Uniformity tests for circular data based on a Parzen–Rosenblatt type estimator

Carlos Tenreiro*

CMUC, Department of Mathematics, University of Coimbra, Coimbra, Portugal

2 June 2025

Abstract

Given an independent and identically distributed sample of angles from some absolutely continuous circular random variable with unknown probability density function f, in this work we study the problem of testing the hypothesis on whether f is the uniform distribution on the circle. For this purpose we consider a Bickel–Rosenblatt type test statistic (L^2 distance) based on the Parzen–Rosenblatt type estimator for circular data. The asymptotic behaviour of the proposed test procedure for fixed and non-fixed bandwidths is studied. From a finite sample point of view the power performance of the tests associated with different bandwidths depends on the considered bandwidth which acts as a tuning parameter. The automatic selection of this tuning parameter, the choice of which is crucial to obtaining a performing test procedure, is also addressed in this work, and comparisons are made with other existing uniformity tests through a simulation study.

KEYWORDS: Circular data; Parzen–Rosenblatt type density estimator; Uniformity tests.

AMS 2020 SUBJECT CLASSIFICATIONS: 62G10, 62H11

*ORCID iD: 0000-0002-5495-6644; E-mail: tenreiro@mat.uc.pt; URL: http://www.mat.uc.pt/∼tenreiro/ Postal address: Department of Mathematics, University of Coimbra, Largo D. Dinis, 3000-143 Coimbra, Portugal.

1 Introduction

Given an independent and identically distributed sample $X_1, \dots, X_n \in [0, 2\pi[$ from some absolutely continuous circular random variable X with unknown density f, the standard kernel estimator of f is defined, for $\theta \in [0, 2\pi[$, by

$$\tilde{f}_n(\theta; g) = \frac{c_g(L)}{n} \sum_{i=1}^{n} L\left(\frac{1 - \cos(\theta - X_i)}{g^2}\right), \tag{1}$$

where $L : [0, \infty[\to \mathbb{R}$ is a bounded function satisfying some additional conditions, $g = g_n$ is a sequence of strictly positive numbers converging to zero as n tends to infinity, and $c_g(L)$ is chosen so that $\tilde{f}_n(\cdot; g)$ integrates to unity (see Beran, 1979, Hall et al., 1987, Bai et al., 1988, Klemelä, 2000, García-Portugués, 2013, García-Portugués et al., 2013). An alternative kernel estimator of the density f, which is close in spirit to the Parzen–Rosenblatt (PR) estimator for data on the real line (see Rosenblatt, 1956, and Parzen, 1962), is defined, for $\theta \in [0, 2\pi[$, by

$$\hat{f}_n(\theta; h) = \frac{d_h(K)}{n} \sum_{i=1}^{n} K_h(\theta - X_i), \tag{2}$$

where $h = h_n$ is a sequence of strictly positive real numbers converging to zero as n tends to infinity, K_h is a real-valued periodic function on $\mathbb{R}$, with period 2π, such that

$$K_h(\theta) = K(\theta/h)/h, \text{ for } \theta \in [-\pi, \pi[,$$

with $h > 0$, K a kernel on $\mathbb{R}$, that is, a bounded and integrable real-valued function on $\mathbb{R}$ with $\int_{\mathbb{R}} K(u)du > 0$, and $d_h(K)$ a normalizing constant which is chosen so that $\hat{f}_n(\cdot; h)$ integrates to unity (see Tenreiro, 2022, 2024, 2025). Other than the estimation of the underlying probability density function, the kernel estimator (1) is used by Boente et al. (2014) and García-Portugués et al. (2018) to test the hypothesis that f belongs to a given parametric class of densities. With the same goal in mind, but confining our study to the simple null hypothesis case, in this paper we are interested in using the PR-type estimator (2) to test the hypothesis $H_0 : f = f_0$, where f_0 is a fixed probability density function on the circle. Special attention will be paid to the important case of testing the uniformity of circular distributions, which is the main goal of this work. In the literature several tests for circular uniformity have been proposed. For a review of the available tests, see Mardia and Jupp (2000, Chap. 6); see also Bogdan et al. (2002), Pycke (2010), García-Portugués et al. (2018, 2021, 2023), Jammalamadaka et al. (2020), and Fernández-de-Marcos and García-Portugués (2023), and the references therein.

As in the pioneering work by Bickel and Rosenblatt (1973), where the problem of goodness of fit was addressed for the first time using kernel density estimators on the real line (goodness-of-fit tests based on the histogram estimator were also suggested by Révész, 1971, and Tusnády, 1973), in this paper we suggest a new class of goodness-of-fit tests based on the $L^2([0, 2\pi[)$ distance between the PR-type estimator (2) and its expectation under the null hypothesis, leading to the rejection of the hypothesis H_0 for large values of the statistic

$$I_n(h) = n \int_0^{2\pi} \big\{\hat{f}_n(\theta;h) - E_0\hat{f}_n(\theta;h)\big\}^2 d\theta, \tag{3}$$

with

$$E_0\hat{f}_n(\theta;h) = d_h(K)K_h * f_0(\theta) = d_h(K)\int_0^{2\pi} K_h(\theta-u)f_0(u)du,$$

for $\theta \in [0,2\pi[$, where f_0 is square integrable on $[0,2\pi[$, and $*$ denotes the convolution product. For the sake of simplicity we also denote by f_0 the periodic extension of f_0 to $\mathbb{R}$ given by $f_0(\theta) = f_0(\theta - 2k\pi)$, whenever $\theta \in [2k\pi, 2(k+1)\pi[$, for some $k \in \mathbb{Z}$. Recall that if α and β are real-valued functions with period 2π defined on $\mathbb{R}$, the convolution of α and β is defined, for $x \in \mathbb{R}$, by

$$(\alpha * \beta)(x) = \int_0^{2\pi} \alpha(x-y)\beta(y)dy,$$

whenever this integral exists. As the integrand is periodic with period 2π, the previous definition does not depend on the considered interval of integration with length 2π. The convolution $(\alpha * \beta)(x)$ exists for almost every $x \in \mathbb{R}$ whenever α and β are integrable on $[0,2\pi[$, and it exists for every $x \in \mathbb{R}$ if in addition one of the functions α or β is bounded. Moreover, it exists and is continuous for every $x \in \mathbb{R}$, whenever α and β are square integrable on $[0,2\pi[$. Obviously, the convolution is a periodic function if it exists (see Butzer and Nessel, 1971, §0.4).

As it is based on the PR-type estimator of f, the statistic $I_n(h) = I_n(h_n)$ given by (3) can be seen as a version of the Bickel–Rosenblatt (BR) statistic for circular data. Just like the BR statistic on the real line (see Bickel and Rosenblatt, 1973, Hall, 1984, and Fan, 1994, 1998), under the null hypothesis $I_n(h)$ has a limit normal distribution when h_n converges to zero as n tends to infinity, and an infinite weighted sum of independent χ_1^2 random variables when the bandwidth is fixed, that is, $h_n = h > 0$, for all $n \in \mathbb{N}$. Similarly to what was pointed out by Fan (1998) for the BR statistic on the real line, under some additional conditions on the kernel K and the bandwidth (see Section 6), the statistic $I_n(h)$ can be written as weighted $L^2([0,2\pi[)$ distance between the empirical characteristic function φ_n and the characteristic function of f_0, that is,

$$n^{-1}I_n(h) = \bigg(\int_{\mathbb{R}} K(u)du\bigg)^{-2} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} |\varphi_n(k) - \varphi_{f_0}(k)|^2 |\varphi_K(kh)|^2, \tag{4}$$

where $\varphi_n(k) = n^{-1}\sum_{j=1}^n \exp(\mathrm{i}kX_j)$ and $\varphi_{f_0}(k) = \int_0^{2\pi}\exp(\mathrm{i}k\theta)f_0(\theta)d\theta$, for $k \in \mathbb{Z}$, and φ_K is the Fourier transform of K defined by $\varphi_K(t) = \int_{\mathbb{R}} e^{\mathrm{i}tu}K(u)du$, for $t \in \mathbb{R}$. The previous alternative representation for $I_n(h)$ enables us to conclude that, as with the BR statistic on the real line (see Fan, 1998, Tenreiro, 2007a), the test based on $I_n(h)$ may be asymptotically consistent even when the bandwidth is fixed. Moreover, from it we also conclude that the class of tests based on $I_n(h)$ considered in this work is strongly connected with the class of goodness-of-fit tests for circular distributions based on trigonometric moments recently introduced in Jammalamadaka et al. (2019).

The rest of this paper is organised as follows. In Sections 2 and 3 the asymptotic behaviour of $I_n(h)$ for fixed and non-fixed bandwidths is studied. More precisely, the limiting null distribution of $I_n(h)$, the consistency of the associated test procedure, and its asymptotic power against sequences of local alternatives are derived. Based on these results, the asymptotic superiority of the tests based on $I_n(h)$ with a fixed bandwidth over those with a non-fixed bandwidth for fixed alternatives is established. Confining ourselves to the tests based on $I_n(h)$ with fixed bandwidth, in Section 4 we address the problem of testing a uniformity hypothesis, this being the main goal of this work. From a practical point of view, it is natural to expect that the finite sample power performance of the test based on $I_n(h)$ may be sensitive to the choice of h which acts as a tuning parameter. The usual strategy for selecting this tuning parameter is to evaluate the test power performance for h varying in a finite set H (say), and then suggesting a selection of h that produces a test with a reasonable power against a wide range of alternative distributions. As this strategy of taking a fixed tuning parameter does not prevent the user from obtaining a test that achieves very low power against some of the considered alternative distributions, we implement here a test methodology studied in Tenreiro (2019) that combines tests associated to different values of the tuning parameter into a single multiple test procedure that could show a good power performance against a wide range of alternative distributions. As a result of a simulation study where the empirical power of the new test is compared with that of the uniformity tests of Kuiper (1960) and Watson (1961), the data driven smooth test of Bogdan et al. (2002), one of the tests suggested by Pycke (2010), the projected Anderson-Darling test of García-Portugués et al. (2023), and the 10-fold smooth maximum test of Fernández-de-Marcos and García-Portugués (2023), we conclude that the proposed test procedure is a serious competitor against all of them. Section 5 includes a brief summary and some conclusions. For convenience of exposition the proofs are deferred to Section 6. The simulation results and plots shown in this article were carried out using the R software (R Development Core Team, 2021).

2 Asymptotic null distribution and consistency

Given a probability density function on the circle f_0, we establish in this section the asymptotic null distribution and the consistency of the test based on the statistic $I_n(h_n)$ given by (3) to test

$$H_0 : f = f_0 \quad \text{against} \quad H_a : f \neq f_0, \tag{5}$$

for a sequence (h_n) of strictly positive real numbers satisfying one of the following conditions (B_h) for some $h \geq 0$:

Assumptions on the bandwidth (h_n)

(B_0) $h_n \to 0$ and $nh_n \to +\infty$, as $n \to +\infty$;

(B_h) $h_n = h > 0$, for all $n \in \mathbb{N}$.

Condition (B_0) is usual in kernel density theory. Under (B_0) the PR-type estimator $\hat{f}_n$ given by (2) is a consistent estimator of the common density f of the observations (Tenreiro, 2022). If the bandwidth sequence (h_n) satisfies (B_h) for some $h > 0$, $\hat{f}_n$ is no more an asymptotically unbiased estimator of f. Regarding the kernel K, which we always assume to be a bounded and integrable real-valued function on $\mathbb{R}$ with $\int_{\mathbb{R}} K(u)du > 0$, some additional assumptions are needed to establish the asymptotic null distribution of $I_n(h_n)$ when the assumption (B_0) is fulfilled, and the consistency of the test procedure based on $I_n(h_n)$ when the assumption (B_h) is fulfilled for some $h > 0$.

Assumptions on the kernel K

(K.1) $K(u) = 0$, for $u \notin [-M, M]$, for some $M > 0$;

(K.2) $\varphi_K(t) \neq 0$, for all $t \in \mathbb{R}$.

A simple example of a family of nonnegative kernels satisfying the previous assumptions is given by the symmetric kernels of the form

$$K_p(u) = (1 - |u|)^p \mathbb{I}_{[-1,1]}(u), \tag{6}$$

with $p \geq 2$. In fact, for $t \neq 0$, we have $\varphi_{K_p}(t) < \varphi_{K_p}(0)$, for all $p \geq 0$, and $\varphi_{K_p}(t) = 2pt^{-2}\big(1 - \varphi_{K_{p-2}}(0)^{-1}\varphi_{K_{p-2}}(t)\big)$, for all $p \geq 2$.

In the proof of the following result, given in Section 6, we show that the null asymptotic behaviour of $I_n(h_n)$ depends on the limit distribution of the degenerated U-statistic with kernel $H(\cdot, \cdot; f_0, h)$ defined by (9) which may a priori be a weighted sum of independent χ_1^2 random variables or a normal distribution (see Hoeffding, 1948, Gregory, 1977, Hall, 1984, and Lee, 1990). A normal limit distribution occurs if (B_0) is satisfied, whereas a weighted sum of independent chi-squares limit distribution arises if (B_h) is satisfied for some $h > 0$. These conclusions agree with previous related results for data on the real line obtained by Bickel and Rosenblatt (1973), Hall (1984), Fan (1994, 1998), and Tenreiro (2007a), among others. We denote by $\stackrel{d}{\longrightarrow}$ the convergence in distribution.

Theorem 1. *Suppose that $f = f_0 \in L^\infty([0, 2\pi[)$.*
(a) Under assumption (B_0), if K satisfies (K.1) then

$$h_n^{1/2}\Big\{ I_n(h_n) - h_n^{-1}\int_{\mathbb{R}} K(u)^2 du \Big(\int_{\mathbb{R}} K(u)du\Big)^{-2}\Big\} \stackrel{d}{\longrightarrow} N(0, \nu^2),$$

where

$$\nu^2 = 2\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} K(v+u)K(u)du\Big)^2 dv \Big(\int_{\mathbb{R}} K(u)du\Big)^{-4}\int_0^{2\pi} f_0(\theta)^2 d\theta. \tag{7}$$

(b) If (B_h) is satisfied for some $h > 0$, then

$$I_n(h) \stackrel{d}{\longrightarrow} \sum_{k=1}^{\infty} \lambda_{k,h} Z_k^2,$$

where $\{Z_k, k \geq 1\}$, are independent and identically distributed standard normal variables and $\{\lambda_{h,k} > 0, k \geq 1\}$, with $\sum_{k=1}^{\infty} \lambda_{h,k} < \infty$, are the strictly positive eigenvalues of the symmetric positive semidefinite Hilbert-Schmidt operator $\mathscr{H}_h$ defined, for $g \in L^2([0, 2\pi[, f_0)$, by

$$\mathscr{H}_h g(u) = \int_0^{2\pi} H(u, v; f_0, h) g(v) f_0(v) dv, \tag{8}$$

where

$$H(u, v; f_0, h) = d_h(K)^2 \int_0^{2\pi} \{K_h(\theta - u) - K_h * f_0(\theta)\}\{K_h(\theta - v) - K_h * f_0(\theta)\} d\theta, \tag{9}$$

for $u, v \in [0, 2\pi[$ and $h > 0$, and $L^2([0, 2\pi[, f_0)$ is the space of real-valued functions g such that $\int_0^{2\pi} g(y)^2 f_0(y) dy < \infty$.

From the previous result we know that the statistic defined by

$$T_n(h_n) = h_n^{1/2} \Big\{ I_n(h_n) - h_n^{-1} \int_{\mathbb{R}} K(u)^2 du \Big(\int_{\mathbb{R}} K(u) du \Big)^{-2} \Big\},$$

if (B_0) is satisfied, and by

$$T_n(h_n) = I_n(h),$$

if (B_h) is satisfied for some $h > 0$, has a non-degenerated limit distribution under H_0. In the following result we establish the asymptotic behaviour of $I_n(h_n)$ under H_a. We denote by $\xrightarrow{p}$ the convergence in probability.

Theorem 2. *Let us assume that $f, f_0 \in L^\infty([0, 2\pi[)$ and that K satisfies (K.1).*

(a) If (B_0) is satisfied, then

$$n^{-1} h_n^{-1/2} T_n(h_n) \xrightarrow{p} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} |\varphi_f(k) - \varphi_{f_0}(k)|^2 = \int_0^{2\pi} (f(\theta) - f_0(\theta))^2 d\theta.$$

(b) If (B_h) is satisfied for some $0 < h \leq \pi/M$, then

$$n^{-1} T_n(h_n) \xrightarrow{p} \Big(\int_{\mathbb{R}} K(u) du \Big)^{-2} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} |\varphi_f(k) - \varphi_{f_0}(k)|^2 |\varphi_K(kh)|^2.$$

For $\alpha \in]0, 1[$, let us consider the test defined by the critical region

$$\mathscr{C}(T_n(h_n), \alpha) = \big\{ T_n(h_n) > q(T_n(h_n), \alpha) \big\}, \tag{10}$$

where $q(T_n(h_n), \alpha)$ denotes the quantile of order $1 - \alpha$ of the distribution of $T_n(h_n)$ under H_0. This quantile is assumed to be a known quantity as it can be well approximated by simulating under the null hypothesis. As stated in the next result, the test based on the critical region $\mathscr{C}(T_n(h_n), \alpha)$ has a level of significance at most equal to α for each sample size n, is asymptotically of level α, and is consistent to test H_0 against H_a whenever the kernel K satisfies the additional assumption (K.2) when (B_h) is satisfied for some $h > 0$.

Theorem 3. *Let us assume that* $f_0 \in L^\infty([0, 2\pi[)$ *and that* K *satisfies (K.1). If* (B_0) *is satisfied or if* (B_h) *is satisfied for some* $0 < h \leq \pi/M$ *and* K *satisfies (K.2), then the test defined by the critical region* $\mathscr{C}(T_n(h_n), \alpha)$*, where* $\alpha \in]0, 1[$*, is such that*

$$P_{f_0}\big(\mathscr{C}(T_n(h_n), \alpha)\big) \leq \alpha, \text{ for all } n \in \mathbb{N},$$

$$\lim_{n\to+\infty} P_{f_0}\big(\mathscr{C}(T_n(h_n), \alpha)\big) = \alpha$$

and

$$\lim_{n\to+\infty} P_f\big(\mathscr{C}(T_n(h_n), \alpha)\big) = 1, \text{ for all } f \in L^\infty([0, 2\pi[)\backslash\{f_0\}.$$

If the kernel K satisfies (K.1)-(K.2), from Theorem 2 we know that the probability order of convergence of the test statistic $T_n(h)$ to $+\infty$ depends on h. More precisely, for all sequences (h_n) satisfying (B_0) and for all $0 < h \leq \pi/M$, we have

$$\frac{T_n(h_n)}{T_n(h)} \xrightarrow{p} 0,$$

for all $f \in L^\infty([0, 2\pi[)\backslash\{f_0\}$. This establishes the asymptotic superiority of the tests based on $I_n(h_n)$ with a fixed bandwidth over those with a non-fixed bandwidth for fixed alternatives.

3 Local power analysis

Local power analysis of testing procedures is based on the research for local alternatives providing a non-degenerate limiting power. To define local alternatives we consider $X_{n1}, X_{n2}, \ldots, X_{nn}, \ldots$ a sequence of independent and identically distributed absolutely continuous circular random variables whose probability density function f_n is such that

$$f_n(\theta) = f_0(\theta)\big(1 + \gamma_n \eta(\theta) + o(\gamma_n)\eta_n(\theta)\big), \tag{11}$$

for $\theta \in [0, 2\pi[$, with η an a.e. (f_0) non-identically null function, (γ_n) a sequence of positive real numbers tending to zero as n tends to infinity, and the functions η and (η_n) are such that

$$\sup_{\theta\in[0,2\pi[} |\eta(\theta)| < +\infty, \quad \sup_{n\in\mathbb{N}} \sup_{\theta\in[0,2\pi[} |\eta_n(\theta)| < +\infty.$$

The goal of this section is to obtain the limiting local power function of the test associated with the critical region given by (10) for the previous sequence of local alternatives. Assuming that condition (B_h) is fulfilled for some $h \geq 0$, it is defined by

$$p_h(\eta, \alpha) = \lim_{n\to+\infty} P_{f_n}\big(\mathscr{C}_n(T_n(h_n), \alpha)\big) = \lim_{n\to+\infty} P_{f_n}\big(T_n(h_n) > q(T_n(h_n), \alpha)\big).$$

Theorem 4. *Let us assume that* $f_0 \in L^\infty([0, 2\pi[)$ *and that* K *satisfies (K.1)-(K.2).*

(a) Under assumption (B_0)*, we have*

$$p_0(\eta,\alpha) = \begin{cases} \alpha, & \gamma_n = o(n^{-1/2}h_n^{-1/4}), \\ 1-\Phi\big(\Phi^{-1}(1-\alpha)-\mu\nu^{-1}\big) > \alpha, & \gamma_n = n^{-1/2}h_n^{-1/4}, \\ 1, & n^{-1/2}h_n^{-1/4} = o(\gamma_n), \end{cases}$$

where

$$\mu = (2\pi)^{-1}\sum_{k=-\infty}^{\infty}|\varphi_{f_0\eta}(k)|^2 > 0,$$

ν^2 *is given by (7) and* Φ *is the cumulative distribution function of the standard normal distribution.*

(b) Under assumption (B_h) *for some* $0 < h \leq \pi/M$*, we have*

$$p_h(\eta,\alpha) = \begin{cases} \alpha, & \gamma_n = o(n^{-1/2}), \\ 1-F_{h,\eta}\big(F_{h,0}^{-1}(1-\alpha)\big) \geq \alpha, & \gamma_n = n^{-1/2}, \\ 1, & n^{-1/2} = o(\gamma_n), \end{cases}$$

where $F_{h,\eta}$ *is the cumulative distribution function of the random variable*

$$\sum_{k=1}^{\infty}\lambda_{h,k}(Z_k + a_{h,k})^2,$$

with

$$a_{h,k} = \int_0^{2\pi} q_{h,k}(\theta)\eta(\theta)f_0(\theta)d\theta,$$

for $k \geq 1$*, where* $\{\lambda_{h,k}, k \geq 1\}$ *and* $\{Z_k, k \geq 1\}$ *are defined in Theorem 1, and* $\{q_{h,k}, k \geq 1\}$ *denotes an orthonormal basis for the orthogonal complement of the eigenspace associated to* $\lambda_{h,0} = 0$*, consisting of eigenfunctions of the operator* $\mathscr{H}_h$ *defined by (8) corresponding to the collection of its strictly positive eigenvalues, that is,* $\int_0^{2\pi} H(\cdot, v; f_0, h)q_{h,k}(v)f_0(v) = \lambda_{h,k}q_{h,k}(\cdot)$*, a.e.* (f_0)*, for all* $k \geq 1$*, where* $H(\cdot,\cdot; f_0, h)$ *is given by (9).*

In view of the degeneracy property of $H(\cdot,\cdot; f_0, h)$, $q_{h,0} = 1$ is an eigenfunction of $\mathscr{H}_h$ corresponding to the eigenvalue $\lambda_{h,0} = 0$. In addition, under assumptions (K_1)-(K_2) it can be proved that the eigenspace associated to $\lambda_{h,0} = 0$ has dimension one, which enables us to conclude that $\mathscr{H}_h$ has a countable infinite number of strictly positive eigenvalues (see Dunford and Schwartz, 1963, Corollary X.3.5, p. 905) and then $\lambda_{h,k} \to 0$, as $k \to \infty$, since $\sum_{k=1}^{\infty}\lambda_{h,k} < +\infty$.

From the previous theorem, we conclude that the tests based on $I_n(h_n)$ with a fixed bandwidth are able to detect all the alternatives of the form (11) that converge to the null hypothesis density function at the rate $\gamma_n = o(n^{-1/2})$, whereas the tests based on $I_n(h_n)$ with a non-fixed bandwidth only detect local alternatives that converge to the null hypothesis density function at the slower rate $\gamma_n = n^{-1/2}h_n^{-1/4}$. This establishes the asymptotic superiority of the tests based on $I_n(h_n)$ with a fixed bandwidth over those with a non-fixed bandwidth for local alternatives.

4 A uniformity goodness-of-fit test

In this section we consider the test of a uniformity hypothesis, that is, we are interested in testing the hypotheses (5) where $f_0(\theta) = (2\pi)^{-1}$, for $\theta \in [0, 2\pi[$. In this case the BR-type statistic $I_n(h)$ defined by (3) takes the form

$$I_n(h) = n \int_0^{2\pi} \big\{\hat{f}_n(\theta; h) - (2\pi)^{-1}\big\}^2 d\theta.$$

Taking into account the asymptotic superiority of the tests based on the BR-type statistic with a fixed bandwidth over those with a non-fixed bandwidth for fixed and local alternatives, but also the fact that no relevant information about the selection of the bandwidth is provided by the asymptotic kernel density estimation theory when the true probability density function is uniform (see Tenreiro, 2022), we will focus our attention to the case where the bandwidth h is fixed. It is interesting to note that in this case the test based on $I_n(h)$ belongs to the class of tests introduced in Beran (1969) as

$$I_n(h) = n^{-1} \sum_{i,j=1}^{n} k(X_i - X_j; h),$$

where

$$\kappa(u; h) = d_h(K)^2 \int_0^{2\pi} K_h(u + v) K_h(v) dv - (2\pi)^{-1}, \; u \in \mathbb{R},$$

is such that $\int_0^{2\pi} k(u; h)^2 du < \infty$ and $\int_0^{2\pi} k(u; h) du = 0$ (on these tests, also called *Sobolev tests*, see Mardia and Jupp, 2000, §6.3.7, p. 110, Pycke, 2010, pp. 81–84, Jammalamadaka et al., 2020, pp. 2228–2229, and García-Portugués et al., 2023, pp. 190–192). The set of orthonormal eigenfunctions of the operator $\mathscr{H}_h$ defined by (8) is the Fourier orthonormal basis of $L^2([0, 2\pi[, f_0)$ given by $\big\{1, \sqrt{2}\cos(k\,\cdot), \sqrt{2}\sin(k\,\cdot), k \in \mathbb{N}\big\}$, and $\lambda_{h,k} = \frac{1}{2\pi}\int_0^{2\pi} \kappa(u; h)\cos(ku)du$, is the eigenvalue associated to both eigenfunctions $\sqrt{2}\cos(k\,\cdot)$ and $\sqrt{2}\sin(k\,\cdot)$, for $k \in \mathbb{N}$. Note that, in this specific case of testing a uniformity hypothesis, the asymptotic behaviour of $I_n(h)$ under the null hypothesis and under fixed alternatives could also have been derived from Proposition 1 in Pycke (2010, p. 83).

From a practical point of view, it is natural to expect that the finite sample power performance of the test based on $I_n(h)$ with critical region

$$\mathscr{C}(I_n(h), \alpha) = \big\{I_n(h) > q(I_n(h), \alpha)\big\}, \tag{12}$$

where $q(I_n(h), \alpha)$ denotes the quantile of order $1-\alpha$ of the null distribution of $I_n(h)$, may be very sensitive to the choice of h, which was confirmed through some preliminar simulation experiments. As with the BR-test with fixed bandwidth for linear data (see Tenreiro, 2007b), the bandwidth h acts as a tuning parameter through which the user can increase the power of the test toward some particular direction along the alternative distributions set. However, as the formulation of a specified alternative hypothesis is not possible in general, the usual practice is to evaluate the test power performance for h varying in a finite set H, and then suggesting a selection of h that produces a test with a reasonable power against a wide range of alternative distributions. As this strategy of taking a single value of the tuning parameter does not prevent the user from obtaining a test that achieves very low power against some of the considered alternative distributions, we will consider a test procedure considered in Klar (2001) and Fromont and Laurent (2006) that combines tests associated to different values of h into a single test procedure that could show a good power performance against a wide range of alternative distributions. The proposed test, studied in Tenreiro (2019), which can be viewed as an improvement of the classical Bonferroni multiple test procedure, leads to the rejection of the null hypothesis if one of the statistics $I_n(h)$, for $h \in H$, is larger than its $(1-u)$ quantile under the null hypothesis, the level u being calibrated so that the resulting multiple test has a level of significance at most equal to α. Thus, the associated critical region is given by

$$\mathscr{C}_n(H,u) = \Big\{ \max_{h \in H} \big(I_n(h) - q(I_n(h),u) \big) > 0 \Big\}, \tag{13}$$

for some $u \in]0,1[$. Unlike the classic Bonferroni multiple testing procedure, that can be obtained by taking $u = \alpha/|H|$, where $|H|$ denotes the cardinality of H, the previous rejection region takes in consideration the dependence structure among the test statistics, $I_n(h)$ for $h \in H$. Taking into account that the previous critical region can be written as

$$\mathscr{C}_n(H,u) = \big\{ I_n(\bar{h}_u) > q(I_n(\bar{h}_u),u) \big\},$$

where

$$\bar{h}_u = \bar{h}_u(X_1,\ldots,X_n) = \operatorname*{argmax}_{h \in H} \big(I_n(h) - q(I_n(h),u) \big),$$

the previous multiple test procedure can be seen as a test based on a data-dependent procedure for selecting the tuning parameter h: for a given sample of size n, one selects the value $h \in H$ for which the test statistic $I_n(h)$ shows strong evidence, at level u, against the null hypothesis.

4.1 The calibration procedure

Given a finite set H of tuning parameters, we know that by selecting the value $u \in]0,1[$ such that $0 < u \le \alpha/|H|$, the Type I error of the test with critical region $\mathscr{C}_n(H,u)$ given by (13) may be put under a preassigned level of significance α (see Tenreiro, 2019, Theorem 2.1). Taking into account that the test should have a level of significance not only less than or equal to α but also as close to α as possible, the practical selection of the level u at which each one of the tests based on $I(h)$, $h \in H$, is performed, will be made by considering a regular grid $G_p = \{u_k, k \in I_p\}$ on the interval $]0,1[$, where $u_1 = p$, $u_{k+1} = u_k + p$, for some $0 < p \le \alpha/|H|$, and $I_p = \{k \in N : kp < 1\}$, and taking for $u = u_{n,\alpha}^{H,p} = u_{n,\alpha}$ (say) the largest value of G_p satisfying $P_{f_0}(\mathscr{C}_n(H,u)) \le \alpha$, that is,

$$u_{n,\alpha} = \max \big\{ u \in G_p : P_{f_0}(\mathscr{C}_n(H,u)) \le \alpha \big\}. \tag{14}$$

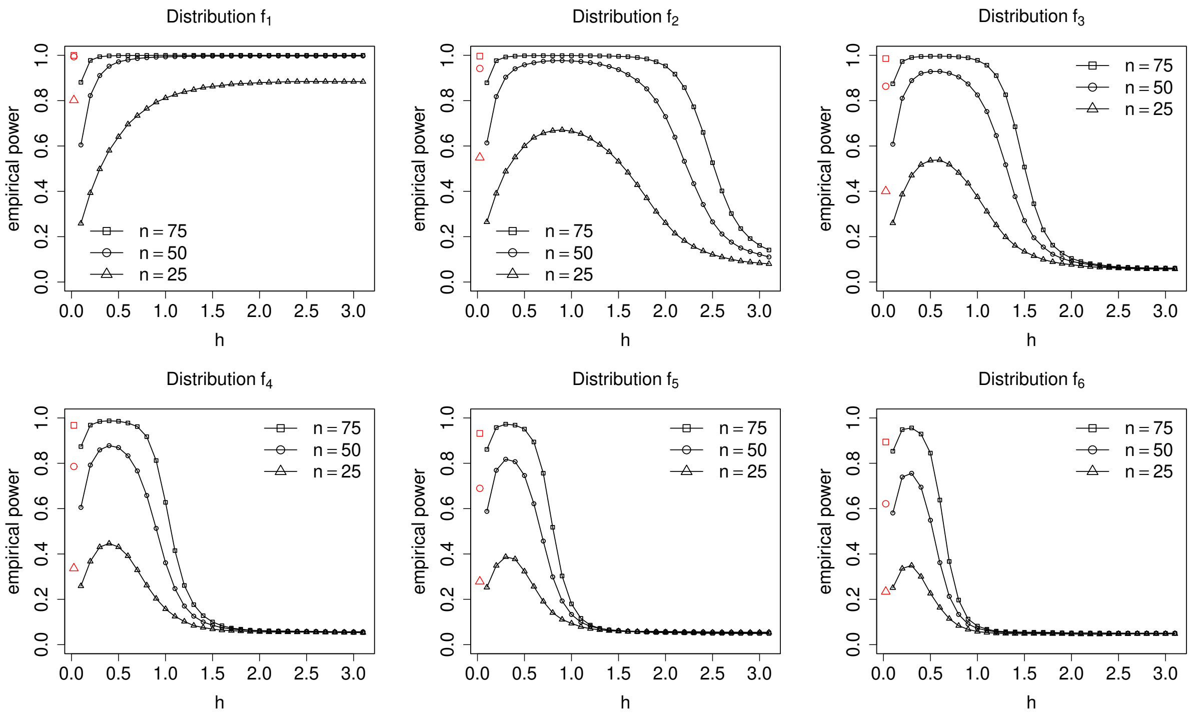

Figure 1: *Empirical power, at level* $\alpha = 0.05$ *and sample sizes* $n = 25, 50, 75$*, of the tests based on* $I_n(h)$*, as a function of* h*, for distributions with densities* f_j*, for* $j = 1, \ldots, 6$*. The red marks indicate the empirical power of the test based on the critical region* $\mathscr{C}_n(H, u_{n,\alpha})$*, with* $H = \{0.2, 0.5, 1.0, 2.0\}$*. The power estimates are based on 10,000 samples from the considered distributions.*

In order to implement this calibration procedure in practice, we used 50,000 simulations under the null hypothesis on the involved test statistics $I_n(h)$, $h \in H$, and the R function quantile(.,type=7) for estimating the $(1-u)$ quantiles $q(I_n(h), u)$, for u varying on G_p with $p = 0.001$. Further 50,000 simulations were used for estimating the probabilities $P_{f_0}(\mathscr{C}_n(H, u))$, for u varying on G_p.

4.2 The selection of H

Taking in mind the choice of the tuning parameters set H, we start by performing some simulation experiments for a large set of alternative distributions in order to analyse the power performance of the tests based on the critical regions $\mathscr{C}(I_n(h), \alpha)$ given by (12) as a function of h, where we take for kernel K the kernel K_2 defined by (6). From now on, this will be the kernel we consider. We concluded that a moderate or large bandwidth h leads to a test especially performing in detecting deviations in the first trigonometric moment from the null hypothesis of uniformity, whereas moderate or small values of h enable us to obtain test procedures highly performing in detecting alternatives with null first trigonometric moments.

H	$\alpha = 0.01$			$\alpha = 0.05$		
	$n = 25$	$n = 50$	$n = 75$	$n = 25$	$n = 50$	$n = 75$
	Distribution f_1					
H_a	0.53	0.95	1.00	0.80	0.99	1.00
H_b	0.51	0.95	1.00	0.78	0.99	1.00
H_c	0.56	0.96	1.00	0.80	0.99	1.00
	Distribution f_2					
H_a	0.24	0.77	0.97	0.66	0.97	1.00
H_b	0.24	0.77	0.97	0.51	0.93	1.00
H_c	0.27	0.79	0.97	0.55	0.94	1.00
	Distribution f_3					
H_a	0.16	0.60	0.92	0.39	0.85	0.98
H_b	0.16	0.59	0.93	0.38	0.85	0.98
H_c	0.17	0.62	0.93	0.40	0.86	0.99
	Distribution f_4					
H_a	0.12	0.47	0.86	0.32	0.77	0.96
H_b	0.12	0.46	0.85	0.32	0.75	0.96
H_c	0.12	0.49	0.86	0.34	0.79	0.97
	Distribution f_5					
H_a	0.10	0.39	0.78	0.27	0.69	0.93
H_b	0.09	0.34	0.75	0.26	0.65	0.92
H_c	0.10	0.38	0.77	0.28	0.69	0.93
	Distribution f_6					
H_a	0.08	0.33	0.72	0.24	0.63	0.90
H_b	0.08	0.31	0.70	0.22	0.59	0.88
H_c	0.08	0.33	0.71	0.23	0.62	0.90

Table 1: *Empirical power results, at levels* $\alpha = 0.01, 0.05$*, and sample sizes* $n = 25, 50, 75$*, of the tests based on the critical region* $\mathscr{C}_n(H, u_{n,\alpha})$*, with* $H_a = \{0.1, 0.2, \ldots, 3.1\}$*,* $H_b = \{0.1, 0.2, 0.5, 1.0, 2.0, 3.0\}$*, and* $H_c = \{0.2, 0.5, 1.0, 2.0\}$*, for the distributions with densities* f_j*, for* $j = 1, \ldots, 6$*. The power estimates are based on 10,000 samples from the considered distributions.*

This behaviour is illustrated in Figure 1 where we present the empirical power, at level $\alpha = 0.05$, of the tests based on $I_n(h)$ as a function of h for a set of distributions with probability densities of the form $f_j(\theta) = (2\pi)^{-1}(1 + \rho\cos(j(\theta - \pi)))$, for $\theta \in [0, 2\pi[$, $\rho = 0.9$ and $j = 1, \ldots, 6$. Data from this type of alternatives are generated by the acceptance-rejection method, taking the uniform circular density as the auxiliary density. The characteristic function of density f_j is such that $|\varphi_{f_j}(j)| = |\rho|/2$, and $|\varphi_{f_j}(p)| = 0$, for $p \in \mathbb{N} \setminus \{j\}$.

After this initial step we implemented the test with critical region $\mathscr{C}_n\big(H, u_{n,\alpha}\big)$ by taking for H the set $\{0.1, 0.2, \ldots, 3.1\}$, as well as several other subsets of this initial large set of tuning

n	25	50	75	100	150	200
			$\alpha = 0.10$			
$u_{n,\alpha}$	0.0468	0.0484	0.0474	0.0465	0.0465	0.0476
EL	0.0975	0.1065	0.1020	0.0953	0.0963	0.0980
			$\alpha = 0.05$			
$u_{n,\alpha}$	0.0221	0.0229	0.0221	0.0216	0.0216	0.0228
EL	0.0470	0.0530	0.0513	0.0474	0.0465	0.0481
			$\alpha = 0.01$			
$u_{n,\alpha}$	0.0040	0.0036	0.0039	0.0043	0.0042	0.0045
EL	0.0102	0.0089	0.0121	0.0095	0.0080	0.0099

Table 2: *Estimated levels $u_{n,\alpha}$ for a preassigned level α, based on regular grids of size $p = 0.0001$ on the interval $]0,1[$, and estimates of the nominal levels of significance (EL) for the test based on the critical region $\mathscr{C}_n\big(H, u_{n,\alpha}\big)$, for $H = \{0.2, 0.5, 1.0, 2.0\}$ and $K = K_2$. For the estimation of the nominal levels, the number of replications for each case is 10,000.*

parameters. We concluded that the power performance of the resulting test does not strongly depend on the set H once it includes values of h for which the test based on $I_n(h)$ reaches a good power performance. In Table 1 we present the results observed for some of the considered sets of tuning parameters for each one of the distributions f_j, $j = 1, \ldots, 6$.

Based on this preliminary analysis, we decided to take $H = \{0.2, 0.5, 1.0, 2.0\}$, which is the set of tuning parameters we always consider from now on. Although the choice of the set H may be based on some preliminar information, the previous set H is meant for the most common situation in practice where no relevant information about the alternative hypothesis is available.

4.3 Nominal level of significance

For $\alpha = 0.01, 0.05, 0.1$, and sample sizes $n = 25, 50, 100, 150, 200$, we present in Table 2 the estimated levels $u_{n,\alpha}$ as well as estimates of the nominal levels of significance for the test based on the critical region $\mathscr{C}_n\big(H, u_{n,\alpha}\big)$, with $H = \{0.2, 0.5, 1.0, 2.0\}$, for which 10,000 simulations under the null hypothesis were used. With three single marginal exceptions the preassigned level α is inside its approximate 95% confidence interval, revealing the effectiveness of the calibration procedure through the selection of the level u as explained before.

4.4 Finite sample power analysis

In order to investigate the performance of the test based on the critical region $\mathscr{C}_n(H, u_{n,\alpha})$, with $H = \{0.2, 0.5, 1.0, 2.0\}$ and $K = K_2$, labelled IH henceforth, and compare its performance with other existing tests, we carried out simulations for a large set of alternative distributions that includes all the models considered in Bogdan et al. (2002) and Oliveira et al. (2012). Here we

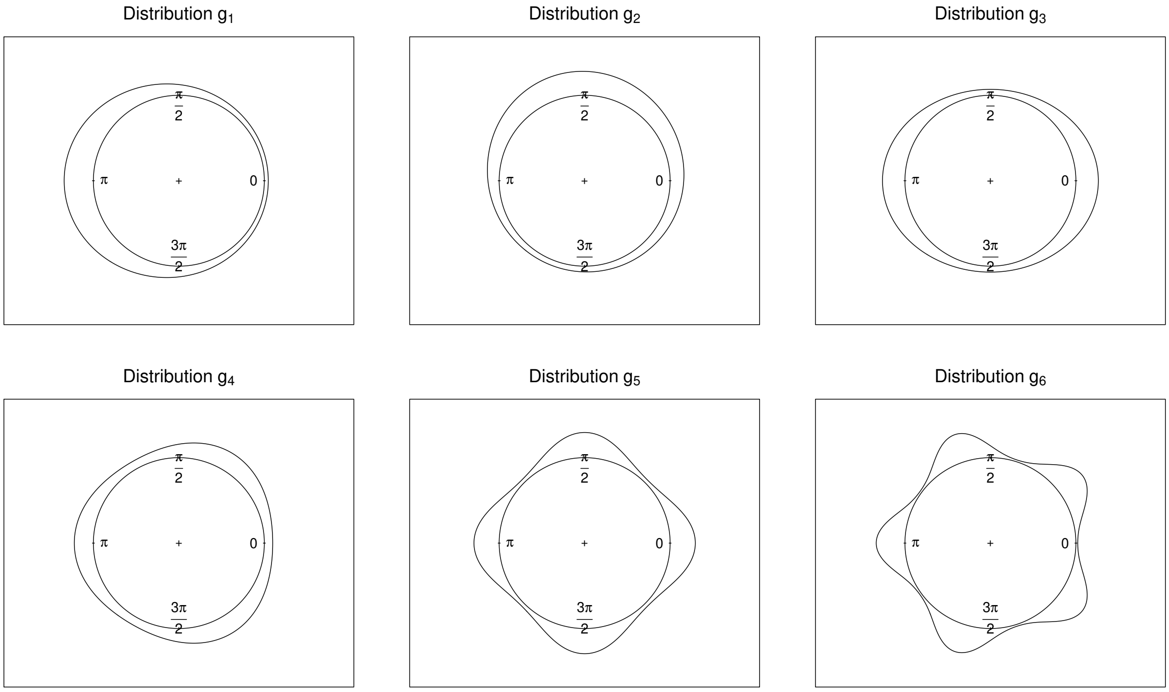

Figure 2: *Probability densities of the mixtures of von Mises distributions whose parameters are given in Table 3.*

	m	$w_1, \ldots, w_m$	$\mu_1, \ldots, \mu_m$	$\kappa_1, \ldots, \kappa_m$
g_1	1	1	π	1
g_2	2	$1/4, 3/4$	$0, \pi/\sqrt{3}$	$1, 1$
g_3	2	$1/2, 1/2$	$0, \pi$	$2, 2$
g_4	3	$1/3, 1/3, 1/3$	$\pi/3, \pi, 5\pi/3$	$3, 3, 3$
g_5	4	$1/4, 1/4, 1/4, 1/4$	$0, \pi/2, \pi, 3\pi/2$	$6, 6, 6, 6$
g_6	5	$1/5, 1/5, 1/5, 1/5, 1/5$	$\pi/5, 3\pi/5, 5\pi/5, 7\pi/5, 9\pi/5$	$18, 18, 18, 18, 18$

Table 3: *Parameters of the considered mixtures of von Mises distributions.*

show the empirical power results observed for some alternatives from the following two families of distributions:

(A) Mixtures of von Mises distributions with probability densities given by

$$g(\theta) = \sum_{\ell=1}^{m} w_\ell f_{\mathrm{vM}}(\theta; \mu_\ell, \kappa_\ell),$$

where

$$f_{\mathrm{vM}}(\theta; \mu, \kappa) = \frac{1}{2\pi I_0(\kappa)} \exp(\kappa \cos(\theta - \mu)),$$

is the von Mises density with mean direction $\mu \in [0, 2\pi[$ and concentration parameter $\kappa \geq 0$, and

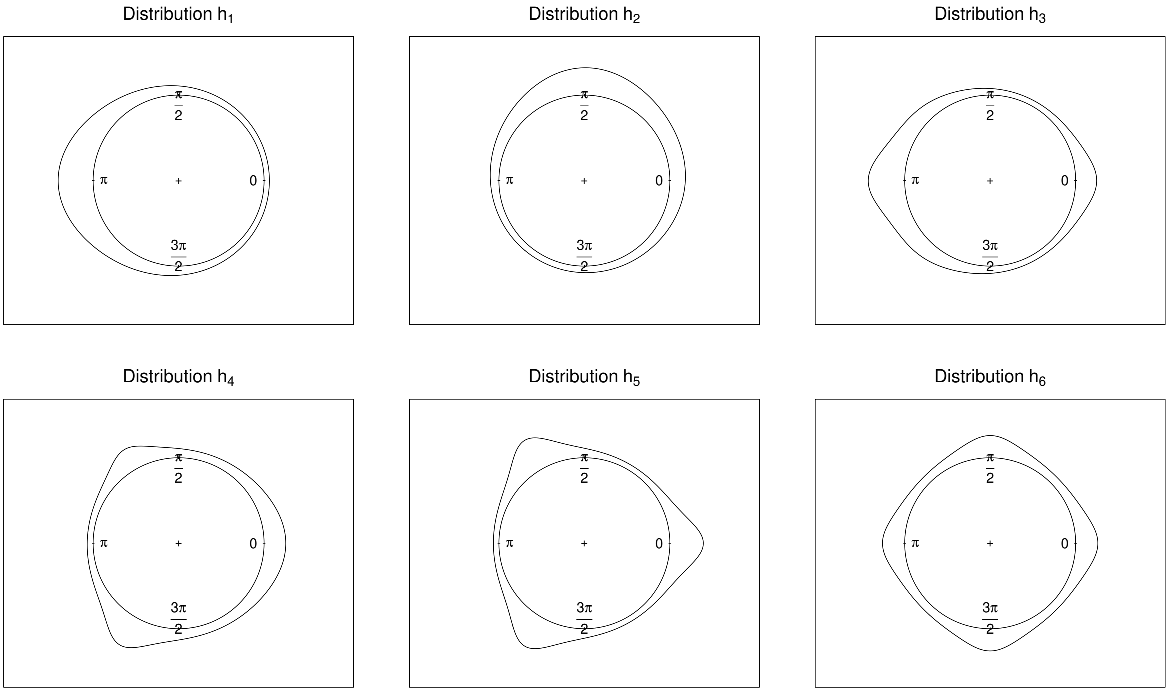

Figure 3: *Probability densities of the mixtures of wrapped Cauchy distributions whose parameters are given in Table 4.*

	m	$w_1, \ldots, w_m$	$\mu_1, \ldots, \mu_m$	$\rho_1, \ldots, \rho_m$
h_1	1	1	π	0.44
h_2	2	$1/4, 3/4$	$0, \pi/2$	$0.42, 0.42$
h_3	4	$0.25, 0.15, 0.4, 0.2$	$0, 3\pi/4, \pi, 5\pi/4$	$0.69, 0.575, 0.69, 0.575$
h_4	3	$0.5, 0.2, 0.3$	$0, 2\pi/3, 4\pi/3$	$1/2, 3/4, 3/4$
h_5	3	$1/3, 1/3, 1/3$	$0, 2\pi/3, 4\pi/3$	$3/4, 3/4, 3/4$
h_6	4	$1/4, 1/4, 1/4, 1/4$	$0, \pi/2, \pi, 3\pi/2$	$0.7, 0.7, 0.7, 0.7$

Table 4: *Parameters of the considered mixtures of wrapped Cauchy distributions.*

$I_r(\nu)$ is, for $\nu \geq 0$ and $r \geq 0$, the modified Bessel function of the first kind and order r defined by

$$I_r(\nu) = \frac{1}{2\pi} \int_0^{2\pi} \cos(r\theta) \exp(\nu \cos\theta) d\theta;$$

(B) Mixtures of wrapped Cauchy distributions with probability densities given by

$$h(\theta) = \sum_{\ell=1}^{m} w_\ell f_{\mathrm{wC}}(\theta; \mu_\ell, \rho_\ell),$$

where

$$f_{\mathrm{wC}}(\theta; \mu, \rho) = \frac{1}{2\pi} \frac{1 - \rho^2}{1 - 2\rho\cos(\theta - \mu) + \rho^2},$$

is the wrapped Cauchy density with mean direction $\mu \in [0, 2\pi[$ and concentration parameter $\rho \in [0, 1[$.

From each one of these families six distributions are chosen. They include densities considered or inspired by some of the models in Bogdan et al. (2002) and Oliveira et al. (2012). The associated parameters are given in Tables 3 and 4, and the corresponding densities are depicted in Figures 2 and 3. We used the package 'circular' (Lund and Agostinelli, 2017) for generating data from the von Mises and the wrapped Cauchy distributions. The characteristic functions of the considered von Mises distribution mixtures are such that $|\varphi_{g_1}(1)| = 0.446$, $|\varphi_{g_2}(1)| = 0.326$, $|\varphi_{g_3}(1)| = 0$, $|\varphi_{g_3}(2)| = 0.302$, $|\varphi_{g_4}(p)| = 0$, $p = 1, 2$, and $|\varphi_{g_4}(3)| = 0.197$, $|\varphi_{g_5}(p)| = 0$, $p = 1, 2, 3$, $|\varphi_{g_5}(4)| = 0.396$, $|\varphi_{g_5}(p)| = 0$, $p = 1, 2, 3, 4$, and $|\varphi_{g_6}(5)| = 0.492$. Regarding the considered mixtures of wrapped Cauchy distributions, we have $|\varphi_{h_1}(1)| = 0.44$, $|\varphi_{h_2}(1)| = 0.332$, $|\varphi_{h_3}(1)| = 0.247$, $|\varphi_{h_4}(1)| = 0.090$, $|\varphi_{h_5}(p)| = 0, p = 1, 2$, $|\varphi_{h_5}(3)| = 0.421$, $|\varphi_{h_6}(p)| = 0, p = 1, 2, 3$, and $|\varphi_{h_6}(4)| = 0.240$. The first two densities of family (A) and the first three densities of family (B) express departures from the null hypothesis of uniformity in the first trigonometric moment, while the remaining densities of each family express deviations from the null hypothesis of uniformity in relation to higher-order trigonometric moments (we include here the fourth density of family (B) because it has a first trigonometric moment close to zero).

As competitor tests we consider the classical tests of Kuiper (1960) and Watson (1961), which are based on two different measures of the discrepancy between the empirical distribution function and the distribution function of the uniform distribution. The Kuiper test statistic V is inspired by the Kolmogorov-Smirnov statistic for linear data, while the Watson test statistic U^2 is an adaptation of the Cramér-von Mises test for linear data. Denoting by $X_{(1)}, \ldots, X_{(n)}$ the ordered observations and letting $U_i = X_{(i)}/(2\pi)$, for $i = 1, \ldots, n$, the Kuiper test statistic can be written as

$$V = \max_{1 \leq i \leq n} \left(U_i - \frac{i}{n} \right) - \min_{1 \leq i \leq n} \left(U_i - \frac{i}{n} \right) + \frac{1}{n},$$

and the Watson test statistic can be expressed as

$$U^2 = \sum_{i=1}^{n} \left(U_i - \bar{U}_n - \frac{i - 1/2}{n} + \frac{1}{2} \right) + \frac{1}{12n},$$

where $\bar{U}_n = (U_1 + \cdots + U_n)/n$ (see Mardia and Jupp, 2000, pp. 99–105). The null hypothesis of uniformity is rejected for large values of these statistics. Note that the Watson test is equivalent to the projected Cramér-von Mises test of uniformity proposed in García-Portugués et al. (2021).

We also include in our study a data driven smooth test suggested by Bogdan et al. (2002), which rejects the null hypothesis of uniformity for large values of the statistic N_{2S}, where

$$N_{2k} = 2n \sum_{p=1}^{k} \left\{ \left(\frac{1}{n} \sum_{i=1}^{n} \cos(pX_i) \right)^2 + \left(\frac{1}{n} \sum_{i=1}^{n} \sin(pX_i) \right)^2 \right\},$$

for $k \in \mathbb{N}$, and $S = \operatorname{argmin}_{1 \leq k \leq 10} L(k)$, with $L(k) = N_{2k} - 2k \log n$, and a uniformity test proposed by Pycke (2010) which is based on the statistic V_q, with $q = \sqrt{1/2}$. This test belongs to a family of tests indexed by $q \in]0,1[$, that reject the null hypothesis of uniformity for large values of the V-statistic

$$V_q = \frac{2}{n} \sum_{i,j=1}^{n} \frac{\cos(X_i - X_j) - q}{1 - 2q\cos(X_i - X_j) + q^2}.$$

The simulation results presented in Bogdan et al. (2002) and Pycke (2010) support the conclusion that these tests performs well for a wide range of alternatives, being in general better or at least quite competitive against other tests such as the test of Rayleigh (see Mardia and Jupp, 2000, pp. 94–99), the test of Ajne (1968), or the tests of Hermans and Rasson (1985). For this reason, none of these tests is included in our study.

Finally, we consider in our simulation study two recently proposed tests of uniformity for circular data. The first one is the projected Anderson-Darling test, proposed in García-Portugués et al. (2023, p. 189), that rejects the null hypothesis of uniformity for large values of the statistic

$$\mathrm{AD} = \frac{2}{n} \sum_{1 \le i < j \le n} \psi\big(\cos^{-1}(\cos(X_i - X_j))\big) + n,$$

where $\psi(\theta) = -2\log(2\pi) + \pi^{-1}\big(\theta \log\theta + (2\pi - \theta)\log(2\pi - \theta)\big)$, for $\theta \in]0,\pi]$, and $\psi(0) = 0$. The second one is the K-fold smooth maximum test, proposed in Fernández-de-Marcos and García-Portugués (2023), which is based on the family of statistics

$$T_\kappa = \frac{2}{n} \sum_{1 \le i < j \le n} \exp\big(\kappa(\cos(X_i - X_j) - 1)\big) - (n-1)I_0(\kappa)\exp(-\kappa),$$

that depends on the parameter $\kappa > 0$, which selection is crucial in practice to ensure a powerful test. In implementing this test procedure, we have followed the description given in Fernández-de-Marcos and García-Portugués (2023, Definition 2, p. 1518) by taking $K = 10$, which is the authors' recommended value for using the K-fold method, and using the gamma-match method in Step 2.(b) to obtain the asymptotic p-values p_k, for $k = 1, \dots, K$. The simulation results presented in the previous works give strong indications that both tests are quite competitive against several other uniformity tests.

In Tables 5 and 6 we show the empirical power of the tests IH, N_{2S}, V_q ($q = \sqrt{1/2}$), V, U^2, AD and T_κ, for the considered alternatives $g_1, \dots, g_6$ and $h_1, \dots, h_6$, respectively. We limit ourselves to present here the results obtained for the nominal level $\alpha = 0.05$ and sample sizes $n = 25, 50, 75, 100$. However, similar conclusions can be drawn for the nominal levels $\alpha = 0.1, 0.01$ also considered in our study. The quantiles of order $1 - \alpha$ of each one of the tests based on N_{2S}, V_q ($q = \sqrt{1/2}$), V, U^2 and AD, are estimated by performing 100,000 simulations under the null hypothesis, and the corresponding power estimates are based on 10,000 samples from the considered alternatives.

Although none of the considered tests presents uniformly better results for the considered set of alternatives, the main conclusion that can be drawn is that the new IH test presents a good overall performance for a wide range of alternative distributions. Although slightly less performing

	Alternatives (A)							
	$n = 25$	$n = 50$	$n = 75$	$n = 100$	$n = 25$	$n = 50$	$n = 75$	$n = 100$
	Distribution g_1				Distribution g_2			
IH	0.77	0.98	1.00	1.00	0.44	0.79	0.94	0.99
N_{2S}	0.42	0.80	0.96	1.00	0.21	0.44	0.66	0.83
V_q	0.76	0.98	1.00	1.00	0.45	0.78	0.93	0.99
V	0.81	0.99	1.00	1.00	0.50	0.82	0.95	0.99
U^2	0.87	0.99	1.00	1.00	0.57	0.87	0.97	1.00
AD	0.84	0.99	1.00	1.00	0.54	0.85	0.96	1.00
T_κ	0.55	0.94	1.00	1.00	0.22	0.60	0.84	0.95
	Distribution g_3				Distribution g_4			
IH	0.23	0.50	0.71	0.86	0.09	0.15	0.23	0.31
N_{2S}	0.18	0.36	0.56	0.74	0.09	0.15	0.23	0.31
V_q	0.26	0.54	0.73	0.90	0.09	0.15	0.23	0.32
V	0.13	0.24	0.39	0.54	0.06	0.08	0.10	0.13
U^2	0.12	0.21	0.37	0.55	0.06	0.07	0.09	0.10
AD	0.13	0.26	0.46	0.66	0.06	0.07	0.09	0.11
T_κ	0.02	0.09	0.21	0.40	0.01	0.01	0.02	0.02
	Distribution g_5				Distribution g_6			
IH	0.25	0.60	0.86	0.96	0.34	0.81	0.97	1.00
N_{2S}	0.30	0.64	0.86	0.91	0.47	0.79	0.63	0.35
V_q	0.21	0.46	0.75	0.92	0.21	0.52	0.84	0.97
V	0.09	0.14	0.21	0.28	0.09	0.15	0.23	0.25
U^2	0.08	0.09	0.13	0.17	0.08	0.09	0.13	0.16
AD	0.08	0.11	0.17	0.26	0.08	0.11	0.19	0.26
T_κ	0.02	0.06	0.20	0.45	0.02	0.12	0.44	0.79

Table 5: *Empirical power results, at level* $\alpha = 0.05$ *and sample sizes* $n = 25, 50, 75, 100$*, for the tests* IH*,* N_{2S}*,* V_q *(*$q = \sqrt{1/2}$*),* V*,* U^2*,* AD *and* T_κ *and the alternatives* $g_1, \ldots, g_6$ *from the family (A). The power estimates are based on 10,000 samples from the considered distributions.*

than the Kuiper, Watson and projected Anderson-Darling tests for some of the alternatives that express departures from the null hypothesis of uniformity in the first trigonometric moment, it is clearly better than these tests for alternatives that do not express deviations from the null hypothesis of uniformity in the first trigonometric moment but in higher-order trigonometric moments. Moreover, it is quite competitive against both N_{2S} and V_q tests for the latter type of alternatives, it outperforms the smooth N_{2S} test and it is similar to the V_q test for the former type of alternatives. For all the considered alternatives the test IH outperforms the 10-fold smooth maximum test T_κ that shows a very low power against some of them. For all the sample sizes considered the estimated significance level of the 10-fold smooth maximum test was much lower than the nominal level α. Let us finally mention the fact that the test based N_{2S} may present an empirical power which is not an increasing function of n. This anomaly, observed for alternative g_6, where the empirical power for $n = 100$ is lower than for $n = 25$, but also for alternative h_5,

	Alternatives (B)							
	$n = 25$	$n = 50$	$n = 75$	$n = 100$	$n = 25$	$n = 50$	$n = 75$	$n = 100$
	Distribution h_1				Distribution h_2			
IH	0.76	0.98	1.00	1.00	0.47	0.81	0.95	0.99
N_{2S}	0.47	0.84	0.97	1.00	0.23	0.49	0.72	0.87
V_q	0.77	0.98	1.00	1.00	0.48	0.80	0.94	0.99
V	0.79	0.98	1.00	1.00	0.52	0.83	0.95	0.99
U^2	0.84	0.99	1.00	1.00	0.59	0.88	0.97	0.99
AD	0.83	0.99	1.00	1.00	0.55	0.86	0.97	0.99
T_κ	0.54	0.92	0.99	1.00	0.23	0.62	0.86	0.95
	Distribution h_3				Distribution h_4			
IH	0.48	0.82	0.95	0.99	0.19	0.41	0.62	0.79
N_{2S}	0.33	0.67	0.87	0.96	0.20	0.41	0.61	0.78
V_q	0.53	0.86	0.97	1.00	0.20	0.42	0.64	0.81
V	0.43	0.74	0.91	0.97	0.12	0.22	0.33	0.46
U^2	0.44	0.76	0.92	0.98	0.11	0.19	0.27	0.38
AD	0.44	0.78	0.94	0.99	0.11	0.20	0.31	0.45
T_κ	0.16	0.48	0.75	0.91	0.02	0.06	0.13	0.26
	Distribution h_5				Distribution h_6			
IH	0.39	0.80	0.96	0.99	0.11	0.19	0.30	0.42
N_{2S}	0.41	0.76	0.88	0.83	0.13	0.24	0.37	0.50
V_q	0.39	0.77	0.95	0.99	0.10	0.15	0.24	0.34
V	0.14	0.23	0.41	0.57	0.07	0.08	0.09	0.12
U^2	0.11	0.18	0.30	0.49	0.07	0.07	0.08	0.09
AD	0.13	0.25	0.48	0.72	0.06	0.07	0.08	0.10
T_κ	0.04	0.22	0.55	0.81	0.01	0.01	0.02	0.03

Table 6: *Empirical power results, at level* $\alpha = 0.05$ *and sample sizes* $n = 25, 50, 75, 100$*, for the tests* IH*,* N_{2S}*,* V_q *(*$q = \sqrt{1/2}$*),* V*,* U^2*,* AD *and* T_κ*, and the alternatives* $h_1, \ldots, h_6$ *from the family (B). The power estimates are based on 10,000 samples from the considered distributions.*

as we can confirm by comparing the empirical powers for sample sizes $n = 75$ and $n = 100$, was also observed for several other, usually multimodal, alternatives (such as models #7, #11, #13, #14, #20 in Oliveira et al., 2012).

Taking into account the excellent performance shown by the IH and V_q tests for some of the alternatives, together with the fact that these tests are among the best of the considered tests for all the considered alternative distributions, if one is going to rely on one and only one of the considered test procedures, one of the tests IH or V_q is recommended.

A function written in R language that implements an approximation for the p-value of the IH uniformity test is available on the author's website.

5 Conclusions

With the aim of testing a uniformity hypothesis on the circle, we consider in this work a Bickel–Rosenblatt type test statistic (L^2 distance), based on the Parzen–Rosenblatt type estimator for circular data, for testing a general simple null hypothesis. The asymptotic behaviour of the proposed test procedure for fixed and non-fixed bandwidths is studied and the asymptotic superiority of the tests with a fixed bandwidth over those with a non-fixed bandwidth is established for fixed and local alternatives. Taking this into account, a multiple test procedure that combines a finite set of Bickel–Rosenblatt type test statistics for uniformity obtained for different values of the fixed bandwidth, that acts as a tuning parameter, is then proposed for testing a uniformity hypothesis on the circle. The results of a simulation study indicate that the new test procedure reveals a good empirical power performance for a wide range of alternative distributions, being quite competitive against all the uniformity tests with which it was compared.

6 Proofs

Proof of equality (4): The Fourier transforms of $\hat{f}_n(\cdot;h)$ and $E_0\hat{f}_n(\cdot;h) = d_h(K)(K_h * f_0)(\cdot)$ are given, for $k \in \mathbb{Z}$, by $\varphi_{\hat{f}_n(\cdot;h)}(k) = d_h(K)\varphi_n(k)\varphi_{K_h}(k) = d_h(K)\varphi_n(k)\varphi_K(kh)$ and $\varphi_{E_0\hat{f}_n(\cdot;h)}(k) = d_h(K)\varphi_{K_h}(k)\varphi_f(k)$, where $\varphi_n(k) = n^{-1}\sum_{j=1}^n \exp(\mathrm{i}kX_j)$ is the empirical characteristic function. Assuming that K satisfies (K.1) and $0 < h \leq \pi/M$, we have

$$\varphi_{K_h}(k) = \int_{-\pi/h}^{\pi/h} K(v)\exp(\mathrm{i}khv)du = \int_{\mathbb{R}} K(v)\exp(\mathrm{i}khv)du = \varphi_K(kh),$$

and $d_h(K)^{-1} = \int_{-\pi/h}^{\pi/h} K(y)dy = \int_{\mathbb{R}} K(u)du$. Therefore, equality (4) follows easily from the Parseval's identity (see Butzer and Nessel, 1971, Proposition 4.2.2, p. 175). ■

Proof of Theorem 1: Given $X_1, \ldots, X_n \in [0, 2\pi[$ independent circular random variables with common probability density function $f_0 \in L^\infty([0, 2\pi[)$, we begin by noticing that the statistic $I_n(h)$ defined by (3) can be written as

$$I_n(h) = EH(X_1, X_1; f_0, h) + J_n(h) + R_n(h), \tag{15}$$

where

$$J_n(h) = \frac{2}{n}\sum_{1\leq i<j\leq n} H(X_i, X_j; f_0, h)$$

and

$$R_n(h) = \frac{1}{n}\sum_{i=1}^n \{H(X_i, X_i; f_0, h) - EH(X_i, X_i; f_0, h)\},$$

with $H(u, v; f_0, h)$ defined by (9) for $u, v \in [0, 2\pi[$ and $h > 0$.

In order to derive the asymptotic behaviour of each one of the terms of the right-hand side of equality (15), we start by establishing two useful properties.

Property 1. *For $u, v \in [0, 2\pi[$ and $h > 0$, we have*

$$d_h(K)^{-2} H(u, v; f_0, h) = \bar{K}_h * K_h(u - v) + r(u, v; f_0, h),\ u, v \in [0, 2\pi[,\ h > 0,$$

where $\bar{K}_h(u) = K_h(-u)$ and

$$C_{f_0} := \sup_{h>0} \sup_{u,v \in [0,2\pi[} |r(u, v; f_0, h)| < \infty.$$

Proof: For $u, v \in [0, 2\pi[$ and $h > 0$ we have

$$\begin{aligned} d_h(K)^{-2} H(u, v; f, h) = & \int_0^{2\pi} K_h(\theta - u) K_h(\theta - v) d\theta - \int_0^{2\pi} K_h(\theta - u)(K_h * f)(\theta) d\theta \\ & - \int_0^{2\pi} K_h(\theta - v)(K_h * f)(\theta) d\theta + \int_0^{2\pi} (K_h * f)(\theta)^2 d\theta, \end{aligned}$$

where $\int_0^{2\pi} K_h(\theta - u) K_h(\theta - v) d\theta = \bar{K}_h * K_h(u - v)$ and

$$|(K_h * f)(\theta)| \leq ||f||_\infty \int_0^{2\pi} |K_h(\theta - u)| du \leq ||f||_\infty \int_{\mathbb{R}} |K(u)| du,$$

which concludes the proof. ■

Property 2. *If K satisfies (K.1) and $0 < h \leq \pi/(3M)$, then*

$$\bar{K}_h * K_h(u) = h^{-1} \bar{K} \star K(h^{-1} u) \mathbb{1}_{[-\pi + hM, \pi - hM]}(u),\ \text{for } u \in [-\pi, \pi],$$

where $\bar{K}(u) = K(-u)$ and $\star$ denotes the convolution product

$$\bar{K} \star K(u) = \int_{\mathbb{R}} \bar{K}(u - v) K(v) dv = \int_{\mathbb{R}} K(u + v) K(v) dv,\ u \in \mathbb{R}.$$

Proof: As $\bar{K}_h * K_h$ is symmetric, it is enough to consider the case $u \in [0, \pi]$. Taking into account that K_h is periodic with period 2π with $K_h(\theta) = K(\theta/h)/h$, for $\theta \in [-\pi, \pi[$, we have

$$\begin{aligned} \bar{K}_h * K_h(u) &= h^{-2} \int_{-\pi}^{\pi} \mathbb{1}_{[-\pi,\pi[}(u - y) \bar{K}(h^{-1}(u - y)) K(h^{-1} y) dy \\ &\quad + h^{-2} \int_{-\pi}^{\pi} \mathbb{1}_{[\pi,2\pi[}(u - y) \bar{K}(h^{-1}(u - y - 2\pi)) K(h^{-1} y) dy \\ &= h^{-1} \int_{(u-\pi)/h}^{\pi/h} \bar{K}(h^{-1} u - z) K(z) dz + h^{-1} \int_{-\pi/h}^{(u-\pi)/h} \bar{K}(h^{-1}(u - 2\pi) - z) K(z) dz. \end{aligned}$$

For $0 < h \leq \pi/M$ and $u \in [0, \pi - hM]$, we have

$$\bar{K}_h * K_h(u) = h^{-1} \int_{\mathbb{R}} \bar{K}(h^{-1} u - z) K(z) dz = h^{-1} \bar{K} \star K(h^{-1} u),$$

as $-\pi/h \leq (u - \pi)/h \leq -M \leq M \leq \pi/h$. For $0 < h \leq \pi/M$ and $u \in]\pi - hM, \pi]$ we have

$$\bar{K}_h * K_h(u) = h^{-1} \bar{K} \star K(h^{-1} u) + h^{-1} \int_{-M}^{(u-\pi)/h} \bar{K}(h^{-1}(u - 2\pi) - z) K(z) dz,$$

as $-\pi/h \le -M \le (u-\pi)/h \le M \le \pi/h$, and $h^{-1}u - z \ge M$ for $z \le (u-\pi)/h$. In order to conclude, it suffices to note that the right-hand side of the previous equality vanishes when $0 < h \le \pi/(3M)$. ■

Taking into account that $K_h(\theta) = K(\theta/h)/h \le h^{-1}||K||_\infty$ and

$$\sup_{u\in[0,2\pi[} |\bar{K}_h * K_h(u)| \le h^{-1}||K||_\infty \int_{\mathbb{R}} |K(u)|du, \tag{16}$$

from Property 1 we deduce that

$$\sup_{u,v\in[0,2\pi[} |H(u,v;f_0,h)| \le d_h(K)^2 \Big(h^{-1}||K||_\infty \int_{\mathbb{R}} |K(u)|du + C_{f_0} \Big),$$

which enables us to rewrite equality (15) as

$$I_n(h) = EH(X_1,X_1;f_0,h) + J_n(h) + O_p\big(n^{-1/2} d_h(K)^2 (1+h^{-1})\big). \tag{17}$$

If the assumption (B_h) is satisfied for some $h > 0$, from this equality we conclude that the convergence in distribution stated in part (b) follows from the limit distribution theorem for degenerate U-statistics with fixed kernel applied to the U-statistic $J_n(h)$ as $E|H(X_1,X_1;f_0,h)| < \infty$, $E(H(X_1,v;f_0,h)) = 0$ for all $v \in [0,2\pi[$, and $E(H(X_1,X_2;f_0,h)^2) < \infty$ (see Gregory, 1977, Theorem 2.1, p. 111), and the fact that $EH(X_1,X_1;f_0,h) = \sum_{k=1}^{\infty} \lambda_{h,k}$, which follows from the integral form of $H(u,v;f_0,h)$.

From now on we assume that the bandwidth $h = h_n$ satisfies the assumption (B_0) and that the kernel K satisfies assumption (K.1). Under these conditions and for large enough n we have

$$EH(X_1,X_1;f_0,h) = h^{-1} \int_{\mathbb{R}} K(u)^2 du \left(\int_{\mathbb{R}} K(u)du \right)^{-2} + O(1),$$

from which together with equality (17) we deduce that

$$h^{1/2} \left\{ I_n(h) - h^{-1} \int_{\mathbb{R}} K(u)^2 du \left(\int_{\mathbb{R}} K(u)du \right)^{-2} \right\} = h^{1/2} J_n(h) + o_p(1).$$

Therefore, in order to establish the convergence in distribution stated in part (a) of Theorem 1 we will prove that $h^{1/2}J_n(h)$ is asymptotically normal with zero mean and variance ν^2. As $J_n(h)$ is a degenerate U-statistic with a kernel depending on n, such asymptotic normality will be derived by using U-statistics techniques introduced in Hall (1984). For that some auxiliary results are established in the following propositions.

Proposition 1. *We have*

a) $hE\big(H(X_1,X_2;f_0,h)^2\big) = \left(\int_{\mathbb{R}} K(u)du \right)^4 \int_{\mathbb{R}} \bar{K} \star K(v)^2 dv \int_0^{2\pi} f_0(\theta)^2 d\theta + o(1);$

b) $h^3 E\big(H(X_1,X_2;f_0,h)^4\big) = O(1).$

Proof: From Property 1 we have

$$h^{1/2}d_h(K)^{-2}H(X_1,X_2;f_0,h) = h^{1/2}\bar{K}_h * K_h(X_1-X_2) + h^{1/2}r(X_1,X_2;f_0,h),$$

where $E\big(hr(X_1,X_2;f_0,h)^2\big) \leq hC_{f_0} = o(1)$. Moreover, from Property 2 and for h small enough ($h \leq \pi/(3M)$) we have

$$\begin{aligned} hE\big(\bar{K}_h * K_h(X_1-X_2)^2\big) &= h\int_0^{2\pi}\int_0^{2\pi} \bar{K}_h * K_h(u-v)^2 f_0(u)f_0(v)dudv \\ &= h\int_0^{2\pi} \bar{K}_h * K_h(w)^2 \bar{f}_0 * f_0(w)dw \\ &= h^{-1}\int_{-\pi+hM}^{\pi-hM} \bar{K} \star K(h^{-1}w)^2 \bar{f}_0 * f_0(w)dw \\ &= \int_{\mathbb{R}} \bar{K} \star K(v)^2 \bar{f}_0 * f_0(hv)dv. \end{aligned}$$

The result stated in a) follows now from the continuity of $\bar{f}_0 * f_0$ and the fact that $d_h(K)^{-1} = \int_{\mathbb{R}} K(u)du$, for $h \leq \pi/M$. Similar arguments can be used to prove b). ■

For $x, y \in [0, 2\pi[$ define

$$G(x,y;f_0,h) = E\big(H(X_1,x;f_0,h)H(X_1,y;f_0,h)\big).$$

Proposition 2. *We have*

$$hE\big(G(X_1,X_2;f_0,h)^2\big) = O(1).$$

Proof: From the definition of $H(u,v;f_0,h)$ and Property 1 we have

$$\begin{aligned} d_h(K)^{-4}G(x,y;f_0,h) &= \int_0^{2\pi} \bar{K}_h * K_h(u-x)\bar{K}_h * K_h(u-y)f_0(u)du \\ &\quad + \int_0^{2\pi} \bar{K}_h * K_h(u-x)r(u,y;f_0,h)f_0(u)du \\ &\quad + \int_0^{2\pi} \bar{K}_h * K_h(u-y)r(u,x;f_0,h)f_0(u)du \\ &\quad + \int_0^{2\pi} r(u,x;f_0,h)r(u,y;f_0,h)f_0(u)du \\ &=: g_{n1}(x,y) + g_{n2}(x,y) + g_{n2}(y,x) + g_{n3}(x,y), \end{aligned}$$

where $|g_{n2}(x,y)| \leq C_{f_0}||f_0||_\infty \int_0^{2\pi} |\bar{K}_h * K_h(v)|dv \leq C_{f_0}||f_0||_\infty \int_{\mathbb{R}} K(u)^2du$, and $|g_{n3}(x,y)| \leq C_{f_0}^2$, for all $x, y \in [0, 2\pi[$. In order to conclude, it suffices to use (16) and the fact that

$$\begin{aligned} E\big(g_{n1}(X_2,X_2)^2\big) &= \int_0^{2\pi}\int_0^{2\pi}\bigg(\int_0^{2\pi} \bar{K}_h * K_h(u-x)\bar{K}_h * K_h(u-y)f_0(u)du\bigg)^2 f_0(x)f_0(y)dxdy \\ &\leq 2\pi||f_0||_\infty^4\bigg(\int_0^{2\pi}|\bar{K}_h * K_h(u)|du\bigg)^3 \sup_{u\in[0,2\pi[} |\bar{K}_h * K_h(u)| \\ &\leq 2\pi||f_0||_\infty^4\bigg(\int_{\mathbb{R}} K(u)^2du\bigg)^3 \sup_{u\in[0,2\pi[} |\bar{K}_h * K_h(u)|. \end{aligned}$$

■

Taking into account the previous propositions we conclude that

$$\frac{E(G_n(X_1,X_2;f_0,h)^2)+n^{-1}E(H_n(X_1,X_2;f_0,h)^4)}{\{E(H_n(X_1,X_2;f_0,h)^2)\}^2}=O\big(n^{-1}h^{-1}+h\big)=o(1).$$

Therefore, from the central limit theorem for degenerate U-statistics of Hall (1984, Theorem 1, pp. 3–4), we conclude that $h^{1/2}J_n(h)\stackrel{d}{\longrightarrow}N(0,\nu^2)$. This ends the proof of Theorem 1. ■

Proof of Theorem 2: Given $X_1,\ldots,X_n\in[0,2\pi[$ independent circular random variables with common probability density function $f\in L^\infty([0,2\pi[)$, consider the following expansion where $I_n(h)$ is the statistic defined by (3):

$$\begin{aligned}n^{-1}I_n(h)&=\int_0^{2\pi}\big\{\hat f_n(\theta;h)-E\hat f_n(\theta;h)\big\}^2d\theta+\int_0^{2\pi}\big\{E\hat f_n(\theta;h)-E_0\hat f_n(\theta;h)\big\}^2d\theta\\&\quad+2\int_0^{2\pi}\big\{\hat f_n(\theta;h)-E_0\hat f_n(\theta;h)\big\}\big\{E\hat f_n(\theta;h)-E_0\hat f_n(\theta;h)\big\}d\theta\\&=:A_n+B_n+2C_n.\end{aligned}$$

Reasoning as in the proof of equality (4), from the Parseval's identity we have

$$B_n=\Big(\int_{\mathbb{R}}K(u)du\Big)^{-2}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}|\varphi_f(k)-\varphi_{f_0}(k)|^2|\varphi_K(kh)|^2,$$

as K satisfies (K.1) and $0<h\le\pi/M$. From Theorem 1 and the previous equality we know that $A_n=o_p(1)$ and $B_n=O(1)$, respectively, and also $C_n=o_p(1)$ from the Cauchy-Schwarz inequality. Therefore, we have

$$n^{-1}I_n(h)=\Big(\int_{\mathbb{R}}K(u)du\Big)^{-2}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}|\varphi_f(k)-\varphi_{f_0}(k)|^2|\varphi_K(kh)|^2+o_p(1),$$

from which we get the asymptotic behaviour of the statistic $T_n(h_n)$ stated in Theorem 2. ■

Proof of Theorem 3: Denoting by F_{n,h_n} the distribution function of $T_n(h_n)$ under the null hypothesis, from the properties of the quantile function (see van der Vaart, 2000, Lemma 21.1, p. 304) we have

$$\begin{aligned}P_{f_0}\big(T_n(h_n)>q(T_n(h_n),\alpha)\big)&=1-P_{f_0}\big(T_n(h_n)\le q(T_n(h_n),\alpha)\big)\\&=1-F_{n,h_n}\big(F_{n,h_n}^{-1}(1-\alpha)\big)\\&\le1-(1-\alpha)=\alpha.\end{aligned}$$

Denoting by F_h the cumulative distribution functions of the limiting distribution of $T_n(h_n)$ under the null hypothesis when the assumption (B_h) is satisfied for some $h\ge0$, we know from Theorem 1 that F_h is continuous on $\mathbb{R}$. Therefore, in order to prove that the test with critical region $\mathscr{C}(T_n(h_n),\alpha)$ is asymptotically of level α it is enough to show that

$$F_{n,h_n}\big(F_{n,h_n}^{-1}(1-\alpha)\big)\to F_h\big(F_h^{-1}(1-\alpha)\big).\qquad(18)$$

We have

$$\begin{aligned}&\big|F_{n,h_n}\big(F^{-1}_{n,h_n}(1-\alpha)\big)-F_h\big(F^{-1}_h(1-\alpha)\big)\big|\\&\le\sup_{x\in\mathbb{R}}\big|F_{n,h_n}(x)-F_h(x)\big|+\big|F_h\big(F^{-1}_{n,h_n}(1-\alpha)\big)-F_h\big(F^{-1}_h(1-\alpha)\big)\big|,\end{aligned}$$

where, from the continuity of F_h, $\sup_{x\in\mathbb{R}}\big|F_{n,h_n}(x)-F_h(x)\big|\to 0$ (van der Vaart, 2000, Lemma 2.11, p. 12). Finally, using the fact that F_h is strictly increasing on $\{x\in\mathbb{R}:F_h(x)>0\}$, we get that F_h^{-1} is continuous on $]0,1[$, from which we deduce that

$$F_h\big(F^{-1}_{n,h_n}(1-\alpha)\big)\to F_h\big(F^{-1}_h(1-\alpha)\big)$$

(van der Vaart, 2000, Lemma 21.2, p. 305). This concludes the proof of (18).

Taking into account that $q(T_n(h),\alpha)=F^{-1}_{n,h_n}(1-\alpha)\to F^{-1}_h(1-\alpha)$, the consistency of the test with critical region $\mathscr{C}(T_n(h_n),\alpha)$ follows from the convergence in probability of the test statistic $T_n(h_n)$ to $+\infty$ for a fixed alternative $f\in L^\infty([0,2\pi[)\backslash\{f_0\}$, that we can deduce from Theorem 2 when (B_0) is satisfied, and under the additional assumption (K.2) on the kernel when (B_h) is satisfied for some $0<h\le\pi/M$. ■

Proof of Theorem 4: Adapting the proofs of Theorems 1 and 2 for local alternatives, and for μ, ν^2, $\{\lambda_{h,k},k\ge 1\}$ and $\{a_{h,k},k\ge 1\}$ defined in their statements, we may conclude that: a) under assumption (B_0), we have $T_n(h_n)\stackrel{d}{\longrightarrow}N(0,\nu^2)$, for $\gamma_n=o(n^{-1/2}h^{-1/4})$, $T_n(h_n)\stackrel{d}{\longrightarrow}N(\mu,\nu^2)$, for $\gamma_n=n^{-1/2}h^{-1/4}$, and $T_n(h_n)\stackrel{p}{\longrightarrow}+\infty$, for $n^{-1/2}h^{-1/4}=o(\gamma_n)$; b) under assumption (B_h) for some $0<h\le\pi/M$, we have $T_n(h_n)\stackrel{d}{\longrightarrow}\sum_{k=1}^\infty\lambda_{k,h}Z_k^2$, for $\gamma_n=o(n^{-1/2})$, $T_n(h_n)\stackrel{d}{\longrightarrow}\sum_{k=1}^\infty\lambda_{h,k}(Z_k+a_{h,k})^2$, for $\gamma_n=n^{-1/2}$, and $T_n(h_n)\stackrel{p}{\longrightarrow}+\infty$, for $n^{-1/2}=o(\gamma_n)$. Theorem 4 follows now from these results and the arguments used in the proof of Theorem 3. ■

Funding

Research partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324.

References

Ajne, B. (1968), ‘A Simple Test for Uniformity of a Circular Distribution’, *Biometrika*, 55, 343–354.

Bai, Z.D., Rao, C.R., and Zhao, L.C. (1988), ‘Kernel Estimators of Density Function of Directional Data’, *Journal of Multivariate Analysis*, 27, 24–39.

Beran, R.J. (1969). ‘Asymptotic Theory of a Class of Tests for Uniformity of a Circular Distribution’, *The Annals of Mathematical Statistics*, 40, 1196–1206.

Beran, R. (1979), 'Exponential Models for Directional Data', *The Annals of Statistics*, 7, 1162–1178.

Bickel, P.J., and Rosenblatt, M. (1973), 'On Some Global Measures of the Deviations of Density Function Estimates', *The Annals of Statistics*, 1, 1071–1095.

Boente, G., Rodríguez, D., and González-Manteiga, W. (2014), 'Goodness-of-fit Test for Directional Data', *Scandinavian Journal of Statistics*, 41, 259–275.

Bogdan, M., Bogdan, K., and Futschik, A. (2002), 'A Data Driven Smooth Test for Circular Uniformity', *Annals of the Institute of Statistical Mathematics*, 54, 29–44.

Butzer, P.L., and Nessel, R.J. (1971), *Fourier Analysis and Approximation. One-Dimensional Theory*, Basel: Birkhauser.

Dunford, N., and Schwartz, J.T. (1963), *Linear Operators. Part II: Spectral Theory*, New York: Interscience Publishers.

Fan, Y. (1994), 'Testing the Goodness of Fit of a Parametric Density Function by Kernel Method', *Econometric Theory*, 10, 316–356.

Fan, Y. (1998), 'Goodness-of-Fit Tests Based on Kernel Density Estimators with Fixed Smoothing Parameters', *Econometric Theory*, 14, 604–621.

Fernández-de-Marcos, A., García-Portugués, E. (2023), 'On New Omnibus Tests of Uniformity on the Hypersphere', *TEST*, 32, 1508–1529.

Fromont, M., and Laurent, B. (2006), 'Adaptive Goodness-of-Fit Tests in a Density Model', *The Annals of Statistics*, 34, 680–720.

García-Portugués, E. (2013), 'Exact Risk Improvement of Bandwidth Selectors for Kernel Density Estimation with Directional Data', *Electronic Journal of Statistics*, 7, 1655–1685.

García-Portugués, E., Crujeiras, R.M., and González-Manteiga, W. (2013), 'Kernel Density Estimation for Directional-Linear Data', *Journal of Multivariate Analysis*, 121, 152–175.

García-Portugués, E., Crujeiras, R.M., González-Manteiga, W. (2018), 'Smoothing-Based Tests with Directional Random Variables', In: Gil, E., Gil, E., Gil, J., Gil, M. (eds) *The Mathematics of the Uncertain*. Studies in Systems, Decision and Control, vol 142. Springer, Cham.

García-Portugués, E., Navarro-Esteban, P., Cuesta-Albertos, J.A. (2021), 'A Cramér-von Mises Test of Uniformity on the Hypersphere', In: Balzano, S., Porzio, G.C., Salvatore, R., Vistocco, D., Vichi, M. (eds) *Statistical Learning and Modeling in Data Analysis*. CLADAG 2019. Studies in Classification, Data Analysis, and Knowledge Organization. Springer, Cham.

García-Portugués, E., Navarro-Esteban, P., Cuesta-Albertos, J.A. (2023), 'On a Projection-Based Class of Uniformity Tests on the Hypersphere', *Bernoulli*, 29, 181–204.

Gregory, G.G. (1977), 'Large Sample Theory for U-Statistics and Tests of Fit', *The Annals of Statistics*, 5, 110–123.

Hall, P. (1984), 'Central Limit Theorem for Integrated Square Error of Multivariate Nonparametric Density Estimators', *Journal of Multivariate Analysis*, 14, 1–16.

Hall, P., Watson, G.S., and Cabrera, J. (1987), 'Kernel Density Estimation with Spherical Data', *Biometrika*, 74, 751–762.

Hermans, M., and Rasson, J.P. (1985). 'A New Sobolev Test for Uniformity on the Circle', *Biometrika*, 72, 698–702.

Hoeffding, W. (1948), 'A Class of Statistics with Asymptotically Normal Distribution', *The Annals of Mathematical Statistics*, 19, 293–325.

Jammalamadak, S.R., Jiménez-Gamero, M.D., and Meintanis, S.G. (2019), 'A Class of Goodness-of-Fit Tests for Circular Distributions Based on Trigonometric Moments', *SORT-Statistics and Operations Research Transactions*, 43, 1–20.

Jammalamadaka, S.R., Meintanis, S., Verdebout, T. (2020), 'On Sobolev Tests of Uniformity on the Circle With an Extension to the Sphere', *Bernoulli*, 26, 2226–2252.

Klar, B. (2001), 'Goodness-of-Fit Tests for the Exponential and the Normal Distribution Based on the Integrated Distribution Function', *Annals of the Institute of Statistical Mathematics*, 53, 338–353.

Klemelä, J. (2000), 'Estimation of Densities and Derivatives of Densities with Directional Data', *Journal of Multivariate Analysis*, 73, 18–40.

Kuiper, N.H. (1960), 'Tests Concerning Random Points on a Circle', *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Series A*, 63, 38–47.

Lee, A.J. (1990), *U-statistics, Theory and Practice*, New York: Marcel Dekker.

Lund, U., and Agostinelli, C. (2017), 'circular: Circular Statistics', R package version 0.4-93. http://CRAN.R-project.org/package=circular.

Mardia, K.V., and Jupp, P.E. (2000), *Directional Statistics*, Chischester: John Wiley.

Oliveira, M., Crujeiras, R.M., and Rodríguez-Casal, A. (2012), 'A Plug-in Rule for Bandwidth Selection in Circular Density Estimation', *Computational Statistics and Data Analysis*, 56, 3898–3908.

Parzen, E. (1962), 'On Estimation of a Probability Density Function and Mode', *The Annals of Mathematical Statistics*, 33, 1065–1076.

Pycke, J.-R. (2010), 'Some Tests for Uniformity of Circular Distributions Powerful Against Multimodal Alternatives', *The Canadian Journal of Statistics*, 38, 80–96.

R Development Core Team (2021), *R: a language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria. URL https://www.R-project.org/

Révész, P. (1971), 'Testing of Density functions', *Periodica Mathematica Hungarica*, 1, 35–44.

Rosenblatt, M. (1956), 'Remarks on Some Non-Parametric Estimates of a Density Function', *The Annals of Mathematical Statistics*, 27, 832–837.

Tenreiro, C. (2007a), 'On the Asymptotic Behaviour of Location-Scale Invariant Bickel-Rosenblatt Tests', *Journal of Statistical Planning and Inference*, 137, 103–116.

Tenreiro, C. (2007b), 'On the Finite Sample Behaviour of Fixed Bandwidth Bickel-Rosenblatt Test for Univariate and Multivariate Uniformity', *Communications in Statistics - Simulation and Computation*, 36, 827–846.

Tenreiro, C. (2019), 'On the Automatic Selection of the Tuning Parameter Appearing in Certain Families of Goodness-of-Fit Tests', *Journal of Statistical Computation and Simulation*, 89, 1780–1797.

Tenreiro, C. (2022), 'Kernel Density Estimation for Circular Data: a Fourier Series-Based Plug-in Approach for Bandwidth Selection', *Journal of Nonparametric Statistics*, 34, 377–406.

Tenreiro, C. (2024), 'A Parzen-Rosenblatt Type Density Estimator for Circular Data: Exact and Asymptotic Optimal Bandwidths', *Communications in Statistics - Theory and Methods*, 53, 7436–7452.

Tenreiro, C. (2025), 'A Note on a Parzen–Rosenblatt Type Density Estimator for Circular Data', In *New Frontiers in Statistics and Data Science (SPE2023, Guimarães, Portugal, October 11-14)* Henriques-Rodrigues, L. et al. (eds), Springer Proceedings in Mathematics & Statistics 469, 1–11.

Tusnády, G. (1973), 'An Asymptotically Efficient Test for Fitting a Distribution', *Periodica Mathematica Hungarica*, 3, 157–165.

van der Vaart, A.W. (2000), *Asymptotic Statistics*, Cambridge: Cambridge University Press.

Watson, G.S. (1961), 'Goodness-of-Fit Tests on a Circle', *Biometrika*, 48, 109–114.