludes

$$u_h \geq -\overline{w}$$

in $\overline{\Omega}_h$. Hence

$$-\frac{1}{\varepsilon}\left(\|g\|_{L^\infty(\partial\Omega)} + 1 + \frac{\|f\|_{L^\infty(\Omega)}}{\varepsilon^{\frac{\theta_2}{2}}} \right) \leq u_h \leq \frac{1}{\varepsilon}\left(\|g\|_{L^\infty(\partial\Omega)} + 1 + \frac{\|f\|_{L^\infty(\Omega)}}{\varepsilon^{\frac{\theta_2}{2}}} \right).$$

Because the bounds above are independent of h, the result follows. □

We have established that the method in (16) is monotone, consistent with (5) and stable. Therefore, we are in a position to prove Theorem 3.

Proof of Theorem 3. Let G_h be defined in (16). By combining Propositions 10, 11 and 12, we conclude that G_h is monotone, consistent with (5), and stable. Therefore, Proposition 1 ensures that $u_h^\varepsilon \to u^\varepsilon$ locally uniformly, where u^ε is the unique viscosity solution to (5).

By letting $\varepsilon \to 0$, u^ε converges locally uniformly to u, a viscosity solution to the free transmission problem (2), and the proof is complete. □

6 Numerical experiments

In this section, we present one-dimensional examples to validate the convergence of our numerical methods. We compute approximate solutions to the regularised problems 3 and 5. These in turn converge to the solution of the original problems (1) and (2), respectively, as $\varepsilon \to 0$.

Specifically, we compute an approximate solution u_h^ε to the regularised problems (3) and (5), and compare it with the exact solution u of the respective main problems (1) and (2). For simplicity, as in the previous sections, we omit the superscript ε in the notation and refer to the numerical solutions simply as u_h. The following examples are constructed so that the exact solution is known a priori, allowing for a direct assessment of accuracy.

We discretise the domain using grid points $x_i = x_0 + ih$, for $i = 0, 1, \dots, N-1, N$. Here, x_0 and x_N denote the left and right boundaries, respectively. The finite difference approximation to the solution at each grid point x_i is denoted with $u_{h,i}$.

We recall the definitions of the discrete operators used to approximate first and second-order derivatives. The approximation for $|Du|^2$ is given by

$$|D_h u_{h,i}|^2 = \frac{1}{h^2} \max\left(u_{h,i} - u_{h,i-1}, u_{h,i} - u_{h,i+1}, 0\right)^2,$$

which is accurate of order $O(h^2)$ for C^2-regular function. We also define the approximation of the second-order derivative as

$$D_h^2 u_{h,i} = \frac{1}{h^2}(u_{h,i+1} - 2u_{h,i} + u_{h,i-1}).$$

Notice the latter is accurate to order $O(h^2)$, for C^4-regular functions.

The numerical scheme is

$$G_h(u_{h,i}, x_i) := (\varepsilon u_{h,i} + F_h(D_h^2 u_{h,i})) - \frac{f_i}{(\varepsilon + |D_h u_{h,i}|^2)^{\theta_i/2}},$$

for $i = 1, \dots, N-1$, where

$$\theta_i := \theta(u_{h,i}) \quad \text{and} \quad f_i = f(x_i).$$

The exponent θ_i is constant in the pure equation setting, whereas it depends on $i = 1, \dots, N-1$ in the free transmission context. The operator G_h is also defined on the boundary as

$$G_h(u_{h,i}, x_i) := u_{h,i} - g_i,$$

for $i = 0$ and $i = N$. Here, g_i is defined in the obvious way. This discretisation leads to a nonlinear system of equations that must be solved to obtain the approximate solution over the discrete domain. To address this, we apply the well-known explicit Euler method. It solves iteratively the nonlinear system represented by $G_h(u_{h,i}, x_i)$, for $i = 1, \ldots, N$. See [28]; see also [3].

For $\rho > 0$, define the Euler map $S_\rho(u_h) := (S_\rho(u_{h,1}) \ldots, S_\rho(u_{h,N}))$, where

$$S_\rho(u_{h,i}) := u_{h,i} - \rho G_h(u_{h,i}, x_i),$$

$i = 1, \ldots, N$. The map S_ρ is a contraction in the ℓ_∞-norm provided we choose $0 < \rho \ll 1$ small enough. That is, the CFL condition found previously appears to ensure a contraction; see Remark 1. As a consequence, there exists a unique fixed point u_h^*, satisfying

$$u_h^* = S_\rho(u_h^*).$$

In the next examples, we consider $F(D^2u) = -D^2u$. Starting with an ansatz $u_h^0(x_i)$, for $i = 1, \ldots, N-1$, where $u_h^0(x_0) = u(x_0)$ and $u_h^0(x_N) = u(x_N)$ we seek the discrete solution u_h^n through the iterative process

$$u_h^n = S_\rho u_h^{n-1}.$$

We run the experiments until an (artificial) fixed terminal instant T is reached. This means the number of iterations n is chosen such that $n\rho = T$. As we refine the spatial step h, the number of iterations required increases accordingly.

At this point, we must also discuss the parameter ε. Our goal is to compare the computed results with the solution u of the original problem. To accurately capture the features of the solution, it is necessary that $\varepsilon \geq h$. Therefore, as ε decreases, h must decrease accordingly. In the following experiments, we set $\varepsilon = h$ to ensure this condition is satisfied.

We begin with a test case for the pure problem (1), followed by two test cases for the transmission problem (2).

Example 1 (Pure equation)**.** In the first example, we consider the domain

$\Omega = (-1, 1)$ and the degeneracy rate $\theta = 2$. The source term in problem (1) is chosen so that the exact solution is $u(x) = (1-x)^2(1+x)^2$. The corresponding boundary conditions are $u(-1) = 0$ and $u(1) = 0$. In Figure 1(a) we display the initial approximation, $u^0(x) = (1-x)(x+1)$, used to start the Euler iteration alongside the exact solution $u(x)$. Figure 1(b) illustrates a sequence of Euler iterations performed until the approximation error falls around h.

The Euler iteration mimics a time-dependent process, with the parameter ρ playing a role analogous to a time-step. To satisfy the stability condition, ρ is chosen to depend on the mesh size h, specifically set as $\rho = 0.01h^2$. For this example, we simulate until the (artificial) final time $T = 1$.

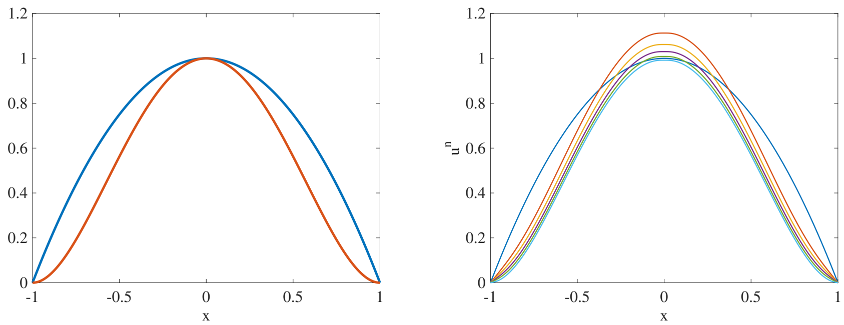

Fig. 1: Example 1 : (a) Exact solution of the pure problem (1) (red line) and the ansatz u_h^0 (blue line); (b) Solutions obtained via Euler iteration with $h = 0.01$ and $n\rho = 1$. The maximum error is $\max_i |u_{h,i} - u_i| = 8.2137 \times 10^{-3}$.

When it comes to the free transmission problem, we present two examples. In the first one, the exact solution has a strictly positive derivative, whereas in the second one, the gradient of the solution vanishes at $x = 0$. We notice our method is capable of bypassing the effects of the gradient-driven degeneracy in both cases.

Example 2 (Transmission problem I – strictly positive gradient)**.** Consider the domain $\Omega = (1/2, 3/2)$ and the degeneracy rates $\theta_1 = 2$ and $\theta_2 = 4$. The source term in problem (2) is chosen so that the exact solution is $u(x) = \log(x)$. The corresponding boundary conditions are $u(1/2) = \log(1/2)$ and $u(3/2) = \log(3/2)$.

The initial estimate for the Euler iteration is

$$u_{h,i}^0(x_i) = x_i, \; i = 1, \dots, N-1.$$

In Figure 2(a), we display the initial approximation, $u^0(x)$, used to start the Euler iteration alongside the exact solution $u(x)$. In Figure 2 (b), we display how Euler's method progresses until the solution approximates the exact solution with an error less than h. To satisfy the stability condition, ρ is chosen to depend on the mesh size h, specifically set as $\rho = 0.01h^2$. For this example, we simulate until the final time $T = 0.3$. We need fewer iterations than in the previous example, due to the geometry of the exact solution.

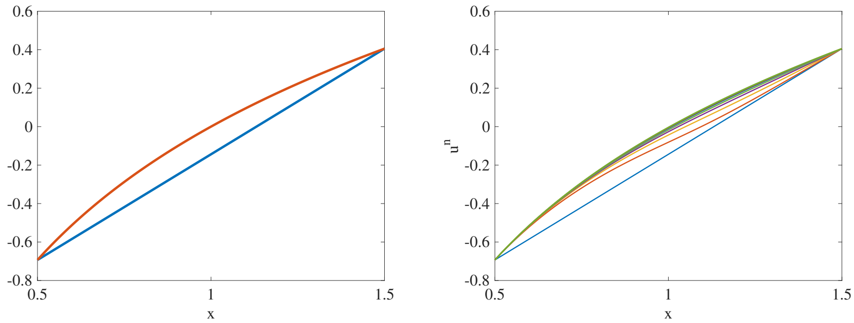

Fig. 2: Example 2 : (a) Exact solution of problem (2) (red line) and the ansatz u_h^0 (blue line); (b) Solution obtained via Euler iteration with $h = 0.01$ and $n\rho = 0.3$. The maximum error is $\max_i |u_{h,i} - u_i| = 4.3047 \times 10^{-3}$.

Example 3 (Transmission problem II – vanishing gradient)**.** In the third example, $\Omega = (-1, 1)$, $\theta_1 = 2$, and $\theta_2 = 4$. The source term in problem (2) is chosen so that the exact solution is $u(x) = x^2$ for $x > 0$ and $u(x) = -x^4$ for $x < 0$. The corresponding boundary conditions are $u(-1) = -1/2$ and $u(1) = 1/2$.

The initial estimate for the Euler iteration is the same as in Example 2; namely,

$$u_{h,i}^0(x_i) = x_i, \; i = 1, \dots, N-1.$$

We chose ρ as in the previous examples, $\rho = 0.01h^2$. In Figure 3(a), we display the initial approximation, $u^0(x)$, used to start the Euler iteration alongside the exact solution $u(x)$. Figure 3(b) shows the Euler method iterations until the solution approximates the exact solution with an error around h.

We kept the final value $n\rho = 0.3$ the same as in the previous example for comparison purposes. This third example requires more iterations to reach a

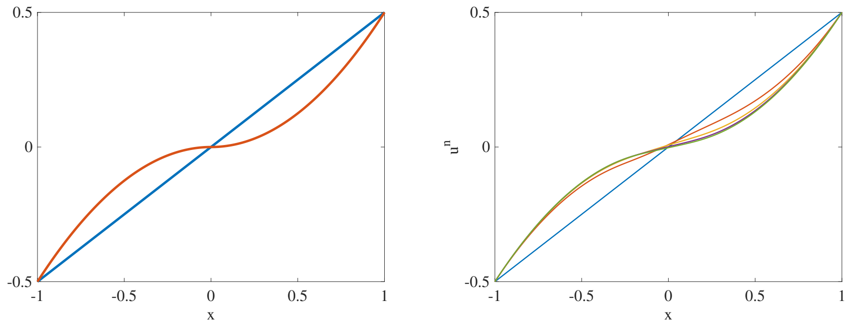

Fig. 3: Example 3: (a) Exact solution of problem (2) (red line) and the ansatz u_h^0 (blue line); (b) Solution obtained via Euler iteration with $h = 0.01$ and $n\rho = 0.3$. The maximum error is $\max_i |u_{h,i} - u_i| = 1.3222 \times 10^{-2}$.

satisfactory approximation of the solution compared to the second example. The most challenging region to approximate appears to be around $x = 0$. The maximum error is slightly larger than that of the second example.

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Declarations

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References

[1] Damião J. Araújo, Gleydson Ricarte, and Eduardo V. Teixeira. Geometric gradient estimates for solutions to degenerate elliptic equations. *Calc. Var. Partial Differential Equations*, 53(3-4):605–625, 2015.

[2] Martino Bardi and Maurizio Falcone. An approximation scheme for the minimum time function. *SIAM J. Control Optim.*, 28(4):950–965, 1990.

[3] Guy Barles and Panagiotis E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, 4(3):271–283, 1991.

[4] Vincenzo Bianca, Edgard A. Pimentel, and José Miguel Urbano. Transmission problems: regularity theory, interfaces and beyond. Preprint, arXiv:2306.15570 [math.AP] (2023), 2023.

[5] Isabeau Birindelli and Françoise Demengel. Comparison principle and Liouville type results for singular fully nonlinear operators. *Ann. Fac. Sci. Toulouse Math. (6)*, 13(2):261–287, 2004.

[6] Isabeau Birindelli and Françoise Demengel. First eigenvalue and maximum principle for fully nonlinear singular operators. *Adv. Differential Equations*, 11(1):91–119, 2006.

[7] Isabeau Birindelli and Françoise Demengel. The Dirichlet problem for singular fully nonlinear operators. *Discrete Contin. Dyn. Syst.*, pages 110–121, 2007.

[8] Isabeau Birindelli and Françoise Demengel. Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators. *Commun. Pure Appl. Anal.*, 6(2):335–366, 2007.

[9] Isabeau Birindelli and Françoise Demengel. Eigenvalue and Dirichlet problem for fully-nonlinear operators in non-smooth domains. *J. Math. Anal. Appl.*, 352(2):822–835, 2009.

[10] Isabeau Birindelli and Françoise Demengel. $C^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations. *ESAIM Control Optim. Calc. Var.*, 20(4):1009–1024, 2014.

[11] Anne C. Bronzi, Edgard A. Pimentel, Giane C. Rampasso, and Eduardo V. Teixeira. Regularity of solutions to a class of variable-exponent fully nonlinear elliptic equations. *J. Funct. Anal.*, 279(12):108781, 31, 2020.

[12] Xavier Cabré and Luis A. Caffarelli. Interior $C^{2,\alpha}$ regularity theory for a class of nonconvex fully nonlinear elliptic equations. *J. Math. Pures Appl. (9)*, 82(5):573–612, 2003.

[13] Luis A. Caffarelli and Xavier Cabré. *Fully nonlinear elliptic equations*, volume 43 of *Colloq. Publ., Am. Math. Soc.* Providence, RI: AMS, 1995.

[14] Filippo Cagnetti, Diogo Gomes, and Hung V. Tran. Convergence of a semi-discretization scheme for the Hamilton-Jacobi equation: a new approach with the adjoint method. *Appl. Numer. Math.*, 73:2–15, 2013.

[15] Italo Capuzzo Dolcetta and Maurzio Falcone. Discrete dynamic programming and viscosity solutions of the Bellman equation. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 6:161–183, 1989.

[16] Italo Capuzzo Dolcetta and Hitoshi Ishii. Approximate solutions of the Bellman equation of deterministic control theory. *Appl. Math. Optim.*, 11:161–181, 1984.

[17] Michael G. Crandall, Lawrence C. Evans, and Pierre-Louis Lions. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Trans. Am. Math. Soc.*, 282:487–502, 1984.

[18] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Am. Math. Soc., New Ser.*, 27(1):1–67, 1992.

[19] Michael G. Crandall and Pierre-Louis Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Am. Math. Soc.*, 277:1–42, 1983.

[20] Gonzalo Dávila, Patricio Felmer, and Alexander Quaas. Alexandroff-Bakelman-Pucci estimate for singular or degenerate fully nonlinear elliptic equations. *C. R. Math. Acad. Sci. Paris*, 347(19-20):1165–1168, 2009.

[21] Maurizio Falcone. A numerical approach to the infinite horizon problem of deterministic control theory. *Appl. Math. Optim.*, 15(1):1–13, 1987.

[22] Xiaobing Feng and Max Jensen. Convergent semi-Lagrangian methods for the Monge-Ampère equation on unstructured grids. *SIAM J. Numer. Anal.*, 55(2):691–712, 2017.

[23] Brittany D. Froese, Adam M. Oberman, and Tiago Salvador. Numerical methods for the 2-Hessian elliptic partial differential equation. *IMA J. Numer. Anal.*, 37(1):209–236, 2017.

[24] Gerardo Huaroto, Edgard A. Pimentel, Giane C. Rampasso, and Andrzej Święch. A fully nonlinear degenerate free transmission problem. *Ann. PDE*, 10(1):30, 2024. Id/No 5.

[25] Cyril Imbert and Luis Silvestre. $C^{1,\alpha}$ regularity of solutions of some degenerate fully non-linear elliptic equations. *Adv. Math.*, 233:196–206, 2013.

[26] Cyril Imbert and Luis Silvestre. Estimates on elliptic equations that hold only where the gradient is large. *J. Eur. Math. Soc. (JEMS)*, 18(6):1321–1338, 2016.

[27] Connor Mooney. Harnack inequality for degenerate and singular elliptic equations with unbounded drift. *J. Differential Equations*, 258(5):1577–1591, 2015.

[28] Adam M. Oberman. Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problems. *SIAM J. Numer. Anal.*, 44(2):879–895, 2006.

[29] Adam M. Oberman and Tiago Salvador. Filtered schemes for Hamilton-Jacobi equations: a simple construction of convergent accurate difference schemes. *J. Comput. Phys.*, 284:367–388, 2015.

[30] Edgard A. Pimentel and Andrzej Święch. Existence of solutions to a fully nonlinear free transmission problem. *J. Differ. Equations*, 320:49–63, 2022.

[31] Panagiotis E. Souganidis. Approximation schemes for viscosity solutions of Hamilton-Jacobi equations. *J. Differ. Equations*, 59:1–43, 1985.

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