

Equivariant deformation problems and homotopy operators

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Abstract

We use homotopy operators for the L_∞ -algebra associated with an equivariant deformation problem in order to describe a smooth parametrization of the space of structures around a given one. Along the way we give new algebraic and explicit proofs of rigidity and unobstructedness theorems.

Contents

1	Introduction	2
2	Rigidity	6
2.1	Deformations	6
2.2	Rigidity	7
2.3	Homotopy operators and rigidity	8
2.4	Rigidity of Lie algebras and parallel transport	9
3	An introduction to L_∞-algebras	11
3.1	The graded symmetric algebra	12
3.2	Plurilinear maps	12
3.3	L_∞ -algebras	13
3.4	Twisted L_∞ -algebras and the Maurer-Cartan equation	14
4	Deformations of Maurer-Cartan elements and obstructions	15
4.1	Deformations and obstructions	15

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4.2	Formal Maurer-Cartan elements	17
5	Integrability	19
5.1	Stable sections	19
5.2	Equivariant deformation problems and L_∞ -algebras	20
5.3	Homotopy operators and integrability	22
5.4	Integrability of N -strict L_∞ -algebras	23
A	Appendix: Combinatorics	25
A.1	Super-Catalan numbers	25
A.2	Partitions of a number	27
B	Appendix: Higher order derivatives	28
	References	33

1 Introduction

When studying an algebraic or geometric structure (e.g. associative algebras, Lie algebras, complex structures), it is fruitful to understand how it changes under small variations. In other words, the goal is to understand a neighbourhood of a given structure inside the space of such structures up to equivalence - roughly a *moduli space*.

By considering *deformations*, i.e. paths in the space of structures starting at a given one, we can approximate infinitesimally such neighbourhoods by the spaces of tangent vectors to deformations. The infinitesimal consequences of the equations that the structures under observation must satisfy (e.g. associativity, the Jacobi identity, Cauchy-Riemann equations) lead to the construction of a deformation cochain complex, such that tangent vectors to deformations represent deformation cocycles.

This approach has been standard since the classical works on the deformation theories of complex structures by Frölicher–Nijenhuis [FN57] and Kodaira–Spencer [KS58, KS60], of associative algebras by Gerstenhaber [Ger64], and of Lie algebras by Nijenhuis–Richardson [NR64], among many others since then.

All these works found descriptions of appropriate deformation complexes. It was also shown in [Kur62, Ger63, NR64], and stressed in [NR66], that the complexes carried additional structure relevant to describing moduli spaces, in the form of compatible Lie brackets (making the deformation complex into a differential graded Lie algebra). This structure is useful, for example, in

order to understand which deformation cocycles can arise as tangent vectors to actual deformations (e.g. in [Kur62]).

From the early observations of Nijenhuis–Richardson, a guiding principle arose, postulated by Deligne [Del86], Drinfeld [Dri14], and in an equivalent formulation by Schlessinger–Stasheff [SS79]: Any reasonable deformation problem in characteristic zero is controlled by a differentiable graded Lie algebra. This principle has been recently turned into a theorem [Pri10, Lur11], stated as an equivalence between the ∞ -categories of formal moduli problems and of differential graded Lie algebras. Nonetheless, there is no general procedure to explicitly describe the Lie brackets for any deformation cochain complex, although there are different methods available for several classes of examples.

In this paper we are interested in studying deformation problems for which the space of structures can be described as the space $\sigma^{-1}(0)$ of zeros of a section $\sigma \in \Gamma(E)$ of a vector bundle $\pi : E \rightarrow M$. In fact, we will consider the extra structure

$$\begin{array}{ccc} E \curvearrowright G & \xrightarrow{\Phi} & F \\ \sigma \nearrow \downarrow & & \nwarrow \\ M \curvearrowright G & & \end{array}$$

consisting of suitable Lie group actions of G on M and on E , an equivariant section $\sigma \in \Gamma_G(E)$ and a vector bundle map Φ satisfying $\Phi \circ \sigma = 0$. We call this structure an *equivariant deformation problem*.

Consider $\sigma^{-1}(0) \subset M$ as a topological space with the subspace topology. Intuitively, M describes the space of ‘almost’ structures, while $\sigma^{-1}(0)$ describes the space of actual structures. For example, σ being an associator or Jacobiator, leads to $\sigma^{-1}(0)$ being the spaces of associative or Lie algebra structures on a fixed vector space. The action of G identifies equivalent structures; the role of F and Φ will become clear later. We are interested in the following:

Main question: For a given solution $x_0 \in \sigma^{-1}(0)$, is there an open $x_0 \in U \cap \sigma^{-1}(0)$ that admits a smooth structure?

If such an open exists we say that the equivariant deformation problem is *integrable* at x_0 . On the other hand, the structure defining an equivariant deformation problem implies that $G(x_0) \subset \sigma^{-1}(0)$, with $G(x_0)$ the orbit. Given that $G(x_0)$ has a smooth structure, another natural question is whether the orbit is open in $\sigma^{-1}(0)$ around x_0 , i.e. $G(x_0) \cap U = \sigma^{-1}(0) \cap U$. In this case we call x_0 *rigid*. Let \mathfrak{g} be the Lie algebra of G , and consider the sequence

$$\mathfrak{g} \xrightarrow{d_{e m_{x_0}}} T_{x_0} M \xrightarrow{d_{x_0}^v \sigma} E_{x_0} \xrightarrow{\Phi_{x_0}} F_{x_0}, \quad (1.1)$$

with $m_{x_0} : G \rightarrow M$ the action at x_0 and $d_{x_0}^v \sigma$ the vertical derivative. In [CSS14] the authors proved that the exactness of the sequence at $T_{x_0} M$ and E_{x_0} implies rigidity of x_0 , and integrability at x_0 , respectively (see Propositions 4.3 and 4.4). Their proof relies, essentially, on the inverse function theorem: it is a clever concatenation of the constant rank theorem and transversality. In fact, their results can be stated as:

$$\text{infinitesimal integrability/rigidity} \Rightarrow \text{integrability/rigidity}.$$

These results did not yet use the additional algebraic structure that we expect the deformation complex 1.1 to carry. In the PhD thesis [Baa19] the author showed that the Taylor expansions of the structures defining the equivariant deformation problem induce a (curved) L_∞ -structure (V, ℓ) on the sequence 1.1 [Baa19, Theorem 5.2.4]. In fact, the equation

$$\sum_{k \geq 0} \frac{1}{k!} d^k \sigma(0)(v, \dots^k, v) = 0,$$

corresponds to the so called *Maurer-Cartan equation*

$$\sum_{k \geq 0} \frac{1}{k!} \ell_k(v, \dots^k, v) = 0.$$

We call the solutions of this equation *Maurer-Cartan elements*. Accordingly, when σ is analytic around x_0 , we get a local correspondence between $\sigma^{-1}(0)$ and Maurer-Cartan elements [Baa19, Theorem 5.2.5]. Given that the sequence 1.1 corresponds to

$$V_{-1} \xrightarrow{\ell_1} V_0 \xrightarrow{\ell_1} V_1 \xrightarrow{\ell_1} V_2,$$

and the exactness of 1.1 at $T_{x_0} M$ and E_{x_0} amounts to the vanishing of the cohomologies $H^0(V, \ell)$ and $H^1(V, \ell)$, then

$$\begin{aligned} \text{infinitesimal rigidity} &\Leftrightarrow H^0(V, \ell) = 0, \\ \text{infinitesimal integrability} &\Leftrightarrow H^1(V, \ell) = 0. \end{aligned}$$

As long as V is finite dimensional, the vanishing $H^i(V, \ell) = 0$ is equivalent to the existence of linear maps $h_1 : V_i \rightarrow V_{i-1}$ and $h_2 : V_{i+1} \rightarrow V_i$ such that

$$\ell_1 \circ h_1 + h_2 \circ \ell_1 = \text{id}_{V_i}.$$

The linear maps h_1, h_2 are called *homotopy operators in degree i* .

Our main contributions are to show how to construct, from homotopy operators in degrees 0 and 1, smooth structures on $\sigma^{-1}(0)$ around x_0 . For rigidity we get a parametrization of the orbit in terms of homotopy operators in degree 0. Additionally, we give a new proof of a rigidity result for Lie algebras, in terms of the parallel transport of a connection defined in terms of homotopy operators.

Integrability is more subtle. We will use the local correspondence between $\sigma^{-1}(0)$ and Maurer-Cartan elements. In fact, we give a differential geometric proof of the well-known fact that the cohomology $H^1(V, \ell)$ of an L_∞ -algebra controls the existence of formal Maurer-Cartan elements. Indeed, we get a recursive process to construct a formal Maurer-Cartan element $u_t = \sum_{k \geq 0} \frac{u_k}{k!} t^k$, by solving the infinite sequence of cohomological equations

$$\ell_1(u_{k+1}) = -Obs^k(u_0, \dots, u_k),$$

where $Obs^k(u_0, \dots, u_k) \in H^1(V, \ell)$ are appropriate obstruction classes. We use homotopy operators in degree 1 to provide the explicit solutions

$$u_{k+1} = -h_1(Obs^k(u_0, \dots, u_k)).$$

Finally we show that, provided that the L_∞ -algebra is N -strict, u_t converges. Using this construction, we describe a smooth structure on $\sigma^{-1}(0)$ around x_0 . These arguments and constructions, in terms of homotopy operators, can be compared with Remark 4.6 of [Cra04], which says that the perturbation lemma corresponds to the algebraic version of Newton's iteration method. Indeed, we use a homotopy operator, recursively, to construct a formal solution and then we prove its convergence.

Outline of the paper

In section 2 we use homotopy operators to give an alternative, algebraic proof of a rigidity result (Theorem 2.6) for equivariant deformation problems:

Theorem 1. Let $x_0 \in \sigma^{-1}(0)$. If x_0 is infinitesimally rigid then it is rigid.

Additionally, we use homotopy operators for the deformation complex of Lie algebras to construct a connection on a tautological bundle of Lie algebras. Using the associated parallel transport we give a new proof of a rigidity result for Lie algebras (see Theorem 2.14).

Theorem 2. If $H_{CE}^1(\mathfrak{g}, \mathfrak{g}) = 0$ then \mathfrak{g} is rigid.

In section 3 we provide an introduction to L_∞ -algebras and Maurer-Cartan elements.

In section 4 we give an explicit description of the obstruction classes Obs^k , and use them to recursively construct formal Maurer-Cartan elements (see Theorem 4.9):

Theorem 3. Let (V, ℓ) be an L_∞ -algebra such that $H^1(V, \ell) = 0$. Then any infinitesimal deformation can be extended to a formal deformation.

In section 5 we use the explicit description of formal Maurer-Cartan elements obtained before, together with homotopy operators, in order to obtain an integrability result (see Theorem 5.18 and Corollary 5.19), i.e. smooth parametrizations for $\sigma^{-1}(0)$ around a structure x_0 .

Theorem 4. Let $(M \xrightarrow{\sigma} E \xrightarrow{\Phi} F) \curvearrowright G$ be a real analytic deformation problem with (V, ℓ) its associated deformation L_∞ -algebra. If (V, ℓ) is N -strict and $H^1(V, \ell) = 0$, then $\sigma^{-1}(0)$ is smooth around x_0 .

We collect in Appendix A the combinatorial background material used in the proofs; in Appendix B we compute higher order derivatives of the differential of an L_∞ -structure.

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2 Rigidity

2.1 Deformations

Let $\pi_E : E \rightarrow M$ be a vector bundle with an action $E \curvearrowright G$ such that the zero section $0_M \cdot G = 0_M$ is G -invariant. Accordingly, M inherits an action of G . Take a G -equivariant section $\sigma \in \Gamma_G(E)$. We want to study the space of solutions of $\sigma(x) = 0$. Fix $x_0 \in M$ such that $\sigma(x_0) = 0$ and consider $\sigma^{-1}(0) \subset M$ with the subspace topology.

Definition 2.1. A smooth path $x_t : I \rightarrow M$ that starts at x_0 and satisfies $\sigma(x_t) = 0$ for all $t \in I$ is called a *deformation of x_0* .

The *vertical derivative* of σ at x_0 is the linear map $d_{x_0}^v \sigma : T_{x_0} M \rightarrow E_{x_0}$ given by

$$T_{x_0} M \xrightarrow{d_{x_0}^v \sigma} T_{0_{x_0}} E \xrightarrow{\text{can}} T_{x_0} M \oplus E_{x_0} \xrightarrow{pr} E_{x_0},$$

where the isomorphism is canonically induced by the zero section 0_M . The following is a standard result in deformation theory.

Proposition 2.2. *If x_t is a deformation of x_0 then $\partial_{t=0} x_t \in \ker d_{x_0}^v \sigma$.*

In this sense we think of $\ker d_{x_0}^v \sigma$ as the model space for the tangent space “ $T_{x_0} \sigma^{-1}(0)$ ”. In other words, the best situation we can expect is that $\sigma^{-1}(0)$ has locally, around x_0 , a manifold structure modelled on $\ker d_{x_0}^v \sigma$.

2.2 Rigidity

Let $G(x_0)$ be the orbit of x_0 under the action $M \curvearrowright G$. By the G -equivariance of σ we have that $G(x_0) \subset \sigma^{-1}(0)$.

Definition 2.3. A solution x_0 is called *rigid* if there exists an open neighbourhood $x_0 \in U \subset M$ such that $G(x_0) \cap U = \sigma^{-1}(0) \cap U$.

In the next Proposition we show that rigidity, seen as openness of the orbit around x_0 , is equivalent to another notion of rigidity found commonly in the literature, for example in Proposition 4.3 of [CSS14].

Proposition 2.4. *A solution x_0 is rigid if and only if there exists an open set $x_0 \in U \subset M$ and smooth map $h : U \rightarrow G$ such that, for every $y \in \sigma^{-1}(0) \cap U$, we have that $y = x_0 \cdot h(y)$.*

Proof. (\Rightarrow) Given that orbits of Lie group actions are locally embedded, there exists an open neighbourhood $x_0 \in U \subset M$ such that $V' := G(x_0) \cap U$ is an open of $G(x_0)$. Take a relatively compact open set V such that $x_0 \in V \subset \bar{V} \subset V' \subset G(x_0)$. Let G_{x_0} be the isotropy of G at x_0 . The diffeomorphism $\mu : G/G_{x_0} \cong G(x_0)$ gives us a relatively compact open set $W = \mu^{-1}(V)$ of $W' = \mu^{-1}(V')$ such that $[e] \in W \subset \bar{W} \subset W' \subset G/G_{x_0}$. Given that $G_{x_0} \subset G$ is a closed subgroup then $G \rightarrow G/G_{x_0}$ is a G_{x_0} -principal bundle. Shrink W' if necessary and take a section $s \in \Gamma_{W'}(G \rightarrow G/G_{x_0})$. It induces a smooth map $h = s \circ \mu^{-1} : V' \rightarrow G$ such that $y = x_0 \cdot h(y)$. The result follows from extending $h : \bar{V} \rightarrow G$ to a smooth map $h : U \subset M \rightarrow G$. Indeed, shrinking U if necessary, since x_0 is rigid, we can assume that $G(x_0) \cap U = \sigma^{-1}(0) \cap U$. (\Leftarrow) We know that $G(x_0) \subset \sigma^{-1}(0)$. By hypothesis $\sigma^{-1}(0) \cap U \subset G(x_0) \cap U$ and the result follows. \square

Let $m_{x_0} : G \rightarrow M$ be $g \mapsto x_0 \cdot g$. By the G -equivariance of σ , the sequence

$$\mathfrak{g} \xrightarrow{d_e m_{x_0}} T_{x_0} M \xrightarrow{d_{x_0}^v \sigma} E_{x_0}. \quad (2.1)$$

is a cochain complex.

Definition 2.5. The solution x_0 is called *infinitesimally rigid* if the sequence 2.1 is exact at $T_{x_0} M$.

Proposition 4.3 of [CSS14] says the following:

Theorem 2.6. *Let $x_0 \in \sigma^{-1}(0)$. If x_0 is infinitesimally rigid then it is rigid.*

2.3 Homotopy operators and rigidity

Let (V^\bullet, δ) be a cochain complex and $H_\delta^\bullet(V)$ its cohomology.

Definition 2.7. The pair of linear maps $h_{k+1} : V^{k+1} \rightarrow V^k$ and $h_k : V^k \rightarrow V^{k-1}$ are called *homotopy operators in degree k* if

$$\delta \circ h_k + h_{k+1} \circ \delta = Id.$$

The following is a standard result:

Proposition 2.8. *Let V^\bullet be finite dimensional. Then $H_\delta^k(V) = 0$ if and only if there exist homotopy operators in degree k .*

Now we are going to give an alternative proof of Theorem 2.6 using homotopy operators. Even though the idea of the proof is in essence the same, i.e. to use the constant rank theorem, we are going to see how homotopy operators provide an explicit parametrization of the orbit. Moreover, they give a clearer frame on how, and why, the constant rank theorem is used. First of all, since the results we want to prove are local, we suppose $E = M \times V$ for some vector space V and $\sigma : M \rightarrow V$. Hence $d_{x_0}^v \sigma$ corresponds to $d_{x_0} \sigma$.

Alternative proof of Theorem 2.6. By hypothesis the sequence 2.1 is exact and so by Proposition 2.8 there exists homotopy operators $h_2 : V \rightarrow T_{x_0} M$ and $h_1 : T_{x_0} M \rightarrow \mathfrak{g}$ such that

$$d_e m_{x_0} \circ h_1 + h_2 \circ d_{x_0} \sigma = Id. \quad (2.2)$$

Let $\psi_h : \ker d_{x_0} \sigma \rightarrow G(x_0) \subset M$ be given by the composition

$$\ker d_{x_0} \sigma \xrightarrow{h_1} \mathfrak{g} \xrightarrow{\exp} G \xrightarrow{m_{x_0}} G(x_0) \subset M.$$

Equation 2.2 implies that $d_0\psi_h = Id$. Additionally, $\dim G(x_0) = \text{rk } d_e m_{x_0} = \dim \ker d_{x_0}\sigma$ and so ψ_h gives a local parametrization of the orbit around x_0 . Moreover, since ψ_h is an embedding, there exists a complement $T_{x_0}M = \ker d_{x_0}\sigma \oplus C$ and a chart $\varphi : U \subset M \rightarrow \ker d_{x_0}\sigma \oplus C$, with $x_0 \mapsto 0$, and such that

$$y \in G(x_0) \iff \varphi(y) = (v, 0) \in \ker d_{x_0}\sigma \oplus C.$$

Hence x_0 is rigid if and only if $\sigma(v, c) = 0$ implies that $c = 0$. To see this, define $\Phi : \ker d_{x_0}\sigma \oplus C \rightarrow \ker d_{x_0}\sigma \oplus V$ by $(v, c) \mapsto (v, \sigma(v, c))$. Given that $\sigma(v, 0) = 0$ for all v and $\text{rk } d_{x_0}\sigma = \dim C$, then $d_0\Phi$ has maximal constant rank and the result follows. \square

Remark 2.9. With respect to the previous structure, the map $h : U \subset M \rightarrow G$ given in Proposition 2.4 corresponds to

$$h := \exp \circ h_1 \circ (Id - h_2 \circ d_{x_0}\sigma) \circ \varphi.$$

2.4 Rigidity of Lie algebras and parallel transport

Let \mathfrak{g} be a finite dimensional vector space and let $C^k(\mathfrak{g}) := \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$. Consider the action $C^k(\mathfrak{g}) \curvearrowright GL(\mathfrak{g})$ given by

$$(\eta \cdot A)(x_1, \dots, x_k) := A^{-1}\eta(Ax_1, \dots, Ax_k).$$

A *Lie algebra structure* on \mathfrak{g} is an element $\mu \in C^2(\mathfrak{g})$ which is a zero of the Jacobiator $Jac : C^2(\mathfrak{g}) \rightarrow C^3(\mathfrak{g})$, given by

$$Jac(\mu)(x, y, z) = \mu(\mu(x, y), z) + \mu(\mu(y, z), x) + \mu(\mu(z, x), y).$$

Accordingly, Lie algebras can be thought as zeros of the equivariant section

$$\begin{array}{c} C^2(\mathfrak{g}) \times C^3(\mathfrak{g}) \curvearrowright GL(\mathfrak{g}) \\ \begin{array}{c} \nearrow \\ Jac \\ \searrow \end{array} \downarrow \\ C^2(\mathfrak{g}) \curvearrowright GL(\mathfrak{g}) \end{array}$$

We denote the space of Lie algebra structures on \mathfrak{g} by $Lie(\mathfrak{g}) := Jac^{-1}(0)$.

The trivial vector bundle $\tau_{C^2(\mathfrak{g})} := C^2(\mathfrak{g}) \times \mathfrak{g} \rightarrow C^2(\mathfrak{g})$ is equipped with a tautological skew-symmetric bilinear operation $[-, -] : \wedge^2 \tau_{C^2(\mathfrak{g})} \rightarrow \tau_{C^2(\mathfrak{g})}$, given by

$$[-, -] : C^\infty(C^2(\mathfrak{g}), \mathfrak{g}) \times C^\infty(C^2(\mathfrak{g}), \mathfrak{g}) \rightarrow C^\infty(C^2(\mathfrak{g}), \mathfrak{g}),$$

$$[\alpha, \beta](\mu) = \mu(\alpha(\mu), \beta(\mu)).$$

We call the pair $(\tau_{C^2(\mathfrak{g})}, [-, -])$ the *tautological bundle of $C^2(\mathfrak{g})$* . The restriction of $\tau_{C^2(\mathfrak{g})}$ to the subspace $Lie(\mathfrak{g}) \subset C^2(\mathfrak{g})$ is a topological vector subbundle; the restriction of $[-, -]$ makes it into a bundle of Lie algebras.

Definition 2.10. The *tautological bundle of \mathfrak{g}* is the (topological) bundle of Lie algebras $\tau_{\mathfrak{g}} := (Lie(\mathfrak{g}) \times \mathfrak{g} \rightarrow Lie(\mathfrak{g}), [-, -])$.

Now fix a $\mu_0 \in Lie(\mathfrak{g})$ and suppose that we have homotopy operators $h_2^{\mu_0} : C^3(\mathfrak{g}) \rightarrow C^2(\mathfrak{g})$ and $h_1^{\mu_0} : C^2(\mathfrak{g}) \rightarrow C^1(\mathfrak{g})$ for the sequence

$$C^1(\mathfrak{g}) \xrightarrow{d_e m_{\mu_0}} C^2(\mathfrak{g}) \xrightarrow{d_{\mu_0} Jac} C^3(\mathfrak{g}).$$

Denote by $H^1(\mu_0)$ the cohomology at the middle term of this complex. It vanishes, because we have the homotopy operators. Let $\mu_0 \in U \subset C^2(\mathfrak{g})$ be an open set such that the maps $H_2 : U \times C^3(\mathfrak{g}) \rightarrow C^2(\mathfrak{g})$ and $H_1 : U \times C^2(\mathfrak{g}) \rightarrow C^1(\mathfrak{g})$, given by

$$\begin{aligned} H_2(\mu, \cdot) &:= h_2^{\mu_0} \circ (1 - (d_{\mu - \mu_0} Jac) \circ h_2^{\mu_0})^{-1}, \\ H_1(\mu, \cdot) &:= h_1^{\mu_0} \circ (1 - (d_{\mu - \mu_0} m) \circ h_2^{\mu_0})^{-1} \end{aligned}$$

are well-defined. Denote $h_2^\mu = H_2(\mu, \cdot)$ and $h_1^\mu = H_1(\mu, \cdot)$. By Proposition 3.4 of [Cra04], whenever $\mu \in Lie(\mathfrak{g}) \cap U$, we have that h_1^μ and h_2^μ are homotopy operators for

$$C^1(\mathfrak{g}) \xrightarrow{d_e m_\mu} C^2(\mathfrak{g}) \xrightarrow{d_\mu Jac} C^3(\mathfrak{g}). \quad (2.3)$$

Consider the restriction of $\tau_{C^2(\mathfrak{g})}$ to U , the bundle $(U \times \mathfrak{g} \rightarrow U, [-, -])$. Given that $U \subset C^2(\mathfrak{g})$ is an open subset of a vector space, we can identify $\mathfrak{X}(U)$ with $C^\infty(U, C^2(\mathfrak{g}))$. Define the connection $\nabla : \mathfrak{X}(U) \times \Gamma(U \times \mathfrak{g}) \rightarrow \Gamma(U \times \mathfrak{g})$ by

$$\nabla_X \xi := \mathcal{L}_X \xi + H_1(X)(\xi).$$

A direct computation shows the following:

Lemma 2.11. *For every $\alpha, \beta \in \Gamma(U \times \mathfrak{g})$ and $X \in \mathfrak{X}(U)$ we have that*

$$\begin{aligned} \nabla_X [\alpha, \beta] &- [\nabla_X \alpha, \beta] - [\alpha, \nabla_X \beta] = \\ &X(\alpha, \beta) + H_1(X)([\alpha, \beta]) - [H_1(X)(\alpha), \beta] - [\alpha, H_1(X)(\beta)]. \end{aligned} \quad (2.4)$$

Proposition 2.12. *Let $\mu_t : I \rightarrow U \cap Lie(\mathfrak{g})$ be a deformation of μ_0 . Then, for every $\alpha, \beta \in \Gamma(U \times \mathfrak{g})$, we have that*

$$\nabla_{\partial_t \mu_t} [\alpha, \beta] = [\nabla_{\partial_t \mu_t} \alpha, \beta] + [\alpha, \nabla_{\partial_t \mu_t} \beta].$$

Proof. By Proposition 2.2 we know that $\partial_t \gamma(t) \in \ker d_{\gamma(t)} Jac$. The fact that $h_1^{\mu_t}$ is an homotopy operator for the sequence 2.3, with $\mu = \mu_t$, implies that

$$(d_e m_{\mu_t} \circ h_1^{\mu_t})(\partial_t \mu_t) = \partial_t \mu_t.$$

But for $A \in C^1(\mathfrak{g})$ we get

$$d_e m_{\mu_t}(A)(x, y) = \mu_t(A(x), y) + \mu_t(x, A(y)) - A(\mu_t(x, y)),$$

and then

$$(\partial_t \mu_t)(x, y) + h_1^{\mu_t}(\partial_t \mu_t)(\mu_t(x, y)) - \mu_t(h_1^{\mu_t}(\partial_t \mu_t)(x), y) - \mu_t(x, h_1^{\mu_t}(\partial_t \mu_t)(y)) = 0.$$

Letting $x = \alpha(\mu_t)$, $y = \beta(\mu_t)$, $X = \partial_t \mu_t$, and recalling that $[-, -](\mu_t) = \mu_t$, we conclude that the right hand side of 2.4 vanishes and the result follows. \square

The deformation μ_t is called *trivial* if $(\mathfrak{g}, \mu_t) \cong (\mathfrak{g}, \mu_0)$ for all t .

Corollary 2.13. *If $H^1(\mu_0) = 0$ then every (small) deformation μ_t of μ_0 is trivial.*

Proof. $\nabla_{\partial_t \mu_t}$ is a derivation of $[-, -]$ and $[-, -](\mu_t) = \mu_t$. Hence, the parallel transport $P_\mu^{t,0}(\nabla) : \mathfrak{g} \rightarrow \mathfrak{g}$ along μ_t gives a Lie algebra isomorphism between (\mathfrak{g}, μ_0) and (\mathfrak{g}, μ_t) . \square

Indeed, the parallel transport can be thought of as a smooth function $P_\mu(\nabla) : I \rightarrow GL(\mathfrak{g})$ and the Lie algebra isomorphism $(\mathfrak{g}, \mu_0) \cong (\mathfrak{g}, \mu_t)$ is given by the change of coordinates induced by the parallel transport, i.e.

$$\mu_0(x, y) = (P_\mu^{t,0}(\nabla))^{-1} \mu_t(P_\mu^{t,0}(\nabla)(x), P_\mu^{t,0}(\nabla)(y)).$$

In other words

$$\mu_0 = \mu_t \cdot P_\mu^{t,0}(\nabla). \quad (2.5)$$

Theorem 2.14. *If $H^1(\mu_0) = 0$ then μ_0 is rigid.*

Proof. The space of Lie algebra structures $Lie(\mathfrak{g})$ is a quadratic affine algebraic variety. Because of [BCR98, Theorem 9.3.6] we have that it is locally path connected.

This fact, together with 2.5, implies the result. \square

3 An introduction to L_∞ -algebras

In this section we follow chapter 2 of [Baa19].

3.1 The graded symmetric algebra

Let $V = \bigoplus_{i \in \mathbb{Z}} V_i$ be a graded vector space. An element $v \in V$ is called *homogeneous* if $v \in V_k$ for some $k \in \mathbb{Z}$. Its *degree* is denoted by $|v| := k$. The tensor algebra of V can be decomposed by degree and rank via

$$\bigotimes V = \bigoplus_{k \in \mathbb{N}} \bigoplus_{i \in \mathbb{Z}} \bigoplus_{n_1 + \dots + n_k = i} \bigotimes_{j=1}^k V_{n_j},$$

with k the rank and i the degree. Accordingly, $(\bigotimes V, \otimes)$ becomes a graded algebra with respect to the degree. Let \mathcal{I} be the graded ideal generated by

$$\mathcal{I} := \langle u \otimes v - (-1)^{|u||v|} v \otimes u \mid u, v \in V \text{ homogeneous} \rangle.$$

The quotient $S(V) := \bigotimes V / \mathcal{I}$ has the induced algebra structure \odot from \otimes and is called the *graded symmetric algebra of V* . It is useful to introduce a sign rule that takes care of these symmetries.

Definition 3.1. Let $\sigma \in S_k$ be a permutation and $v_1, \dots, v_k \in V$ homogeneous elements. The *Koszul sign rule* is given by

$$\epsilon_\sigma(v_1, \dots, v_k) := \prod_{\substack{i < j \\ \sigma(i) > \sigma(j)}} (-1)^{|v_i||v_j|}.$$

We would write ϵ_σ when there is no risk of confusion. The sign rule is such that

$$v_{\sigma(1)} \odot \dots \odot v_{\sigma(k)} = \epsilon_\sigma v_1 \odot \dots \odot v_k.$$

3.2 Plurilinear maps

Let V and W be two graded vector spaces.

Definition 3.2. A linear map $f : \bigotimes^k V \rightarrow W$ is called *graded symmetric* if, for any two consecutive indices i and $i + 1$, we have that

$$f(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = (-1)^{|v_i||v_{i+1}|} f(v_1, \dots, v_{i+1}, v_i, \dots, v_k).$$

f is called *homogeneous of degree $d \in \mathbb{Z}$* if, for every k -tuple of homogeneous elements $v_1, \dots, v_k \in V$, we have that

$$f(v_1, \dots, v_k) \in W_{|v_1| + \dots + |v_k| + d}.$$

Let $\text{Lin}(S^k(V), W)_d$ be the vector space of graded symmetric k -multilinear maps of degree d .

Definition 3.3. The space of *plurilinear maps of degree d* is the vector space given by

$$\text{Lin}(S(V), W)_d := \prod_{k \in \mathbb{N}} \text{Lin}(S^k(V), W)_d.$$

An element $\ell \in \text{Lin}(S(V), W)_d$ is given by a sequence $(\ell_k)_{k \in \mathbb{N}}$ such that $\ell_k \in \text{Lin}(S^k(V), W)_d$. In particular $\ell_0 : \mathbb{R} \rightarrow V_d$ is identified with its image $\ell_0(1) \in V_d$.

3.3 L_∞ -algebras

Recall that a permutation $\sigma \in S_{p+q}$ is a (p, q) -*unshuffle* if $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$. We denote the *space of (p, q) -unshuffles* by $S_{p,q}$.

Definition 3.4. The map $\text{Jac}_n : \text{Lin}(S(V), V)_d \rightarrow \text{Lin}(S^n(V), V)_{2d}$, given by

$$\begin{aligned} \text{Jac}_n(\ell)(v_1, \dots, v_n) = \\ \sum_{\substack{i+j=n+1 \\ i \leq j}} \sum_{\sigma \in S_{i,j-1}} \epsilon_\sigma \ell_j(\ell_i(v_{\sigma(1)}, \dots, v_{\sigma(i)}), v_{\sigma(i+1)}, \dots, v_{\sigma(n)}), \end{aligned} \quad (3.1)$$

is called the n -*Jacobiator*.

Example 3.5. Let $\ell \in \text{Lin}(S(V), V)_1$ be such that $\ell_k = 0$ for $k \neq 1, 2$. Then:

1. $\text{Jac}_1(\ell) = 0$ if and only if (V, ℓ_1) is a cochain complex.
2. $\text{Jac}_3(\ell) = 0$ if and only if (V, ℓ_2) is a graded Lie algebra.
3. $\text{Jac}_1(\ell) = \text{Jac}_2(\ell) = \text{Jac}_3(\ell) = 0$ if and only if (V, ℓ_1, ℓ_2) is a differential graded Lie algebra.

Definition 3.6. A *curved L_∞ -algebra structure on V* is a plurilinear map of degree one $\ell \in \text{Lin}(S(V), V)_1$ such that $\text{Jac}_n(\ell) = 0$ for all $n \in \mathbb{N}$. Its *curvature* is the element $\ell_0 \in V_1$. An *L_∞ -algebra* is a curved L_∞ -algebra with zero curvature.

Remark 3.7. L_∞ -algebras were introduced in [LS93]. The definition given in this paper is usually encountered as $L_\infty[1]$ -algebra structures. Using the shifted graded vector space $(V[1])_d := V_{d+1}$ and the décalage isomorphism, one can show that the definitions in *loc. cit.* and in this paper are equivalent. A textbook account of the use of L_∞ -algebras in deformation theory can be found in [Man22].

3.4 Twisted L_∞ -algebras and the Maurer-Cartan equation

Let (V, ℓ) be an L_∞ -algebra and $u \in V_0$.

Definition 3.8. The *twisting* of ℓ by u is the graded map $\ell^u \in \text{Lin}(S(V), V)_1$ given by

$$\ell_p^u(v_1, \dots, v_p) := \sum_{k \in \mathbb{N}} \frac{1}{k!} \ell_{p+k}(\odot^k u \odot v_1 \odot \dots \odot v_p).$$

Let $C_\ell = \{u \in V_0 \mid \ell^u \text{ converges}\}$.

Definition 3.9. The *domain of convergence* of a curved L_∞ -algebra (V, ℓ) is the interior of C_ℓ . We denote it by $D_\ell := \text{Int}(C_\ell)$.

The following result is given by Propositions 2.4.3 and 2.4.4 of [Baa19].

Proposition 3.10. For every curved L_∞ -algebra (V, ℓ) , the map

$$\mathfrak{l} : D_\ell \rightarrow \text{Lin}(S(V), V)_1, \quad u \mapsto \ell^u,$$

is real analytic and its derivatives are given by

$$D^k \mathfrak{l}(u)(\odot^k \dot{u})(v_1, \dots, v_p) = \ell_{k+p}^u(\odot^k \dot{u} \odot v_1 \odot \dots \odot v_p).$$

Moreover, $(V, \mathfrak{l}(u))$ is a curved L_∞ -algebra.

Corollary 3.11. For every smooth path $u_t \subset D_\ell$ we have that

$$\begin{aligned} \partial_t \ell_i^{u_t}(\partial_t^{r_1} u_t \odot \dots \odot \partial_t^{r_i} u_t) &= \ell_{i+1}^{u_t}(\partial_t u_t \odot \partial_t^{r_1} u_t \odot \dots \odot \partial_t^{r_i} u_t) \\ &+ \sum_{j=1}^i \ell_i^{u_t}(\partial_t^{r_1} u_t \odot \dots \odot \partial_t^{r_j+1} u_t \odot \dots \odot \partial_t^{r_i} u_t). \end{aligned}$$

Proof. We can write $\ell_i^{u_t}(\partial_t^{r_1} u_t \odot \dots \odot \partial_t^{r_i} u_t)$ as $\mathfrak{l}(u_t)(\partial_t^{r_1} u_t \odot \dots \odot \partial_t^{r_i} u_t)$. Using the chain rule the result follows. \square

The *Maurer-Cartan equation* measures how far the twisted curved L_∞ -algebra ℓ^u is from being an L_∞ -algebra. Explicitly, $MC : D_\ell \subset V_0 \rightarrow V_1$ is defined by $MC(u) := \ell_0^u$.

Definition 3.12. $u \in D_\ell$ is a *Maurer-Cartan element* if $u \in MC^{-1}(0)$. Explicitly

$$MC(u) = \ell_0^u = \sum_{k \in \mathbb{N}} \frac{1}{k!} \ell_k(\odot^k u) = 0.$$

We denote by $MC(V, \ell)$ the set of Maurer-Cartan elements of (V, ℓ) .

4 Deformations of Maurer-Cartan elements and obstructions

4.1 Deformations and obstructions

Let (V, ℓ) be an L_∞ -algebra and let $u_0 \in MC(V, \ell)$.

Definition 4.1. A *deformation* of u_0 is a smooth path $u_t \subset D_\ell$ that starts at u_0 and satisfies $MC(u_t) = 0$.

We want to find obstructions to the existence of deformations u_t of u_0 . Let $u_t \subset D_\ell$ be any smooth path starting at u_0 . Natural necessary conditions for $MC(u_t) = 0$ to hold are the differential consequences of this equation, namely $\partial_{t=0}^k MC(u_t) = 0$. Denote $\partial_{t=0}^k u_t = u_0^k$ and use the notation $\vec{r}_i = k$ of Appendix A.2 for partitions of a number.

Proposition 4.2. *For every $k \in \mathbb{N}$ we have the equation*

$$\partial_{t=0}^k MC(u_t) = \ell_1^{u_0}(u_0^k) + \sum_{i=2}^k \sum_{\vec{r}_i=k} \binom{k}{\vec{r}_i} \frac{1}{i!} \ell_i^{u_0}(u_0^{r_1} \odot \cdots \odot u_0^{r_i}).$$

Proof. Since $MC(u_t) = \ell_0^{u_t}$ then $\partial_t(MC(u_t)) = \ell_1^{u_t}(\partial_t u_t)$. We now make use of Proposition B.1, taking $v_t = \partial_t u_t$, to conclude that

$$\begin{aligned} \partial_{t=0}^k MC(u_t) &= \ell_1^{u_0}(u_0^k) + \\ &\sum_{i=1}^{k-1} \sum_{j=0}^{k-1-i} \sum_{\vec{r}_i=k-1-j} \binom{k-1}{\vec{r}_i} \frac{1}{i!j!} \ell_{i+1}^{u_0}(u_0^{r_1} \odot \cdots \odot u_0^{r_i} \odot u_0^{j+1}). \end{aligned}$$

Reorganizing the terms we get

$$\partial_{t=0}^k MC(u_t) = \ell_1^{u_0}(u_0^k) + \sum_{i=2}^k \sum_{\vec{r}_i=k} \binom{k-1}{\vec{r}_i} \frac{r_i}{(i-1)!} \ell_i^{u_0}(u_0^{r_1} \odot \cdots \odot u_0^{r_i}).$$

Moreover, for every pair of summands r_p and r_q of $r_1 + \cdots + r_i = k$, we have that

$$\sum_{\vec{r}_i=k} \binom{k-1}{\vec{r}_i} \frac{r_p}{(i-1)!} \ell_i^{u_0}(u_0^{r_1} \odot \cdots \odot u_0^{r_i}) = \sum_{\vec{r}_i=k} \binom{k-1}{\vec{r}_i} \frac{r_q}{(i-1)!} \ell_i^{u_0}(u_0^{r_1} \odot \cdots \odot u_0^{r_i}).$$

Thus

$$\begin{aligned}
& i \cdot \sum_{\vec{r}_i=k} \binom{k-1}{\vec{r}_i} \frac{r_i}{(i-1)!} \ell_i^{u_0}(u_0^{r_1} \odot \cdots \odot u_0^{r_i}) \\
&= \sum_{\vec{r}_i=k} \binom{k-1}{\vec{r}_i} \frac{r_1 + \cdots + r_i}{(i-1)!} \ell_i^{u_0}(u_0^{r_1} \odot \cdots \odot u_0^{r_i}) \\
&= \sum_{\vec{r}_i=k} \binom{k}{\vec{r}_i} \frac{1}{(i-1)!} \ell_i^{u_0}(u_0^{r_1} \odot \cdots \odot u_0^{r_i}),
\end{aligned}$$

and the result follows. \square

We will focus on building paths u_t that satisfy this sequence of necessary conditions.

Definition 4.3. A k -deformation of u_0 is a path $u_t \subset D_\ell$ that starts at u_0 and satisfies $\partial_{t=0}^j MC(u_t) = 0$ for all $0 \leq j \leq k$. It is called $(k+1)$ -extendable if there exists a $(k+1)$ -deformation v_t of u_0 such that $v_0^j = u_0^j$ for all $0 \leq j \leq k$.

The problem of $(k+1)$ -extending a k -deformation u_t , i.e. finding an appropriate v_0^{k+1} , amounts to solving the equation

$$0 = \ell_1^{u_0}(v_0^{k+1}) + Obs^k(u_t), \quad (4.1)$$

where the last term of the equation is the following obstruction.

Definition 4.4. Let $k \in \mathbb{N}$. The *obstruction to $(k+1)$ -extendability* of a k -deformation u_t is the element $Obs^k(u_t) \in V_1$ given by

$$Obs^k(u_t) = \sum_{i=2}^{k+1} \sum_{\vec{r}_i=k+1} \binom{k+1}{\vec{r}_i} \frac{1}{i!} \ell_i^{u_0}(u_0^{r_1} \odot \cdots \odot u_0^{r_i}). \quad (4.2)$$

Recall that, by the first Jacobi identity 3.1, $(V, \ell_1^{u_0})$ is a cochain complex. We will now see that the obstructions are cocycles.

Proposition 4.5. If u_t is a k -deformation of u_0 then $\ell_1^{u_0}(Obs^k(u_t)) = 0$. In particular it defines a class in $H^1(V, \ell^{u_0})$.

In order to prove this proposition we make use of the following lemma.

Lemma 4.6. For every $k \in \mathbb{N}$ we have the following equation

$$\begin{aligned}
& \ell_1^{u_t}(\partial_t^k MC(u_t)) \\
&+ \sum_{i=1}^k \sum_{j=0}^{k-i} \sum_{\vec{r}_i=k-j} \binom{k}{\vec{r}_i} \frac{1}{i!j!} \ell_{i+1}^{u_t}(\partial_t^{r_1} u_t \odot \cdots \odot \partial_t^{r_i} u_t \odot \partial_t^j MC(u_t)) = 0.
\end{aligned}$$

Proof. By Proposition B.1, letting $v_t = MC(u_t)$, the left-hand side equals

$$\begin{aligned} \partial_t^k \ell_1^{u_t} MC(u_t) &= \ell_1^{u_t} (\partial_t^k MC(u_t)) + \\ &\quad \sum_{i=1}^k \sum_{j=0}^{k-i} \sum_{\vec{r}_i=k-j} \binom{k}{\vec{r}_i} \frac{1}{i!j!} \ell_{i+1}^{u_t} (\partial_t^{r_1} u_t \odot \cdots \odot \partial_t^{r_i} u_t \odot \partial_t^j MC(u_t)). \end{aligned}$$

By the 0-th Jacobi identity 3.1, $\ell_1^{u_t} MC(u_t) = \ell_1^{u_t} (\ell_0^{u_t}) = 0$. \square

Proof of Proposition 4.5. By hypothesis $\partial_{t=0}^k MC(u_t) = 0$. Using Lemma 4.6, we get that

$$\ell_1^{u_0} (\partial_{t=0}^{k+1} MC(u_t)) = 0.$$

By Proposition 4.2,

$$\partial_{t=0}^{k+1} MC(u_t) = \ell_1^{u_0} (u_0^{k+1}) + Obs^k(u_t), \quad (4.3)$$

and the result follows. \square

We can now recast Equation 4.1 as an equation in cohomology.

Theorem 4.7. *Let $u_0 \in MC(V, \ell)$ be a Maurer-Cartan element and $u_t \subset D_\ell$ a deformation of u_0 . Then*

$$Obs^k(u_t) = 0 \quad \text{in } H^1(V, \ell^{u_0}),$$

for every $k \in \mathbb{N}$.

Proof. If u_t is a deformation of u_0 , it is in particular a k -deformation and $k+1$ -extendable, for all k . The result follows from Proposition 4.5 and Equation 4.1. \square

4.2 Formal Maurer-Cartan elements

Now we consider the formal power series $V[[t]] := \oplus_{i \in \mathbb{Z}} V_i[[t]]$. We can extend the L_∞ -algebra framework to the formal setting by letting all the structure maps be t -linear.

Definition 4.8. A formal Maurer-Cartan element of (V, ℓ) is a Maurer-Cartan element of $(V[[t]], \ell)$.

Let $u[[t]] \in V_0[[t]]$ be $u[[t]] = \sum_{k \geq 0} \frac{u_k}{k!} t^k$. By the Equations 4.3 we know that $u[[t]]$ is a formal Maurer-Cartan element if and only if

$$\ell_1^{u_0} (u_{k+1}) + Obs^k(u[[t]]) = 0, \quad \text{for all } k. \quad (4.4)$$

By definition, $Obs^k(u[[t]])$ depends only on u_0, \dots, u_k . Therefore we can construct a formal Maurer-Cartan element by solving a recursive sequence of cohomological equations.

Theorem 4.9. *Let v_0 be a Maurer-Cartan element of (V, ℓ) . If $H^1(V, \ell^{v_0}) = 0$ then, for every $v_1 \in \ker \ell_1^{v_0}$, there exists a formal Maurer-Cartan element $u[[t]]$ such that $u_0 = v_0$ and $u_1 = v_1$.*

Proof. Let $u_0 = v_0$ and $u_1 = v_1 \in \ker \ell_1^{v_0}$ and suppose we have u_0, \dots, u_k such that 4.4 holds for $1 \leq j \leq k-1$. By Proposition 4.5, letting

$$u^k[[t]] = \sum_{j=0}^k \frac{u_j}{j!} t^j,$$

we have that $Obs^k(u^k[[t]]) \in H^1(V, \ell^{v_0})$. Consequently, there exists u_{k+1} such that

$$\ell_1^{v_0}(u_{k+1}) + Obs^k(u^k[[t]]) = 0.$$

Therefore, we can extend the k -deformation $u^k[[t]]$ to the $(k+1)$ -deformation

$$u^{k+1}[[t]] = \sum_{j=0}^{k+1} \frac{u_j}{j!} t^j.$$

□

Remark 4.10. The same result, when (V, ℓ) is a differential graded Lie algebra, is classic and can be found for example in [DMZ07, Theorem 3.25] (in the context of deformations of associative algebras). Although the result for general (V, ℓ) could then be obtained via rectification (see [Qui69, Appendix B6], or [Hin01, Section 2.2]) that process leads to infinite dimensional differential graded Lie algebras.

We have chosen to offer a construction of the obstructions directly in terms of (V, ℓ) . When V is finite dimensional, the obstructions turn out to be convenient for the study of convergence of formal Maurer-Cartan elements, explicitly realizing them as smooth families of Maurer-Cartan elements. We will focus on that task in the next Section.

5 Integrability

We come back to the set up of Section 2:

$$\begin{array}{ccc} E & \hookrightarrow & G \\ \sigma \downarrow & & \\ M & \hookrightarrow & G \end{array} \quad (5.1)$$

Definition 5.1. We say that the problem 5.1 is *integrable* at x_0 if there exists an open subset $x_0 \in U \subset M$ and a (immersed) submanifold $S_{x_0} \subset M$ such that $S_{x_0} \cap U = \sigma^{-1}(0) \cap U$.

Remark 5.2. If we have integrability at x_0 then $G(x_0) \cap U \subset S_{x_0} \cap U$. Furthermore, by Proposition 2.2

$$\dim G(x_0) \leq \dim S_{x_0} \leq \dim \ker d_{x_0}^v \sigma.$$

Definition 5.3. The problem 5.1 is called *maximally integrable* at x_0 if it is integrable of maximal dimension, i.e. $\dim S_{x_0} = \dim \ker d_{x_0}^v \sigma$.

5.1 Stable sections

In order to motivate the definition of stable sections, and of equivariant deformation problems, we recall one of the main elements of the proof of Proposition [CSS14, Proposition 4.4 (1)].

Integrability is a local question, therefore we consider $E = M \times V$, with V some vector space and $\sigma : M \rightarrow V$. We want $\sigma^{-1}(0)$ to have a manifold structure. Of course, the first thing that comes to mind is to use the regular value theorem. However, being a submersion is very restrictive and we want to relax this hypothesis. An alternative is to use transversality. In fact, picking a complement $V = \text{Im } d_{x_0} \sigma \oplus B$ gives us a manifold structure for $\sigma^{-1}(B)$. How far is $\sigma^{-1}(B)$ from being $\sigma^{-1}(0)$? The following extra structure controls this question.

Definition 5.4. Let $\pi_F : F \rightarrow M$ be a vector bundle and let $\Phi : E \rightarrow F$ be a vector bundle map over the identity. The section $\sigma : M \rightarrow E$ is called *stabilized by* Φ if $\Phi \circ \sigma = 0$.

Indeed, since $\Phi \circ \sigma = 0$, we know that $\sigma(M) \subset \ker \Phi$. Accordingly, the following cochain complex controls how far is $\sigma^{-1}(B)$ from being equal to $\sigma^{-1}(0)$:

$$T_{x_0} M \xrightarrow{d_{x_0}^v \sigma} E_{x_0} \xrightarrow{\Phi_{x_0}} F_{x_0}. \quad (5.2)$$

Definition 5.5. The solution x_0 is called *infinitesimally stable* if the sequence 5.2 is exact at E_{x_0} .

In [CSS14], the authors prove the following.

Proposition 5.6. *Let $x_0 \in \sigma^{-1}(0)$. If x_0 is infinitesimally stable, the problem is maximally integrable at x_0 .*

We will use homotopy operators for a suitable L_∞ -algebra to give a new proof of this result in Corollary 5.19.

Remark 5.7. A section can be stabilized in different ways. For example, one can always take $F \rightarrow M$ an arbitrary vector bundle and $\Phi = 0$. In this case being infinitesimally stable just means that 0 is a regular value of σ .

5.2 Equivariant deformation problems and L_∞ -algebras

Definition 5.8. An (analytic) equivariant deformation problem is given by

$$\begin{array}{ccc} E \curvearrowright G & \xrightarrow{\Phi} & F \\ \sigma \downarrow \wr & \swarrow & \\ M \curvearrowright G & & \end{array}$$

where:

- i. $E \rightarrow M$ is an (analytic) vector bundle whose zero section 0_M is G -invariant.
- ii. $\sigma \in \Gamma_G(E)$ is an (analytic) G -equivariant section.
- iii. $\pi_F : F \rightarrow M$ is an (analytic) vector bundle.
- iv. $\Phi : E \rightarrow F$ is an (analytic) vector bundle map such that $\Phi \circ \sigma = 0$.

We denote an (analytic) deformation problem by $(M \xrightarrow{\sigma} E \xrightarrow{\Phi} F) \curvearrowright G$.

Example 5.9. Lie algebra structures: Let \mathfrak{g} be a finite dimensional vector space. Recall that $C^k(\mathfrak{g}) = \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$ and that $GL(\mathfrak{g})$ acts on $C^k(\mathfrak{g})$ by

$$(\eta \cdot A)(x_1, \dots, x_k) := A^{-1} \eta(Ax_1, \dots, Ax_k).$$

Then

$$\begin{array}{ccc} C^2(\mathfrak{g}) \times C^3(\mathfrak{g}) \curvearrowright GL(\mathfrak{g}) & \xrightarrow{\Phi} & C^2(\mathfrak{g}) \times C^4(\mathfrak{g}) \\ \text{Jac} \downarrow \wr & \swarrow & \\ C^2(\mathfrak{g}) \curvearrowright GL(\mathfrak{g}) & & \end{array}$$

is an analytic equivariant deformation problem with Jac the Jacobiator and $\Phi(\mu, \eta) := \delta_\mu \eta$, where δ_μ is the Chevalley-Eilenberg operator

$$\begin{aligned} \delta_\mu \eta(v_1, \dots, v_4) &= \sum_{i=1}^4 (-1)^{i+1} \mu(v_i, \eta(v_1, \dots, \hat{v}_i, \dots, v_4)) \\ &\quad + \sum_{1 \leq i < j \leq 4} (-1)^{i+j} \eta(\mu(v_i, v_j), v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_4). \end{aligned}$$

The space $Jac^{-1}(0) = Lie(\mathfrak{g})$ is the space of Lie algebra structures on \mathfrak{g} .

Equivariant deformation problems and curved L_∞ -algebras are closely related. To see this, take a chart $\varphi : U \subset M \rightarrow \mathbb{R}^n$, with $x_0 \mapsto 0$, such that the vector bundles $E|_U \cong U \times A$ and $F|_U \cong U \times B$ are trivialized. Consider the graded vector space $V = \mathfrak{g}[-1] \oplus \mathbb{R}^n[0] \oplus A[1] \oplus B[2]$, where $[i]$ indicates the grading. In the PhD thesis [Baa19] the author proved (a more general version) of the following:

Theorem 5.10 ([Baa19], Theorem 5.25). *Let $(M \xrightarrow{\sigma} E \xrightarrow{\Phi} F) \curvearrowright G$ be an analytic equivariant deformation problem and let φ , U and V be as above. Then V admits a curved L_∞ -algebra structure $\ell \in Lin(\bigodot V, V)_1$ such that there exists an open subset $U' \subset U$ for which*

$$\varphi : U' \cap \sigma^{-1}(0) \rightarrow \varphi(U') \cap MC(V, \ell)$$

is a bijection.

Remark 5.11. The curved L_∞ -algebra (V, ℓ) is determined by the Taylor series at x_0 of: the section σ , the actions of G on M and on E , the Lie bracket on \mathfrak{g} , and the vector bundle map Φ [Baa19, Theorem 5.2.4]. Moreover, if we choose different trivializations we get isomorphic curved L_∞ -algebras.

In particular, the Maurer-Cartan equation $MC : \mathbb{R}^n \rightarrow A$ is given by

$$MC(v) = \sum_{k \geq 0} \frac{1}{k!} d_0^k \tilde{\sigma}(v, \cdot^{\cdot^{\cdot^k}}, v),$$

with $\tilde{\sigma} : \mathbb{R}^n \rightarrow A$ the local description of the section. Accordingly

$$\sigma(x_0) = 0 \Leftrightarrow MC(0) = 0 \Leftrightarrow (V, \ell) \text{ is an } L_\infty\text{-algebra.}$$

5.3 Homotopy operators and integrability

Let $(M \xrightarrow{\sigma} E \xrightarrow{\phi} F) \curvearrowright G$ be an analytic deformation problem, with $\sigma(x_0) = 0$, and let (V, ℓ) be the L_∞ -structure associated to it by Theorem 5.10. The chart $\varphi : U \subset M \rightarrow \mathbb{R}^n$ gives an equivalence between integrability of the problem at x_0 and smoothness of an open neighbourhood $0 \in W \cap MC(V, \ell)$ of the Maurer-Cartan elements. We will use the algebraic structure on V to construct an explicit smooth structure on $W \cap MC(V, \ell)$. Indeed, Theorem 4.9 tells us how the cohomology $H^1(V, \ell)$ controls formal Maurer-Cartan elements u_t . Letting

$$u_t = \sum_k \frac{u_k}{k!} t^k,$$

we showed that the coefficients of u_t can be built recursively by solving the infinite sequence of cohomological equations

$$\ell_1(u_{k+1}) + \text{Obs}^k(u[t]) = 0.$$

Now, by Proposition 2.8, the vanishing of $H^1(V, \ell) = 0$ gives us homotopy operators $h_2 : B \rightarrow A$ and $h_1 : A \rightarrow \mathbb{R}^n$. They provide explicit solutions for the coefficients of u_t .

Proposition 5.12. *Let $u_0 = 0$ and $u_1 \in \ker \ell_1$. If $H^1(V, \ell) = 0$ then $u_t = \sum_k \frac{u_k}{k!} t^k \in \mathbb{R}^n[[t]]$, given by*

$$u_{k+1} := -h_1(\text{Obs}^k(u_t)),$$

is a formal Maurer-Cartan element.

Remark 5.13. When (V, ℓ) is a differential graded Lie algebra, a similar recurrence has recently been used to construct formal Maurer-Cartan elements, in the context of perturbative quantum mechanics [LS24].

Our objective now is to use these explicit solutions to prove that u_t converges for small values of t and u_1 . To see this, take a norm $\|\cdot\|$ on \mathbb{R}^n . By the equation 4.2 we get that

$$\frac{\|u_{k+1}\|}{(k+1)!} \leq \|h_1\| \sum_{i=2}^{k+1} \frac{\|\ell_i\|}{i!} \sum_{\vec{r}_{i+1}=k+1} \frac{\|u_{r_1}\|}{(r_1)!} \cdots \frac{\|u_{r_{i+1}}\|}{(r_{i+1})!}. \quad (5.3)$$

Lemma 5.14. *Suppose that there exists $\alpha > 0$ satisfying*

$$\sum_{i=1}^{\infty} \frac{\|\ell_i\|}{i!} \leq \alpha.$$

Then, for every k we have that

$$\frac{\|u_k\|}{k!} \leq \|u_1\|^k (\|h_1\|\alpha)^{k-1} C_k,$$

with

$$C_k = \sum_{i=2}^k \sum_{\vec{r}_i=k} C_{r_1} \cdots C_{r_i}.$$

Proof. If $\|h_1\|\alpha \leq 1$ then the result follows by Inequality 5.3. For $\|h_1\|\alpha \geq 1$, the proof is by strong induction. By Proposition 5.12 we know that

$$u_2 = -h_1 \circ \ell_2(u_1, u_1),$$

and letting $C_1 = C_2 = 1$ the base cases holds. Assume now that the result holds for every $p \leq k-1$. By Inequality 5.3, we conclude that

$$\frac{\|u_k\|}{k!} \leq (\|u_1\| \|h_1\| \alpha)^k \sum_{i=2}^k \left(\frac{1}{\|h_1\| \alpha} \right)^i \sum_{\vec{r}_i=k} C_{r_1} \cdots C_{r_i}.$$

But $\left(\frac{1}{\|h_1\| \alpha} \right)^i \leq \frac{1}{\|h_1\| \alpha}$ for all i and the result follows. \square

5.4 Integrability of N -strict L_∞ -algebras

Definition 5.15. We say that an L_∞ -algebra is N -strict if $\ell_k = 0$ for all $k \geq N$.

For example, any nilpotent L_∞ -algebra is N -strict [Man22, Section 10.5]. Given an N -strict L_∞ -algebra, and any norm, let

$$\alpha_\ell = \sum_{i=1}^N \frac{\|\ell_i\|}{i!}.$$

Theorem 5.16. Let (V, ℓ) be an N -strict L_∞ -algebra such that $H^1(V, \ell) = 0$. Then, for every $u_1 \in \ker \ell_1$ with

$$\|u_1\| < \frac{1}{12\|h_1\|\alpha_\ell},$$

there exists a deformation $u_t : [0, 2) \rightarrow MC(V, \ell) \subset \mathbb{R}^n$ of 0 by Maurer-Cartan elements, such that $\partial_{t=0} u_t = u_1$.

Proof. Let u_t be given as in Proposition 5.12. By Lemma 5.14 we know that

$$\frac{\|u_k\|}{k!} \leq \|u_1\|^k (\|h_1\|_{\alpha_\ell})^{k-1} C_k.$$

Moreover, the numbers C_k are the super-Catalan numbers (see Proposition A.5) and by the asymptotic growth A.6 we know that, for $k \gg 1$

$$\frac{\|u_k\|}{k!} \leq \frac{1}{\|h_1\|_{\alpha_\ell}} (6\|u_1\| \|h_1\|_{\alpha_\ell})^k.$$

Therefore, there exists a constant $M > 0$ such that

$$\|u_t\| \leq M + \frac{1}{\|h_1\|_{\alpha_\ell}} \sum_{k \gg 1} (6\|u_1\| \|h_1\|_{\alpha_\ell})^{k+1},$$

and u_t converges. Finally, since u_t is analytic by construction and the twisting of an L_∞ -algebra is analytic by Proposition 3.10, then $MC(u_t)$ is an analytic map. Thus $MC(u_t) = 0$ if and only if $\partial_{t=0}^k MC(u_t) = 0$ for all k . We conclude that $u_t : [0, 2) \rightarrow MC(V, \ell)$ takes values in the Maurer-Cartan elements. \square

Let

$$B_{h_1, \ell} := \left\{ v \in \ker \ell_1 \mid \|v\| < \frac{1}{12\|h_1\|_{\alpha_\ell}} \right\},$$

and define

$$\begin{aligned} \psi : B_{h_1, \ell} &\rightarrow C^\infty([0, 2), \mathbb{R}^n) \\ v &\mapsto v_t \end{aligned}$$

the map that associates to each cocycle v the path v_t given by Theorem 5.16.

Lemma 5.17. *For every $s \in [0, 1]$ and $v \in B_{h_1, \ell}$ we have that $\psi(sv)(t) = \psi(v)(st)$.*

Proof. Let $v_t = \psi_h(v)(t)$ and $(sv)_t = \psi_h(sv)(t)$. Denote their coefficients by v_k and $(sv)_k$ respectively. We claim that $(sv)_{k+1} = s^{k+1}v_k$. Indeed, by Proposition 5.12 and definition 4.2 it is easy to see that this holds. Consequently

$$\psi_h(v)(st) = \sum_{k \geq 0} v_k(st)^k = \sum_{k \geq 0} (sv)_k t^k = \psi(sv)(t). \quad \square$$

Theorem 5.18. *Let $\Psi : B_{h_1, \ell} \rightarrow MC(V, \ell)$ be given by $\Psi(v) = \psi(v)(1)$. Then Ψ is smooth. In fact, there exists an open subset $0 \in U \subset B_{h_1, \ell}$ such that $\Psi : U \subset \ker \ell_1 \rightarrow MC(V, \ell) \subset \mathbb{R}^n$ is an embedding.*

Proof. By Theorem 5.16 and the previous lemma we can compute

$$\partial_{s=0}\Psi(sv) = \partial_{s=0}\Psi_h(v)(s) = v,$$

and then $d_0\Psi$ exists and is the identity, which implies the result. \square

Corollary 5.19. *Let $(M \xrightarrow{\sigma} E \xrightarrow{\Phi} F) \curvearrowright G$ be an analytic deformation problem with (V, ℓ) the associated L_∞ -structure described by Theorem 5.10. Assume that (V, ℓ) is N -strict. If $H^1(V, \ell) = 0$, the problem is maximally integrable at x_0 . In fact, around x_0 , $\sigma^{-1}(0)$ has the smooth parametrization given by $\varphi^{-1} \circ \Psi : U \subset \ker \ell_1 \rightarrow M$.*

Proof. The map $\varphi^{-1} \circ \Psi : U \subset \ker \ell_1 \rightarrow \sigma^{-1}(0) \subset M$ gives an embedding. The same argument as the one given at the end of the alternative proof of Theorem 2.6 proves that this embedding parametrizes, locally, all the possible zeros of σ . \square

Remark 5.20. In fact, we can also obtain the results of this section when (V, ℓ) satisfies a weaker condition than N -strictness. It is enough that $\ell|_{\odot^k V_0} = 0$ for all $k \geq N$. For example, this condition is satisfied for the L_∞ -algebras controlling simultaneous deformations of associative algebras and their morphisms (see [FMY09, Corollary 6.5]), and of Lie algebras and their morphisms (see [FMY09, Corollary 8.5] and [FZ15, Lemma 2.6]).

A Appendix: Combinatorics

In this appendix we collect some results which are useful in dealing with the coefficients of the higher order derivatives of the Maurer-Cartan equation.

A.1 Super-Catalan numbers

We will now study the sequence of numbers

$$C_k = \sum_{i=2}^k \sum_{r_1+\dots+r_i=k} C_{r_1} \cdots C_{r_i},$$

with initial condition $C_1 = 1$. Let S be a non-empty set.

Definition A.1. A word of length n in S is a sequence $w = w_1 \dots w_n$, with $w_i \in S$. The alphabet generated by S is the free monoid generated by the words

$$Alf(S) = \langle w_1 \dots w_m \mid m \in \mathbb{N} \text{ and } w_j \in S \rangle,$$

with multiplication given by concatenation

$$v \cdot w := v_1 \dots v_n w_1 \dots w_m,$$

and identity given by the empty word.

We call an element $w \in Alf(S)$ a *word* and denote its length by $|w| = n$. If $|w| = 1$ we call it a *letter*.

Definition A.2. A *bracketing* of a word $w \in Alf(S)$ is obtained by expressing w as the product of 2 or more non-empty words $w = u_1 \dots u_k$, unless w is a letter, and then inductively bracketing each u_i , until we get letters.

Example A.3. Let $1\ 2\ 3 \in Alf(\mathbb{N})$. The possible bracketings are given by

$$(1)((2)(3)), ((1)(2))(3) \text{ and } (1)(2)(3).$$

Definition A.4. The *super-Catalan* k -number, denoted by C_k , is given by the number of different ways of bracketing a word $w \in Alf(S)$ with $|w| = k$.

By the previous example $C_3 = 3$. Indeed, the super-Catalan numbers satisfy the following recursive formula:

Proposition A.5. Let C_k be the super-Catalan k -number. Then

$$C_k = \sum_{i=2}^k \sum_{r_1+\dots+r_i=k} C_{r_1} \dots C_{r_i}.$$

Proof. The proof is by strong induction. Clearly the only bracketings of 1 and 1 2 are (1) and (1)(2), i.e. $C_1 = C_2 = 1$. Accordingly

$$C_3 = C_1 C_2 + C_2 C_1 + C_1 C_1 C_1 = 3,$$

and the base case follows. Suppose the result holds for $1 \leq p \leq k-1$ and take the word $w \in Alf(\mathbb{N})$ given by

$$w = 1\ 2 \dots k.$$

Fix $2 \leq i \leq k$ and a partition $r_1 + \dots + r_i = k$. Decompose w by

$$w = u_1 u_2 \dots u_i,$$

with $|u_i| = r_i$. All the possible bracketings of this decomposition are given by the product of all the possible bracketings of each of the factors, i.e. the number of bracketings of this decomposition is $C_{r_1} \dots C_{r_i}$. The number i controls

in how many different words can be decomposed w , i.e. from 2 to k . The partitions $r_1 + \dots + r_i = k$ give all the possible decomposition into i words of a word with k letters. Since this covers all the possible decomposition of w into smaller words, we get a bijection between the number of bracketings and the formula above. \square

The following asymptotic growth can be found at the Online Encyclopedia of Integer Sequences (*OEIS*, A001003, 2025)[OEI].

Proposition A.6. *The super-Catalan numbers have the asymptotic growth*

$$C_k \sim W \frac{(3 + \sqrt{8})^k}{k^{3/2}},$$

$$\text{where } W = \frac{1 + \sqrt{2}}{2^{7/4} \sqrt{\pi}}.$$

A.2 Partitions of a number

Let $k, i \in \mathbb{N}$ with $i < k$.

Definition A.7. An i -partition of k is a sequence $(r_1, \dots, r_i) \in \mathbb{N}^i$ such that $r_1 + \dots + r_i = k$. We denote it by $\vec{r}_i = k$ or \vec{r}_i when there is no risk of confusion. An *ordered i -partition* of k is a partition \vec{r}_i such that $r_1 \leq \dots \leq r_i$.

Definition A.8. The *multinomial coefficient* of k associated with \vec{r}_i is defined to be

$$\binom{k}{\vec{r}_i} := \binom{k}{r_1, \dots, r_i} = \frac{k!}{r_1! \dots r_i!}.$$

The permutation group $\sigma \in S_i$ acts on partitions \vec{r}_i by

$$(\sigma \cdot r)_j = r_{\sigma(j)}.$$

Two partitions that are on the same orbit are called *equivalent*. Let $S_{i, \vec{r}}$ be the isotropy of the S_i action on \vec{r}_i and $S_i(\vec{r})$ its orbit. Denote by $\#$ the cardinality of a set. Then

$$\#S_i(\vec{r}) = \frac{i!}{\#S_{i, \vec{r}}}. \quad (\text{A.1})$$

We want an explicit description for $\#S_i(\vec{r})$. To do so, notice that every partition \vec{r}_i can be decomposed as

$$r_{a_1} + \overset{b_1}{\cdot} \dots + r_{a_1} + \dots + r_{a_s} + \overset{b_s}{\cdot} \dots + r_{a_s} = k, \quad (\text{A.2})$$

with $r_{a_1} < \dots < r_{a_s}$ and $b_1 + \dots + b_s = i$. We call it the *factorization* of \vec{r}_i into repeated factors and denote it by $\vec{r}_{b_1, \dots, b_s}$. The following lemma follows by construction.

Lemma A.9. *Let \vec{r}_i be a i -partition of k and $\vec{r}_{b_1, \dots, b_s}$ its factorization into repeated factors. Then $\#S_{i, \vec{r}} = b_1! \cdots b_s!$.*

By Equation A.1 and Lemma A.9 we know that

$$\#S_i(\vec{r}) = \frac{i!}{b_1! \cdots b_s!}. \quad (\text{A.3})$$

B Appendix: Higher order derivatives

In this Appendix we prove a formula which is used in Proposition 4.2 to find obstructions in terms of higher order derivatives of the Maurer-Cartan equation; it is also used in Proposition 4.5 to prove that those obstructions are cohomological.

Let (V, ℓ) be an L_∞ -algebra with domain of convergence D_ℓ and take smooth paths $u_t, v_t : I \rightarrow D_\ell \subset V_0$.

Proposition B.1. *For every $k \in \mathbb{N}$ the following formula holds*

$$\begin{aligned} & \partial_t^k \ell_1^{u_t}(v_t) \\ &= \ell_1^{u_t}(\partial_t^k v_t) + \sum_{i=1}^k \sum_{j=0}^{k-i} \sum_{\vec{r}_i = k-j} \binom{k}{\vec{r}_i} \frac{1}{i!j!} \ell_{i+1}^{u_t}(\partial_t^{r_1} u_t \odot \cdots \odot \partial_t^{r_i} u_t \odot \partial_t^j v_t). \end{aligned} \quad (\text{B.1})$$

Proof. The proof is by induction. The case $k = 1$ follows from corollary 3.11. Assume that the result is true for k . Then

$$\begin{aligned} & \partial_t^{k+1} \ell_1^{u_t}(v_t) = \partial_t \partial_t^k \ell_1^{u_t}(v_t) \\ &= \partial_t \ell_1^{u_t}(\partial_t^k v_t) + \sum_{i=1}^k \sum_{j=0}^{k-i} \sum_{\vec{r}_i = k-j} \binom{k}{\vec{r}_i} \frac{1}{i!j!} \partial_t \ell_{i+1}^{u_t}(\partial_t^{r_1} u_t \odot \cdots \odot \partial_t^{r_i} u_t \odot \partial_t^j v_t). \end{aligned} \quad (\text{B.2})$$

By Corollary 3.11, and after a careful verification, we have that B.2 can be written as

$$\ell_1^{u_t}(\partial_t^{k+1} v_t) + \sum_{m=1}^{k+1} \sum_{n=0}^{k+1-m} \sum_{\substack{\vec{p}_m = k+1-n \\ p_1 \leq \dots \leq p_m}} C_{\vec{p}_m, n} \ell_{m+1}^{u_t}(\partial_t^{p_1} u_t \odot \cdots \odot \partial_t^{p_m} u_t \odot \partial_t^n v_t),$$

for some coefficients $C_{\vec{p}_m, n}$ to be determined. Notice that in the previous formula we are considering the sum over ordered partitions \vec{p}_m . On the other

hand, by the symmetry of the r_1, \dots, r_i factors on the Equation B.1, we get that, for $k+1$, the Equation B.1 is alternatively given by

$$\begin{aligned} \partial_t^{k+1} \ell_1^{u_t}(v_t) &= \ell_1^{u_t}(\partial_t^{k+1} v_t) \\ &+ \sum_{m=1}^{k+1} \sum_{n=0}^{k+1-m} \sum_{\substack{\vec{p}_m = k+1-n \\ p_1 \leq \dots \leq p_m}} \binom{k+1}{\vec{p}_m} \frac{\#S_m(\vec{p})}{m!n!} \ell_{m+1}^{u_t}(\partial_t^{p_1} u_t \odot \dots \odot \partial_t^{p_m} u_t \odot \partial_t^n v_t), \end{aligned}$$

with $\#S_m(\vec{p})$ given by A.1. Accordingly, for the induction step we need to show that

$$C_{\vec{p}_m, n} = \binom{k+1}{\vec{p}_m} \frac{\#S_m(\vec{p})}{m!n!}.$$

Thus, we need to find which summands of B.2 contribute to each coefficient $C_{\vec{p}_m, n}$ and to determine that contribution. The strategy is the following:

1. Fix $1 \leq m \leq k+1$; $0 \leq n \leq k+1-m$ and an ordered partition $p_1 + \dots + p_m = k+1-n$.
2. Find all possible $1 \leq i \leq k$, $0 \leq j \leq k-i$ and ordered partitions $r_1 + \dots + r_i = k-j$ such that

$$\partial_t \ell_{i+1}^{u_t}(\partial_t^{r_1} u_t \odot \dots \odot \partial_t^{r_i} u_t \odot \partial_t^j v_t) = \ell_{m+1}^{u_t}(\partial_t^{p_1} u_t \odot \dots \odot \partial_t^{p_m} u_t \odot \partial_t^n v_t) + \dots \quad (\text{B.3})$$

Denote by $S(m, n, \vec{p}_m)$ the set of all ordered partitions $\vec{r}_i = k-j$ satisfying B.3.

3. Take $\vec{r}_i \in S(m, n, \vec{p}_m)$. Use the Corollary 3.11 to find the number of times that $\ell_{m+1}^{u_t}(\partial_t^{p_1} u_t \odot \dots \odot \partial_t^{p_m} u_t \odot \partial_t^n v_t)$ appears in the expression $\partial_t \ell_{i+1}^{u_t}(\partial_t^{r_1} u_t \odot \dots \odot \partial_t^{r_i} u_t \odot \partial_t^j v_t)$. Explicitly, if we denote this number by $A_{\vec{r}_i}$, we have that

$$\begin{aligned} \partial_t \ell_{i+1}^{u_t}(\partial_t^{r_1} u_t \odot \dots \odot \partial_t^{r_i} u_t \odot \partial_t^j v_t) \\ = A_{\vec{r}_i} \cdot \ell_{m+1}^{u_t}(\partial_t^{p_1} u_t \odot \dots \odot \partial_t^{p_m} u_t \odot \partial_t^n v_t) + \dots \quad (\text{B.4}) \end{aligned}$$

On the other hand, by the Equation B.2, the coefficient of the partition \vec{r}_i in B.2 is $\frac{1}{i!j!} \cdot \binom{k}{\vec{r}_i}$. Hence, letting $C_{\vec{r}_i, j}$ be the contribution of \vec{r}_i to $C_{\vec{p}_m, n}$, we conclude that

$$C_{\vec{r}_i, j} = \frac{A_{\vec{r}_i}}{i!j!} \cdot \binom{k}{\vec{r}_i}.$$

4. Notice that if $\vec{r}_i \in S(m, n, \vec{p}_m)$ then $\vec{r}_i \cdot \sigma$ also contributes to $C_{\vec{p}_m, n}$ for all $\sigma \in S_i$. Moreover, $A_{\vec{r}_i \cdot \sigma} = A_{\vec{r}_i}$ and the coefficients of the partitions $\vec{r}_i \cdot \sigma$ in B.2 are all equal. Hence, the contribution of all the possible partitions $\vec{r}_i \cdot \sigma$ to $C_{\vec{p}_m, n}$ is given by

$$\#\vec{r}_i := \sum_{\sigma \in S_i} C_{\vec{r}_i \cdot \sigma, j} = \#S_i(\vec{r}) \cdot C_{\vec{r}_i, j}.$$

Accordingly, we need to find $\#S_i(\vec{r})$.

5. Finally, we need to show that

$$\sum_{(i, j, \vec{r}_i) \in S(m, n, \vec{p}_m)} \#\vec{r}_i = C_{\vec{p}_m, n} = \binom{k+1}{\vec{p}_m} \frac{\#S_m(\vec{p})}{m!n!}.$$

Let us now elaborate each of the previous steps:

1. Take m, n, \vec{p}_m and let

$$b_1 p_{a_1} + \dots + b_s p_{a_s} = k + 1 - n,$$

be its factorization into repeated factors (see A.2). Thus, by the Equation A.3, we conclude that

$$\binom{k+1}{\vec{p}_m} \frac{\#S_m(\vec{p})}{m!n!} = \frac{1}{b_1! \dots b_s! n!} \binom{k+1}{\vec{p}_m}. \quad (\text{B.5})$$

2. By the Corollary 3.11, we have four possible situations for $\vec{r}_i = k - j$ satisfying B.3:

- (a) $p_{a_1} = 1$ and \vec{r}_i is obtained by subtracting 1 to p_1 :

$$\vec{r}_{m-1} = (p_2, \dots, p_m),$$

so $(m-1, n, \vec{r}_{m-1}) \in S(m, n, \vec{p}_m)$.

- (b) $p_{a_1} > 1$ and \vec{r}_i is obtained by subtracting 1 to p_1 :

$$\vec{r}_m = (p_1 - 1, p_2, \dots, p_m),$$

so $(m, n, \vec{r}_m) \in S(m, n, \vec{p}_m)$.

- (c) \vec{r}_i is obtained by subtracting 1 to p_{a_d} , for each $2 \leq d \leq s$: define $1 \leq q(d) \leq m$ by

$$q(d) = b_1 + \dots + b_{d-1} + 1.$$

We need to consider all the

$$\vec{r}_m = (p_1, \dots, p_{q(d)} - 1, \dots, p_m),$$

$$\text{so } (m, n, \vec{r}_m) \in S(m, n, \vec{p}_m).$$

(d) $j = n - 1$, forcing

$$\vec{r}_m = \vec{p}_m,$$

$$\text{so } (m, n - 1, \vec{r}_m) \in S(m, n, \vec{p}_m).$$

These four cases cover all possible elements $(i, j, \vec{r}_i) \in S(m, n, \vec{p}_m)$.

3. Recall that $A_{\vec{r}_i}$ is defined by Equation B.4. Using Corollary 3.11 it follows that $A_{\vec{r}_i}$, in each of the previous four cases, is given by:

(a) $A_{\vec{r}_i} = 1$ and then

$$C_{\vec{r}_i, j} = \frac{1}{(m-1)!n!} \binom{k}{p_2, \dots, p_m}.$$

(b) $A_{\vec{r}_i} = 1$ and then

$$C_{\vec{r}_i, j} = \frac{1}{m!n!} \binom{k}{p_1 - 1, p_2, \dots, p_m}.$$

(c) We have two cases:

i) If $p_{a_{q(d)}-1} + 1 = p_{a_{q(d)}}$ then $A_{\vec{r}_i} = b_{q(d)-1} + 1$. Thus

$$C_{\vec{r}_i, j} = \frac{b_{q(d)-1} + 1}{m!n!} \binom{k}{p_1, \dots, p_{a_{q(d)}} - 1, \dots, p_m}.$$

ii) If $p_{a_{q(d)}-1} + 1 < p_{a_{q(d)}}$ then $A_{\vec{r}_i} = 1$. Thus

$$C_{\vec{r}_i, j} = \frac{1}{m!n!} \binom{k}{p_1, \dots, p_{a_{q(d)}} - 1, \dots, p_m}.$$

(d) $A_{\vec{r}_i} = 1$ and then

$$C_{\vec{r}_i, j} = \frac{1}{m!(n-1)!} \binom{k}{p_1, \dots, p_m}.$$

4. Using the equation A.3, with respect to the previous cases, we have that:

(a)

$$\#S_i(\vec{r}) = \frac{(m-1)!}{(b_1-1)!b_2!\cdots b_s!},$$

and then

$$\#\vec{r}_i = \frac{1}{(b_1-1)!b_2!\cdots b_s!n!} \binom{k}{p_2, \dots, p_m}.$$

(b)

$$\#S_i(\vec{r}) = \frac{m!}{(b_1-1)!b_2!\cdots b_s!},$$

and then

$$\#\vec{r}_i = \frac{1}{(b_1-1)!b_2!\cdots b_s!n!} \binom{k}{p_1-1, \dots, p_m}.$$

(c) i)

$$\#S_i(\vec{r}) = \frac{m!}{b_1 \cdots (b_{q(d)-1}+1)!(b_{q(d)}-1)! \cdots b_s!}.$$

ii)

$$\#S_i(\vec{r}) = \frac{m!}{b_1 \cdots (b_{q(d)}-1)! \cdots b_s!}.$$

In both cases we get that

$$\#\vec{r}_i = \frac{1}{b_1! \cdots (b_{q(d)}-1)! \cdots b_s!n!} \binom{k}{p_1, \dots, p_{a_{q(d)}}-1, \dots, p_m}$$

(d)

$$\#S_i(\vec{r}) = \frac{1}{m!(n-1)!} \binom{k}{p_1, \dots, p_m},$$

and then

$$\#\vec{r}_i = \frac{1}{b_1! \cdots b_s!(n-1)!} \binom{k}{p_1, \dots, p_m}.$$

5. We are going to compute

$$\sum_{(i,j,\vec{r}_i) \in S(m,n,\vec{p}_m)} \#\vec{r}_i,$$

where $\vec{p}_m = b_1 p_{a_1} + \dots + b_s p_{a_s} = k + 1 - n$, with $p_{a_1} > 1$ and $n > 0$. The other cases follow by similar computations. By 2. the possible triples $(\vec{r}_i, i, j) \in S(\vec{p}_m, m, n)$ are the cases b), c) and d). Hence

$$\begin{aligned}
& \sum_{(i,j,\vec{r}_i) \in S(m,n,\vec{p}_m)} \# \vec{r}_i = \frac{1}{(b_1 - 1)! b_2! \dots b_s! n!} \binom{k}{p_1 - 1, \dots, p_m} \\
& + \sum_{d=2}^s \frac{1}{b_1! \dots (b_{q(d)} - 1)! \dots b_s! n!} \binom{k}{p_1, \dots, p_{a_{q(d)}} - 1, \dots, p_m} \\
& + \frac{1}{b_1! \dots b_s! (n - 1)!} \binom{k}{p_1, \dots, p_m} \\
& \stackrel{(*)}{=} \frac{1}{b_1! \dots b_s! n!} \left(\sum_{q=1}^m \binom{k}{p_1, \dots, p_q - 1, \dots, p_m} + n \binom{k}{p_1, \dots, p_m} \right) \\
& = \frac{1}{b_1! \dots b_s! n!} \binom{k+1}{p_1, \dots, p_m}.
\end{aligned}$$

Equality (*) is obtained using that

$$\begin{aligned}
b_1 \binom{k}{p_1 - 1, \dots, p_m} &= b_1 \binom{k}{p_{a_1} - 1, \dots, p_{a_1}, \dots, p_{a_s}, \dots, p_{a_s}} \\
&= \sum_{q=1}^{b_1} \binom{k}{p_1, \dots, p_q - 1, \dots, p_m},
\end{aligned}$$

and similar arguments for $b_{q(d)}$. By Equation B.5 it follows that

$$\sum_{(i,j,\vec{r}_i) \in S(m,n,\vec{p}_m)} \# \vec{r}_i = \binom{k+1}{\vec{p}_m} \frac{\# S_m(\vec{p})}{m! n!} = C_{\vec{p}_m, n}. \quad \square$$

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