Arithmetic for octonionic-closed balls

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Abstract

With inspiration given by circular complex interval arithmetic, an algebraic framework for octonionic-closed balls is proposed. More concretely, an addition and two multiplications are defined on octonionic-closed balls and analyzed. Diverse algebraic properties, such as power-associativity, inclusion monotonicity and (sub)distributivity of the multiplications, are explored. Throughout the manuscript, the composition structure of the involved octonion algebra plays a key role in the results. These highlight how interval-like objects behave in the non-associative setting of octonions, giving rise to rich algebraic structures with potential applications in generalized interval analysis.

Keywords. octonionic-closed ball, composition algebra, operation, property, algebraic structure.

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1 Motivation and structure

Long known for producing bounds on computational errors, as highlighted by Geréb and Sándor in [10], interval analysis can be found in applications to: robotics and robust control, [12]; rigorous algebraic computation with real numbers, in the context of ball arithmetic, in Johansson's article [13]; solving ordinary differential equations, in Valença's first work [22] on the subject. One of the fundamental parts of interval analysis is interval arithmetic, which studies, namely, properties of operations on intervals. Within interval analysis, the term "interval" refers originally to a closed interval of real numbers, as in the pioneering thesis [15] of Moore who analised real-valued computational errors. The work of Moore was rapidly extended to complex numbers by Boche, [7], in the form of Cartesian and polar products, and later on by Hansen, [11], who introduced a generalized interval arithmetic.

In a broad sense, the term "interval" refers to a generalized interval. For instance, among others, in the works: [17] of Ohta, Gong and Haneda, on polygon interval arithmetic, it means a polygon in the complex plane; [9] of Gargantini and Henrici, devoted to circular complex interval arithmetic, it refers to a closed ball in $\mathbb{C}$. Recent research related to interval analysis for computation purposes can be found in articles of the journals *Reliable Computing* (previously, *Interval Computations*), [10], and *Granular Computing*, devoted to the computing paradigm of information with the same name, [18]. In addition to published articles, books on interval analysis, in particular containing content on interval arithmetic, were written by: Alefeld and Herzberger, [2]; Petković and Petković, [19]; Jaulin, Kieffer, Didrit and Walter, [12]; Moore, Kearfott and Cloud, [16]; Dawood, [8]; Mayer, [14].

Inspired by circular complex interval arithmetic, an arithmetic for closed balls in $\mathbb{R}^n$ was pursued by Beites, Nicolás and Vitória in [4]. More concretely, the properties of certain operations (an addition, a subtraction, and several multiplications) on closed balls in $\mathbb{R}^n$, some of which related either to the Hadamard product of vectors or to the 2-fold vector cross product when $n \in \{3, 7\}$, were studied. In particular, known results for operations on closed balls in $\mathbb{C}$, which can be identified with $\mathbb{R}^2$, were extended to closed balls in $\mathbb{R}^n$. With the same motivation, in [5], the cited authors considered operations on closed balls in $\mathbb{C}^n$. In the latter reference, the properties of possible multiplications for closed balls in $\mathbb{C}^n$, related to the mentioned products of vectors, were studied. In addition, certain equations involving the defined multiplications were solved.

In the present work, after some preliminaries in section 2, operations on octonionic-closed balls are considered – section 3 –, starting with subsection 3.1 where an addition for these closed balls is examined. In subsections 3.2 and 3.3, properties of two possible multiplications for octonionic-closed balls, both related to the multiplication of the real (non-split) octonion algebra $\mathbb{O} = (\mathbb{R}^8, *)$, are established. In particular, some results are obtained taking advantage of the underlying structure of composition algebra of $\mathbb{O}$. (Anti-)Commutativity, (power-)associativity, existence of neutral element and reciprocal of each element, and also its square root(s), are studied. Inclusion monotonicity – the basis for diverse applications of interval arithmetic, [2] – and the (sub)distributivity of each multiplication relative to the addition are analysed. Furthermore, certain algebraic structures are highlighted.

2 Preliminaries

The present section is devoted to preliminaries on the octonion algebra $\mathbb{O}$, based on [3] and references therein, and also on octonionic-closed balls, adapting definitions in [2] and in [1] for the current context.

Consider the usual real vector space $\mathbb{R}^8$, and its canonical basis denoted by $\{e_0, \ldots, e_7\}$. Let us equip $\mathbb{R}^8$ with the multiplication $*$ given by

$$e_i * e_i = -e_0, i \in \{1, \ldots, 7\},$$

being e_0 the identity, and the Fano plane

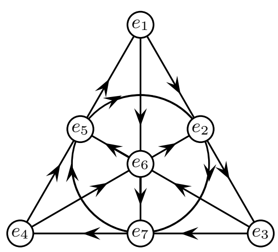

Figure 1: Fano plane for $\mathbb{O}$.

where the cyclic ordering of each three elements lying on the same line is shown by the arrows. Recall that $\mathbb{O} = (\mathbb{R}^8, *)$ is the real (non-split) octonion algebra.

Moreover, $\mathbb{O}$ is a *composition algebra* since it is endowed with a nondegenerate quadratic form (the *norm*) $n : \mathbb{O} \to \mathbb{R}_0^+$ which is *multiplicative*, i.e., for any $x, y \in \mathbb{O}$,

$$n(x * y) = n(x)n(y).$$

The form n being nondegenerate means that the associated symmetric bilinear form

$$n(x, y) = \frac{1}{2}(n(x + y) - n(x) - n(y))$$

is nondegenerate. The latter form can also be written as

$$n(x, y) = \frac{1}{2}(x * \overline{y} + y * \overline{x}),$$

where $^- : x \mapsto \overline{x}$ is the usual involution of $\mathbb{O} = (\mathbb{O}, *)$. The norm $n(x)$ and the trace $t(x)$ of $x \in \mathbb{O}$ are, respectively, given by

$$n(x)e_0 = x * \overline{x} = \overline{x} * x, \; t(x)e_0 = x + \overline{x}.$$

Now consider $\|\cdot\| : \mathbb{O} \to \mathbb{R}_0^+$ given by $x \mapsto \|x\| = \sqrt{n(x)}$.

Definition 2.1. *Let $c \in \mathbb{O}$ and let $r \in \mathbb{R}_0^+$. The closed ball in $\mathbb{O}$, called octonionic-closed ball, with center c and radius r is*

$$\boldsymbol{a} = \langle c; r \rangle = \{x \in \mathbb{O} : \|x - c\| \leq r\}.$$

The set of closed balls in $\mathbb{O}$ is denoted by $\mathcal{B}$, and by $\mathcal{B}^+$ or $\mathcal{B}^0$ if, respectively, $r \in \mathbb{R}^+$ or $r = 0$.

Definition 2.2. *Let $\boldsymbol{a} = \langle c_1; r_1 \rangle, \boldsymbol{b} = \langle c_2; r_2 \rangle \in \mathcal{B}$. The octonionic-closed balls $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal ($\boldsymbol{a} = \boldsymbol{b}$) if set-theoretic equality holds, that is, $c_1 = c_2$ and $r_1 = r_2$; $\boldsymbol{a}$ is contained in $\boldsymbol{b}$ ($\boldsymbol{a} \subseteq \boldsymbol{b}$) if set-theoretic inclusion is valid.*

Definition 2.3. *Let $*_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be a binary operation. The operation $*_{\mathcal{B}}$ is inclusion monotonic if, for all $\boldsymbol{a}_m, \boldsymbol{b}_m \in \mathcal{B}$ such that $\boldsymbol{a}_m \subseteq \boldsymbol{b}_m$, $m \in \{1, 2\}$, $\boldsymbol{a}_1 *_{\mathcal{B}} \boldsymbol{a}_2 \subseteq \boldsymbol{b}_1 *_{\mathcal{B}} \boldsymbol{b}_2$.*

Definition 2.4. *Let $*_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be a binary operation. The operation $*_{\mathcal{B}}$ is power-associative if, for all $\boldsymbol{a} \in \mathcal{B}$ and for all $m, s \in \mathbb{N}$, $\boldsymbol{a}^s *_{\mathcal{B}} \boldsymbol{a}^m = \boldsymbol{a}^{s+m}$.*

3 Operations

The present section is devoted to operations on octonionic-closed balls, and their properties. We start with an auxiliary result for some of the following subsections.

Lemma 3.1. *Let $\boldsymbol{a} = \langle c_1; r_1 \rangle, \boldsymbol{b} = \langle c_2; r_2 \rangle \in \mathcal{B}$. Then $\boldsymbol{a} \subseteq \boldsymbol{b}$ if and only if $\|c_1 - c_2\| \leq r_2 - r_1$. In particular, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are concentric then $\boldsymbol{a} \subseteq \boldsymbol{b}$ if and only if $r_1 \leq r_2$.*

Proof. ($\Rightarrow$) Suppose that $\boldsymbol{a} \subseteq \boldsymbol{b}$. Assume that $\|c_1 - c_2\| = \sqrt{n(c_1 - c_2)} > r_2 - r_1$. Consider the line passing through c_1 and c_2. This line intersects the border of $\boldsymbol{a}$ at a point x such that $\|x - c_2\| = \|c_1 - c_2\| + \|x - c_1\| > r_2 - r_1 + r_1 = r_2$, which leads to the contradiction $x \notin \boldsymbol{b}$.

($\Leftarrow$) Let $x \in \boldsymbol{a}$. Then $\|x - c_1\| = \sqrt{n(x - c_1)} \leq r_1$. Hence, $x \in \boldsymbol{b}$ since

$$\|x - c_2\| = \sqrt{n(x - c_1 + c_1 - c_2)} \leq \sqrt{n(x - c_1)} + \sqrt{n(c_1 - c_2)} \leq r_2.$$

The particular result for concentric balls is now immediate. □

3.1 Addition

In the current subsection, results related to properties of the operation $+_{\mathcal{B}}$ are established.

Definition 3.2. *The binary operation $+_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, hereinafter called addition $+_{\mathcal{B}}$, is given by*

$$\boldsymbol{a} +_{\mathcal{B}} \boldsymbol{b} = \langle c_1; r_1 \rangle +_{\mathcal{B}} \langle c_2; r_2 \rangle := \langle c_1 + c_2; r_1 + r_2 \rangle.$$

Theorem 3.3. *The addition $+_{\mathcal{B}}$ is commutative and associative. Moreover, $\langle 0; 0 \rangle$ is the neutral element relative to $+_{\mathcal{B}}$.*

Proof. Owing to the commutativity and to the associativity of the addition in $\mathbb{O}$, as well as to the commutativity and to the associativity of the addition in $\mathbb{R}$, it is straightforward to prove that, for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{B}$, $\boldsymbol{a} +_{\mathcal{B}} \boldsymbol{b} = \boldsymbol{b} +_{\mathcal{B}} \boldsymbol{a}$ and $(\boldsymbol{a} +_{\mathcal{B}} \boldsymbol{b}) +_{\mathcal{B}} \boldsymbol{c} = \boldsymbol{a} +_{\mathcal{B}} (\boldsymbol{b} +_{\mathcal{B}} \boldsymbol{c})$. Taking into account the neutral elements of $\mathbb{O}$ and $\mathbb{R}$ relative to the respective additions, it is also direct to prove that $\langle 0; 0 \rangle$ is the neutral element relative to $+_{\mathcal{B}}$. □

Corollary 3.4. *$(\mathcal{B}, +_{\mathcal{B}})$ is a commutative monoid.*

Proof. A direct consequence of Theorem 3.3. □

Corollary 3.5. *The set of elements of $\mathcal{B}$ which possess reciprocal relative to the addition $+_{\mathcal{B}}$ is $\mathcal{B}^0$. Furthermore, the reciprocal of $\boldsymbol{a} = \langle c_1; 0 \rangle \in \mathcal{B}^0$ relative to $+_{\mathcal{B}}$ is $\langle -c_1; 0 \rangle$.*

Proof. Let $\boldsymbol{e} = \langle 0; 0 \rangle$. Let $\boldsymbol{a} = \langle c_1; r_1 \rangle \in \mathcal{B}$. Suppose that $\boldsymbol{a'} = \langle c_1'; r_1' \rangle \in \mathcal{B}$ is the reciprocal of $\boldsymbol{a}$ relative to $+_{\mathcal{B}}$. From $\boldsymbol{a} +_{\mathcal{B}} \boldsymbol{a'} = \boldsymbol{e}$, we have $c_1' = -c_1$ and $r_1' = -r_1$. Thus, $r_1 = 0$. □

Lemma 3.6. *Let $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{B}$. Then $\boldsymbol{a} +_{\mathcal{B}} \boldsymbol{b} = \{x + y : x \in \boldsymbol{a} \wedge y \in \boldsymbol{b}\}$.*

Proof. Let $\boldsymbol{a} = \langle c_1; r_1 \rangle$, $\boldsymbol{b} = \langle c_2; r_2 \rangle \in \mathcal{B}$.

($\subseteq$) Let $u \in \boldsymbol{a} +_{\mathcal{B}} \boldsymbol{b} = \langle c_1 + c_2; r_1 + r_2 \rangle$. Then $\|u - (c_1 + c_2)\| = \sqrt{n(u - (c_1 + c_2))} \leq r_1 + r_2$. If $r_1 + r_2 = 0$ then the inclusion holds since $u = c_1 + c_2$. If $r_1 + r_2 \neq 0$ then the inclusion also holds since $u = v + (u - v)$ with $v = \alpha u + (1 - \alpha)(c_1 + c_2) - c_2 \in \boldsymbol{a}$, $\alpha = \frac{r_1}{r_1 + r_2}$, and $u - v \in \boldsymbol{b}$. In fact,

$$\begin{aligned} \|v - c_1\| &= \sqrt{n(\alpha(u - c_1 - c_2))} \\ &= \sqrt{\alpha^2 n(u - c_1 - c_2)} \\ &= \alpha \sqrt{n(u - c_1 - c_2)} \\ &= \alpha \|u - (c_1 + c_2)\| \\ &\leq r_1 \end{aligned}$$

and $\|u - v - c_2\| = (1-\alpha)\|u - (c_1 + c_2)\| \leq r_2$.

($\supseteq$) Let $x \in \boldsymbol{a}$ and $y \in \boldsymbol{b}$. Then $\|x - c_1\| = \sqrt{n(x - c_1)} \leq r_1$ and $\|y - c_2\| = \sqrt{n(y - c_2)} \leq r_2$. Observe that

$$\begin{aligned} n(x - c_1 + y - c_2) &= n(x - c_1) + n(y - c_2) + 2n(x - c_1, y - c_2) \\ &\leq n(x - c_1) + n(y - c_2) + 2|n(x - c_1, y - c_2)| \\ &\leq n(x - c_1) + n(y - c_2) + 2\sqrt{n(x - c_1)}\sqrt{n(y - c_2)} \\ &= (\sqrt{n(x - c_1)} + \sqrt{n(y - c_2)})^2, \end{aligned}$$

which implies

$$\begin{aligned} \|x + y - (c_1 + c_2)\| &= \sqrt{n(x - c_1 + y - c_2)} \\ &\leq \sqrt{n(x - c_1)} + \sqrt{n(y - c_2)} \\ &\leq r_1 + r_2. \end{aligned}$$

Therefore, $x + y \in \boldsymbol{a} +_{\mathcal{B}} \boldsymbol{b} = \langle c_1 + c_2; r_1 + r_2 \rangle$. □

Theorem 3.7. *The addition $+_{\mathcal{B}}$ is inclusion monotonic.*

Proof. Let $\boldsymbol{a}_m, \boldsymbol{b}_m \in \mathcal{B}$ such that $\boldsymbol{a}_m \subseteq \boldsymbol{b}_m$, $m \in \{1, 2\}$. By Lemma 3.6, $\boldsymbol{a}_1 +_{\mathcal{B}} \boldsymbol{a}_2 = \{x + y : x \in \boldsymbol{a}_1 \wedge y \in \boldsymbol{a}_2\} \subseteq \{x + y : x \in \boldsymbol{b}_1 \wedge y \in \boldsymbol{b}_2\} = \boldsymbol{b}_1 +_{\mathcal{B}} \boldsymbol{b}_2$. □

3.2 Multiplication $*_{\mathcal{B},r}$

In the current subsection, results related to properties of the operation $*_{\mathcal{B},r}$ are established.

Definition 3.8. *The binary operation $*_{\mathcal{B},r} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, hereinafter called multiplication $*_{\mathcal{B},r}$, is given by*

$$\boldsymbol{a} *_{\mathcal{B},r} \boldsymbol{b} = \langle c_1; r_1 \rangle *_{\mathcal{B},r} \langle c_2; r_2 \rangle := \langle c_1 * c_2 + r_1 c_2 + r_2 c_1; r_1 r_2 \rangle.$$

Despite the fact that commutativity, anti-commutativity and associativity do not hold, $*_{\mathcal{B},r}$ satisfies the subsequent properties.

Theorem 3.9. *The neutral element relative to $*_{\mathcal{B},r}$ is $\langle 0; 1 \rangle$.*

Proof. Let $\boldsymbol{a} = \langle c; r \rangle \in \mathcal{B}$. Then we get

$$\langle c; r \rangle *_{\mathcal{B},r} \langle 0; 1 \rangle = \langle c; r \rangle = \langle 0; 1 \rangle *_{\mathcal{B},r} \langle c; r \rangle.$$

Assuming that $\langle u; s \rangle$ is another neutral element relative to $*_{\mathcal{B},r}$, it is straightforward that $\langle u; s \rangle = \langle u; s \rangle *_{\mathcal{B},r} \langle 0; 1 \rangle = \langle 0; 1 \rangle$. □

Corollary 3.10. $(\mathcal{B}, *_{\mathcal{B},r})$ *is a unital magma.*

Proof. A straightforward consequence of Theorem 3.9. $\square$

Corollary 3.11. *The set of elements of $\mathcal{B}$ which possess reciprocal relative to the multiplication $*_{\mathcal{B},r}$ is $\{\langle c; r\rangle \in \mathcal{B}^+ : c + re_0 \neq 0\}$. Furthermore, the reciprocal of $\langle c; r\rangle \in \mathcal{B}^+$ relative to $*_{\mathcal{B},r}$ is $\langle -r^{-1}(c+re_0)^{-1} * c; r^{-1}\rangle$.*

Proof. Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. Suppose that $\boldsymbol{a}' = \langle u; \alpha\rangle \in \mathcal{B}$ is the right reciprocal of $\boldsymbol{a}$ relative to $*_{\mathcal{B},r}$. From $\langle c; r\rangle *_{\mathcal{B},r} \langle u; \alpha\rangle = \langle 0; 1\rangle$ we get $\alpha = r^{-1}$ whenever $r \neq 0$ and, demanding $c + re_0 \neq 0$,

$$c * u + \alpha c + ru = 0 \Leftrightarrow (c + re_0) * u = -\alpha c \Leftrightarrow u = -\alpha(c + re_0)^{-1} * c.$$

Now suppose that $\boldsymbol{a}'' = \langle v; \beta\rangle \in \mathcal{B}$ is the left reciprocal of $\boldsymbol{a}$ relative to $*_{\mathcal{B},r}$. From $\langle v; \beta\rangle *_{\mathcal{B},r} \langle c; r\rangle = \langle 0; 1\rangle$, through a similar reasoning, we arrive at the left reciprocal $\langle -r^{-1}c * (c + re_0)^{-1}; r^{-1}\rangle$ of $\boldsymbol{a}$ relative to $*_{\mathcal{B},r}$. Finally, as $(c + re_0)^{-1} = \frac{\bar{c}+re_0}{n(c+re_0)}$, observe that $(c + re_0)^{-1} * c = c * (c + re_0)^{-1}$. $\square$

Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. We define the powers of $\boldsymbol{a} \neq \langle 0; 0\rangle$ relative to $*_{\mathcal{B},r}$ by

$$\boldsymbol{a}^0 = \langle 0; 1\rangle \text{ and } \boldsymbol{a}^k = \boldsymbol{a}^{k-1} *_{\mathcal{B},r} \boldsymbol{a} \text{ for } k \in \mathbb{N}.$$

Denote e_0 by c^{*0} and $c^{*(k-1)} * c$ by c^{*k} for $k \in \mathbb{N}$. If $\boldsymbol{a} = \langle 0; 0\rangle$ then, for all $k \in \mathbb{N}$, $\boldsymbol{a}^k = \langle 0; 0\rangle$.

Theorem 3.12. *The multiplication $*_{\mathcal{B},r}$ is power-associative.*

Proof. Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. On the one hand, we have

$$\begin{aligned}
\boldsymbol{a}^2 *_{\mathcal{B},r} \boldsymbol{a} &= \langle c * c + 2rc; r^2\rangle *_{\mathcal{B},r} \langle c; r\rangle \\
&= \langle (c * c) * c + 3rc * c + 3r^2c; r^3\rangle \\
&= \langle c * (c * c) + 3rc * c + 3r^2c; r^3\rangle \\
&= \langle c; r\rangle *_{\mathcal{B},r} \langle c * c + 2rc; r^2\rangle \\
&= \boldsymbol{a} *_{\mathcal{B},r} \boldsymbol{a}^2.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
(\boldsymbol{a}^2 *_{\mathcal{B},r} \boldsymbol{a}) *_{\mathcal{B},r} \boldsymbol{a} &= \langle (c * c) * c + 3rc * c + 3r^2c; r^3\rangle *_{\mathcal{B},r} \langle c; r\rangle \\
&= \langle ((c * c) * c) * c + 4r(c * c) * c + 6r^2c * c + 4r^3c; r^4\rangle
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{a}^2 *_{\mathcal{B},r} \boldsymbol{a}^2 &= \langle c * c + 2rc; r^2\rangle *_{\mathcal{B},r} \langle c * c + 2rc; r^2\rangle \\
&= \langle (c * c) * (c * c) + 4r(c * c) * c + 6r^2c * c + 4r^3c; r^4\rangle.
\end{aligned}$$

As $\boldsymbol{a}^2 *_{\mathcal{B},r} \boldsymbol{a} = \boldsymbol{a} *_{\mathcal{B},r} \boldsymbol{a}^2$ and $(\boldsymbol{a}^2 *_{\mathcal{B},r} \boldsymbol{a}) *_{\mathcal{B},r} \boldsymbol{a} = \boldsymbol{a}^2 *_{\mathcal{B},r} \boldsymbol{a}^2$, invoking [1], the result follows. $\square$

Theorem 3.13. *Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. Relative to the multiplication $*_{\mathcal{B},r}$, for all $k \in \mathbb{N}$, $\boldsymbol{a}^k = \langle \sum_{j=1}^{k} \binom{k}{j-1} r^{j-1} c^{*(k+1-j)}; r^k\rangle$.*

Proof. Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. We use induction on k. The equality obviously holds for $k = 1$. Suppose that it is true for k. Then we have

$$\begin{aligned}
\boldsymbol{a}^{k+1} &= \boldsymbol{a}^k *_{\mathcal{B},r} \boldsymbol{a} \\
&= \langle \textstyle\sum_{j=1}^{k} \binom{k}{j-1} r^{j-1} c^{*(k+1-j)}; r^k\rangle *_{\mathcal{B},r} \langle c; r\rangle \\
&= \langle \textstyle\sum_{j=1}^{k} \binom{k}{j-1} r^{j-1} c^{*(k+2-j)} + \sum_{j=1}^{k} \binom{k}{j-1} r^{j} c^{*(k+1-j)} + r^k c; r^{k+1}\rangle \\
&= \langle \textstyle\sum_{j=1}^{k} \binom{k}{j-1} r^{j-1} c^{*(k+2-j)} + \sum_{j=1}^{k-1} \binom{k}{j-1} r^{j} c^{*(k+1-j)} + (k+1) r^k c; r^{k+1}\rangle \\
&= \langle c^{*(k+1)} + \textstyle\sum_{j=2}^{k} (\binom{k}{j-1} + \binom{k}{j-2}) r^{j-1} c^{*(k+2-j)} + (k+1) r^k c; r^{k+1}\rangle \\
&= \langle c^{*(k+1)} + \textstyle\sum_{j=2}^{k} \binom{k+1}{j-1} r^{j-1} c^{*(k+2-j)} + (k+1) r^k c; r^{k+1}\rangle \\
&= \langle \textstyle\sum_{j=1}^{k} \binom{k+1}{j-1} r^{j-1} c^{*(k+2-j)} + \sum_{j=k+1}^{k+1} \binom{k+1}{j-1} r^{j-1} c^{*(k+2-j)}; r^{k+1}\rangle \\
&= \langle \textstyle\sum_{j=1}^{k+1} \binom{k+1}{j-1} r^{j-1} c^{*(k+2-j)}; r^{k+1}\rangle.
\end{aligned}$$

□

Theorem 3.14. *Let $\boldsymbol{a} = \langle 0; r\rangle \in \mathcal{B}$. The square roots of $\boldsymbol{a}$ relative to the multiplication $*_{\mathcal{B},r}$ are given by $\boldsymbol{a}^{1/2} = \langle -2\sqrt{r} e_0; \sqrt{r}\rangle$.*

Proof. Let $\boldsymbol{a} = \langle 0; r\rangle \in \mathcal{B}$. Let $\boldsymbol{b} = \langle v; s\rangle \in \mathcal{B}$ such that $\boldsymbol{a} = \boldsymbol{b}^2$. As $\langle 0; r\rangle = \langle v * v + 2sv; s^2\rangle$, we have

$$s^2 = r \text{ and } v * v + 2sv = 0.$$

Thus, $s = \sqrt{r}$. It is clear that $v = 0$ is a solution of $v * v + 2sv = 0$. When $v \neq 0$, as any two elements of $\mathbb{O}$ generate an associative subalgebra, $v * v + 2sv = 0$ leads to $v * (v * v^{-1}) + 2sv * v^{-1} = 0$, that is, $v = -2se_0$. □

Theorem 3.15. *The multiplication $*_{\mathcal{B},r}$ is not inclusion monotonic.*

Proof. Let $\boldsymbol{a}_1 = \langle e_1; 1\rangle$, $\boldsymbol{a}_2 = \langle e_3; 1\rangle$, $\boldsymbol{b}_1 = \langle e_2; 3\rangle$, $\boldsymbol{b}_2 = \langle e_3; 1\rangle \in \mathcal{B}$. As $\|e_1 - e_2\| = \sqrt{n(e_1 - e_2)} = \sqrt{2} \leq 2$ and $\|e_3 - e_3\| = \sqrt{n(e_3 - e_3)} = 0 \leq 0$ then, by Lemma 3.1, $\boldsymbol{a}_m \subseteq \boldsymbol{b}_m$, $m \in \{1, 2\}$. However, as $\| -2e_2 - 2e_3\| = \sqrt{n(-2e_2 - 2e_3)} = 2\sqrt{2} \not\leq 2$, then, again by Lemma 3.1,

$$\boldsymbol{a}_1 *_{\mathcal{B},r} \boldsymbol{a}_2 = \langle e_1 - e_2 + e_3; 1\rangle \not\subseteq \boldsymbol{b}_1 *_{\mathcal{B},r} \boldsymbol{b}_2 = \langle e_1 + e_2 + 3e_3; 3\rangle.$$

□

Theorem 3.16. *The multiplication $*_{\mathcal{B},r}$ is distributive with respect to the addition $+_{\mathcal{B}}$.*

Proof. Owing to the distributivity of $*$ with respect to the addition in $\mathbb{O}$, and to the distributivity of the multiplication with respect to the addition in $\mathbb{R}$, it is straightforward to prove that, for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{B}$, $\boldsymbol{a} *_{\mathcal{B},r} (\boldsymbol{b} +_{\mathcal{B}} \boldsymbol{c}) = (\boldsymbol{a} *_{\mathcal{B},r} \boldsymbol{b}) +_{\mathcal{B}} (\boldsymbol{a} *_{\mathcal{B},r} \boldsymbol{c})$ and $(\boldsymbol{b} +_{\mathcal{B}} \boldsymbol{c}) *_{\mathcal{B},r} \boldsymbol{a} = (\boldsymbol{b} *_{\mathcal{B},r} \boldsymbol{a}) +_{\mathcal{B}} (\boldsymbol{c} *_{\mathcal{B},r} \boldsymbol{a})$. $\square$

Corollary 3.17. *$(\mathcal{B}, +_{\mathcal{B}}, *_{\mathcal{B},r})$ is a ringoid.*

Proof. A straightforward consequence of Theorem 3.16. $\square$

3.3 Multiplication $*_{\mathcal{B},c}$

In the current subsection, results related to properties of the operation $*_{\mathcal{B},c}$ are established.

Definition 3.18. *The binary operation $*_{\mathcal{B},c} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, hereinafter called multiplication $*_{\mathcal{B},c}$, is given by*

$$\boldsymbol{a} *_{\mathcal{B},c} \boldsymbol{b} = \langle c_1; r_1\rangle *_{\mathcal{B},c} \langle c_2; r_2\rangle := \langle c_1 * c_2; r_1\|c_2\| + r_2\|c_1\| + r_1 r_2\rangle.$$

Despite the fact that commutativity, anti-commutativity and associativity do not hold, $*_{\mathcal{B},c}$ satisfies the subsequent properties.

Theorem 3.19. *The neutral element relative to $*_{\mathcal{B},c}$ is $\langle e_0; 0\rangle$.*

Proof. Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. Then we get

$$\langle c; r\rangle *_{\mathcal{B},c} \langle e_0; 0\rangle = \langle c * e_0; r\|e_0\|\rangle = \langle c; r\rangle = \langle e_0; 0\rangle *_{\mathcal{B},c} \langle c; r\rangle.$$

Assuming that $\langle u; s\rangle$ is another neutral element relative to $*_{\mathcal{B},c}$, it is straightforward that $\langle u; s\rangle = \langle u; s\rangle *_{\mathcal{B},c} \langle e_0; 0\rangle = \langle e_0; 0\rangle$. $\square$

Corollary 3.20. *$(\mathcal{B}, *_{\mathcal{B},c})$ is a unital magma.*

Proof. A straightforward consequence of Theorem 3.19. $\square$

Corollary 3.21. *The set of elements of $\mathcal{B}$ which possess reciprocal relative to the multiplication $*_{\mathcal{B},c}$ is $\mathcal{B}^0 \backslash \{\langle 0; 0\rangle\}$. Furthermore, the reciprocal of $\langle c; 0\rangle \in \mathcal{B}^0 \backslash \{\langle 0; 0\rangle\}$ relative to $*_{\mathcal{B},c}$ is $\langle c^{-1}; 0\rangle = \langle \frac{\bar{c}}{\|c\|^2}; 0\rangle$.*

Proof. Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. Suppose that $\boldsymbol{a}' = \langle u; \alpha\rangle \in \mathcal{B}$ is the right reciprocal of $\boldsymbol{a}$ relative to $*_{\mathcal{B},c}$. From $\langle c; r\rangle *_{\mathcal{B},c} \langle u; \alpha\rangle = \langle e_0; 0\rangle$ we get $c*u = e_0$, which implies $c \neq 0$ and $u \neq 0$, and

$$r\|u\| + \alpha\|c\| + r\alpha = 0 \Leftrightarrow r = \alpha = 0.$$

The former equality also implies $u = c^{-1} = \frac{\bar{c}}{n(c)}$. It is straightforward to see that $\langle c^{-1}; 0\rangle$ is also the left reciprocal of $\boldsymbol{a}$ relative to $*_{\mathcal{B},c}$. $\square$

Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. We define the powers of $\boldsymbol{a} \neq \langle 0; 0\rangle$ relative to $*_{\mathcal{B},c}$ by

$$\boldsymbol{a}^0 = \langle e_0; 0\rangle \text{ and } \boldsymbol{a}^k = \boldsymbol{a}^{k-1} *_{\mathcal{B},c} \boldsymbol{a} \text{ for } k \in \mathbb{N}.$$

Denote e_0 by c^{*0} and $c^{*(k-1)} * c$ by c^{*k} for $k \in \mathbb{N}$. If $\boldsymbol{a} = \langle 0; 0\rangle$ then, for all $k \in \mathbb{N}$, $\boldsymbol{a}^k = \langle 0; 0\rangle$.

Theorem 3.22. *The multiplication $*_{\mathcal{B},c}$ is power-associative.*

Proof. Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. On the one hand, we have

$$\begin{aligned}
\boldsymbol{a}^2 *_{\mathcal{B},c} \boldsymbol{a} &= \langle c * c; r^2 + 2r\|c\|\rangle *_{\mathcal{B},c} \langle c; r\rangle \\
&= \langle (c * c) * c; r^3 + 3r^2\|c\| + 3r\|c\|^2\rangle \\
&= \langle c * (c * c); r^3 + 3r^2\|c\| + 3r\|c\|^2\rangle \\
&= \langle c; r\rangle *_{\mathcal{B},c} \langle c * c; r^2 + 2r\|c\|\rangle \\
&= \boldsymbol{a} *_{\mathcal{B},c} \boldsymbol{a}^2.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
(\boldsymbol{a}^2 *_{\mathcal{B},c} \boldsymbol{a}) *_{\mathcal{B},c} \boldsymbol{a} &= \langle (c * c) * c; (r + \|c\|)^3 - \|c\|^3\rangle *_{\mathcal{B},c} \langle c; r\rangle \\
&= \langle ((c * c) * c) * c; (r + \|c\|)^4 - \|c\|^4\rangle
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{a}^2 *_{\mathcal{B},c} \boldsymbol{a}^2 &= \langle c * c; (r + \|c\|)^2 - \|c\|^2\rangle *_{\mathcal{B},c} \langle c * c; (r + \|c\|)^2 - \|c\|^2\rangle \\
&= \langle (c * c) * (c * c); (r + \|c\|)^4 - \|c\|^4\rangle.
\end{aligned}$$

As $\boldsymbol{a}^2 *_{\mathcal{B},c} \boldsymbol{a} = \boldsymbol{a} *_{\mathcal{B},c} \boldsymbol{a}^2$ and $(\boldsymbol{a}^2 *_{\mathcal{B},c} \boldsymbol{a}) *_{\mathcal{B},c} \boldsymbol{a} = \boldsymbol{a}^2 *_{\mathcal{B},c} \boldsymbol{a}^2$, invoking [1], the result follows.

□

Theorem 3.23. *Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. Relative to the multiplication $*_{\mathcal{B},c}$, for all $k \in \mathbb{N}$, $\boldsymbol{a}^k = \langle c^{*k}; (\|c\| + r)^k - \|c\|^k\rangle$.*

Proof. Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. We use induction on k. The equality obviously holds for $k = 1$. Suppose that it is true for k. Then we have

$$\begin{aligned}
\boldsymbol{a}^{k+1} &= \boldsymbol{a}^k *_{\mathcal{B},c} \boldsymbol{a} \\
&= \langle c^{*k}; (r + \|c\|)^k - \|c\|^k\rangle *_{\mathcal{B},c} \langle c; r\rangle \\
&= \langle c^{*k} * c; ((r + \|c\|)^k - \|c\|^k)\|c\| + r\|c\|^k + r((r + \|c\|)^k - \|c\|^k)\rangle \\
&= \langle c^{*(k+1)}; (r + \|c\|)^k(r + \|c\|) - \|c\|^{k+1}\rangle \\
&= \langle c^{*(k+1)}; (r + \|c\|)^{k+1} - \|c\|^{k+1}\rangle.
\end{aligned}$$

□

Theorem 3.24. *Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. The square roots of $\boldsymbol{a}$ relative to the multiplication $*_{\mathcal{B},c}$ are given by*

$$\boldsymbol{a}^{1/2} = \begin{cases} \langle 0; \sqrt{r}\rangle & \textit{if } c = 0; \\ \langle v; \sqrt{\|c\| + r} - \sqrt{\|c\|}\rangle, & \\ \textit{with } \|v\| = \|c\|^{1/2} \textit{ and } t(v) = 0, & \textit{if } c = -\|c\|e_0 \neq 0; \\ \left\langle \pm\frac{c+\|c\|e_0}{\sqrt{t(c)+2\|c\|}}; \sqrt{\|c\| + r} - \sqrt{\|c\|}\right\rangle & \textit{otherwise.} \end{cases}$$

Proof. Let $\boldsymbol{a} = \langle c; r\rangle \in \mathcal{B}$. Let $\boldsymbol{b} = \langle v; s\rangle \in \mathcal{B}$ such that $\boldsymbol{a} = \boldsymbol{b}^2$. Hence,

$$\langle c; r\rangle = \langle v * v; (\|v\| + s)^2 - \|v\|^2\rangle,$$

and we have $c = v * v$ and $r = (\|v\| + s)^2 - \|v\|^2$. From the first equality, we obtain $\|v\|^2 = \|c\|$, which, from the second equality, leads to $s = \sqrt{\|c\| + r} - \sqrt{\|c\|}$. If $c = 0$, then $s = \sqrt{r}$. Now consider $c \neq 0$. As $v \in \mathbb{O}$ is a solution of the quadratic equation $x*x-t(x)x+n(x)e_0 = 0$, we have $t(v)v = c+\|c\|e_0$. If $c = -\|c\|e_0$ then $t(v) = 0$ and since $c = v*v = -v*\overline{v} = -n(v)e_0$, every $v \in \mathbb{O}$ with $\|v\| = \|c\|^{1/2}$ satisfies $\langle v; \sqrt{\|c\| + r} - \sqrt{\|c\|}\rangle^2 = \boldsymbol{a}$. If $c \neq -\|c\|e_0$ then v lies in the linear subspace generated by $c + \|c\|e_0 \neq 0$. So, $v = \alpha(c + \|c\|e_0)$ with $\alpha \in \mathbb{R} \setminus \{0\}$, and we get

$$\|c\| = n(v) = \alpha^2 n(c + \|c\|e_0) = \alpha^2(2\|c\|^2 + 2\|c\|n(c, e_0)).$$

Thus, $\alpha = \pm\frac{1}{\sqrt{t(c)+2\|c\|}}$. □

Theorem 3.25. *The multiplication $*_{\mathcal{B},c}$ is inclusion monotonic.*

Proof. Let $\boldsymbol{a}_m = \langle a_m; r_m\rangle$, $\boldsymbol{b}_m = \langle b_m; s_m\rangle \in \mathcal{B}$ such that $\boldsymbol{a}_m \subseteq \boldsymbol{b}_m$, $m \in \{1, 2\}$. We aim to prove that $\boldsymbol{a}_1 *_{\mathcal{B},c} \boldsymbol{a}_2 \subseteq \boldsymbol{b}_1 *_{\mathcal{B},c} \boldsymbol{b}_2$. From Lemma 3.1, $\|a_m - b_m\| \leq s_m - r_m$, $m \in \{1, 2\}$. We also have

$$\boldsymbol{a}_1 *_{\mathcal{B},c} \boldsymbol{a}_2 = \langle a_1 * a_2; r_2\|a_1\| + r_1\|a_2\| + r_1 r_2\rangle$$

and

$$\boldsymbol{b}_1 *_{\mathcal{B},c} \boldsymbol{b}_2 = \langle b_1 * b_2; s_2\|b_1\| + s_1\|b_2\| + s_1 s_2\rangle.$$

As

$$\begin{aligned} &\|a_1 * a_2 - b_1 * b_2\| \\ &= \|(a_1 - b_1) * b_2 + b_1 * (a_2 - b_2) + (a_1 - b_1) * (a_2 - b_2)\| \\ &\leq \|a_1 - b_1\|\|b_2\| + \|b_1\|\|a_2 - b_2\| + \|a_1 - b_1\|\|a_2 - b_2\| \\ &\leq (s_1 - r_1)\|b_2\| + \|b_1\|(s_2 - r_2) + (s_1 - r_1)(s_2 - r_2) \end{aligned}$$

and

$$-\|b_m\| \leq -\|a_m\| + \|a_m - b_m\| \leq -\|a_m\| + s_m - r_m, m \in \{1, 2\},$$

we obtain $\|a_1 * a_2 - b_1 * b_2\| \leq \beta - \alpha$, where $\beta = s_2\|b_1\| + s_1\|b_2\| + s_1 s_2$ and $\alpha = r_2\|a_1\| + r_1\|a_2\| + r_1 r_2$. Once again by Lemma 3.1, the result follows. $\square$

Theorem 3.26. *The multiplication $*_{\mathcal{B},c}$ is subdistributive with respect to the addition $+_{\mathcal{B}}$.*

Proof. Let $\boldsymbol{a} = \langle a; r_1 \rangle, \boldsymbol{b} = \langle b; r_2 \rangle, \boldsymbol{c} = \langle c; r_3 \rangle \in \mathcal{B}$. Lemma 3.1 allows to arrive at

$$\begin{aligned}
\boldsymbol{a} *_{\mathcal{B},c} (\boldsymbol{b} +_{\mathcal{B}} \boldsymbol{c}) &= \langle a; r_1 \rangle *_{\mathcal{B},c} \langle b + c; r_2 + r_3 \rangle \\
&= \langle a * (b + c); (r_2 + r_3)\|a\| + r_1\|b + c\| + r_1(r_2 + r_3) \rangle \\
&\subseteq \langle a * b + a * c; r_2\|a\| + r_1\|b\| + r_1 r_2 + r_3\|a\| + r_1\|c\| + r_1 r_3 \rangle \\
&= \boldsymbol{a} *_{\mathcal{B},c} \boldsymbol{b} +_{\mathcal{B}} \boldsymbol{a} *_{\mathcal{B},c} \boldsymbol{c}.
\end{aligned}$$

Thus, left subdistributivity holds. An analogous reasoning leads to the right subdistributivity. $\square$

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