PROFINITE APPROACH TO S-ADIC SHIFT SPACES I: SATURATING DIRECTIVE SEQUENCES.

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ABSTRACT. This paper is the first in a series of three, about (relatively) free profinite semigroups and S-adic representations of minimal shift spaces. We associate to each primitive S-adic directive sequence $\boldsymbol{\sigma}$ a *profinite image* in the free profinite semigroup over the alphabet of the induced minimal shift space $X(\boldsymbol{\sigma})$. When this profinite image contains a maximal subgroup of the free profinite semigroup, we say that $\boldsymbol{\sigma}$ is *saturating*. We show that if $\boldsymbol{\sigma}$ is recognizable, then it is saturating. Conversely, we use the notion of saturating sequence to obtain several sufficient conditions for $\boldsymbol{\sigma}$ to be recognizable: $\boldsymbol{\sigma}$ consists of pure encodings; or $\boldsymbol{\sigma}$ is eventually recognizable, saturating and consists of encodings; or $\boldsymbol{\sigma}$ is eventually recognizable, recurrent, bounded and consists of encodings. For the most part, we do not assume that $\boldsymbol{\sigma}$ has finite alphabet rank.

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1. Introduction

This article is the first in a series of three papers linking minimal shift spaces, via their S-adic representations, with free profinite semigroups (cf. [16, 17] for the ensuing two papers). Finitely generated free profinite semigroups are completions of free semigroups by a natural metric. The elements of free profinite semigroups are called *pseudowords*. In this first paper, we apply methods relying on a sort of "algebraic combinatorics on pseudowords" to obtain necessary and sufficient conditions for a primitive S-adic representation to be recognizable.

S-adic representations of minimal shift spaces are an important subject of symbolic dynamics, that in the past few decades has received a lot of attention (as seen, for example, in the books [45, 41] and in the survey [26]). Symbolic dynamics has strong connections with the theory of automata and formal languages [21, 56, 45, 57, 22, 18]. Free profinite semigroups were involved in major advancements in that theory since the 1980s [2, 66, 13, 67]. Hence, it is not surprising there has been an emergence of direct links between symbolic dynamics and free profinite semigroups. The first time that methods from symbolic dynamics were systematically employed in the theory of profinite semigroups was in [4], where they were used to establish a strong decidability property of the pseudovariety of all finite p-groups. Shortly thereafter, the first author introduced a systematic connection between symbolic dynamics and free profinite semigroups, allowing him to associate to each irreducible shift space X a profinite group $G(X)$, naturally realized as a maximal subgroup of the free profinite semigroup over the alphabet of X [5, 6, 8, 7]. The group $G(X)$, called the *Schützenberger group of* X since the paper [6], has dynamical significance: it is a flow invariant [35]. If X is sofic and non-periodic, then $G(X)$ is a free profinite group of rank $\aleph_0$ [34]. Besides the sofic case, the computation of $G(X)$ has only been made for minimal shift spaces; mostly substitutive spaces [7, 11, 46, 47], but not always [12]. The landscape of possibilities for $G(X)$ when X is minimal seems rich, and remains largely unexplored.

In the series of three papers here initiated, we go beyond the substitutive case by systematically expanding to minimal S-adic shift spaces our study of the interplay between free profinite semigroups and symbolic dynamical systems, specially through the Schützenberger groups of the latter. Fixing an S-adic representation for a shift space allows us to see it as a sort of limit of substitutive spaces, thus suggesting a way to approach spaces that are non-substitutive by adapting what was done for the substitutive ones. This kind of approach is sketched in [6] to determine the Schützenberger groups of Arnoux–Rauzy shift spaces.

In this article we focus on connections with the notion of *recognizable* directive sequence. Mossé's celebrated theorem, stating that every aperiodic primitive substitution is recognizable [59, 60], is crucial for the deduction, by the first two authors, of presentations for $G(X)$ when X is defined by a primitive substitution [11]. Her theorem was extended and refined by several authors [28, 36, 52]. This led to far-reaching generalizations by Berthé et. al. [27] concerning primitive directive S-adic sequences, which, in conjunction with past study of the group $G(X)$, motivated the work in the present paper. Further generalizations of Berthé et al.'s results appear in work by Béal et al. [23]. Also testifying their importance, we mention that recognizable directive sequences provide representations of S-adic shift spaces by Bratteli–Vershik systems [27, Theorem 6.5].

At this point, it is convenient to provide some technical context. An S-adic directive sequence is a sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ of substitutions (i.e., homomorphisms $\sigma_n \colon A_{n+1}^+ \to A_n^+$ between free semigroups) defining in a natural way a minimal shift space $X(\boldsymbol{\sigma})$; alternatively, $\boldsymbol{\sigma}$ is called an S-adic representation of $X(\boldsymbol{\sigma})$. Roughly speaking, $\boldsymbol{\sigma}$ is recognizable when, denoting by $\boldsymbol{\sigma}^{(k)}$ the subsequence $(\sigma_n)_{n\geq k}$, every element of $X(\boldsymbol{\sigma}^{(k)})$ has a unique "de-substitution", via σ_k, as an element of $X(\boldsymbol{\sigma}^{(k+1)})$, for every $k \in \mathbb{N}$. Quite often, one needs to assume that $\boldsymbol{\sigma}$ is *bounded*, meaning that the sequence of cardinalities $\mathrm{Card}(A_n)$ is bounded; or at least that $\boldsymbol{\sigma}$ has *finite alphabet rank*, meaning that $\liminf \mathrm{Card}(A_n) < \infty$. One of the most remarkable results from [27], generalizing Mossé's theorem, is that if $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is a primitive directive sequence with finite alphabet rank, such that $X(\boldsymbol{\sigma})$ is aperiodic, then $\boldsymbol{\sigma}$ is *eventually recognizable* (i.e., $\boldsymbol{\sigma}^{(k)}$ is recognizable for some $k \in \mathbb{N}$).

The main contribution from this paper is the introduction and exploration of the notion of *saturating* directive sequence, and its associated machinery, to obtain new results about recognizable directive sequences. Briefly speaking, a primitive directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is saturating when some natural realization of $G(X(\boldsymbol{\sigma}))$ is contained in the intersection of the images of the profinite extensions of the homomorphisms σ_n; we call such intersection the *profinite image* of $\boldsymbol{\sigma}$. The profinite image of $\boldsymbol{\sigma}$ is a group if $\boldsymbol{\sigma}$ is proper (Theorem 7.9); it is a simple semigroup if, and only if, all limit words of $\boldsymbol{\sigma}$ belong to $X(\boldsymbol{\sigma})$ (Theorem 7.1).

The next theorem collects several of our main applications of saturation to recognizability (cf. Theorems 10.10, 10.17, 10.22, and Corollary 10.21); by saying that $\boldsymbol{\sigma}$ is an *encoding* we mean that σ_n is an injective homomorphism for each $n \in \mathbb{N}$, and by saying that it is *pure* we mean that it is an encoding such that, for each $n \in \mathbb{N}$, the image of σ_n is a pure code.

Theorem 1.1. *Let $\boldsymbol{\sigma}$ be an eventually recognizable primitive directive sequence. The following statements hold:*

(i) *if $\boldsymbol{\sigma}$ is recognizable, then it is saturating;*
(ii) *if $\boldsymbol{\sigma}$ is pure, then $\boldsymbol{\sigma}$ is recognizable;*
(iii) *if $\boldsymbol{\sigma}$ is saturating and encoding, then $\boldsymbol{\sigma}$ is recognizable;*
(iv) *if $\boldsymbol{\sigma}$ is recurrent, bounded, and encoding, then $\boldsymbol{\sigma}$ is recognizable.*

Other sufficient conditions for recognizability of $\boldsymbol{\sigma}$ were obtained before. Berthé et al. showed that $\boldsymbol{\sigma}$ is *fully recognizable* (a property stronger than being recognizable) if $X(\boldsymbol{\sigma})$ is aperiodic and for each $n \in \mathbb{N}$ the homomorphism $\sigma_n \colon A_{n+1}^+ \to A_n^+$ satisfies one of the following conditions: $\mathrm{Card}(A_{n+1}) = 2$, the rank of the incidence matrix of σ_n is $\mathrm{Card}(A_{n+1})$, or σ_n is rotationally conjugate to a left or right permutative homomorphism [27, Theorem 4.6]. Bustos-Gajardo et al. showed, again assuming aperiodicity of $X(\boldsymbol{\sigma})$, that $\boldsymbol{\sigma}$ is recognizable if each term σ_n appears infinitely often in $\boldsymbol{\sigma}$ and is a constant-length encoding, cf. [30, Lemma 3.4 and Theorem 3.6].

Theorem 1.1 links symbolic dynamics and free profinite semigroups. The latter are applied in the proofs of all four statements included in the theorem, with the notion of saturating sequence playing a key role in all of them. Concerning the opposite direction, from symbolic dynamics to free profinite semigroups, we mention that Theorem 1.1 is used to obtain upper bounds for the rank of the profinite group

$G(X)$, when X has finite alphabet rank (cf. Corollaries 10.13, 10.14 and 10.15). Other applications, to the computation of Schützenberger groups of minimal shift spaces, appear in the two papers following this one [16, 17].

When delving in the proof of Theorem 1.1, the reader will notice our option to refine the concept of saturation by considering free profinite semigroups relatively to pseudovarieties of finite semigroups. A pseudovariety of finite semigroups is a class of finite semigroups that is closed under taking finite products, subsemigroups, and quotients. This type of class provides one of the main frameworks for the study of finite semigroups and formal languages, particularly via Eilenberg's correspondence theorem [42]. This is enough as motivation to also consider the image $G_{\mathsf{V}}(X)$ of $G(X)$ in the free pro-V semigroup over the alphabet of X, when V is a pseudovariety of finite semigroups; we say that $G_{\mathsf{V}}(X)$ is the V-*Schützenberger group* of X. Note that V-Schützenberger groups are an extension of the original notion of Schützenberger groups, in view of the equality $G(X) = G_{\mathsf{S}}(X)$, where S is the pseudovariety of all finite semigroups. Going back to saturation, the corresponding refinement for the notion of saturating directive sequence is that of V-*saturating* directive sequence; the saturating sequences mentioned in Theorem 1.1 are precisely the S-saturating sequences. Considering V-saturating sequences, for V other than S, allows for more clarity in the proof of Theorem 1.1 and enlarges its scope. It also prepares the path to results about V-Schützenberger groups in the subsequent papers [16, 17].

We proceed by detailing how this paper is organized, highlighting some of the content spread along it. Preliminaries about symbolic dynamics and profinite semigroups are respectively given in the two sections following this introduction. Immediately afterwards, we have a section dedicated to profinite categories. There, we improve Hunter's theorem stating that the monoid of continuous endomorphisms of a finitely generated profinite semigroup is itself a profinite monoid, for the pointwise topology: we extend it to any category of continuous homomorphisms between finitely many finitely generated profinite semigroups (cf. Proposition 4.1). This improvement is necessary for Sections 8 and 9, and for the proofs of Theorem 10.7 and its closely related Theorem 10.22. We also introduce free profinite categories and some of its properties, also needed for the same latter parts of the paper.

In Section 5 we recapitulate existing results connecting minimal shift spaces with profinite semigroups, improving some of them and establishing new ones. Part of the novelty comes from a more systematic consideration in this study of all pseudovarieties of semigroups containing all finite local semilattices, and of the corresponding relatively free profinite semigroups.

In Section 6 we introduce the profinite image of an S-adic directive sequence $\boldsymbol{\sigma}$, moreover establishing and studying a natural inverse limit of the profinite images of the tails $\boldsymbol{\sigma}^{(n)}$. In Section 7 we relate the algebraic structure of the profinite image of $\boldsymbol{\sigma}$ with combinatorial and dynamical aspects of $\boldsymbol{\sigma}$. In Section 8 we see that if the primitive directive sequence $\boldsymbol{\sigma}$ is bounded, then the profinite image of $\boldsymbol{\sigma}$ is the image of primitive continuous homomorphisms between free profinite semigroups, obtained as cluster points of the sequence of homomorphisms $\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_n$. Intuitively, this approximates even more the bounded case to the case of substitutive shift spaces. Section 9 further develops the material of the preceding section by, among other things, associating to each bounded primitive directive sequence a certain set of continuous idempotent endomorphisms (of finitely generated profinite semigroups), which we call *kernel endomorphisms*. The kernel endomorphisms play a key role in the ensuing papers [16, 17].

The last section (Section 10) contains the main results of the paper. Here we introduce and study saturating directive sequences, leading to the deduction of necessary or sufficient conditions, summarized in Theorem 1.1, for an S-adic directive sequence to be recognizable.

2. Symbolic dynamics

This section aims to provide some background on symbolic dynamics; the reader is referred to [45] for a more in-depth introduction, particularly in what concerns substitutions.

2.1. **Basic notions.** Let A be an alphabet, that is, a nonempty set whose elements we call *letters*. We denote by A^* the set of all words over A, including the empty word ε, and we let $A^+ = A^* \setminus \{\varepsilon\}$; under the operation of word concatenation, A^* is the free monoid on A, and A^+ is the free semigroup. The length of a word $w \in A^*$ is denoted $|w|$, while the number of occurrences of a letter $a \in A$ in w is denoted $|w|_a$. We are considering w as an element of $A^{\{0,\dots,|w|-1\}}$, and so we let $w[i]$ be the letter of w in position i, for $0 \leq i < |w|$. For $0 \leq i \leq j \leq |w|$, we let $w[i,j) = w[i] \cdots w[j-1]$. Note that $w[i,i) = \varepsilon$. We denote by $\mathrm{fac}(w)$ the set of *nonempty* factors of w, that is

$$\mathrm{fac}(w) = \{w[i,j) : 0 \leq i < j \leq |w|\}.$$

Factors of the form $w[0,j)$ are further called *prefixes*, while those of the form $w[i,|w|)$ are called *suffixes*. We make the choice of excluding the empty word from $\mathrm{fac}(w)$ because it will often be more convenient to work with free semigroups rather than free monoids.

Let $A^{\mathbb{Z}}$ be the set of two-sided infinite words over A. Given $x \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, we let $x[i]$ be the letter of x on position i. If $i, j \in \mathbb{Z}$ are such that $i \leq j$, we may consider the word $x[i,j) = x[i] \cdots x[j-1]$. Observe that $x[i,i) = \varepsilon$. The set

$$\mathrm{fac}(x) = \{x[i,j) : i < j\}$$

is the set of *nonempty* factors of x. *Mutatis mutandis*, we make similar definitions for *right infinite words* and *left infinite words*, that is elements of $A^{\mathbb{N}}$ and $A^{\mathbb{Z}_-}$, where $\mathbb{N}$ and $\mathbb{Z}_-$ respectively stand for the set of nonnegative and the set of negative integers. For $x \in A^{\mathbb{Z}_-}$ and $y \in A^{\mathbb{N}}$, we denote by $x \cdot y$ the element z of $A^{\mathbb{Z}}$ such that $z[i] = x[i]$ if $i < 0$ and $z[i] = y[i]$ if $i \geq 0$.

At this point, we assume that A is finite and we endow it with the discrete topology, and $A^{\mathbb{Z}}$ with the corresponding product topology. The *shift map* is the homeomorphism $T \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $T(x) = (x[i+1])_{i \in \mathbb{Z}}$. A *shift space* over the alphabet A is a nonempty closed subset X of $A^{\mathbb{Z}}$ that satisfies $T(X) = X$. Note that the pair (X, T) is a topological dynamical system, and so one may apply to shift spaces terminology from the theory of dynamical systems, such as that of *topological conjugacy*, which is the notion of isomorphism for dynamical systems.

We focus primarily on shift spaces that are *minimal* (for the inclusion order). An infinite word $x \in A^{\mathbb{Z}}$ is *periodic* if it has a finite T-orbit, and aperiodic otherwise; a shift space is called *periodic* if it is the orbit of a periodic infinite word, and *aperiodic* if it contains no periodic shift space.

The *language* of a subset $X \subseteq A^{\mathbb{Z}}$ is the subset of A^+ defined by

$$L(X) = \bigcup_{x \in X} \operatorname{fac}(x).$$

It is well known that for two shift spaces $X, Y \subseteq A^{\mathbb{Z}}$, we have $L(X) \subseteq L(Y)$ if and only if $X \subseteq Y$. The language of a shift space X is both *factorial* ($\operatorname{fac}(w) \subseteq L(X)$ for every $w \in L(X)$) and *extendable* (if $w \in L(X)$, then $awb \in L(X)$ for some $a, b \in A$); conversely, every nonempty, factorial and extendable language is the language of a unique shift space. Minimal shift spaces have the simple characterization in terms of their languages that follows. A language $L \subseteq A^+$ is called *uniformly recurrent* if it is factorial, extendable, and for every $u \in L$, there exists $n \in \mathbb{N}$ such that $u \in \operatorname{fac}(v)$ for every $v \in L$ with $|v| \geq n$. Then, a shift space X is minimal if and only if the language $L(X)$ is uniformly recurrent.

Consider a semigroup homomorphism $\sigma\colon A^+ \to B^+$. For each $x \in A^{\mathbb{Z}}$, the element $\sigma(x)$ of $B^{\mathbb{Z}}$ is defined by the equality

$$\sigma(x) = \cdots \sigma(x[-2])\sigma(x[-1]) \cdot \sigma(x[0])\sigma(x[1])\sigma(x[2]) \cdots .$$

2.2. **S-adic representations.** A common way of defining shift spaces is to use so-called S-adic representations, which we proceed to introduce.

Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a sequence of homomorphisms of free semigroups $\sigma_n\colon A_{n+1}^+ \to A_n^+$, where A_n is a finite alphabet for every $n \in \mathbb{N}$. We say that $\boldsymbol{\sigma}$ a *directive sequence*. The *alphabet rank* of $\boldsymbol{\sigma}$ is the limit $\liminf_{n\to\infty} \operatorname{Card}(A_n)$ where $\operatorname{Card}(S)$ denotes the cardinal of a set S. For such a directive sequence $\boldsymbol{\sigma}$ and natural numbers $n \leq m$, let $\sigma_{n,m}$ be the homomorphism $A_m^+ \to A_n^+$ given by the composition

$$\sigma_{n,m} = \sigma_n \circ \cdots \circ \sigma_{m-1},$$

with the convention that $\sigma_{n,n}$ is the identity on A_n^+. Consider the factorial language

$$L(\boldsymbol{\sigma}) = \bigcup_{n\geq 0} \bigcup_{a \in A_n} \operatorname{fac}(\sigma_{0,n}(a)).$$

Let $X(\boldsymbol{\sigma})$ be the set of elements x of $A_0^{\mathbb{Z}}$ such that $\operatorname{fac}(x) \subseteq L(\boldsymbol{\sigma})$. The set $X(\boldsymbol{\sigma})$ is a shift space when it is nonempty. We say that a shift space X is *represented* by the directive sequence $\boldsymbol{\sigma}$, or that $\boldsymbol{\sigma}$ is an *S-adic representation of* X, when $X = X(\boldsymbol{\sigma})$.

Remark 2.1. One has $X(\boldsymbol{\sigma}) \neq \emptyset$ precisely when $L(\boldsymbol{\sigma})$ is infinite, which happens if and only if $\lim_{n\to\infty} \max_{a\in A_{n+1}} |\sigma_{0,n}(a)| = \infty$ (in some publications, this limit is part of the definition of directive sequence, e.g. [25]). The inclusion $L(X(\boldsymbol{\sigma})) \subseteq L(\boldsymbol{\sigma})$ clearly holds, but it may be strict, even if $X(\boldsymbol{\sigma}) \neq \emptyset$ (cf. [41, Example 1.4.5]).

Remark 2.2. In the book of Durand and Perrin [41] the terminology *directive sequence* is reserved for sequences $\boldsymbol{\sigma}$ such that $L(\boldsymbol{\sigma}) = L(X(\boldsymbol{\sigma}))$. On the other hand, our usage is adopted in many other relevant publications (cf. [40, 26, 25, 27]).

Let $k \in \mathbb{N}$. We denote by $\boldsymbol{\sigma}^{(k)}$ the *tail sequence* given by $\boldsymbol{\sigma}^{(k)} = (\sigma_{n+k})_{n\in\mathbb{N}}$. A proof of the following fact is found in [27, Lemma 4.2].

Lemma 2.3. *For every $m, n \in \mathbb{N}$ such that $m \geq n$, the shift space $X(\boldsymbol{\sigma}^{(n)})$ is the smallest one containing the set $\sigma_{n,m}(X(\boldsymbol{\sigma}^{(m)}))$.*

Let φ be a *substitution* over the alphabet A, by which we mean an endomorphism of A^+. In the special case where $\sigma_n = \varphi$ for all $n \in \mathbb{N}$, we denote $L(\boldsymbol{\sigma})$ and $X(\boldsymbol{\sigma})$ respectively by $L(\varphi)$ and $X(\varphi)$. Assuming moreover that $X(\varphi) \neq \emptyset$, we say that

$X(\varphi)$ is a *substitutive* shift space. We mention that in some sources the equality $L(\varphi) = L(X(\varphi))$ is included in the definition of substitution (e.g. [41]).

When studying minimal shift spaces, it is often useful to focus on S-adic representations subject to special conditions, some of which we introduce next. We start with conditions on homomorphisms.

Definition 2.4. A homomorphism $\varphi\colon A^+ \to B^+$ is called:

(i) *expansive* if $|\varphi(a)| \geq 2$ for every $a \in A$;
(ii) *positive* if $B \subseteq \mathrm{fac}(\varphi(a))$ for every $a \in A$, and φ is expansive;
(iii) *circular* if it is injective and $uv, vu \in \varphi(A^+) \implies u, v \in \varphi(A^+)$ for every $u, v \in B^+$;
(iv) *left proper* if there is a letter $b \in B$ such that $\varphi(a) \in bB^*$ for every $a \in A$;
(v) *right proper* if there is a letter $b \in B$ such that $\varphi(a) \in B^*b$ for every $a \in A$;
(vi) *proper* if it is right proper and left proper.

In turn, when we say that a directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is *circular*, *right proper*, *left proper*, *proper*, or *positive*, we mean that σ_n has that property for every $n \in \mathbb{N}$. We also say that $\boldsymbol{\sigma}$ is *encoding* if σ_n is injective for every $n \in \mathbb{N}$.

A slightly more subtle notion is that of primitivity.

Definition 2.5. A directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is *primitive* if, for every $n \in \mathbb{N}$, there exists $m > n$ such that $\sigma_{n,m}$ is positive.

In some papers, for instance [26, 55], primitive directive sequences are called instead *weakly primitive.* The following theorem is well known within the community studying minimal shift spaces and their S-adic representations. A proof can be found in [41, Section 6.4.2].

Theorem 2.6. *Let X be a shift space. The following conditions are equivalent:*

(i) *X is a minimal shift space;*
(ii) *$X = X(\boldsymbol{\sigma})$ for some primitive directive sequence $\boldsymbol{\sigma}$;*
(iii) *$X = X(\boldsymbol{\sigma})$ for some proper, primitive and circular directive sequence $\boldsymbol{\sigma}$.*

Moreover, if $\boldsymbol{\sigma}$ is a primitive directive sequence, then the equality $L(X(\boldsymbol{\sigma})) = L(\boldsymbol{\sigma})$ holds.

Remark 2.7. In [41, Proposition 6.4.5] it is stated that if $\boldsymbol{\sigma}$ is a primitive directive sequence, then the equality $L(X(\boldsymbol{\sigma})) = L(\boldsymbol{\sigma})$ holds under the extra assumption that the sequence is *without bottleneck*, that is, $\mathrm{Card}(A_n) \geq 2$ for all $n \in \mathbb{N}$. We avoid this extra assumption in Theorem 2.6 because we force positive homomorphisms to be expansive also when the alphabet in the image has only one letter.

Remark 2.8. Theorem 2.6 is essentially Proposition 6.4.5 from the book of Durand and Perrin [41], with two notable differences. First, our statement includes periodic shift spaces because we allow bottleneck (cf. Remark 2.7). Second, in the statement provided in the book there is no explicit reference to circular homomorphisms; but the proof found there gives what we write here, since the pertinent homomorphisms are encodings by *return words*, well known to be circular encodings (cf. [39, Lemma 17]). In our companion paper [16] one finds a more detailed discussion about the representation by a proper, primitive and circular directive sequence that follows from that proof.

A useful operation on directive sequences is that of contraction, which consists in grouping consecutive homomorphisms in the sequence. More precisely, a *contraction* (also called a *telescoping* in many sources) of a sequence of homomorphisms $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is a sequence of the form $\boldsymbol{\tau} = (\sigma_{n_k,n_{k+1}})_{k\in\mathbb{N}}$, for some strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that $n_0 = 0$. Note that, if $\boldsymbol{\sigma}$ is primitive, then $\boldsymbol{\sigma}$ has a contraction which is positive; moreover, every contraction of $\boldsymbol{\sigma}$ is primitive. As seen next, under a very mild condition[1], satisfied by primitive directive sequences, the shift space $X(\boldsymbol{\sigma})$ remains unchanged when passing to a contraction. The reader should bear this fact in mind.

Lemma 2.9. *Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a primitive directive sequence with a contraction $\boldsymbol{\tau} = (\sigma_{n_k,n_{k+1}})_{k\in\mathbb{N}}$. Suppose that $A_n \subseteq \mathrm{fac}(\sigma_n(A_{n+1}))$ for every $n \in \mathbb{N}$. Then, the equalities $L(\boldsymbol{\sigma}^{(n_k)}) = L(\boldsymbol{\tau}^{(k)})$ and $X(\boldsymbol{\sigma}^{(n_k)}) = X(\boldsymbol{\tau}^{(k)})$ hold for every $k \in \mathbb{N}$.*

Lemma 2.9, whose proof is an easy exercise, does not hold if we drop some inclusion $A_n \subseteq \mathrm{fac}(\sigma_n(A_{n+1}))$ (cf. Exercise 1.27, and its solution, in the book [41]).

2.3. **Recognizability.** We proceed to give the necessary background on the important notion of *recognizable* directive sequence, following the monograph [41] and the paper [27].

Let $\sigma\colon A^+ \to B^+$ be a homomorphism, where A and B are finite alphabets. A *σ-representation* of a point $y \in B^{\mathbb{Z}}$ is a pair (k, x), where $k \in \mathbb{N}$ and $x \in A^{\mathbb{Z}}$, satisfying $T^k\sigma(x) = y$. We say that it is *centered* if, additionally, $k < |\sigma(x[0])|$.

Definition 2.10 (Dynamical recognizability)**.** Given $X \subseteq A^{\mathbb{Z}}$, we say that σ is *(dynamically) recognizable in X* if every $y \in B^{\mathbb{Z}}$ has at most one centered σ-representation (k, x) with $x \in X$.

In case $X = A^{\mathbb{Z}}$, we say instead that σ is *fully recognizable*. Full recognizability has the following characterization (cf. [41, Proposition 1.4.32]).

Proposition 2.11. *A homomorphism is fully recognizable if and only if it is circular.*

We say that a directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is *recognizable* if the homomorphism σ_n is recognizable in $X(\boldsymbol{\sigma}^{(n+1)})$ for every $n \in \mathbb{N}$; and *eventually recognizable* if this holds only for all but finitely many $n \in \mathbb{N}$. One should bear in mind the following remarkable result of Berthé et al. [27, cf. Theorem 5.2].

Theorem 2.12. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence with finite alphabet rank. If $X(\boldsymbol{\sigma})$ is aperiodic, then $\boldsymbol{\sigma}$ is eventually recognizable.*

There is also a pointwise version of recognizability. Fix $x \in A^{\mathbb{Z}}$ and a homomorphism $\sigma\colon A^+ \to B^+$; define the set of *σ-cutting points of x* by:

$$C_\sigma(x) = \{-|\sigma(x[i, 0))| : i < 0\} \cup \{0\} \cup \{|\sigma(x[0, i))| : i > 0\}.$$

The following definition was introduced in a seminal paper by Mossé [59].

Definition 2.13 (Mossé's recognizability)**.** Let $x \in A^{\mathbb{Z}}$ and write $y = \sigma(x)$. We say that σ is *recognizable for x in Mossé's sense* when, for some positive integer ℓ (called the *constant of recognizability*), the following holds for every $m \in C_\sigma(x)$ and $n \in \mathbb{Z}$:

$$y[m - \ell, m + \ell) = y[n - \ell, n + \ell) \implies n \in C_\sigma(x).$$

[1]This mild condition appears to be implicit in several sources where it is stated that taking a contraction does not change the shift being represented by the directive sequence (e.g. [27, Section 5.2] and [41, Section 6.4.1])

Under mild conditions, dynamical recognizability implies Mossé's recognizability.

Proposition 2.14 ([27, Theorem 2.5(1)])**.** *Let $\sigma\colon A^+ \to B^+$ be a homomorphism, $X \subseteq A^{\mathbb{Z}}$ be a shift space and $x \in X$ be such that $L(X) = \mathrm{fac}(x)$. If σ is recognizable in X, then it is recognizable in Mossé's sense for x.*

In particular, if X is a minimal shift space, then σ is recognizable in Mossé's sense for every $x \in X$.

3. Profinite semigroups

We move on to review some elements of semigroup theory, with a focus on profinite semigroups. We follow the definition of a semigroup as being a *nonempty* set endowed with an associative binary operation (in some sources, such as the book of Rhodes and Steinberg [66], the empty set is considered to be a semigroup).

3.1. **Green's relations.**

We briefly recall a few standard facts about Green's relations; a thorough account may be found in any book covering basic semigroup theory, for instance [31, 49, 53].

Let S be a semigroup, and S^1 be the smallest monoid containing S (obtained by adjoining to S an identity element, generically denoted 1, if needed). For $s, t \in S$, write:

- $s \leq_{\mathcal{R}} t$ (or say that t is a *prefix* of s) when $sS^1 \subseteq tS^1$;
- $s \leq_{\mathcal{L}} t$ (or say that t is a *suffix* of s) when $S^1 s \subseteq S^1 t$;
- $s \leq_{\mathcal{H}} t$ when $sS^1 \subseteq tS^1$ and $S^1 s \subseteq S^1 t$;
- $s \leq_{\mathcal{J}} t$ (or say that t is a *factor* of s) when $S^1 s S^1 \subseteq S^1 t S^1$.

These are quasi-orders known as *Green's quasi-orders*. They induce four equivalence relations, respectively denoted $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{J}$, called *Green's equivalences*. By a classical theorem of Green, the maximal subgroups (maximal for inclusion) of S are precisely the $\mathcal{H}$-classes of its idempotent elements. We may write H_s for the $\mathcal{H}$-class of s and similarly for other Green's equivalences. For any Green's relation $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{K}\}$, we may write $\mathcal{K}_S$ instead of $\mathcal{K}$, whenever we want to emphasise that we are considering the relation $\mathcal{K}$ in the semigroup S; this may be needed when reasoning with different semigroups at the same time.

In this paper, we deal mostly with *compact semigroups*: semigroups endowed with a compact topology for which the multiplication is continuous (we include the Hausdorff property in the definition of compactness). Note that finite semigroups equipped with the discrete topology are compact semigroups. In compact semigroups, all of Green's relations (quasi-orders and equivalences) are closed; in particular, so are the equivalence classes of Green's equivalences. When S is a compact semigroup which is not a monoid, then S^1 is viewed as a compact semigroup by considering the topological sum of S and of the discrete space $\{1\}$.

A useful property of compact semigroups is that they are *stable*, that is,

$$(s \leq_{\mathcal{R}} t \text{ and } s \mathrel{\mathcal{J}} t) \implies s \mathrel{\mathcal{R}} t, \qquad (s \leq_{\mathcal{L}} t \text{ and } s \mathrel{\mathcal{J}} t) \implies s \mathrel{\mathcal{L}} t.$$

Stable semigroups S enjoy several useful properties:

- Two elements s and t are $\mathcal{J}$-equivalent if and only if there is u such that $s \mathrel{\mathcal{R}} u \mathrel{\mathcal{L}} t$, if and only if there is v such that $s \mathrel{\mathcal{L}} v \mathrel{\mathcal{R}} t$.
- A $\mathcal{J}$-class J contains an idempotent if and only if each of its $\mathcal{L}$-classes contains an idempotent; the same holds for $\mathcal{R}$-classes. This is also equivalent to every element of J being *regular*, where an element s of S being regular

means that $s \in sSs$. Whenever J satisfies these equivalent conditions, we call it a *regular $\mathcal{J}$-class*. Its maximal subgroups are then isomorphic to one another, continuously so in the compact case.
- The intersection of every $\mathcal{R}$-class with every $\mathcal{L}$-class contained in the same $\mathcal{J}$-class is an $\mathcal{H}$-class.

A subset F of a semigroup S is said to be *factorial* if it is an upset for the quasi-order $\leq_{\mathcal{J}}$, that is, it is closed under taking factors.

3.2. **Pseudovarieties of semigroups.** A *pseudovariety* of semigroups is a class of finite semigroups closed under taking subsemigroups, homomorphic images, and finite direct products. Examples include:
- the class S of all finite semigroups;
- the class G of all finite groups;
- the *trivial pseudovariety* I (with only the one-element semigroup);
- the class A of all finite *aperiodic semigroups* (semigroups whose subgroups are trivial);
- the class Sl of all finite *semilattices* (commutative semigroups whose elements are idempotent);
- the class CS of finite *simple semigroups*, that is, semigroups where the relation $\mathcal{J}$ is universal;[2]
- the class N of finite *nilpotent* semigroups (a semigroup is nilpotent if it has a zero 0 and $S^k = \{0\}$ for some $k \geq 1$).

A pseudovariety V is said to be *generated* by a class $\mathcal{C}$ of finite semigroups if V is the smallest pseudovariety that contains $\mathcal{C}$.

There are several operators of interest on pseudovarieties. For this paper and the ensuing companions [16, 17], the following are relevant.
- If H is a pseudovariety consisting of finite groups, then the class $\overline{\mathsf{H}}$ of all finite semigroups whose subgroups belong to H is a pseudovariety. Note that $\bar{\mathsf{I}} = \mathsf{A}$ and $\overline{\mathsf{G}} = \mathsf{S}$.
- Given a pseudovariety of semigroups V, the class LV of all finite semigroups S such that $eSe \in \mathsf{V}$ for all idempotents $e \in S$ is also a pseudovariety, called the *local of* V. (In this paper we need to consider the pseudovarieties LI and LSl.)
- For two pseudovarieties of semigroups V and W, their *semidirect product* $\mathsf{V} * \mathsf{W}$ is the pseudovariety generated by the class of all semidirect products of the form $S * R$ with $S \in \mathsf{V}$ and $R \in \mathsf{W}$.

3.3. **Relatively free profinite semigroups.** This subsection serves to introduce profinite semigroups, and in particular relatively free profinite semigroups. For more details on this topic, see [2, 66] and the shorter [8].

Let V be a pseudovariety of finite semigroups. In this paper, we always consider finite semigroups to be equipped with the discrete topology. A *pro-*V *semigroup* is a compact semigroup S which is also *residually* V, in the sense that any two distinct elements $x, y \in S$ take distinct values under some continuous homomorphism $\varphi\colon S \to R$ where $R \in \mathsf{V}$. In particular, members of V are pro-V; we also say that a

[2]The reader unfamiliar with Semigroup Theory is cautioned that in the literature one finds also the class of *completely simple semigroups*, whose intersection with the class of finite semigroups consists precisely of the finite simple semigroups. Note also that the notion of simple semigroup is distinct from the classical notion of simple group: as a semigroup, every group is simple.

pro-S semigroup is *profinite*. Other such specialized terminology will be introduced as needed. Let **Pro** denote the category of profinite semigroups, where morphisms are continuous semigroup homomorphisms. For each class $\mathcal{C}$ of profinite semigroups, denote by $\mathbf{Pro}[\mathcal{C}]$ the full subcategory of **Pro** whose objects are the elements of $\mathcal{C}$.

For each pseudovariety of semigroups V, the category $\mathbf{Pro}[\mathsf{V}]$ has free objects. They can be constructed as follows. Given an alphabet A (not necessarily finite), let Θ_{V} be the set of congruences θ on the free semigroup A^+ such that A^+/θ belongs to V. The hypothesis that V is a pseudovariety guarantees that Θ_{V} is a directed set for the reverse inclusion $\supseteq$, and so we may consider the inverse limit $\overline{\Omega}_A\mathsf{V}$ of the system formed by the finite quotients of the free semigroup A^+ which belong to V:

$$\overline{\Omega}_A\mathsf{V} = \varprojlim_{\theta\in\Theta_{\mathsf{V}}} A^+/\theta.$$

We call $\overline{\Omega}_A\mathsf{V}$ the *free pro-V semigroup*. Let $\iota_{\mathsf{V}}\colon A \to \overline{\Omega}_A\mathsf{V}$ be the mapping sending each letter a of A to $(a/\theta)_{\theta\in\Theta}$. With respect to this mapping, $\overline{\Omega}_A\mathsf{V}$ has the following universal property: for every mapping $f\colon A \to S$ with S a pro-V semigroup, there exists a unique continuous homomorphism $f^{\mathsf{V}}\colon \overline{\Omega}_A\mathsf{V} \to S$ such that $f^{\mathsf{V}} \circ \iota_{\mathsf{V}} = f$; see the diagram below.

$$\begin{tikzcd} A \arrow[r, "\iota_{\mathsf{V}}"] \arrow[dr, "f"'] & \overline{\Omega}_A\mathsf{V} \arrow[d, "f^{\mathsf{V}}"] \\ & S. \end{tikzcd}$$

Semigroups of the form $\overline{\Omega}_A\mathsf{V}$ for some pseudovariety of semigroups V are called *relatively free*.

If the alphabet A is finite, then the topology of $\overline{\Omega}_A\mathsf{V}$ is metrizable, for every pseudovariety V, since Θ_{V} is at most countable in that case. A metric generating the topology of $\overline{\Omega}_A\mathsf{V}$ is the following: for $u, v \in \overline{\Omega}_A\mathsf{V}$ such that $u \neq v$, their distance, denoted $d(u, v)$, is given by the equality $d(u, v) = 2^{-r(u,v)}$ where $r(u, v)$ is the smallest possible cardinal for a semigroup S from V for which there is a continuous homomorphism $\varphi\colon \overline{\Omega}_A\mathsf{V} \to S$ satisfying $\varphi(u) \neq \varphi(v)$.

On the other hand, $\overline{\Omega}_A\mathsf{S}$ is not metrizable if A is infinite (cf. [19]).

Let V, W be two pseudovarieties of semigroups with $\mathsf{W} \subseteq \mathsf{V}$. For a given set A, the universal property of $\overline{\Omega}_A\mathsf{V}$ applied to the mapping $\iota_{\mathsf{W}}\colon A \to \overline{\Omega}_A\mathsf{W}$ gives a continuous onto homomorphism $p_{\mathsf{V},\mathsf{W}}\colon \overline{\Omega}_A\mathsf{V} \to \overline{\Omega}_A\mathsf{W}$ such that $p_{\mathsf{V},\mathsf{W}} \circ \iota_{\mathsf{V}} = \iota_{\mathsf{W}}$ (cf. Diagram 1). We call this the *natural projection* of $\overline{\Omega}_A\mathsf{V}$ onto $\overline{\Omega}_A\mathsf{W}$.

$$\begin{tikzcd} A \arrow[r, "\iota_{\mathsf{V}}"] \arrow[dr, "\iota_{\mathsf{W}}"'] & \overline{\Omega}_A\mathsf{V} \arrow[d, "p_{\mathsf{V},\mathsf{W}}"] \\ & \overline{\Omega}_A\mathsf{W} \end{tikzcd} \tag{1}$$

Whenever V contains the pseudovariety N of finite nilpotent semigroups, or the pseudovariety G of finite groups, the mapping ι_{V} extends to an injective homomorphism $\iota_{\mathsf{V}}^+\colon A^+ \to \overline{\Omega}_A\mathsf{V}$. In particular, for such a pseudovariety V, we may identify A^+ with the (dense) subspace $\iota_{\mathsf{V}}^+(A^+) \subseteq \overline{\Omega}_A\mathsf{V}$ whenever convenient. We denote by $\mathrm{Cl}_{\mathsf{V}}(L)$ the closure in $\overline{\Omega}_A\mathsf{V}$ of a language $L \subseteq A^+$ viewed as a subset of $\overline{\Omega}_A\mathsf{V}$, whenever V is a pseudovariety of semigroups containing N or G. These facts reinforce our perception that the elements of $\overline{\Omega}_A\mathsf{V}$ may be seen as generalizations of words, for which reason they are called *pseudowords*.

Consider a homomorphism $\varphi\colon A^+ \to B^+$ of free semigroups. Assuming that V contains N or G, it follows from the universal property of free pro-V semigroups that for every homomorphism $\varphi\colon A^+ \to B^+$ there is a unique continuous homomorphism $\varphi^{\mathsf{V}}\colon \overline{\Omega}_A\mathsf{V} \to \overline{\Omega}_B\mathsf{V}$ whose restriction to A^+ coincides with φ, that is to say, Diagram (2) is commutative.

$$\begin{tikzcd} \overline{\Omega}_A\mathsf{V} \arrow[r, "\varphi^{\mathsf{V}}"] & \overline{\Omega}_B\mathsf{V} \\ A^+ \arrow[u, hook] \arrow[r, "\varphi"] & B^+ \arrow[u, hook] \end{tikzcd} \tag{2}$$

We say that φ^{V} is the *pro-*V *extension* of φ. By the uniqueness of the pro-V extension of homomorphisms between free semigroups, the correspondence $\varphi \mapsto \varphi_{\mathsf{V}}$ is functorial; in other words, $(\varphi \circ \psi)^{\mathsf{V}} = \varphi^{\mathsf{V}} \circ \psi^{\mathsf{V}}$ whenever φ and ψ are composable homomorphisms of free semigroups, and the pro-V extension of the identity on A^+ is the identity on $\overline{\Omega}_A\mathsf{V}$.

Recall that a *Stone space* is a topological space that is both compact and totally disconnected. Note that closed subspaces of Stone spaces are also Stone spaces.

For every pseudovariety of semigroups V, a language $L \subseteq A^+$ is said to be V-*recognizable* if there are a semigroup $S \in \mathsf{V}$ and a homomorphism $\varphi\colon A^+ \to S$ such that $L = \varphi^{-1}(\varphi(L))$. The *syntactic semigroup* of the language L is the quotient of A^+ by the least congruence saturating L. We have the following alternative characterization of the notion of V-recognizable language: L is V-recognizable if and only if the syntactic semigroup of L belongs to V. The following basic result gives a topological characterization of the same notion, which amounts to the fact that the topological space $\overline{\Omega}_A\mathsf{V}$ is the Stone dual of the Boolean algebra of all V-recognizable subsets of A^+.

Theorem 3.1 ([2, Theorem 3.6.1])**.** *Let* V *be a pseudovariety of semigroups containing* N*. Then a language* $L \subseteq A^+$ *over a finite alphabet* A *is* V*-recognizable if and only if* $\mathrm{Cl}_{\mathsf{V}}(L)$ *is open.*

Endow the set $\mathbb{N}_+$ of positive integers with the semigroup operation of addition. For this structure, the length mapping $\ell\colon A^+ \to \mathbb{N}_+$, defined by $\ell(u) = |u|$ is a semigroup homomorphism. We extend the length homomorphism to pseudowords in the following way. Let $\mathbb{N}_+ \cup \{\infty\}$ be the Alexandroff compactification of $\mathbb{N}_+$, and extend the addition operation on $\mathbb{N}_+$ to $\mathbb{N}_+ \cup \{\infty\}$ by making ∞ an absorbing element of $\mathbb{N}_+ \cup \{\infty\}$. In this way, $\mathbb{N}_+ \cup \{\infty\}$ is a pro-N semigroup (it is in fact a free pro-N semigroup on the single generator 1). Therefore, provided V contains N, the length homomorphism $\ell\colon A^+ \to \mathbb{N}_+$ extends uniquely to a continuous homomorphism $\ell^{\mathsf{V}}\colon \overline{\Omega}_A\mathsf{V} \to \mathbb{N}_+ \cup \{\infty\}$. We use the notation $|u|$ for $\ell^{\mathsf{V}}(u)$, for every $u \in \overline{\Omega}_A\mathsf{V}$. An element $u \in \overline{\Omega}_A\mathsf{V}$ has *infinite length* if $|u| = \infty$, and *finite length* otherwise. Clearly, infinite-length pseudowords form a closed ideal of $\overline{\Omega}_A\mathsf{V}$, a fact included in the next proposition, whose complete proof can be found in [15, Section 3], which extends earlier results for the case when A is finite; see e.g. [2].

Proposition 3.2. *Let* A *be an arbitrary alphabet and let* V *be a pseudovariety of semigroups containing* N*. Then the following hold:*

(i) *the elements of* A^+ *are isolated points in* $\overline{\Omega}_A\mathsf{V}$*;*
(ii) *the set* $\{u \in \overline{\Omega}_A\mathsf{V} : |u| = \infty\}$ *is an ideal of the free pro-*V *semigroup* $\overline{\Omega}_A\mathsf{V}$*;*
(iii) *the set* $\overline{\Omega}_A\mathsf{V} \setminus A^+$ *is an ideal of the free pro-*V *semigroup* $\overline{\Omega}_A\mathsf{V}$*.*

Remark 3.3. Since the elements of A^+ are isolated points in $\overline{\Omega}_A\mathsf{V}$, the equality $\mathrm{Cl}_{\mathsf{V}}(L) \cap A^+ = L$ holds for every language $L \subseteq A^+$, whenever $\mathsf{V} \supseteq \mathsf{N}$.

Remark 3.4. In case A is finite, the equality $\overline{\Omega}_A\mathsf{V} \setminus A^+ = \{u \in \overline{\Omega}_A\mathsf{V} : |u| = \infty\}$ holds whenever $\mathsf{V} \supseteq \mathsf{N}$, since there are only finitely many words of A^+ of a given length. It no longer holds if A is an infinite alphabet: in that case, the topological closure of A in $\overline{\Omega}_A\mathsf{V}$, being compact, contains some element not in A, and any such element has length 1 by continuity of the length homomorphism.

Let us suppose that V contains the pseudovariety LI, bearing in mind that LI contains N. In that case, for every nonnegative integer n, every pseudoword $w \in (\overline{\Omega}_A\mathsf{V})^1$ such that $|w| \geq n$ has a unique prefix of length of n, denoted $w[0, n)$, and a unique suffix of length n, denoted $w[-n, -1]$ (for further details, we refer to the discussion in [15, Section 6]).

In the case of the pseudovariety S, we have the following property, which amounts to saying that in an equality of pseudowords we may cancel equal finite-length prefixes, or suffixes.

Proposition 3.5. *Let* A *be any alphabet. If* $x, y, u, v \in (\overline{\Omega}_A\mathsf{S})^1$ *are pseudowords such that* $xu = yv$ *or* $ux = vy$*, and* $|x| = |y| \in \mathbb{N}$*, then* $x = y$ *and* $u = v$*.*

Let w be an infinite-length pseudoword on $\overline{\Omega}_A\mathsf{S}$, and let x be its prefix of length n, where $n \in \mathbb{N}$. By Proposition 3.5 there is a unique pseudoword $u \in \overline{\Omega}_A\mathsf{S}$ such that $w = xu$. We may denote u by $x^{-1}w$, and sometimes, alternatively, by $w^{(n)}$.

3.4. **Codes.** Let C be a subset of A^+. Recall that C is a *code* if the subsemigroup of A^+ generated by C is free with basis C.

The code C is called *circular* if the implication

$$uv, vu \in C^+ \implies u, v \in C^+$$

holds for every $u, v \in A^+$. Therefore, an injective homomorphism $\varphi\colon B^+ \to A^+$ is circular if and only if $\varphi(A)$ is a circular code.

The code C is called *pure* if it is closed under extraction of roots, that is, if for every $u \in A^+$ and integer $n \geq 1$, the following implication holds:

$$u^n \in C^+ \implies u \in C^+.$$

Every circular code is pure; the code $C = \{\mathtt{ab}, \mathtt{ba}\}$ is an example of a pure code that is not circular.

It turns out that a finite code C over the alphabet A is pure if and only if the syntactic semigroup of C^+ belongs to the pseudovariety A of all finite aperiodic semigroups (cf. [63, Theorem 3.1]; see also [53, Chapter 7, Exercise 8]). Hence, pure codes are often called *aperiodic* codes.

The property of being closed under root extraction carries through for pro-V closures of pure codes, in the following sense.

Lemma 3.6. *Let* C *be a finite pure code over a finite alphabet* A*,* V *a pseudovariety containing* A*, and* u *an element of* $\overline{\Omega}_A\mathsf{V}$*. If* $\mathrm{Cl}_{\mathsf{V}}(C^+) \cap \mathrm{Cl}_{\mathsf{V}}(u^+) \neq \emptyset$*, then* u *belongs to* $\mathrm{Cl}_{\mathsf{V}}(C^+)$*.*

Proof. Take $x \in \mathrm{Cl}_{\mathsf{V}}(C^+) \cap \mathrm{Cl}_{\mathsf{V}}(u^+)$. Let $(u_i)_{i\in\mathbb{N}}$ be a sequence of finite words such that $u = \lim u_i$ and $(u^{n_i})_{i\in\mathbb{N}}$ be a sequence of powers of u converging to x such that n_i is a positive integer for every $i \in \mathbb{N}$. We claim that $(u_i^{n_i})_{i\in\mathbb{N}}$ converges to x. Let $\varphi\colon \overline{\Omega}_A\mathsf{V} \to S$ be an arbitrary continuous homomorphism where $S \in \mathsf{V}$; because $\overline{\Omega}_A\mathsf{V}$ is residually V, we are reduced to showing that there exists $j \in \mathbb{N}$ such that $\varphi(u_i^{n_i}) = \varphi(x)$ for every $i \geq j$. Since S is discrete, there exists $i_1 \in \mathbb{N}$ such that $\varphi(u_i) = \varphi(u)$ for every $i \geq i_1$. Likewise, there exists $i_2 \in \mathbb{N}$ such that $\varphi(u^{n_i}) = \varphi(x)$ for all $i \geq i_2$. Then, for all $i \geq \max\{i_1, i_2\}$, we find that

$$\varphi(u_i^{n_i}) = \varphi(u_i)^{n_i} = \varphi(u)^{n_i} = \varphi(u^{n_i}) = \varphi(x).$$

This concludes the proof of the claim.

Since $\mathsf{A} \subseteq \mathsf{V}$ and the syntactic semigroup of C^+ is in A, it follows that $\mathrm{Cl}_{\mathsf{V}}(C^+)$ is clopen. Therefore, there exists $j \in \mathbb{N}$ such that $u_i^{n_i} \in \mathrm{Cl}_{\mathsf{V}}(C^+)$ for all $i \geq j$. As $\mathrm{Cl}_{\mathsf{V}}(C^+) \cap A^+ = C^+$, this means that $u_i^{n_i} \in C^+$ for all $i \geq j$. By purity, it follows that $u_i \in C^+$ for all $i \geq j$, whence $u = \lim u_i \in \mathrm{Cl}_{\mathsf{V}}(C^+)$. □

Using this lemma, we deduce the following key property of pure codes.

Proposition 3.7. *Let C be a finite pure code over a finite alphabet A and V be a pseudovariety of finite semigroups containing A. For every subgroup $H \subseteq \overline{\Omega}_A\mathsf{V}$, the following implication holds:*

$$H \cap \mathrm{Cl}_{\mathsf{V}}(C^+) \neq \emptyset \implies H \subseteq \mathrm{Cl}_{\mathsf{V}}(C^+).$$

For the proof of this proposition, we use the following: in a profinite finite semigroup S, given $s \in S$, the sequence $(s^{n!})_{n\in\mathbb{N}}$ converges to an idempotent, denoted s^ω, which is the unique idempotent in the closed subsemigroup of S generated by s [18, cf. Proposition 3.9.2]. At some point in the paper we also use the notation $s^{\omega-1}$ for the inverse of $s^{\omega+1} = s^\omega \cdot s$ in the maximal subgroup of S containing the idempotent s^ω.

Proof of Proposition 3.7. Let $u \in H$. Take $h \in H \cap \mathrm{Cl}_{\mathsf{V}}(C^+)$. Observe that $u^\omega = h^\omega$, as H is a subgroup. Since $\mathrm{Cl}_{\mathsf{V}}(C^+)$ is a closed semigroup, it follows that $u^\omega \in \mathrm{Cl}_{\mathsf{V}}(C^+) \cap \mathrm{Cl}_{\mathsf{V}}(u^+)$. This yields $u \in \mathrm{Cl}_{\mathsf{V}}(C^+)$ by Lemma 3.6. □

Let V be a pseudovariety of finite semigroups and $C \subseteq A^+$ be a code. If the syntactic semigroup of C^+ is in V, then we say that C is a V-*code*. In particular the finite A-codes are exactly the finite pure codes.

An injective homomorphism $\sigma\colon A^+ \to B^+$ is called an *encoding*; equivalently, σ is injective on A and $\sigma(A)$ is a code. The encoding σ is *pure* if $\sigma(A)$ is a pure code. We say that an encoding $\sigma\colon A^+ \to B^+$ is a V-*encoding* if $\sigma(A)$ is a V-code.

The following theorem may be attributed to Margolis, Sapir and Weil [58]. Since they did not explicitly state the theorem in this form, we give a short proof for the sake of completeness.

Theorem 3.8. *Let $\sigma\colon A^+ \to B^+$ be a homomorphism and H, K be pseudovarieties of groups. Suppose that:*

(i) $\mathsf{H} * \mathsf{K} \subseteq \mathsf{H}$;
(ii) *σ is a $\overline{\mathsf{K}}$-encoding.*

Then the pro-$\overline{\mathsf{H}}$ extension $\sigma^{\overline{\mathsf{H}}}\colon \overline{\Omega}_A\overline{\mathsf{H}} \to \overline{\Omega}_B\overline{\mathsf{H}}$ is injective.

Proof. We refer the reader to [58] for the definitions of *sagittal semigroup* and of *unambiguous product of semigroups*, which are used in this proof. By [68, Corollary 1 of Theorem 4.9] (which is based on [54, Theorem 3]), we know that the sagittal semigroup of $\sigma(A)$ is in $\overline{\mathsf{K}}$, and by [58, Proposition 2.1], if σ is injective then the extension $\sigma^{\overline{\mathsf{H}}}$ is injective provided the unambiguous product of every semigroup of $\overline{\mathsf{H}}$ with the sagittal semigroup of $\sigma(A)$ is still in $\overline{\mathsf{H}}$. By [58, Lemma 1.3], such unambiguous product indeed belongs to $\overline{\mathsf{H}}$ whenever $\mathsf{H} * \mathsf{K} \subseteq \mathsf{H}$. □

When K is the trivial pseudovariety, then condition (i) in the statement of the theorem holds trivially, while (ii) means that σ is a pure encoding. Thus, if σ is pure, then $\sigma^{\overline{\mathsf{H}}}$ is injective for every pseudovariety of groups H.

In Theorem 3.8, condition (i) cannot be omitted. We illustrate this with the following example.

Example 3.9. Let H be a nontrivial locally finite pseudovariety of groups (*locally finite* means that all finitely generated pro-V semigroups are finite). Let $A = \{\mathsf{a}\}$ be a one-letter alphabet and n be the order of $\overline{\Omega}_A\mathsf{H}$. The free pro-$\overline{\mathsf{H}}$ semigroup $\overline{\Omega}_A\overline{\mathsf{H}}$ consists of all powers a^k with k a positive integer, which are all distinct powers, together with a group of order n. Consider the homomorphism $\sigma\colon A^+ \to A^+$ defined by $\sigma(\mathsf{a}) = \mathsf{a}^n$. The syntactic semigroup of $\sigma(A^+)$ is $\mathbb{Z}/n\mathbb{Z}$, so σ is an H-encoding, hence also an $\overline{\mathsf{H}}$-encoding. The unique idempotent e of $\overline{\Omega}_A\overline{\mathsf{H}}$ satisfies $\sigma^{\overline{\mathsf{H}}}(e) = \sigma^{\overline{\mathsf{H}}}(e\mathsf{a})$, whereas $e \neq e\mathsf{a}$; hence, $\sigma^{\overline{\mathsf{H}}}\colon \overline{\Omega}_A\overline{\mathsf{H}} \to \overline{\Omega}_A\overline{\mathsf{H}}$ is not injective. Note however that $\mathsf{H} * \mathsf{H}$ is not contained in H.

3.5. **Finitely generated profinite semigroups and categories.** A profinite semigroup is called *finitely generated* when it contains a finite subset which generates a dense subsemigroup. This subsection collects a number of useful facts on finitely generated profinite semigroups. The most important one, Theorem 3.10 below, is a slight generalization of a theorem of Hunter.

Given two profinite semigroups S and R, let $\mathrm{Hom}(S, R)$ be the set of continuous semigroup homomorphisms $S \to R$. The monoid of continuous endomorphisms of a profinite semigroup S is denoted $\mathrm{End}(S)$.

Hunter proved that, when S is finitely generated, the monoid $\mathrm{End}(S)$ is a profinite semigroup for the pointwise topology [50, Proposition 1]. This was rediscovered by the first author, who moreover observed the equality between the pointwise and compact-open topologies [8, Proposition 4.13]. The next result is a generalization of this to general hom-sets $\mathrm{Hom}(S, R)$.

Theorem 3.10. *Let S and R be finitely generated profinite semigroups. Then, the compact-open topology on* $\mathrm{Hom}(S, R)$ *agrees with the pointwise topology. Under this topology,* $\mathrm{Hom}(S, R)$ *is a Stone space.*

Before the proof, we need to set up some notation. Let X and Y be two topological spaces and let F be a set of functions from X to Y. Given a compact subset K of X and an open subset O of Y, we denote $[K, O]_F$ the set of all $f \in F$ such that $f(K) \subseteq O$; these sets are a subbasis for the *compact-open* topology on F. When K runs only over singleton subsets of X, we obtain a subbasis for the pointwise topology instead.

Proof of Theorem 3.10. Let U be the topological coproduct of S, R, and $\{0\}$ and extend the multiplications of S and R by declaring all other products in U to be 0.

In this way, U becomes a finitely generated profinite semigroup. We extend each element φ of $\mathrm{Hom}(S,R)$ to a continuous endomorphism $\xi(\varphi)$ of U by mapping $R\cup\{0\}$ to 0. Note that ξ is an injective mapping.

For a compact subset $K\subseteq S$ and an open subset $O\subseteq R$, we have

$$\xi([K,O]_{\mathrm{Hom}(S,R)}) = [K,O]_{\mathrm{End}(U)}\cap \mathrm{Im}(\xi).$$

This shows that ξ is a topological embedding, both when the compact-open and pointwise topologies are considered in both the domain and range of ξ. Since the two topologies coincide on $\mathrm{End}(U)$ as observed above, it follows that the two topologies also coincide on $\mathrm{Hom}(S,R)$. Since the image of ξ has complement the union $\bigcup_{s\in S}[\{s\}, S\cup\{0\}]$, it is a Stone space as so is $\mathrm{End}(U)$ by [50, Proposition 1]. $\square$

For each pair of profinite semigroups S and R, denote by $\mathrm{Eval}_{S,R}$ the *evaluation mapping* $\mathrm{Hom}(S,R)\times S\to R$, defined by the formula $\mathrm{Eval}_{S,R}(\varphi,s)=\varphi(s)$ for every $\varphi\in\mathrm{Hom}(S,R)$ and $s\in S$. We need the following corollary of Theorem 3.10 for several of our proofs.

Corollary 3.11. *Let S and R be finitely generated profinite semigroups. Consider in $\mathrm{Hom}(S,R)$ the pointwise topology. Then the evaluation mapping $\mathrm{Eval}_{S,R}$ is continuous.*

Proof. The evaluation mapping $\mathrm{Eval}_{S,R}$ is continuous under the compact-open topology of $\mathrm{Hom}(S,R)$ [29, Corollary X.3.1], which agrees with the pointwise topology by Theorem 3.10. $\square$

4. Profinite categories

In this paper, a *graph* is a structure Γ which consists of two disjoint sets $V(\Gamma)$ and $E(\Gamma)$, called the *vertex set* and the *edge set*, together with two *adjacency mappings* $\alpha_\Gamma,\omega_\Gamma\colon V(\Gamma)\to E(\Gamma)$, called the *domain mapping* and *range mapping* respectively. The set of *composable edges*, also called *consecutive edges*, is the subset of $E(\Gamma)\times E(\Gamma)$ given by

$$D(\Gamma)=\{(u,v)\in E(\Gamma)\times E(\Gamma) : \alpha_\Gamma(u)=\omega_\Gamma(v)\}.$$

When the graph Γ is clear from the context, we may simply write V, E, D, α and ω. We say that u is an edge *from* $\alpha(u)$ *to* $\omega(u)$. An edge u from a vertex q to the same vertex q is called a *loop* at q.

Every small category C is a graph: the set V is the set of objects of C, the set E is the set of morphisms of C, and the adjacency mappings $\alpha,\omega\colon E\to V$ send a morphism to its domain and codomain respectively. The loops of C are the endomorphisms of objects of C.

The *consolidation* of a small category C is the semigroup $C_{cd}=E(C)\uplus\{0\}$ such that 0 is an element not in $E(C)$, which is a zero of C_{cd}, with the semigroup operation on C_{cd} being the following natural extension of the composition on C:

$$fg=\begin{cases} f\circ g & \text{if } (f,g) \text{ is a pair of composable edges of } C,\\ 0 & \text{otherwise.}\end{cases}$$

Green's relations on the consolidation of C restrict on $E(C)$ to relations which are called *Green's relations* of C.

A *graph homomorphism* $\Gamma \to \Delta$ is a mapping $\varphi\colon V(\Gamma) \cup E(\Gamma) \to V(\Delta) \cup E(\Delta)$ which maps vertices to vertices, edges to edges, and satisfies the following equations for every $u \in E(\Gamma)$:

$$\alpha_\Delta(\varphi(u)) = \varphi(\alpha_\Gamma(u)), \quad \omega_\Delta(\varphi(u)) = \varphi(\omega_\Gamma(u)).$$

Note that, under these conditions, $(\varphi(u), \varphi(v)) \in D(\Delta)$ for all $(u, v) \in D(\Gamma)$. When Γ and Δ are categories, we may say that φ is a *category homomorphism* when φ is a functor, which means that φ is a graph homomorphism such that $\varphi(uv) = \varphi(u)\varphi(v)$ whenever $(u, v) \in D(\Gamma)$ and $\varphi(1_q) = 1_{\varphi(q)}$ whenever $q \in V(\Gamma)$, where 1_p stands for the local identity on object p.

For a graph Γ, a *path* is a word $u \in E(\Gamma)^+$ such that $(u[i], u[i+1]) \in D_\Gamma$ for all $0 \leq i < |u| - 1$ (see Figure 1). To each vertex $q \in V(\Gamma)$ we associate an empty path 1_q. In this way, we form the free category over Γ, denoted Γ^*, given by the following data: one has $V(\Gamma^*) = V(\Gamma)$, the set $E(\Gamma^*)$ is the set of all paths (including empty paths) in Γ, the relations $\alpha(u) = \alpha(u[n-1])$ and $\omega(u) = \omega(u[0])$ hold for every nonempty path u of length n, the empty path 1_q is the local identity at q for every $q \in V(\Gamma)$, and the composition of two consecutive paths u, v is the path uv.

$$q_0 \xleftarrow{u[0]} q_1 \xleftarrow{u[1]} q_2 \longleftarrow \cdots \longleftarrow q_{n-1} \xleftarrow{u[n-1]} q_n$$

FIGURE 1. Path u of length n, seen as a composition of edges $u[i]$ from q_{i+1} to q_i, for $0 \leq i < n$.

A *topological graph* is a graph Γ with topologies on V and E such that the incidence mappings α, ω are continuous; a *topological category* is a small category, as an algebraic structure, with topologies on V and E making continuous the incidence mappings and the category operations (i.e., the composition mapping and the mapping $q \in V(C) \mapsto 1_q \in E(C)$). We say that a topological graph or category is *compact* when both V and E are compact spaces (recall that we include the Hausdorff property in the definition of compactness).

A graph or category is called *finite-vertex* when V is finite, and *finite* when both V and E are finite. A *profinite graph* is an inverse limit of finite discrete graphs; a *profinite category* is likewise an inverse limit of finite categories. When equipped with continuous category homomorphisms (i.e., both vertex and edge components of the homomorphism are continuous), profinite categories form a category in the usual sense of Category Theory.

Let $\mathcal{F}$ be a set of profinite semigroups. Let us say that the category $\mathbf{Pro}[\mathcal{F}]$ *is equipped with the pointwise topology* if it is equipped with the following topological structure:

(i) $\mathcal{F}$ has the discrete topology;
(ii) each hom-set $\mathrm{Hom}(S, R)$, with $S, R \in \mathcal{F}$, is endowed with the pointwise topology;
(iii) the set of morphisms of $\mathbf{Pro}[\mathcal{F}]$ is endowed with the coproduct topology of the spaces of the form $\mathrm{Hom}(S, R)$, with $S, R \in \mathcal{F}$.

Proposition 4.1. *Let $\mathcal{F}$ be a set of finitely generated profinite semigroups. When equipped with the pointwise topology, the category $\mathbf{Pro}[\mathcal{F}]$ is a topological category. If moreover $\mathcal{F}$ is finite, then $\mathbf{Pro}[\mathcal{F}]$ is a profinite category.*

Proof. Since the topology of the set of objects of $\mathbf{Pro}[\mathcal{F}]$ is discrete, the mapping $S \mapsto 1_S$, with domain $\mathcal{F}$, is clearly continuous. For every $S, R \in \mathcal{F}$, the set $\mathrm{Hom}(S, R)$ has the compact-open topology, by Theorem 3.10. The composition is continuous for compact-open topologies (see [29, Proposition X.3.9]). Therefore, $\mathbf{Pro}[\mathcal{F}]$ is indeed a topological category.

A coproduct of finitely many Stone spaces is itself a Stone space. Hence the morphisms of $\mathbf{Pro}[\mathcal{F}]$ form a Stone space by Theorem 3.10. Peter Jones showed that a finite-vertex topological category whose space of morphisms is a Stone space[3] is in fact a profinite category (cf. [51, Theorem 4.1]). Therefore, if $\mathcal{F}$ is finite, then $\mathbf{Pro}[\mathcal{F}]$ is profinite. □

Let Cat be the class of all finite categories. Given a graph Γ, we let $\overline{\Omega}_\Gamma\mathsf{Cat}$ denote the free profinite category on Γ; a description of $\overline{\Omega}_\Gamma\mathsf{Cat}$ may be found in a paper by the first two authors [9]. It is equipped with an inclusion mapping $\iota\colon \Gamma \to \overline{\Omega}_\Gamma\mathsf{Cat}$ with the following universal property: for every profinite category Σ and graph homomorphism $\varphi\colon \Gamma \to \Sigma$, there exists a unique continuous category homomorphism $\widehat{\varphi}\colon \overline{\Omega}_\Gamma\mathsf{Cat} \to \Sigma$ such that $\widehat{\varphi} \circ \iota = \varphi$. In the present paper, we deal exclusively with free profinite categories over finite-vertex graphs.

When Γ is finite-vertex, the canonical mapping ι extends to an inclusion mapping $\iota^*\colon \Gamma^* \to \overline{\Omega}_\Gamma\mathsf{Cat}$ which identifies Γ^* with a dense discrete subcategory of $\overline{\Omega}_\Gamma\mathsf{Cat}$; this is similar to how A^+ is identified to a dense discrete subset of $\overline{\Omega}_A\mathsf{V}$ when V is a pseudovariety of semigroups containing N. The edges of $\overline{\Omega}_\Gamma\mathsf{Cat}$ are called *pseudopaths.*

Every (profinite) monoid is viewed as a (profinite) category with only one object. Therefore, given a finite-vertex graph Γ, we may consider the continuous category homomorphism

$$\chi_\Gamma\colon \overline{\Omega}_\Gamma\mathsf{Cat} \to (\overline{\Omega}_{E(\Gamma)}\mathsf{S})^1$$

which collapses all vertices and maps each $b \in E(\Gamma)$ to the corresponding generator of $\overline{\Omega}_{E(\Gamma)}\mathsf{S}$. The next proposition goes back to [3] and is discussed in full generality in [15, Section 3 and Remark 6.13].

Proposition 4.2. *Let Γ be a finite-vertex graph. The following properties hold:*

(i) *the homomorphism χ_Γ is a faithful functor;*
(ii) *one has $\chi(w) = 1$ if and only if w is a local identity;*
(iii) *the restriction of χ_Γ to the edges of $\overline{\Omega}_\Gamma\mathsf{Cat}$ that are not local identities is injective.*

Following the property expressed in itens (ii) and (iii) of Proposition 4.2, we may view the pseudopaths of $\overline{\Omega}_\Gamma\mathsf{Cat}$ that are not local identities as elements of $\overline{\Omega}_{E(\Gamma)}\mathsf{S}$: that is, if w is a pseudopath of $\overline{\Omega}_\Gamma\mathsf{Cat}$ that is not a local identity, we may identify w with $\chi(w)$. We already apply this convention in the following statement (for a proof, see [15, Proposition 3.19]).

Proposition 4.3. *Let Γ be a finite-vertex graph. If w is a pseudopath of $\overline{\Omega}_\Gamma\mathsf{Cat}$ and $u, v \in \overline{\Omega}_{E(\Gamma)}\mathsf{S}$ are pseudowords such that the equality $w = uv$ holds in $\overline{\Omega}_{E(\Gamma)}\mathsf{S}$, then u and v are consecutive pseudopaths of $\overline{\Omega}_\Gamma\mathsf{Cat}$, and the equality $w = uv$ also holds in $\overline{\Omega}_\Gamma\mathsf{Cat}$.*

[3] Peter Jones used the term *Boolean category* to refer to a topological category whose underlying topological space is a Stone space.

Remark 4.4. Propositions 4.2 and 4.3 allow us to immediately extend to pseudopaths several definitions, properties and notations that we already introduced for pseudowords. For example, we consider the length of a pseudopath w as just being the length of w seen as a pseudoword; we know that every pseudopath of infinite length has a unique finite prefix in $\overline{\Omega}_\Gamma\mathsf{Cat}$ of length n, denoted $w[0,n)$; we may consider the pseudopath $w^{(n)} = (w[0,n))^{-1}w$; etc.

From hereon, we apply liberally the generalizations from pseudowords to pseudopaths mentioned in Remark 4.4.

Definition 4.5 (Prefix accessible pseudopaths)**.** Let Γ be an arbitrary graph. A *right-infinite* path of Γ is an element w of $E(\Gamma)^{\mathbb{N}}$ such that $w[0,n)$ is a path of Γ, for every $n \in \mathbb{N}$. Let w be a right-infinite path of a finite-vertex graph Γ. A cluster point in $\overline{\Omega}_\Gamma\mathsf{Cat}$ of the sequence $(w[0,n))_n$ is said to be *prefix accessible* by w.

Let A be an arbitrary alphabet. A right-infinite word $w \in A^{\mathbb{N}}$ is said to be *recurrent* if for every $n \in \mathbb{N}$, there is some $m > n$ such that $w[0,n) = w[m, m+n)$. Equivalently, $w \in A^{\mathbb{N}}$ is recurrent when every finite factor of w occurs infinitely often in w. For a proof of the following proposition, see [15, Corollary 6.14].

Proposition 4.6. *Let w be a right-infinite path over a finite-vertex graph Γ. Then w is recurrent if and only if there is an idempotent pseudopath in $\overline{\Omega}_\Gamma\mathsf{Cat}$ that is prefix accessible by w.*

Let C be a small category. For an edge x of C, the *right stabilizer* in C is the set

$$\mathrm{Stab}_C(x) = \{y \in E : xy = x\},$$

which we denote simply by $\mathrm{Stab}(x)$ when C is clear from context. Note that $\mathrm{Stab}(x)$ is a submonoid of the monoid of loops at $\alpha(x)$. Moreover, $\mathrm{Stab}(x)$ is a profinite semigroup when C is a profinite category. Since we view monoids as one-vertex categories, the definition of right stabilizer applies to a semigroup S as well, by considering the monoid S^1.

When S is a profinite semigroup, there is a unique $\mathcal{J}$-class $K(S)$ which has all elements of S as factors. The $\mathcal{J}$-class $K(S)$ is frequently called the *kernel of S*.

Theorem 4.7. *Let Γ be an arbitrary finite-vertex graph. For every pseudopath $x \in \overline{\Omega}_\Gamma\mathsf{Cat}$, the kernel of* $\mathrm{Stab}(x)$ *is a left-zero semigroup.*

A proof of Theorem 4.7 may be found in [15, Corollary 7.7], and in the same paper we find a discussion about other similar results going back to work of Rhodes and Steinberg [64].

The following characterization of the right stabilizer of an edge of $\overline{\Omega}_\Gamma\mathsf{Cat}$, extracted from [15, Corollary 7.8], is used in the proof of Proposition 9.9. We adopt throughout the paper the topological definition of net and subnet given by Willard [69, Definition 11.2].

Theorem 4.8. *Let x be a prefix accessible pseudopath of $\overline{\Omega}_\Gamma\mathsf{Cat}$, with Γ being a finite-vertex graph. Then an edge y of $\overline{\Omega}_\Gamma\mathsf{Cat}$ belongs to the kernel of* $\mathrm{Stab}(x)$ *if and only if there is a net $(x_i)_{i\in I}$ of finite-length prefixes of x such that $x_i \to x$ and $x_i^{-1}x \to y$.*

5. Free profinite semigroups and symbolic dynamics

In the 2000s, the first author gave a natural bijection associating to each minimal shift space X of $A^{\mathbb{Z}}$ a regular $\mathcal{J}$-class of $\overline{\Omega}_A\mathsf{S}$ [7], whenever A is a finite alphabet. This subsection aims to provide sufficient background on this mapping, which is at the core of the present paper. When checking the literature, the reader may notice that the bijection is frequently established in the larger realm of irreducible shift spaces (cf. [8]), but here we only deal with the case of minimal shift spaces.

For a proof of the next proposition, see [7, Lemma 2.3] or [34, Section 3] (in the first source, only the pseudovariety S is explicitly mentioned, but the arguments extend to all pseudovarieties containing N).

Proposition 5.1. *For every pseudovariety of semigroups V containing N and every minimal shift space $X \subseteq A^{\mathbb{Z}}$, the set $\mathrm{Cl}_{\mathsf{V}}(L(X)) \setminus L(X)$ is contained in a regular $\mathcal{J}$-class of $\overline{\Omega}_A\mathsf{V}$.*

For each minimal shift space $X \subseteq A^{\mathbb{Z}}$ and pseudovariety V containing N, we denote by $J_{\mathsf{V}}(X)$ the $\mathcal{J}$-class of $\overline{\Omega}_A\mathsf{V}$ containing $\mathrm{Cl}_{\mathsf{V}}(L(X)) \setminus L(X)$. Since $J_{\mathsf{V}}(X)$ is a regular $\mathcal{J}$-class, it contains maximal subgroups of $\overline{\Omega}_A\mathsf{V}$, and all these maximal subgroups are isomorphic profinite groups; we denote by $G_{\mathsf{V}}(X)$ a profinite group representing their isomorphism class. We say that $G_{\mathsf{V}}(X)$ is the V-*Schützenberger group of X*.

The S-Schützenberger group of X is a topological conjugacy invariant [32]. In fact, it is a flow invariant (flow equivalence is an important relation between shift spaces that is strictly coarser than topological conjugacy [56, Section 13.6]) with the same holding for many other pseudovarieties, as seen in the next theorem [35].

Theorem 5.2. *Let H be a pseudovariety of groups. If X and Y are flow equivalent minimal shift spaces, then $G_{\overline{\mathsf{H}}}(X)$ and $G_{\overline{\mathsf{H}}}(Y)$ are isomorphic profinite groups.*

Recall that if $\boldsymbol{\sigma}$ is a primitive directive sequence, then $X(\boldsymbol{\sigma})$ is a minimal shift space (Theorem 2.6). We denote the $\mathcal{J}$-class $J_{\mathsf{V}}(X(\boldsymbol{\sigma}))$ and the profinite group $G_{\mathsf{V}}(X(\boldsymbol{\sigma}))$ respectively by $J_{\mathsf{V}}(\boldsymbol{\sigma})$ and $G_{\mathsf{V}}(\boldsymbol{\sigma})$. In case φ is a primitive substitution, we also write $J_{\mathsf{V}}(\varphi)$ and $G_{\mathsf{V}}(\varphi)$ instead of, respectively, $J_{\mathsf{V}}(X(\varphi))$ and $G_{\mathsf{V}}(X(\varphi))$.

For a pseudoword $w \in \overline{\Omega}_A\mathsf{V}$, we denote by $\mathrm{fac}(w)$ the set of all words $u \in A^+$ such that u is a factor of w, assuming $\mathsf{N} \subseteq \mathsf{V}$ so that A^+ embeds in $\overline{\Omega}_A\mathsf{V}$.

Proposition 5.3. *Let X be a minimal shift space of $A^{\mathbb{Z}}$ and V be a pseudovariety of semigroups containing LSl. Every infinite-length factor of an element of $J_{\mathsf{V}}(X)$ also belongs to $J_{\mathsf{V}}(X)$. More precisely, for every infinite-length pseudoword $w \in \overline{\Omega}_A\mathsf{V}$, we have $w \in J_{\mathsf{V}}(X)$ if and only if $\mathrm{fac}(w) \subseteq L(X)$.*

This proposition is from [7, Lemma 2.3]; alternatively, it is found in [9, Theorem 6.3] with a very different proof. In the first of these two references only the case $\mathsf{V} = \mathsf{S}$ is explicitly mentioned, but the arguments hold whenever $\mathsf{V} \supseteq \mathsf{LSl}$.

Remark 5.4. The hypothesis in Proposition 5.3 that V contains LSl is necessary to guarantee that $\mathrm{Cl}_{\mathsf{V}}(A^*uA^*)$ is open when $u \in A^+$, a property crucially used in the proof. The reason why $\mathrm{Cl}_{\mathsf{V}}(A^*uA^*)$ is then open is that A^*uA^* is an LSl-recognizable language [61, Theorem 5.2.1], which entails the desired topological property by Theorem 3.1. Proposition 5.3 fails if $\mathsf{V} = \mathsf{LI}$, for example; indeed, if $X \subseteq A^{\mathbb{Z}}$ is any minimal shift space, then $e = eue$ for every idempotent $e \in \overline{\Omega}_A\mathsf{LI}$ and word $u \in A^+$, entailing $\mathrm{fac}(e) = A^+$.

Corollary 5.5. *Let X be a minimal shift space of $A^{\mathbb{Z}}$. If V, W are pseudovarieties of semigroups such that $\mathsf{LSI} \subseteq \mathsf{W} \subseteq \mathsf{V}$, then $p_{\mathsf{V},\mathsf{W}}^{-1}(J_{\mathsf{W}}(X)) = J_{\mathsf{V}}(X)$.*

Proof. Take $u \in J_{\mathsf{W}}(X)$. Since $p_{\mathsf{V},\mathsf{W}}$ is onto, we may consider $\hat{u} \in \overline{\Omega}_A\mathsf{V}$ such that $u = p_{\mathsf{V},\mathsf{W}}(\hat{u})$. Because V, W contain LSI, we know that $\mathrm{fac}(u) = \mathrm{fac}(\hat{u})$. Since both u and $\hat{u}$ are infinite pseudowords, it follows from Proposition 5.3 that $\hat{u} \in J_{\mathsf{V}}(X)$. $\square$

The property stated in the next proposition is new in its full generality. The special case of the so called pseudovarieties *closed under concatenation* is treated in the last section of the paper [9]. We point out that the proof of the proposition uses the property, first shown independently in the papers [9, 48], that $\mathrm{Cl}_{\mathsf{S}}(L)$ is factorial in $\overline{\Omega}_A\mathsf{S}$ whenever L is a factorial subset of A^+.

Proposition 5.6. *Let X be a minimal shift space of $A^{\mathbb{Z}}$. If the pseudovariety of semigroups V contains LSI, then the equality*

$$J_{\mathsf{V}}(X) = \mathrm{Cl}_{\mathsf{V}}(L(X)) \setminus A^+$$

holds.

Proof. Recall that $\mathrm{Cl}_{\mathsf{V}}(L(X)) \setminus A^+ \subseteq J_{\mathsf{V}}(X)$ by definition of $J_{\mathsf{V}}(X)$. Conversely, let $u \in J_{\mathsf{V}}(X)$. By Corollary 5.5, there is $\hat{u} \in J_{\mathsf{S}}(X)$ such that $u = p_{\mathsf{S},\mathsf{V}}(\hat{u})$. Since $L(X)$ is a factorial subset of A^+, the topological closure of $\mathrm{Cl}_{\mathsf{S}}(L(X))$ is factorial in $\overline{\Omega}_A\mathsf{S}$, by [9, Proposition 2.4]. Because $J_{\mathsf{S}}(X)$ intersects $\mathrm{Cl}_{\mathsf{S}}(L(X))$, it follows that $\hat{u} \in \mathrm{Cl}_{\mathsf{S}}(L(X))$. As $p_{\mathsf{S},\mathsf{V}}$ restricts to the identity on A^+, and by continuity of $p_{\mathsf{S},\mathsf{V}}$, we conclude that $u \in \mathrm{Cl}_{\mathsf{V}}(L(X))$. $\square$

Remark 5.7. By Proposition 5.6, if X is a minimal shift space of $A^{\mathbb{Z}}$ and V is a pseudovariety of semigroups containing LSI, then $\mathrm{Cl}_{\mathsf{V}}(L(X))$ is factorial in $\overline{\Omega}_A\mathsf{V}$, as the finite factors of elements of $J_{\mathsf{V}}(X)$ belong to $L(X)$ by Proposition 5.3. But one may have a pseudovariety V containing LSI and a shift space X not minimal such that $\mathrm{Cl}_{\mathsf{V}}(L(X))$ is not a factorial subset of $\overline{\Omega}_A\mathsf{V}$ (cf. [35, Example 3.4]).

Recall that if V contains the pseudovariety LI, then every infinite-length pseudoword $w \in \overline{\Omega}_A\mathsf{V} \setminus A^+$ has a well-defined right infinite prefix $\overrightarrow{w} \in A^{\mathbb{N}}$ and left-infinite suffix $\overleftarrow{x} \in A^{\mathbb{Z}_-}$ (see Section 3.3). Consider the mapping $\Bbbk \colon \overline{\Omega}_A\mathsf{V} \setminus A^+ \to A^{\mathbb{Z}}$ defined by $\Bbbk(x) = \overleftarrow{x} \cdot \overrightarrow{x}$. The next result shows that this mapping characterizes the $\mathcal{H}$-classes of $J_{\mathsf{V}}(X)$; it was originally proved by the first author [6, Theorem 3.3] (see also [9, Lemma 6.6]).

Lemma 5.8. *Let X be a minimal shift space and V be a pseudovariety of semigroups containing LI. Then, for every $u, v \in J_{\mathsf{V}}(X)$, the equality $\Bbbk(u) = \Bbbk(v)$ holds if and only if $u \mathrel{\mathcal{H}} v$. More precisely, for every $u, v \in J_{\mathsf{V}}(X)$ we have $\overrightarrow{u} = \overrightarrow{v}$ if and only if $u \mathrel{\mathcal{R}} v$, and $\overleftarrow{u} = \overleftarrow{v}$ if and only if $u \mathrel{\mathcal{L}} v$.*

It follows that the mapping $\bar{\Bbbk}(u/\mathcal{H}) = \Bbbk(u)$, with $u \in J_{\mathsf{V}}(X)$, is well defined. For an element $x \in A^{\mathbb{Z}}$, let

$$x(-\infty, 0) = \cdots x[-2]x[-1] \in A^{\mathbb{Z}_-}, \quad x[0, \infty) = x[0]x[1] \cdots \in A^{\mathbb{N}}.$$

Lemma 5.8 says in particular that the mapping $\bar{\Bbbk}$ is a bijection between the $\mathcal{H}$-classes of $J_{\mathsf{V}}(X)$ and the following set:

$$\{y(-\infty, 0) \cdot x[0, \infty) : x, y \in X\}.$$

The next result locates the maximal subgroups in $J_{\mathsf{V}}(X)$.

Proposition 5.9 ([10, Lemma 5.3])**.** *Let X be a minimal shift space and V be a pseudovariety of semigroups containing LSI. An $\mathcal{H}$-class H of $J_{\mathsf{V}}(X)$ contains an idempotent if and only if $\bar{\Bbbk}(H) \in X$. Moreover, every element of X is of the form $\Bbbk(e)$ for a unique idempotent e of $J_{\mathsf{V}}(X)$.*

According to the following corollary, the shape of the $\mathcal{J}$-class $J_{\mathsf{V}}(X)$ is independent of the pseudovariety V, provided $\mathsf{LSI} \subseteq \mathsf{V}$.

Corollary 5.10. *Let V, W be pseudovarieties of semigroups such that $\mathsf{LSI} \subseteq \mathsf{W} \subseteq \mathsf{V}$. Let X be a minimal shift space of $A^{\mathbb{Z}}$. The following properties hold:*

(i) *If H is an $\mathcal{H}$-class of $J_{\mathsf{V}}(X)$, then the set $p_{\mathsf{V},\mathsf{W}}(H)$ is an $\mathcal{H}$-class of $J_{\mathsf{W}}(X)$.*
(ii) *If K is an $\mathcal{H}$-class of $J_{\mathsf{W}}(X)$, then the set $p_{\mathsf{V},\mathsf{W}}^{-1}(K)$ is an $\mathcal{H}$-class of $J_{\mathsf{V}}(X)$.*

Proof. Note that $\Bbbk(p_{\mathsf{V},\mathsf{W}}(u)) = \Bbbk(u)$ for every $u \in \overline{\Omega}_A\mathsf{V}$. The corollary now follows immediately from Corollary 5.5. $\square$

We apply again Proposition 5.9 to show the following lemma.

Lemma 5.11. *Let X be a minimal shift space of $A^{\mathbb{Z}}$ and V be a pseudovariety of semigroups containing LSI. Let $u, v \in \overline{\Omega}_A\mathsf{V}$ and $x, y \in A^*$ be such that $|x| = |y|$.*

(i) *If $xu \in J_{\mathsf{V}}(X)$ and $xu = yv$, then we have $x = y$ and $u = v$.*
(ii) *If $ux \in J_{\mathsf{V}}(X)$ and $ux = vy$, then we have $x = y$ and $u = v$.*

Proof. Suppose that $xu = yv \in J_{\mathsf{V}}(X)$. As V contains LI, every pseudoword of $\overline{\Omega}_A\mathsf{V}$ of length at least n has a unique prefix and a unique suffix of length n, whenever $n \in \mathbb{N}$. In particular, we have $x = y$. Since x has finite length and xu has infinite length, both u, v and have infinite length, whence $u, v \in J_{\mathsf{V}}(X)$ by Proposition 5.3. As $\overline{\Omega}_A\mathsf{V}$ is stable, it follows that $u \mathrel{\mathcal{L}} xu = xv \mathrel{\mathcal{L}} v$. Also because x has finite length, we have

$$x \cdot \overrightarrow{u} = \overrightarrow{xu} = \overrightarrow{xv} = x \cdot \overrightarrow{v},$$

thus $\overrightarrow{u} = \overrightarrow{v}$. We deduce from Lemma 5.8 that $u \mathrel{\mathcal{H}} v$.

By Green's Lemma, the mapping $H_u \to H_{xu}$ sending each element w in the $\mathcal{H}$-class H_u to xw is a bijection (see [66, Lemma A.3.1]). In particular, since $u, v \in H_u$, it follows from the equality $xu = xv$ that $u = v$. This shows (i), and the proof of (ii) follows by symmetry of arguments. $\square$

Remark 5.12. When $\mathsf{V} = \mathsf{S}$, Lemma 5.11 is a special case of Proposition 3.5. While Proposition 3.5 still holds if we replace S by many other pseudovarieties V [15, Proposition 6.4], it does not hold for all V containing LSI (cf. [14, Proposition 6.2]).

The following proposition is used in the proof of Theorem 10.17.

Proposition 5.13. *Let X be a minimal shift space and V be a pseudovariety of semigroups containing LSI. Let e, f be idempotents in $J_{\mathsf{V}}(X)$. Let n be a positive integer. The following conditions are equivalent:*

(i) *the equality $\Bbbk(e) = T^n(\Bbbk(f))$ holds;*
(ii) *one has $pe = fp$ for some word p of length n;*
(iii) *one has $pe \mathrel{\mathcal{H}} fp$ for some word p of length n.*

Moreover, if p is a word of length n such that $pe \mathrel{\mathcal{H}} fp$, then pe and fp belong to $J_{\mathsf{V}}(X)$, and the equalities $p = f[0, n) = e[-n, -1]$ and $pe = fp$ hold.

Proof. (ii)⇒(iii). This implication is trivial.

(iii)⇒(i). From $pe \mathrel{\mathcal{H}} fp$ we get, on one hand, the equalities $\overrightarrow{pe} = \overrightarrow{fp} = \overrightarrow{f}$, whence

$$e[0,\infty) = f[n,\infty); \tag{3}$$

and, on the other hand, the equalities $\overleftarrow{e} = \overleftarrow{pe} = \overleftarrow{fp}$, thus

$$e(-\infty,-1] = f(-\infty,n-1]. \tag{4}$$

Combining (3) and (4), we obtain $\Bbbk(e) = T^n(\Bbbk(f))$.

(i)⇒(ii). Assuming that $\Bbbk(e) = T^n(\Bbbk(f))$, we have $e[-n,-1] = f[0,n)$. Set $p = e[-n,-1]$, and consider the factorization $f = pt$. By Proposition 5.3, the infinite-length pseudoword t belongs to $J_{\mathsf{V}}(X)$, and so $f \mathrel{\mathcal{L}} t$ because profinite semigroups are stable. As f is idempotent, we then have $t = tf$.

Consider the infinite-length pseudoword $g = tp$. Since $f = f^2 = ptpt = pgp$, we know that $g \in J_{\mathsf{V}}(X)$ by Proposition 5.3. Note that $pg = fp$. Hence, it suffices to show that $g = e$ to conclude the proof of the implication (i)⇒(ii). We first check that g is idempotent: indeed, as $pt = f$ and $t = tf$, we have $g^2 = tptp = tfp = tp = g$. Then, it follows from the equality $pg = fp$ and the already established implication (iii)⇒(i) (with g playing the role of e in that implication) that

$$\Bbbk(g) = T^n(\Bbbk(f)) = \Bbbk(e).$$

This implies $g \mathrel{\mathcal{H}} e$ by Lemma 5.8, which means that $g = e$ as g, e are idempotents.

We have therefore established the chain of equivalences (i)⇔(ii)⇔(iii). It remains to justify why the last sentence in the proposition is valid. Suppose that $pe \mathrel{\mathcal{H}} fp$ for a word p of length n. Then in fact we have $pe = fp$, as we already proved the implication (iii)⇒(ii). Note that f is a prefix of pe, whence $p = f[0,n)$, by the unicity of the prefix of length n in any infinite-length pseudoword of $\overline{\Omega}_A\mathsf{V}$. Similarly, p is the suffix $e[-n,-1]$ of e. Let t be such that $e = tp$. Then $e = e^2 = tpe$. Since pe is an infinite-length factor of e, it follows from Proposition 5.3 that $pe \in J_{\mathsf{V}}(X)$. Similarly, we have $fp \in J_{\mathsf{V}}(X)$. □

6. Profinite images of directive sequences

In this section, we consider a directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$, with $\sigma_n \colon A_{n+1}^+ \to A_n^+$, and a pseudovariety of semigroups V containing N.

Recall that $\sigma_n^{\mathsf{V}} \colon \overline{\Omega}_{A_{n+1}}\mathsf{V} \to \overline{\Omega}_{A_n}\mathsf{V}$ and $\sigma_{m,n}^{\mathsf{V}} \colon \overline{\Omega}_{A_m}\mathsf{V} \to \overline{\Omega}_{A_n}\mathsf{V}$ are the unique continuous homomorphisms extending $\sigma_n \colon A_{n+1}^+ \to A_n^+$ and $\sigma_{m,n} \colon A_{m+1}^+ \to A_n^+$, respectively, and that $\sigma_{m,n}^{\mathsf{V}} = \sigma_m^{\mathsf{V}} \circ \cdots \circ \sigma_{n-1}^{\mathsf{V}}$ (cf. Subsection 3.3).

Definition 6.1. The V-*image* of $\boldsymbol{\sigma}$, denoted $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, is the intersection

$$\bigcap_{n\in\mathbb{N}} \mathrm{Im}(\sigma_{0,n}^{\mathsf{V}}).$$

By a *profinite image* of $\boldsymbol{\sigma}$ we mean a set of the form $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ for some pseudovariety V.

Remark 6.2. Since the sequence of sets $\mathrm{Im}(\sigma_{0,n}^{\mathsf{V}})$ is a chain for the reverse inclusion, we have

$$\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) = \bigcap_{k\in\mathbb{N}} \mathrm{Im}(\sigma_{0,n_k}^{\mathsf{V}})$$

for every strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of nonnegative integers.

Remark 6.3. The set $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a closed subsemigroup of $\overline{\Omega}_{A_0}\mathsf{V}$; indeed, $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a nonempty compact space by the finite intersection property of compact spaces.

We next register that a contraction does not change the V-image.

Lemma 6.4. *If $\boldsymbol{\tau}$ is a contraction of $\boldsymbol{\sigma}$, then* $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\tau}) = \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$.

Proof. Let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers with $n_0 = 0$ such that $\boldsymbol{\tau} = (\sigma_{n_k,n_{k+1}})_{k\in\mathbb{N}}$. Set $\tau_k = \sigma_{n_k,n_{k+1}}$ for each $k \in \mathbb{N}$. As $\tau_{0,k} = \sigma_{0,n_k}$, we have $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\tau}) = \bigcap_{k\in\mathbb{N}} \mathrm{Im}(\sigma^{\mathsf{V}}_{0,n_k})$, and so $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\tau}) = \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ in view of Remark 6.2. $\square$

The next lemma is used in several subsequent proofs.

Lemma 6.5. *Consider the following sets:*

(i) *the set C of cluster points, in the space $\overline{\Omega}_{A_0}\mathsf{V}$, of sequences $(w_n)_{n\in\mathbb{N}}$ of pseudowords such that $w_n \in \mathrm{Im}(\sigma^{\mathsf{V}}_{0,n})$ for every $n \in \mathbb{N}$;*

(ii) *the set D of cluster points, in the space $\overline{\Omega}_{A_0}\mathsf{V}$, of sequences $(w_n)_{n\in\mathbb{N}}$ of words such that $w_n \in \mathrm{Im}(\sigma_{0,n})$ for every $n \in \mathbb{N}$.*

Then the equalities $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) = C = D$ *hold.*

Proof. We first establish the inclusion $C \subseteq \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. Let $w \in C$. Take a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers and a sequence $(w_k)_{k\in\mathbb{N}}$ of elements of $\overline{\Omega}_{A_0}\mathsf{V}$ converging to w such that $w_k \in \mathrm{Im}(\sigma^{\mathsf{V}}_{0,n_k})$ for each $k \in \mathbb{N}$. Fix $r \in \mathbb{N}$. If $k \geq r$, then $n_k \geq r$ and so the inclusion $\mathrm{Im}(\sigma^{\mathsf{V}}_{0,n_k}) \subseteq \mathrm{Im}(\sigma^{\mathsf{V}}_{0,r})$ holds. Hence, $w = \lim_{k\geq r} w_k$ is in the closed subspace $\mathrm{Im}(\sigma^{\mathsf{V}}_{0,r})$. Since r is arbitrary, this shows that $w \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$.

We next prove the inclusion $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq D$. Let $w \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. Let n be a positive integer. Since w is in the topological closure of $\sigma_{0,n}(A_n^+)$, there is $u_n \in (A_n)^+$ such that $d(w, \sigma_{0,n}(u_n)) < \frac{1}{n}$. Then we have $\sigma_{0,n}(u_n) \to u$. This yields $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq D$.

Finally, since $\mathrm{Im}(\sigma_{0,n}) \subseteq \mathrm{Im}(\sigma^{\mathsf{V}}_{0,n})$, the inclusion $D \subseteq C$ is trivial. This establishes the lemma. $\square$

There is a natural relationship between profinite images of $\boldsymbol{\sigma}$ relative to comparable pseudovarieties.

Proposition 6.6. *Let V, W be pseudovarieties of semigroups such that $\mathsf{N} \subseteq \mathsf{W} \subseteq \mathsf{V}$. The following equality holds:*

$$\mathrm{Im}_{\mathsf{W}}(\boldsymbol{\sigma}) = p_{\mathsf{V},\mathsf{W}}(\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})).$$

Proof. As $\sigma^{\mathsf{W}}_{0,n} \circ p_{\mathsf{V},\mathsf{W}} = p_{\mathsf{V},\mathsf{W}} \circ \sigma^{\mathsf{V}}_{0,n}$, we clearly have $\mathrm{Im}(\sigma^{\mathsf{W}}_{0,n}) = p_{\mathsf{V},\mathsf{W}}(\mathrm{Im}(\sigma^{\mathsf{V}}_{0,n}))$ and so we immediately obtain

$$p_{\mathsf{V},\mathsf{W}}(\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})) \subseteq \bigcap_{n\in\mathbb{N}} p_{\mathsf{V},\mathsf{W}}(\mathrm{Im}(\sigma^{\mathsf{V}}_{0,n})) = \mathrm{Im}_{\mathsf{W}}(\boldsymbol{\sigma}).$$

Conversely, let $w \in \mathrm{Im}_{\mathsf{W}}(\boldsymbol{\sigma})$. For each $n \in \mathbb{N}$, we may take $v_n \in \mathrm{Im}(\sigma^{\mathsf{V}}_{0,n})$ such that $w = p_{\mathsf{V},\mathsf{W}}(v_n)$. Let v be a cluster point of the sequence $(v_n)_n$. By continuity, we get $w = p_{\mathsf{V},\mathsf{W}}(v)$. On the other hand, we have $v \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ by Lemma 6.5. This establishes the inclusion $\mathrm{Im}_{\mathsf{W}}(\boldsymbol{\sigma}) \subseteq p_{\mathsf{V},\mathsf{W}}(\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}))$, thus concluding the proof. $\square$

Denote by $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ the set of pseudowords of $\overline{\Omega}_{A_0}\mathsf{V}$ that are cluster points of some sequence $(w_n)_{n\in\mathbb{N}}$ such that $w_n \in \sigma_{0,n}(A_n)$ for every $n \in \mathbb{N}$. A standard argument, which we include here for the sake of completeness, yields the following fact.

Lemma 6.7. *The set* $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ *is a closed subspace of* $\mathrm{Im}_{\mathsf{W}}(\boldsymbol{\sigma})$.

Proof. By Lemma 6.5, the set $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ is contained in $\mathrm{Im}_{\mathsf{W}}(\boldsymbol{\sigma})$. Since $\mathrm{Im}_{\mathsf{W}}(\boldsymbol{\sigma})$ is closed in $\overline{\Omega}_{A_0}\mathsf{V}$, what we want to show is that $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ is a closed subspace of $\overline{\Omega}_{A_0}\mathsf{V}$.

Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ converging to a pseudoword w. For every positive integer k, let $r_k \in \mathbb{N}$ be such that $d(w, w_{r_k}) < \frac{1}{2k}$. We recursively build a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers, together with a sequence $(a_k)_{k\geq 1}$ of letters such that $a_k \in A_{n_k}$, as follows:

- $n_1 = 1$ and a_1 is some element of A_1;
- if $k > 1$, then $n_k \in \mathbb{N}$ and $a_k \in A_{n_k}$ are chosen such that $n_k > n_{k-1}$ and $d(w_{r_k}, \sigma_{n_k}(a_k)) < \frac{1}{2k}$ (such n_k and a_k must exist by the definition of the set $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$, to which w_{r_k} belongs).

Then we have $d(w, \sigma_{n_k}(a_k)) \leq d(w, w_{r_k}) + d(w_{r_k}, \sigma_{n_k}(a_k)) < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}$ for every $k \geq 1$. It follows that $\lim \sigma_{n_k}(a_k) = w$, whence $w \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$. This proves that $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ is a closed subspace of $\overline{\Omega}_{A_0}\mathsf{V}$. □

We next establish some properties of the set $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ in the case in which we focus: the case where $\boldsymbol{\sigma}$ is primitive.

Proposition 6.8. *Let* $\boldsymbol{\sigma}$ *be a primitive directive sequence. The following properties hold:*

(i) $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$;
(ii) $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) = \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \cdot \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) = \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \cdot \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$;
(iii) $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ *is contained in a regular* $\mathcal{J}$*-class of the semigroup* $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$.

Proof. (i). Since the set $\sigma_{0,n}(A_n)$ is contained in $L(\boldsymbol{\sigma})$ for every $n \in \mathbb{N}$, we clearly have $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq \mathrm{Cl}_{\mathsf{V}}(L(\boldsymbol{\sigma}))$. Moreover, the fact that $\boldsymbol{\sigma}$ is primitive also ensures that $\lim_{n\to\infty} \min\{|\sigma_{0,n}(a)| : a \in A_n\} = \infty$. Therefore, and in view of Lemma 6.7, we have indeed $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$.

(ii). We show the equality $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) = \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \cdot \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$.

The inclusion $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \cdot \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ clearly holds as $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ and $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a semigroup. Conversely, let $w \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. Then, by Lemma 6.5, we have $w = \lim \sigma_{0,n_k}(u_k)$ for some strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers and a sequence $(u_k)_{k\in\mathbb{N}}$ such that $u_k \in (A_{n_k})^+$ for every $k \in \mathbb{N}$. Since $\boldsymbol{\sigma}$ is primitive, for each $k \in \mathbb{N}$ we may choose some $r(k) \in \mathbb{N}$ such that the word $w_k = \sigma_{n_k, n_{r(k)}}(u_{r(k)})$ has length at least two. Moreover, we may build the sequence $(r(k))_{k\in\mathbb{N}}$ so that it is strictly increasing. For such a sequence, we have $\lim_{k\to\infty} \sigma_{0,n_k}(w_k) = \lim_{k\to\infty} \sigma_{0,n_{r(k)}}(u_{r(k)}) = w$.

For each $k \in \mathbb{N}$, since $|w_k| \geq 2$, there are $a_k \in A_{n_k}$ and $s_k \in (A_{n_k})^+$ such that $u_k = a_k s_k$. Let (a, s) be an accumulation point in $\overline{\Omega}_{A_0}\mathsf{V} \times \overline{\Omega}_{A_0}\mathsf{V}$ of the sequence $(\sigma_{0,n_k}(a_k), \sigma_{0,n_k}(s_k))_k$. Since $\lim \sigma_{0,n_k}(a_k)\sigma_{0,n_k}(s_k) = \lim \sigma_{0,n_k}(u_k) = w$, we have $w = as$ by continuity of the multiplication. Note also that $(a, s) \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \times \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ by the definition of $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ and by Lemma 6.5. Therefore, we have $w \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \cdot \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. This concludes the proof of the equality $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) = \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \cdot \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. The proof of the equality $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) = \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \cdot \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ is entirely similar.

(iii). Let $a, b \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$. Then there are strictly increasing sequences $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ of positive integers such that

$$a = \lim \sigma_{0,n_k}(a_k) \qquad \text{and} \qquad b = \lim \sigma_{0,m_k}(b_k)$$

for some sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ for which we have $a_k \in A_{n_k}$ and $b_k \in B_{m_k}$ for every $k \in \mathbb{N}$. Since $\boldsymbol{\sigma}$ is primitive, for each $k \in \mathbb{N}$ we may choose some $r(k) > k$ such that $n_{r(k)} > m_k$ and $\mathrm{fac}(\sigma_{m_k,n_{r(k)}}(A_{n_{r(k)}})) \supseteq A_{m_k}$. Moreover, the sequence $(r(k))_{k\in\mathbb{N}}$ may chosen to be strictly increasing. Going on with such a choice, we have, for each $k \in \mathbb{N}$, a factorization

$$\sigma_{m_k,n_{r(k)}}(a_{n_{r(k)}}) = p_k b_k s_k$$

with $p_k, s_k \in (A_{n_k})^*$, and with at least one of the words p_k, s_k being nonempty. By compactness, we may extract from the sequence $(\sigma_{0,m_k}(p_k), \sigma_{0,m_k}(s_k))_{k\in\mathbb{N}}$ a subsequence $(\sigma_{0,m_{k_i}}(p_{k_i}), \sigma_{0,m_{k_i}}(s_{k_i}))_{i\in\mathbb{N}}$ converging in $(\overline{\Omega}_{A_0}\mathsf{V})^1 \times (\overline{\Omega}_{A_0}\mathsf{V})^1$ to some pair (p, s). We then have

$$a = \lim_{i\in\mathbb{N}} \sigma_{0,n_{r(k_i)}}(a_{n_{r(k_i)}}) = \lim_{i\in\mathbb{N}} \Big(\sigma_{0,m_{k_i}}(p_{k_i}) \cdot \sigma_{0,m_{k_i}}(b_{k_i}) \cdot \sigma_{0,m_{k_i}}(s_{k_i})\Big) = pbs. \tag{5}$$

Note that $p, s \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \cup \{\varepsilon\}$ by Lemma 6.5. This shows that $a \leq_{\mathcal{J}} b$ in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. Since a, b are arbitrary elements of $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$, we then get $a \mathrel{\mathcal{J}} b$ in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. Going back to (5), and taking $a = b$, we get $a = pas$. Since $ps = \lim \sigma_{0,m_k}(p_k s_k)$ and $p_k s_k \neq \varepsilon$ for every $k \in \mathbb{N}$, at least one of the pseudowords p, s is not the empty word. Without loss of generality, assume that $p \neq \varepsilon$. From $a = pas$, we obtain $a = p^k a s^k$ for every $k \in \mathbb{N}$, whence $a = p^\omega a s^\omega$, which in turn yields $a \leq_{\mathcal{J}} p^\omega$. On the other hand, for some $c \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ we have $p^\omega \leq_{\mathcal{J}} c$ in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, by the already shown item (ii). But we already proved that all elements of $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ are $\mathcal{J}$-equivalent in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. Joining all pieces, we see that $p^\omega \mathrel{\mathcal{J}} a$ in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. This shows that a is regular in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, concluding the proof that $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ is contained in a regular $\mathcal{J}$-class of the profinite semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. □

The set $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ is used in the proof of several results, starting with the next one.

Theorem 6.9. *Let* V *be a pseudovariety of semigroups containing* N*, and let* $\boldsymbol{\sigma}$ *be a primitive directive sequence. The set* $J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ *is a regular* $\mathcal{J}$*-class of the semigroup* $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$*.*

Proof. By Proposition 6.8, the set $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ is contained in a regular $\mathcal{J}$-class J of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. We also know by Proposition 6.8 that $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$, and so we already know that $J \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$.

Conversely, let u be an element of $J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. By Proposition 6.8, there are idempotents $e \in J$ and $f \in J$ such that $u = eu = uf$. In particular, it suffices to show that $e \in u\,\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ to get $u \in J$.

As u and e belong to the same $\mathcal{J}$-class of the stable semigroup $\overline{\Omega}_{A_0}\mathsf{V}$, the equality $u = eu$ yields the existence of some pseudoword x such that $ux = e$. We may assume that $x = fxe$, because e and f are idempotents and $u = uf$. Under such assumption, the pseudowords x and e belong the same $\mathcal{L}$-class of $\overline{\Omega}_{A_0}\mathsf{V}$. Therefore, the equality

$$uH_x = H_e$$

holds by Green's Lemma (cf. [66, Lemma A.3.1]), where H_s denotes the $\mathcal{H}$-class in $\overline{\Omega}_{A_0}\mathsf{V}$ of the pseudoword s.

From the equalities $x = fxe$ and $e = ux$, and from $e \mathrel{\mathcal{J}_{\overline{\Omega}_{A_0}\mathsf{V}}} f$, we obtain $f \mathrel{\mathcal{R}_{\overline{\Omega}_{A_0}\mathsf{V}}} x \mathrel{\mathcal{L}_{\overline{\Omega}_{A_0}\mathsf{V}}} e$, by stability of $\overline{\Omega}_{A_0}\mathsf{V}$; on the other hand, since $e, f \in J$, there is $w \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ such that $f \mathrel{\mathcal{R}_{\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})}} w \mathrel{\mathcal{L}_{\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})}} e$. This implies that $w \in H_x$, thus $uw \in H_e$. Since H_e is a profinite group with identity e, it follows that $(uw)^\omega = e$. That is, for the pseudoword $z = w(uw)^{\omega-1}$, we have $e = uz$. Since $u, w \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, the pseudoword z belongs to $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. This shows that indeed $e \in u\,\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, which, as already noted, yields $u \in J$. This establishes the inclusion $J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J$. $\square$

Corollary 6.10. *Let* V *be a pseudovariety of semigroups containing* N*, and let* $\boldsymbol{\sigma}$ *be a primitive directive sequence. The inclusion* $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$ *holds if and only if the profinite semigroup* $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ *is simple.*

Proof. By Theorem 6.9, the intersection $J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a $\mathcal{J}$-class of the semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. Therefore, the $\mathcal{J}$-relation of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is universal in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ if and only if the inclusion $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$ holds. $\square$

In the next example we see that $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ may not be contained in $J_{\mathsf{V}}(\boldsymbol{\sigma})$.

Example 6.11. Let σ be the primitive substitution on the alphabet $A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ defined by

$$\sigma \colon \mathtt{a} \mapsto \mathtt{ac},\ \mathtt{b} \mapsto \mathtt{bcb},\ \mathtt{c} \mapsto \mathtt{ba},$$

and consider the constant directive sequence $\boldsymbol{\sigma} = (\sigma, \sigma, \ldots)$. Let w be any cluster point in $\overline{\Omega}_A\mathsf{S}$ of the sequence $\sigma^{2n}(\mathtt{a})$. Note that $w \in \Lambda_{\mathsf{S}}(\boldsymbol{\sigma})$, and so we have $w \in J_{\mathsf{S}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$. Since $\sigma^2(\mathtt{a}) = \mathtt{acba}$, the pseudoword w starts and ends with $\mathtt{a}$, thus $\mathtt{a}^2$ is a factor of w^2. On the other hand, since $\mathtt{a}^2$ is not a factor of any of the words $\sigma^{2n}(\mathtt{a})$, we also know that $\mathtt{a}^2$ is not a factor of w. Therefore the element w^2 of $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$ does not belong to $J_{\mathsf{S}}(\boldsymbol{\sigma})$.

We proceed to see how the V-images of the tails of a directive sequence are related with each other.

Lemma 6.12. *The equality* $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)}) = \sigma^{\mathsf{V}}_{k,k+n}\big(\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(k+n)})\big)$ *holds for all* $k, n \in \mathbb{N}$.

Proof. As $\boldsymbol{\sigma}^{(k+n)} = (\boldsymbol{\sigma}^{(k)})^{(n)}$, it is enough to show the lemma for the case $k = 0$.

Since $\sigma^{\mathsf{V}}_{0,n}(\mathrm{Im}(\sigma^{\mathsf{V}}_{n,n+r})) = \mathrm{Im}(\sigma^{\mathsf{V}}_{0,n+r})$ for every $r \in \mathbb{N}$, one clearly has

$$\sigma^{\mathsf{V}}_{0,n}(\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})) = \sigma^{\mathsf{V}}_{0,n}\left(\bigcap_{r\in\mathbb{N}} \mathrm{Im}(\sigma^{\mathsf{V}}_{n,n+r})\right) \subseteq \bigcap_{r\in\mathbb{N}} \mathrm{Im}(\sigma^{\mathsf{V}}_{0,n+r})$$

for each $r \in \mathbb{N}$. This establishes the inclusion $\sigma^{\mathsf{V}}_{0,n}(\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})) \subseteq \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$.

Conversely, let $u \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. For each $r \in \mathbb{N}$, let $u_r \in \overline{\Omega}_{A_r}\mathsf{S}$ be such that $u = \sigma^{\mathsf{V}}_{0,r}(u_r)$. By compactness of $\overline{\Omega}_{A_n}\mathsf{V}$, there is a strictly increasing sequence $(r_m)_{m\in\mathbb{N}}$ of integers greater than n such that the sequence $(\sigma^{\mathsf{V}}_{n,r_m}(u_{r_m}))_{m\in\mathbb{N}}$ converges to some pseudoword w of $\overline{\Omega}_{A_n}\mathsf{V}$. It follows that $w \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ by Lemma 6.5. From the continuity of $\sigma^{\mathsf{V}}_{0,n}$, we then get

$$\sigma^{\mathsf{V}}_{0,n}(w) = \lim_{m\to\infty} \sigma^{\mathsf{V}}_{0,n}(\sigma^{\mathsf{V}}_{n,r_m}(u_{r_m})) = \lim_{m\to\infty} \sigma^{\mathsf{V}}_{0,r_m}(u_{r_m}) = u.$$

This shows $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq \sigma^{\mathsf{V}}_{0,n}(\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})^{(n)})$, concluding the proof of the lemma. $\square$

The proof of the next lemma is very similar to that of the previous lemma.

Lemma 6.13. *The equality* $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)}) = \sigma^{\mathsf{V}}_{k,k+n}(\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(k+n)}))$ *holds for all* $k, n \in \mathbb{N}$.

Proof. It suffices to consider the case $k = 0$.

Let $a \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$. Then a is a cluster point in $\overline{\Omega}_{A_n}\mathsf{V}$ of a sequence $(\sigma_{n,k}(a_k))_{k>n}$ such that $a_k \in A_k$ for every $k > n$. As $\sigma_{0,k}(a_k) = \sigma_{0,n}(\sigma_{n,k}(a_k))$, it then follows by continuity of $\sigma_{0,n}^{\mathsf{V}}$ that $\sigma_{0,n}^{\mathsf{V}}(a)$ is a cluster point of the sequence $(\sigma_{0,k}(a_k))_{k>n}$. Hence $\sigma_{0,n}^{\mathsf{V}}(a) \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$, thus showing the inclusion $\sigma_{0,n}^{\mathsf{V}}(\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})) \subseteq \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$.

Conversely, let $a \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$. We may pick a strictly increasing sequence $(m_r)_{r\in\mathbb{N}}$ of integers greater than n and a sequence $(a_r)_{r\in\mathbb{N}}$ such that $a_r \in A_{m_r}$, for every $r \in \mathbb{N}$, and $a = \lim \sigma_{0,m_r}(a_r)$. By compactness of $\overline{\Omega}_{A_n}\mathsf{V}$, the sequence $(\sigma_{n,m_r}(a_r))_{r\in\mathbb{N}}$ has a subsequence $(\sigma_{n,m_{r_s}}(a_{r_s}))_{s\in\mathbb{N}}$ converging in $\overline{\Omega}_{A_n}\mathsf{V}$ to a pseudoword b. Note that $b \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$. By continuity of $\sigma_{0,n}^{\mathsf{V}}$, we have

$$\sigma_{0,n}^{\mathsf{V}}(b) = \lim_{s\to\infty} \sigma_{0,n}^{\mathsf{V}}(\sigma_{n,m_{r_s}}(a_{r_s})) = \lim_{s\to\infty} \sigma_{0,m_{r_s}}(a_{r_s}) = a.$$

This establishes the inclusion $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq \sigma_{0,n}^{\mathsf{V}}(\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)}))$, finishing the proof. $\square$

Lemma 6.12 allows us to consider the following inverse system of onto continuous homomorphisms of pro-V semigroups:

$$\mathcal{F} = \{\sigma_{n,m}^{\mathsf{V}} \colon \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)}) \to \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)}) \mid m, n \in \mathbb{N},\, m \geq n\}.$$

Similarly, Lemmas 6.13 and 6.7 yield the following inverse system of onto continuous functions between compact spaces:

$$\mathcal{G} = \{\sigma_{n,m}^{\mathsf{V}} \colon \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)}) \to \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)}) \mid m, n \in \mathbb{N},\, m \geq n\}.$$

We denote the inverse limits $\varprojlim \mathcal{F}$ and $\varprojlim \mathcal{G}$ respectively by $\operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ and $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$. By compactness, these sets are nonempty [43, Theorem 3.2.13], with $\operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ moreover being a pro-V semigroup and $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ a closed subspace of $\operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$. The corresponding projections $\operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) \to \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ are onto continuous homomorphisms [43, Theorem 3.2.15], which we denote by $\sigma_{n,\infty}^{\mathsf{V}}$, for every $n \in \mathbb{N}$.

The next proposition is deduced from Proposition 6.14 with routine arguments.

Proposition 6.14. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence. The following properties hold:*

(i) $\operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) = \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) \cdot \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) = \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) \cdot \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$;

(ii) *the set $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ is contained in a regular $\mathcal{J}$-class of $\operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$.*

Proof. (i): Let $u \in \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$. For each $n \in \mathbb{N}$, consider the set

$$Y_n = \{(v, w) \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) \times \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) \mid \sigma_{n,\infty}(v) \cdot \sigma_{n,\infty}(w) = \sigma_{n,\infty}(u)\}.$$

As $\sigma_{n,\infty}(u) \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)}) \cdot \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ by Proposition 6.8, the set Y_n is nonempty in view of the surjectivity of the canonical projections $\operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) \to \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ and $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) \to \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$. By continuity of $\sigma_{n,\infty}$, the set Y_n is a closed subspace of $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) \times \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$. Moreover, if $m, n \in \mathbb{N}$ are such that $n \leq m$, then we have $Y_m \subseteq Y_n$. Hence, by compactness, the intersection $Y = \bigcap_{n\in\mathbb{N}} Y_n$ is nonempty. Let $(v, w) \in Y$. We then have $\sigma_{n,\infty}(vw) = \sigma_{n,\infty}(u)$ for every $n \in \mathbb{N}$, that is to say, $vw = u$. This establishes the equality $\operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) = \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) \cdot \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$. The equality $\operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) = \Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) \cdot \operatorname{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ follows by symmetric arguments.

(ii). It is folklore, whose proof is an easy exercise, the fact that in a inverse limit $S = \lim_{i\in I} S_i$ of compact semigroups, and for all elements $s = (s_i)_{i\in I}$ and $t = (t_i)_{i\in I}$ of S, one has $s \mathrel{\mathcal{J}} t$ if and only if $s_i \mathrel{\mathcal{J}} t_i$ for every $i \in I$; and that s is regular if and only if s_i is regular for every $i \in I$ (e.g., cf. [64, Propositions 9.1 and 9.3] or [2, Corollary 5.6.2]). With this on hand, the second item follows immediately from Proposition 6.8. $\square$

Denote by $J_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ the regular $\mathcal{J}$-class of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ containing the set $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$.

Corollary 6.15. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence. Then the following hold, for every $n, m \in \mathbb{N}$, with $n \leq m$:*

(i) $\sigma^{\mathsf{V}}_{n,\infty}(J_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$.

(ii) $\sigma^{\mathsf{V}}_{n,m}\Big(J_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)})\Big) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$.

Proof. In view of the definition of the inverse limits $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ and $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$, it suffices to note that $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ and $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ are respectively contained in the $\mathcal{J}$-classes $J_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ and $J_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$, for every $n \in \mathbb{N}$ (cf. Proposition 6.8 (i).) $\square$

Corollary 6.16. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence. There is a sequence $(e_n)_{n\in\mathbb{N}}$ of idempotent pseudowords satisfying $e_n \in J_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ and $e_n = \sigma^{\mathsf{V}}_{n,m}(e_m)$ for every $n, m \in \mathbb{N}$ such that $n \leq m$.*

Proof. We may take an idempotent e in the regular $\mathcal{J}$-class $J_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ and consider, for each $k \in \mathbb{N}$, the idempotent $e_k = \sigma^{\mathsf{V}}_{k,\infty}(e)$. By the definition of the inverse limit $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$, this immediately yields the equality $e_n = \sigma^{\mathsf{V}}_{n,m}(e_m)$ for every $n, m \in \mathbb{N}$ such that $n \leq m$. The remaining of the statement follows from Corollary 6.15 (ii). $\square$

7. Simple profinite images of directive sequences

Consider a directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$, with $\sigma_n \colon A^+_{n+1} \to A^+_n$ for each $n \in \mathbb{N}$. In this section, we investigate more systematically necessary and sufficient conditions for $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ to be a simple profinite semigroup (cf. Corollary 6.10). It turns out that being left or right proper is such a sufficient condition (cf. Theorems 7.6 and 7.9).

A *limit word* of $\boldsymbol{\sigma}$ is an element of $\bigcap_{n\geq 1} \sigma_{0,n}(A^{\mathbb{Z}}_n)$. For a discussion about the significance of this notion, see the introductory paragraphs of Section 4 from [27].

Theorem 7.1. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence. Let V be a pseudovariety of semigroups containing LSl. The following statements are equivalent:*

(i) *the profinite semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is simple;*

(ii) *the inclusion $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$ holds;*

(iii) *all limit words of $\boldsymbol{\sigma}$ belong to $X(\boldsymbol{\sigma})$.*

Proof. (i)$\Leftrightarrow$(ii). This equivalence holds by Corollary 6.10.

(ii)$\Rightarrow$(iii). Suppose that $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$. Let x be a limit word of $\boldsymbol{\sigma}$. Take $k \in \mathbb{N}$. We want to show that $x[-k,k] \in L(\boldsymbol{\sigma})$. For each $n \in \mathbb{N}$, take $x_n \in A^{\mathbb{Z}}_n$ such that $x = \sigma_{0,n}(x_n)$. Let w be a cluster point in $\overline{\Omega}_{A_0}\mathsf{V}$ of the sequence $(\sigma_{0,n}(x_n[-1,0]))_{n\in\mathbb{N}}$. Since $\boldsymbol{\sigma}$ is primitive, the word $x[-k,k]$ is a factor of $\sigma_{0,n}(x_n[-1,0])$ for every sufficiently large n. Therefore, $x[-k,k]$ is also a factor of w. Note that $w \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, by Lemma 6.5. By the assumption $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$, all finite-length factors of w belong to $L(\boldsymbol{\sigma})$, by Proposition 5.3. In particular, we have $x[-k,k] \in L(\boldsymbol{\sigma})$, for every $k \in \mathbb{N}$. This means that $x \in X(\boldsymbol{\sigma})$.

(iii)$\Rightarrow$(ii). Suppose that all limit words of $\boldsymbol{\sigma}$ belong to $X(\boldsymbol{\sigma})$. Let $u \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. By Lemma 6.5, we know that there is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers and a sequence $(u_k)_{k\in\mathbb{N}}$ of words, with $u_k \in (A_{n_k})^+$, such that $\sigma_{0,n_k}(u_k) \to u$. In particular, the pseudoword u has infinite length.

Let v be a finite-length factor of w. We claim that $v \in L(\boldsymbol{\sigma})$. Note that the set $(\overline{\Omega}_A\mathsf{V})^1 v (\overline{\Omega}_A\mathsf{V})^1$ is clopen, as V contains LSl. Hence, taking subsequences, we may suppose that v is a factor of $\sigma_{0,n_k}(u_k)$ for every $k \in \mathbb{N}$. Since $\boldsymbol{\sigma}$ is primitive, we may further assume that all words in $\sigma_{0,n_k}(A_{n_k})$ have length greater than that of v. Therefore, for each $k \in \mathbb{N}$, we may take letters $a_k, b_k \in A_{n_k}$ such that v is a factor of $\sigma_{0,n_k}(a_k b_k)$. If v is a factor of $\sigma_{0,n_k}(a_k)$ or of $\sigma_{0,n_k}(b_k)$ for some k, then $v \in L(\boldsymbol{\sigma})$ and the claim is proved. Therefore, we may as well suppose that for every $k \in \mathbb{N}$ there is a factorization $v = s_k p_k$ such that s_k is a nonempty suffix of $\sigma_{0,n_k}(a_k)$ and p_k is a nonempty prefix of $\sigma_{0,n_k}(b_k)$. In fact, again by taking subsequences, we are reduced to the case where (s_k, p_k) has constant value (s, p). Repeating the process of taking subsequences, we may as well suppose that the sequence $(\sigma_{0,n_k}(a_k), \sigma_{0,n_k}(b_k))_{k\geq 1}$ converges in $\overline{\Omega}_{A_0}\mathsf{V} \times \overline{\Omega}_{A_0}\mathsf{V}$ to some pair (α, β) of elements of $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$. Note that s is a finite-length suffix of α and p is a finite-length prefix of β. Consider the element x of $A_0^{\mathbb{Z}}$ such that, for every positive integer n, the words $x[-n, -1]$ and $x[0, n)$ are respectively the suffix of length of n of α and the prefix of length n of β. Then we have $v = x[-|s|, |p|)$. Therefore, to show that $v \in L(\boldsymbol{\sigma})$, it suffices to show that x is a limit word of $\boldsymbol{\sigma}$.

Let $r \in \mathbb{N}$. As $\alpha, \beta \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, we may take infinite-length pseudowords $\alpha', \beta' \in \overline{\Omega}_{A_r}\mathsf{V}$ such that $\alpha = \sigma_{0,r}^{\mathsf{V}}(\alpha')$ and $\beta = \sigma_{0,r}^{\mathsf{V}}(\beta')$. Let y be the element of $A_r^{\mathbb{Z}}$ such that, for every positive integer n, the words $y[-n, -1]$ and $y[0, n)$ are respectively the suffix of length of n of α' and the prefix of length n of β'. Then $\sigma_{0,r}(y[-n, -1])$ and $\sigma_{0,r}(y[0, n))$ are respectively a suffix of length at least n of α and a prefix of length at least n of β. This shows that $x = \sigma_{0,r}(y)$. Since r is an arbitrary element of $\mathbb{N}$, we conclude that $x \in \bigcap_{r\in\mathbb{Z}} \sigma_{0,r}(A_r^{\mathbb{Z}})$, that is to say, that x is a limit word of $\boldsymbol{\sigma}$. By assumption, we therefore have $x \in X(\boldsymbol{\sigma})$, establishing the claim that $v \in L(\boldsymbol{\sigma})$.

Since v is an arbitrary finite-length factor of u, by Proposition 5.3 we deduce that $u \in J_{\mathsf{V}}(\boldsymbol{\sigma})$. This establishes the inclusion $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$. □

Remark 7.2. In view of item (iii) in Theorem 7.1, the choice of V plays no role in the statement of the theorem. More precisely, one has $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{S}}(\boldsymbol{\sigma})$ if and only if $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$, when V is a pseudovariety of semigroups containing LSl. This equivalence also follows directly from Corollary 5.5 and Proposition 6.6.

Denote by $\mathrm{fac}_n(w)$ the set of factors of length n of a pseudoword w, for each $n \in \mathbb{N}$.

Definition 7.3. We say that the directive sequence $\boldsymbol{\sigma}$ is *stable* if for every $n \in \mathbb{N}$ and every $a, b \in A_n$, the inclusion $\mathrm{fac}_2(\sigma_n(ab)) \subseteq L(\boldsymbol{\sigma}^{(n)})$ holds. It is called *contraction stable* if it has a contraction which is stable.

In other words: $\boldsymbol{\sigma}$ is contraction stable if and only if there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $n_0 = 0$ and for all $k \geq 1$, and all $u \in A_{n_{k+1}}^2$ (equivalently, all $u \in A_{n_{k+1}}^+$), every factor of length two of $\sigma_{n_k,n_{k+1}}(u)$ belongs to $L(\boldsymbol{\sigma}^{(n_k)})$. Note that a contraction of a contraction stable directive sequence is also contraction stable (cf. Lemma 2.3).

Example 7.4. Set $A = \{\mathtt{a}, \mathtt{b}\}$. Consider the Prouhet-Thue-Morse substitution

$$\tau \colon \mathtt{a} \mapsto \mathtt{ab},\ \mathtt{b} \mapsto \mathtt{ba}.$$

Then we have $\tau(A^2) \subseteq L(\tau)$, and so the constant primitive directive sequence $\boldsymbol{\tau} = (\tau, \tau, \tau, \ldots)$ is stable.

The following theorem improves a similar result that the second author obtained for the special case of primitive substitutive directive sequences [33, Lemma 3.12].

Theorem 7.5. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence. Let V be a pseudovariety of semigroups containing LSl. The following statements are equivalent:*

(i) $\boldsymbol{\sigma}$ *is contraction stable;*
(ii) $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)})$ *is a simple semigroup, for every* $k \geq 0$*;*
(iii) $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ *is a simple semigroup.*

Proof. (i)$\Rightarrow$(ii). If $\boldsymbol{\sigma}$ is contraction stable, then so is $\boldsymbol{\sigma}^{(k)}$ for every $k \geq 0$. Therefore, it suffices to show that the inclusion $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$ holds, which, by Corollary 6.10, means that the semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is simple.

Let $u \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. By Lemma 6.5, we may take a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers and a sequence $(u_k)_{k\in\mathbb{N}}$ of words, with $u_k \in (A_{n_k})^+$, such that $\sigma_{0,n_k}(u_k) \to u$. If $|u_k| \to 1$, then $u \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$, and so $u \in J_{\mathsf{V}}(\boldsymbol{\sigma})$ by Proposition 6.8. Therefore, we may as well assume that $|u_k| \geq 2$ for every $k \in \mathbb{N}$. Let w be a finite-length factor of $u = \lim \sigma_{0,n_k}(u_k)$. Since V contains LSI, by taking subsequences, we may further assume that w is a factor of $\sigma_{0,n_k}(u_k)$ for every $k \in \mathbb{N}$. Because $\boldsymbol{\sigma}$ is primitive, there is $k_0 \in \mathbb{N}$ such that, for every $k > k_0$ and every letter a that is a factor of u_k, we have $|\sigma_{0,n_k}(a)| > |w|$. Hence, for every $k > k_0$, there are letters $c_k, d_k \in A_{n_k}$ such that w is a factor of $\sigma_{0,n_k}(c_k d_k)$. Since $\boldsymbol{\sigma}$ is stable, we have $\sigma_{n_k}(c_k d_k) \in L(\boldsymbol{\sigma}^{(n_k)})$. It follows that $\sigma_{0,n_k}(c_k d_k) \in L(\boldsymbol{\sigma})$ (cf. Lemma 2.3), whence $w \in L(\boldsymbol{\sigma})$. Since w is an arbitrary finite-length factor of u, we conclude that $u \in J_{\mathsf{V}}(\boldsymbol{\sigma})$ by Proposition 5.3. This establishes the inclusion $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$.

(ii)$\Rightarrow$(i). Suppose that property (ii) holds for $\boldsymbol{\sigma}$. Then the same property holds for $\boldsymbol{\sigma}^{(n)}$ for every $n \in \mathbb{N}$, as $(\boldsymbol{\sigma}^{(n)})^{(k)} = \boldsymbol{\sigma}^{(n+k)}$ for every $n, k \in \mathbb{N}$. Therefore, to establish the implication (ii)$\Rightarrow$(i), it suffices to establish the inclusion $\mathrm{fac}_2(\sigma_{0,n}(A_n^2)) \subseteq L(\boldsymbol{\sigma})$ for some positive integer n.

Suppose, on the contrary, that we have $\mathrm{fac}_2(\sigma_{0,n}(A_n^2)) \nsubseteq L(\boldsymbol{\sigma})$ for every positive integer n. Then, for each $n \geq 1$, we may take letters $a_n, b_n \in A_n$ and a word $w_n \in A_0^2 \setminus L(\boldsymbol{\sigma})$ such that $\sigma_{0,n}(a_n b_n) \in A^* w_n A^*$. Let (α, β, w) be a cluster point in $(\overline{\Omega}_{A_0}\mathsf{V})^3$ of the sequence $(\sigma_{0,n}(a_n), \sigma_{0,n}(b_n), w_n)_{n\geq 1}$. Note that w is a factor of $\alpha\beta$. Moreover, since A_0 is a finite alphabet, one must have $w = w_m$ for infinitely many integers m. Therefore, $\alpha\beta$ has a finite-length factor (namely w) not in $L(\boldsymbol{\sigma})$. This implies that $\alpha\beta \notin J_{\mathsf{V}}(\boldsymbol{\sigma})$ by Proposition 5.3. On the other hand, we have $\alpha, \beta \in J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ by Proposition 6.8. It follows that $\alpha\beta \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \setminus J_{\mathsf{V}}(\boldsymbol{\sigma})$, which contradicts the assumption that $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is simple. To avoid the contradiction, we indeed must have $\mathrm{fac}_2(\sigma_{0,n}(A_n^2)) \subseteq L(\boldsymbol{\sigma})$ for some positive integer n.

(iii)$\Rightarrow$(ii). Immediately after defining the inverse limit $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) = \varprojlim \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)})$, we observed that $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)})$ is a homomorphic image of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$, for every natural number k. This gives the implication, as the homomorphic image of a simple semigroup is also simple.

(ii)$\Rightarrow$(iii). Any inverse limit of profinite simple semigroups is a simple profinite semigroup (folklore, cf. [64, Corollary 9.2]). $\square$

A semigroup S is said to be *right simple* if the relation $\mathcal{R}$ on S is the universal relation (cf. [66, Section A.1]).

Theorem 7.6. *Let* $\boldsymbol{\sigma}$ *be a primitive directive sequence. Let* V *be a pseudovariety of semigroups containing* LSI*. The following statements are equivalent:*

(i) $\boldsymbol{\sigma}$ *has a left proper contraction;*
(ii) $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)})$ *is a right simple semigroup for every* $k \geq 0$*;*
(iii) $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ *is a right simple semigroup.*

Before showing Theorem 7.6, we state the following lemma used in its proof.

Lemma 7.7. *Let* V *be a pseudovariety of finite semigroups that is contained in* LSl*. Consider a finite alphabet* A*. Let* $u, v \in \overline{\Omega}_A\mathsf{V}$*. If* x *is a finite-length factor of* uv*, then either* x *is a factor of* u*, or of* v*, or* $x = sp$ *for some suffix* s *of* u *and some prefix* p *of* v*. In particular, if* x *is a finite factor of* uwv *and* $w \in \overline{\Omega}_A\mathsf{V} \setminus A^+$*, then* x *is a factor of* uw *or of* wv*.*

This lemma, whose proof is an easy exercise, is subsumed into Lemma 8.2 of the paper [20]. The statement in [20] is made for the pseudovariety S of all finite semigroups, but the proof given there holds for all pseudovarieties containing LSl.

Proof of Theorem 7.6. (ii) $\Rightarrow$ (iii). An inverse limit of right simple finite semigroups is itself a right simple semigroup (folklore, cf. [64, Corollary 9.2]).

(iii) $\Rightarrow$ (ii). The homomorphic image of a right simple semigroup is right simple.

(i) $\Rightarrow$ (ii). Note that $\boldsymbol{\sigma}$ has a left proper contraction if and only $\boldsymbol{\sigma}^{(k)}$ has a left proper contraction, for every $k \geq 0$. Therefore, it suffices to show that $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is right simple. We may as well assume that σ_n is left proper for every $n \in \mathbb{N}$, thanks to Lemma 6.4.

For each $n \in \mathbb{N}$, let $b_n \in A_n$ be such that $\sigma_n(A_{n+1}) \subseteq b_n A_n^*$. By compactness of $\overline{\Omega}_{A_0}\mathsf{V}$, we may pick a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $(\sigma_{0,n_k}(b_{n_k}))_{k\in\mathbb{N}}$ converges to some pseudoword β of $\overline{\Omega}_{A_0}\mathsf{V}$. Note that $\beta \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$. Hence, β is a regular element of the semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ by Proposition 6.8, and so we may select an idempotent e that is $\mathcal{R}$-equivalent in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ to β. In particular, the equality $\beta = e\beta$ holds.

We claim that

$$\forall z \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}), \quad z = ez. \tag{6}$$

Let $u \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. By Lemma 6.5, there is a strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ of positive integers such that $u = \lim \sigma_{0,m_k}(u_k)$ for some sequence $(u_k)_{k\in\mathbb{N}}$ of words. By taking a subsequence of $(m_k)_{k\in\mathbb{N}}$, we may as well assume that $m_k > n_k$ for every $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $s_k \in (A_{n_k})^*$ be a word such that $\sigma_{n_k,m_k}(u_k) = b_{n_k}s_k$. Further taking subsequences, we may assume that the sequence $(\sigma_{0,n_k}(s_k))_k$ converges to some pseudoword $s \in \overline{\Omega}_{A_0}\mathsf{V}$. We then have

$$u = \lim \sigma_{0,m_k}(u_k) = \lim \sigma_{0,n_k}(b_{n_k})\sigma_{0,n_k}(s_k) = \beta s = e\beta s = eu.$$

This establishes the claim that (6) holds.

We proceed to show that $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$. In what follows, bear in mind that $e \in J_{\mathsf{V}}(\boldsymbol{\sigma})$, as e is $\mathcal{R}$-equivalent to an element of $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$, namely β, and $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$ by Proposition 6.8.

Let us continue with the arbitrary element u of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ we were considering. If $\lim_{k\to\infty} |u_k| = 1$, then $u \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$, and so $u \in J_{\mathsf{V}}(\boldsymbol{\sigma})$ by Proposition 6.8. Therefore, we may as well assume that $|u_k| \geq 2$ for every $k \in \mathbb{N}$. Let w be a finite-length factor of $u = \lim \sigma_{0,m_k}(u_k)$. By taking subsequences, we may further assume that w is a factor of $\sigma_{0,m_k}(u_k)$ for every $k \in \mathbb{N}$. Because $\boldsymbol{\sigma}$ is primitive, there is $k_0 \in \mathbb{N}$ such that, for every $k > k_0$ and for every factor z of length one of u_k, we have $|\sigma_{0,m_k}(z)| > |w|$. Hence, for every $k > k_0$, there are letters $c_k, d_k \in A_{m_k}$ such that $c_k d_k$ is a factor of u_k and w is a factor of $\sigma_{0,m_k}(c_k d_k)$. Let (γ, δ) be a cluster point of the sequence $(\sigma_{0,m_k}(c_k), \sigma_{0,m_k}(d_k))_{k>k_0}$. Note that $\gamma, \delta \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$, and that w is a factor of $\gamma\delta$.

By Proposition 6.8, there is some idempotent f in $J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ such that $\gamma = \gamma f$. Since $\gamma\delta = \gamma f \delta$, the word w is a factor of $\gamma f = \gamma$ or of $f\delta$ by Lemma 7.7.

On the other hand, we know that $f = ef$ by (6). Since $e \mathcal{J} f$, it follows from profinite semigroups being stable that $e \mathcal{R} f$; whence $e = fe$ as f is idempotent. Therefore, since $\delta = e\delta$ again by (6), we have $f\delta = fe\delta = e\delta = \delta$. Hence w is a factor of γ or of δ. As both γ and δ belong to $J_{\mathsf{V}}(\boldsymbol{\sigma})$, we deduce that $w \in L(\boldsymbol{\sigma})$. Since w is an arbitrary finite-length factor of u, we conclude that $u \in J_{\mathsf{V}}(\boldsymbol{\sigma})$ by Proposition 5.3. This establishes the inclusion $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$, which means that $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is simple, by Theorem 6.9. Going back to the equality $u = eu$, it now follows from the stability of the profinite semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ that u is in the $\mathcal{R}$-class of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ containing e. Since u is an arbitrary element of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, this establishes that $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a right simple semigroup.

(ii) $\Rightarrow$ (i). Since $\boldsymbol{\sigma}$ has a left proper contraction if and only if every tail of $\boldsymbol{\sigma}$ has a left proper contraction, it suffices to show that there is a positive integer n such that $\sigma_{0,n}$ is left proper. Suppose that, on the contrary, for every positive integer n the homomorphism $\sigma_{0,n}$ is not left proper. Then, for each $n \geq 1$, there are letters $a_n, b_n \in A_0$, with $a_n \neq b_n$, and $c_n, d_n \in A_n$ such that the words $\sigma_{0,n}(c_n)$ and $\sigma_{0,n}(d_n)$ respectively start with a_n and b_n. Let $(\gamma, \delta, a, b)_{n\geq 1}$ be a cluster point in $\overline{\Omega}_{A_0}\mathsf{V}$ of the sequence $(\sigma_{0,n}(c_n), \sigma_{0,n}(d_n), a_n, b_n)_{n\geq 1}$. We have $(a, b) = (a_m, b_m)$ for infinitely many integers m, and so a and b are distinct letters of A_0. Moreover, the pseudowords γ and δ respectively start with a and b. If two elements of $\overline{\Omega}_{A_0}\mathsf{V}$ are $\mathcal{R}$-equivalent, then they have the same finite-length prefixes. Hence, γ and δ are not $\mathcal{R}$-equivalent. But this contradicts our assumption that $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is right simple. Therefore, there is indeed a positive integer n such that $\sigma_{0,n}$ is left proper. $\square$

Combining Theorems 7.5 and 7.6, we instantly get the following fact, thus avoiding a direct combinatorial proof of it.

Corollary 7.8. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence. If $\boldsymbol{\sigma}$ has a left proper contraction, then it is stable.*

We end this section with the analog of Theorem 7.6 for proper directive sequences.

Theorem 7.9. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence. Let V be a pseudovariety of semigroups containing LSI. The following statements are equivalent:*

(i) *$\boldsymbol{\sigma}$ has a proper contraction;*
(ii) *$\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)})$ is a group for every $k \geq 0$;*
(iii) *$\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ is a group.*

Proof. (i)$\Rightarrow$(ii). This implication follows from Theorem 7.6 and its dual, because a semigroup is a group if and only if it is both left and right simple [66, Lemma A.3.1].

(ii)$\Rightarrow$(iii). Any inverse limit of profinite groups is a profinite group.

(iii)$\Rightarrow$(i). By Theorem 7.6 and its dual, $\boldsymbol{\sigma}$ is simultaneously left proper and right proper, which means that it is proper (indeed, if $\sigma_{n,m}$ and $\sigma_{r,s}$ are respectively left proper and right proper, then $\sigma_{n,k}$ is proper, for every $n, m, r, s, k \in \mathbb{N}$ such that $n < m < r < s < k$). $\square$

8. The case of bounded directive sequences

Let us say that the directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$, with $\sigma_n \colon A_{n+1}^+ \to A_n^+$, is *bounded* when the set $\{A_n : n \in \mathbb{N}\}$ of its alphabets is finite.

Remark 8.1. A directive sequence has finite alphabet rank if and only if it has some contraction that is, up to relabeling of its alphabets, bounded. Moreover, if $\boldsymbol{\sigma}'$ is a contraction of $\boldsymbol{\sigma}$, the relabeled directive sequence $\boldsymbol{\sigma}''$ obtained from $\boldsymbol{\sigma}'$ may be chosen, by not relabeling A_0, such that $X(\boldsymbol{\sigma}) = X(\boldsymbol{\sigma}'')$ and $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) = \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}'')$ for every pseudovariety of semigroups V containing N.

For technical reasons, related with the convenience of using finite-vertex profinite categories, we mostly prefer to work directly with bounded directive sequences, although they have the same expressive power of directive sequences with finite alphabet rank, as seen in the previous remark.

A way of thinking about the directive sequence $\boldsymbol{\sigma}$ is to visualize it as a left-infinite path

$$A_0^+ \xleftarrow{\sigma_0} A_1^+ \xleftarrow{\sigma_1} A_2^+ \xleftarrow{\sigma_2} A_3^+ \xleftarrow{\sigma_3} \cdots \tag{7}$$

over the graph $\Gamma(\boldsymbol{\sigma})$ whose vertices are the free semigroups A_n^+ and where the arrows from A_k^+ to A_l^+ are the homomorphisms from A_k^+ to A_l^+. Note that $\boldsymbol{\sigma}$ being bounded means that $\Gamma(\boldsymbol{\sigma})$ has a finite number of vertices.

From hereon, let V be a pseudovariety of finite semigroups containing the pseudovariety N of finite nilpotent semigroups. Consider the following set of finitely generated profinite semigroups:

$$\mathcal{F}_{\mathsf{V}}(\boldsymbol{\sigma}) = \{\overline{\Omega}_{A_n}\mathsf{V} : n \in \mathbb{N}\}.$$

Let $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$ denote the category $\mathbf{Pro}[\mathcal{F}_{\mathsf{V}}(\boldsymbol{\sigma})]$, consisting of continuous homomorphisms between elements of $\mathcal{F}_{\mathsf{V}}(\boldsymbol{\sigma})$. Closely associated to the left-infinite path (7) in $\Gamma(\boldsymbol{\sigma})$, we also have the following left-infinite path

$$\overline{\Omega}_{A_0}\mathsf{V} \xleftarrow{\sigma_0^{\mathsf{V}}} \overline{\Omega}_{A_1}\mathsf{V} \xleftarrow{\sigma_1^{\mathsf{V}}} \overline{\Omega}_{A_2}\mathsf{V} \xleftarrow{\sigma_2^{\mathsf{V}}} \overline{\Omega}_{A_3}\mathsf{V} \xleftarrow{\sigma_3^{\mathsf{V}}} \cdots, \tag{8}$$

which is a path in the graph $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$.

The set $\mathcal{F}_{\mathsf{V}}(\boldsymbol{\sigma})$ is finite precisely when $\boldsymbol{\sigma}$ is bounded. Therefore, assuming that $\boldsymbol{\sigma}$ is bounded, as we shall always do in this section from hereon, the category $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a finite-vertex profinite category (cf. Proposition 4.1).

Let us say that a continuous homomorphism $\psi\colon \overline{\Omega}_A\mathsf{V} \to \overline{\Omega}_B\mathsf{V}$ is *primitive* when every element of B is a factor of every element of $\psi(A)$. In particular, if $\varphi\colon A^+ \to B^+$ is a primitive substitution, then its extension $\varphi^{\mathsf{V}}\colon \overline{\Omega}_A\mathsf{V} \to \overline{\Omega}_B\mathsf{V}$ is a primitive homomorphism.

Lemma 8.2. *The set of primitive homomorphisms between elements of $\mathcal{F}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a closed subspace of $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$.*

Proof. Let $(\varphi_i)_{i\in I}$ be a net of primitive homomorphisms between elements of $\mathcal{F}_{\mathsf{V}}(\boldsymbol{\sigma})$ converging in $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$ to a homomorphism φ from $\overline{\Omega}_A\mathsf{V}$ to $\overline{\Omega}_B\mathsf{V}$. Since the space of vertices in the category $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a finite discrete space, we may assume that φ_i is always a homomorphism from $\overline{\Omega}_A\mathsf{V}$ to $\overline{\Omega}_B\mathsf{V}$. Let $a \in A$ and $b \in B$. As φ_i is primitive, we have $\varphi_i(a) \leq_{\mathcal{J}} b$ for every $i \in I$. Since $\leq_{\mathcal{J}}$ is a closed relation in $\overline{\Omega}_B\mathsf{V}$ and we are dealing with the pointwise topology of $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$, we conclude that $\varphi(a) \leq_{\mathcal{J}} b$ for every $a \in A$ and $b \in B$. This means that φ is primitive, concluding the proof. $\square$

Definition 8.3. Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a bounded directive sequence, and let V be a pseudovariety of semigroups containing N. A V-*compression of* $\boldsymbol{\sigma}$ is a cluster point of the sequence $\left(\sigma_{0,n}^{\mathsf{V}}\right)_{n\in\mathbb{N}}$, in the profinite category $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$.

A V-compression ξ of $\boldsymbol{\sigma}$ must be a continuous homomorphism from $\overline{\Omega}_{A_k}\mathsf{V}$ to $\overline{\Omega}_{A_0}\mathsf{V}$, for some $k \geq 0$. If $\boldsymbol{\sigma}$ is primitive, then ξ is primitive, by Lemma 8.2.

Example 8.4. Let $\sigma\colon A^+ \to A^+$ be a substitution. Consider the constant directive sequence $\boldsymbol{\sigma} = (\sigma, \sigma, \ldots)$. Then $(\sigma^{\mathsf{V}})^\omega$ is a V-compression of $\boldsymbol{\sigma}$, for every pseudovariety of semigroups V containing N.

The next theorem says, in particular, that when the directive sequence $\boldsymbol{\sigma}$ is bounded primitive, the profinite semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is generated by elements of $J_{\mathsf{V}}(\boldsymbol{\sigma})$. A similar result, concerning primitive directive sequences of substitutions over a constant alphabet, appeared in earlier work by the first author [6, Theorem 3.7].

Theorem 8.5. *Let $\xi\colon \overline{\Omega}_B\mathsf{V} \to \overline{\Omega}_{A_0}\mathsf{V}$ be a V-compression of a bounded directive sequence $\boldsymbol{\sigma}$. The equality $\mathrm{Im}(\xi) = \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ holds. If, moreover, $\boldsymbol{\sigma}$ is primitive, then the inclusion $\xi(B) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$ holds.*

Proof. We first only assume that $\boldsymbol{\sigma}$ is bounded. We may take a subnet $\big(\sigma^{\mathsf{V}}_{0,n_i}\big)_{i\in I}$ of $\big(\sigma^{\mathsf{V}}_{0,n}\big)_{n\in\mathbb{N}}$ such that $\xi = \lim_{i\in I} \sigma^{\mathsf{V}}_{0,n_i}$ in $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$ and $A_{n_i} = B$ for all $i \in I$, as the profinite category $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$ has a discrete vertex space.

Let $u \in \overline{\Omega}_B\mathsf{V}$. Since we are dealing with the pointwise topology of $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$, we have $\xi(u) = \lim_{i\in I} \sigma^{\mathsf{V}}_{0,n_i}(u)$. This implies that $\xi(u) \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ by Lemma 6.5, thus establishing the inclusion $\mathrm{Im}(\xi) \subseteq \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$.

Conversely, let $w \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. Then, for each $i \in I$, there is $u_i \in \overline{\Omega}_B\mathsf{V}$ such that $w = \sigma^{\mathsf{V}}_{n_i}(u_i)$. Let u be a cluster point of the net $(u_i)_{i\in I}$. By continuity of the evaluation mapping $\mathrm{Eval}_{\overline{\Omega}_B\mathsf{V},\overline{\Omega}_{A_0}\mathsf{V}}$, seen in Corollary 3.11, it follows that $w = \xi(u)$, thus establishing the inclusion $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq \mathrm{Im}(\xi)$.

Finally, assume that, moreover, $\boldsymbol{\sigma}$ is primitive. When $u \in B$, from the equality $\xi(u) = \lim_{i\in I} \sigma^{\mathsf{V}}_{0,n_i}(u)$ we get $\xi(u) \in \Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$ by the definition of $\Lambda_{\mathsf{V}}(\boldsymbol{\sigma})$. It then follows from Proposition 6.8 that $\xi(B) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma})$. □

Corollary 8.6. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence with finite alphabet rank n. The profinite semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is generated by a finite subset of $J_{\mathsf{V}}(\boldsymbol{\sigma})$ with at most n elements.*

Proof. We may as well suppose that $\boldsymbol{\sigma}$ is bounded with alphabet rank n (cf. Remark 8.1). We may pick an alphabet B such that $\mathrm{Card}(B) = n$ and $B = A_k$ for infinitely many values of k. Then, by compactness of the category $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$, there exists a V-compression $\xi\colon \overline{\Omega}_B\mathsf{V} \to \overline{\Omega}_{A_0}\mathsf{V}$ of $\boldsymbol{\sigma}$. By Theorem 8.5, the profinite semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is generated by the set $\xi(B)$, and this set is contained in $J_{\mathsf{V}}(\boldsymbol{\sigma})$. □

Example 8.7. Let $\sigma\colon A^+ \to A^+$ be a substitution. Consider the constant directive sequence $\boldsymbol{\sigma} = (\sigma, \sigma, \ldots)$. Then the equality $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) = \mathrm{Im}((\sigma^{\mathsf{V}})^\omega)$ holds (cf. Example 8.4), and so $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is generated by $\mathrm{Card}(A)$ elements of $J_{\mathsf{V}}(\boldsymbol{\sigma})$.

A finitely generated profinite semigroup S is said to have *rank* k, if k is the smallest positive integer n such that S is generated, as a profinite semigroup, by n elements.

Corollary 8.8. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence with finite alphabet rank n. If $\boldsymbol{\sigma}$ is contraction stable, then $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is an n-generated simple profinite semigroup whose maximal subgroups have rank at most $n^2 - n + 1$.*

Proof. If $\boldsymbol{\sigma}$ is contraction stable, then $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a profinite simple semigroup, by Theorems 7.5 and 7.1. Moreover, $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is n-generated, by Corollary 8.6. Therefore, $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a continuous homomorphic image of $\overline{\Omega}_n\mathsf{CS}$, the n-generated free profinite semigroup over the pseudovariety CS of finite simple semigroups. The first author showed that the maximal subgroups of $\overline{\Omega}_n\mathsf{CS}$ are free profinite groups of rank $n^2 - n + 1$ (cf. [1, Theorem 3.3]). In every continuous homomorphic image S of $\overline{\Omega}_n\mathsf{CS}$, the image of a maximal subgroup of $\overline{\Omega}_n\mathsf{CS}$ is a maximal subgroup of S. Hence, the maximal subgroups of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ have rank at most $n^2 - n + 1$. □

9. Models and kernel endomorphisms for bounded directive sequences

When arguing about a V-compression $\xi = \lim_{i\in I} \sigma^{\mathsf{V}}_{0,n_i}$, where the limit of the net is being taken in $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$, it will be convenient to keep track of the path $(\sigma^{\mathsf{V}}_0, \sigma^{\mathsf{V}}_1, \ldots, \sigma^{\mathsf{V}}_{n_i-1})$ of the graph $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$, which originates the homomorphism $\sigma^{\mathsf{V}}_{0,n_i}$ by multiplication of its edges. Further abstracting, for the sake of clarity of thought, we are lead to Definition 9.1 below. In what follows, for any mapping ψ and element x of the domain of ψ, we may denote $\psi(x)$ by ψ_x.

Definition 9.1 (Model of directive sequence)**.** Let $\boldsymbol{\sigma}$ be a bounded directive sequence.

- A V-*model* of $\boldsymbol{\sigma}$ is a triple $\boldsymbol{\psi} = (\Gamma, \psi, x)$ consisting of:
  (i) a finite-vertex graph Γ;
  (ii) a continuous category homomorphism $\psi\colon \overline{\Omega}_\Gamma\mathsf{Cat} \to \mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$;
  (iii) a prefix accessible pseudopath x of $\overline{\Omega}_\Gamma\mathsf{Cat}$ such that $\psi_{x[n]} = \sigma^{\mathsf{V}}_n$ for all $n \in \mathbb{N}$.
- A *standard* V-*model* of $\boldsymbol{\sigma}$ is any V-*model* of $\boldsymbol{\sigma}$ of the form $\boldsymbol{\psi} = (\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma}), \psi, x)$ such that ψ restricts to the identity on the graph $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$.

Remark 9.2. Every bounded directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ has a standard V-model. Indeed, the graph $\Gamma = \mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$ is finite because $\boldsymbol{\sigma}$ is bounded; the identity mapping on Γ extends to a unique continuous homomorphism of categories $\psi\colon \overline{\Omega}_\Gamma\mathsf{Cat} \to \mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$; and any cluster point x in $\overline{\Omega}_\Gamma\mathsf{Cat}$ of the sequence of finite paths

$$(\sigma^{\mathsf{V}}_0, \ldots, \sigma^{\mathsf{V}}_{n-1})_{n\geq 1}$$

over Γ is a prefix accessible pseudopath that satisfies $x[n] = \sigma^{\mathsf{V}}_n$, for every $n \geq 0$.

Proposition 9.3. *Let $\boldsymbol{\sigma}$ be a bounded directive sequence. Let ξ be a morphism of the category $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$. The following conditions are equivalent:*

(i) *ξ is a V-compression of $\boldsymbol{\sigma}$;*
(ii) *$\xi = \psi_x$ for some V-model (Γ, ψ, x);*
(iii) *$\xi = \psi_x$ for some standard V-model $(\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma}), \psi, x)$.*

Proof. (iii) ⇒ (ii): This implication is trivial.

(ii) ⇒ (i): Consider a V-model (Γ, ψ, x). For each $n \in \mathbb{N}$, let x_n be the prefix of length n of x. Note that

$$\psi_{x_n} = \psi_{x[0]} \circ \cdots \circ \psi_{x[n-1]} = \sigma^{\mathsf{V}}_0 \circ \cdots \circ \sigma^{\mathsf{V}}_{n-1} = \sigma^{\mathsf{V}}_{0,n}.$$

Since x is a prefix accessible pseudopath, there is a net $(x_{n_i})_{i\in I}$ converging in $\overline{\Omega}_\Gamma\mathsf{Cat}$ to x. Then

$$\psi_x = \lim \sigma^{\mathsf{V}}_{0,n_i}$$

is a V-compression of $\boldsymbol{\sigma}$.

(i) $\Rightarrow$ (iii): Suppose that $\xi = \lim_{i\in I} \sigma^{\mathsf{V}}_{0,n_i}$, where the limit of the net is being taken in $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$. Let ψ be the unique continuous homomorphism of profinite categories $\overline{\Omega}_{\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})}\mathsf{Cat} \to \mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$ extending the identity on $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$. For each $n \in \mathbb{N}$, consider the following path in the graph $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$:

$$x_n = (\sigma^{\mathsf{V}}_0, \sigma^{\mathsf{V}}_1, \dots, \sigma^{\mathsf{V}}_{n-1}).$$

We then have $\psi_{x_n} = \sigma^{\mathsf{V}}_{0,n}$. By compactness, we may consider a cluster point x of the net $(x_{n_i})_{i\in I}$ in $\overline{\Omega}_{\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})}\mathsf{Cat}$. Note that x is a prefix accessible pseudopath of $\overline{\Omega}_{\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})}\mathsf{Cat}$, whence $(\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma}), \psi, x)$ is a standard V-model of $\boldsymbol{\sigma}$. Moreover, by continuity of ψ, we must have $\psi_x = \xi$. □

Corollary 9.4. *Let $\boldsymbol{\sigma}$ be a bounded directive sequence. If (Γ, ψ, x) is a V-model of $\boldsymbol{\sigma}$, then* $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) = \mathrm{Im}(\psi_x)$.

Proof. This follows from combining Theorem 8.5 with Proposition 9.3. □

The following may be convenient to deal with tails of a directive sequence, as it often occurs. Recall that if x is a pseudopath with prefix u of finite length k, then $x^{(k)}$ is the unique pseudopath w such that $x = uw$.

Lemma 9.5. *Let $\boldsymbol{\sigma}$ be a bounded directive sequence. Let $k \in \mathbb{N}$. If (Γ, ψ, x) is a V-model of $\boldsymbol{\sigma}$, then $(\Gamma, \psi, x^{(k)})$ is a V-model of $\boldsymbol{\sigma}^{(k)}$.*

Proof. For every infinite-length pseudopath x, and every $n \in \mathbb{N}$, the equality $(x^{(k)})[n] = x[k+n]$ holds. If moreover x is a prefix accessible pseudopath, then $x^{(k)}$ is also a prefix accessible pseudopath [15, Proposition 6.10]. □

Remark 9.6. If (Γ, ψ, x) is a standard V-model of $\boldsymbol{\sigma}$, then $(\Gamma, \psi, x^{(k)})$ may not be a standard V-model of $\boldsymbol{\sigma}^{(k)}$: indeed, the graph $\mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)})$ may have less vertices than the graph $\Gamma = \mathcal{C}_{\mathsf{V}}(\boldsymbol{\sigma})$.

Definition 9.7 (Kernel endomorphism of a directive sequence)**.** Let $\boldsymbol{\sigma}$ be a bounded directive sequence. A V-*kernel endomorphism for $\boldsymbol{\sigma}$* is an endomorphism of $\overline{\Omega}_{A_{\alpha(y)}}\mathsf{V}$ of the form ψ_y for some V-model (Γ, ψ, x) and some element y of the kernel of the right stabilizer $\mathrm{Stab}_{\overline{\Omega}_\Gamma\mathsf{Cat}}(x)$.

Lemma 9.8. *Every V-kernel endomorphism for $\boldsymbol{\sigma}$ is an idempotent continuous homomorphism. Moreover, if ξ is a V-compression of $\boldsymbol{\sigma}$, then $\xi = \xi \circ \zeta$ for some V-kernel endomorphism ξ for $\boldsymbol{\sigma}$.*

Proof. If (Γ, ψ, x) is V-model of $\boldsymbol{\sigma}$, and y is an element of the kernel of $\mathrm{Stab}(x)$, then y is idempotent by Theorem 4.7, and so ψ_y is an idempotent endomorphism.

Moreover, if ξ is a V-compression of $\boldsymbol{\sigma}$, then $\xi = \psi_x$ for some V-model (Γ, ψ, x) of $\boldsymbol{\sigma}$, by Proposition 9.3. For y in the kernel of $\mathrm{Stab}(x)$, set $\zeta = \psi_y$. Then we have $\xi = \psi_{xy} = \psi_x \circ \psi_y = \xi \circ \zeta$. □

Proposition 9.9. *Let V be a pseudovariety of semigroups containing N and let $\boldsymbol{\sigma}$ be a bounded primitive directive sequence. Suppose that $\zeta\colon \overline{\Omega}_B\mathsf{V} \to \overline{\Omega}_B\mathsf{V}$ is a V-kernel endomorphism for $\boldsymbol{\sigma}$. Then the following hold:*

(i) *ζ is primitive;*
(ii) *the set $\zeta(B)$ is contained in a regular $\mathcal{J}$-class of the semigroup* $\mathrm{Im}(\zeta)$*;*
(iii) *the profinite semigroups* $\mathrm{Im}(\zeta)$ *and* $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ *are isomorphic.*

Proof. By definition of V-kernel endomorphism, there is a V-model (Γ, ψ, x) of $\boldsymbol{\sigma}$ and some loop y in the kernel of $\mathrm{Stab}(x)$ such that $\xi = \psi_y$. By Theorem 4.8, there is a net $(x_i)_{i\in I}$ of finite-length prefixes of x such that $x_i \to x$ and $x_i^{-1}x \to y$. Since the space of vertices of the category $\overline{\Omega}_\Gamma\mathsf{Cat}$ is discrete, we may as well assume that $x_i^{-1}x$ is a loop at $\alpha(y) = \omega(y)$.

For every $i \in I$, the triple $(\Gamma, \psi, x_i^{-1}x)$ is a V-model of $\boldsymbol{\sigma}^{(|x_i|)}$ by Lemma 9.5, whence $\psi_{x_i^{-1}x}$ is a V-compression of $\boldsymbol{\sigma}^{(|x_i|)}$ (cf. Proposition 9.3). Therefore, by Lemma 8.2, the continuous endomorphism $\psi_{x_i^{-1}x}$ is primitive for every $i \in I$, and so is $\zeta = \lim \psi_{x_i^{-1}x}$. This establishes item (i) in the statement.

Note that, since ζ is primitive by (i), every element of $\mathrm{Im}(\zeta)$ admits every element of $\zeta(B)$ as a factor. Hence, to prove item (ii), it suffices to show that the semigroup has $\mathrm{Im}(\zeta)$ has a unique maximal $\mathcal{J}$-class, which is regular. Since, by Proposition 6.14, the semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)})$ has that property, item (ii) follows immediately from item (iii), which we proceed to show.

Consider the set $M = \{|x_i| : i \in I\}$. For each $n \in M$, let Ψ_n denote the continuous endomorphism $\psi_{x_i^{-1}x} \colon \overline{\Omega}_B\mathsf{V} \to \overline{\Omega}_B\mathsf{V}$ when $i \in I$ is such that x_i is the prefix of x with length n. Since $\psi_{x_i^{-1}x}$ is a V-compression of $\boldsymbol{\sigma}^{(|x_i|)}$, by Lemma 8.5 the equality $\mathrm{Im}(\Psi_n) = \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ holds for every $n \in M$.

As $\lim |x_i| = \infty$, the set M is cofinal in $\mathbb{N}$, and so the profinite semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(\infty)}) = \varprojlim_{n\in\mathbb{N}} \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ is isomorphic to the inverse limit $\varprojlim_{n\in M} \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$.

Consider the mapping $\Psi \colon \mathrm{Im}(\zeta) \to \prod_{n\in M} \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ defined by $\Psi(u) = (\Psi_n(u))_{n\in M}$ for every $u \in \mathrm{Im}(\zeta)$. Note that Ψ is a continuous homomorphism, as all the mappings Ψ_n are continuous homomorphisms. In view of the remark made in the preceding paragraph, to prove item (iii) in the statement of the proposition, it suffices to show that $\mathrm{Im}(\Psi) = \varprojlim_{n\in M} \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ and that Ψ is injective.

Let $n \in M$, and take $i \in I$ such that $n = |x_i|$. Let $m \in M$ be such that $m > n$, and take $j \in I$ such that $|x_j| = m$. Then we have $x_j = x_i z$ for a path z of length $m - n$, with $\psi_z = \sigma^{\mathsf{V}}_{n,m}$. Since

$$x_i z(x_j^{-1}x) = x_j(x_j^{-1}x) = x = x_i(x_i^{-1}x),$$

canceling the finite-length prefix x_i we obtain $z(x_j^{-1}x) = x_i^{-1}x$ (cf. Proposition 3.5 and Remark 4.4). Therefore, for every $u \in \mathrm{Im}(\zeta)$, we have

$$\sigma^{\mathsf{V}}_{n,m}(\Psi_m(u)) = \psi_z\psi_{x_j^{-1}x}(u) = \psi_{x_i^{-1}x}(u) = \Psi_n(u).$$

This shows that the inclusion $\mathrm{Im}(\Psi) \subseteq \varprojlim_{n\in M} \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ holds.

We claim that $\Psi_n = \Psi_n \circ \zeta$ for every $n \in M$. Letting $i \in I$ be such that $|x_i| = n$, one has

$$x_i(x_i^{-1}x)y = xy = x = x_i(x_i^{-1}x),$$

thus $(x_i^{-1}x)y = x_i^{-1}x$ by cancellation of the finite-length prefix x_i. As $\Psi_n = \psi_{x_i^{-1}x}$ and $\Psi_y = \zeta$, this establishes the claim that $\Psi_n = \Psi_n \circ \zeta$. Therefore, we have $\Psi_n(\mathrm{Im}(\zeta)) = \mathrm{Im}(\Psi_n) = \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$ for every $n \in \mathbb{N}$. This entails the equality $\mathrm{Im}(\Psi) = \varprojlim_{n\in M} \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(n)})$, by well known properties of the continuous mappings involving inverse systems of compact spaces [43, Theorem 3.2.14].

It remains to show that Ψ is injective. Let $u, v \in \mathrm{Im}(\zeta)$ be such that $\Psi(u) = \Psi(v)$. Then we have $\Psi_n(u) = \Psi_n(v)$ for every $n \in \mathbb{N}$. This is the same to say that

$\psi_{x_i^{-1}x}(u) = \psi_{x_i^{-1}x}(u)$ for every $i \in I$. Since we are endowing $\mathrm{Hom}(\overline{\Omega}_B\mathsf{V}, \overline{\Omega}_B\mathsf{V})$ with the pointwise topology, we get

$$\zeta(u) = \psi_y(u) = \lim_{i\in I} \psi_{x_i^{-1}x}(u) = \lim_{i\in I} \psi_{x_i^{-1}x}(v) = \psi_y(v) = \zeta(v).$$

But ζ is idempotent (cf. Lemma 9.8), and so it restricts to the identity on $\mathrm{Im}(\zeta)$. Hence we have $u = v$. This establishes the injectivity of Ψ and finishes the proof of item (iii) of the proposition. $\square$

In the setting of Proposition 9.9 we denote by $J_{\mathsf{V}}(\zeta)$ the regular $\mathcal{J}$-class of $\overline{\Omega}_B\mathsf{V}$ containing the set $\zeta(B)$. If $\varphi\colon B^+ \to B^+$ is a primitive substitution, then we know that the $\mathcal{J}$-class $J_{\mathsf{V}}(\varphi)$ is $\leq_{\mathcal{J}}$-maximal among the regular $\mathcal{J}$-classes of $\overline{\Omega}_B\mathsf{V}$, whenever V contains LSl (cf. Proposition 5.3). Hence, as ζ is a primitive continuous endomorphism of $\overline{\Omega}_B\mathsf{V}$, it is natural to ask whether the $\mathcal{J}$-class $J_{\mathsf{V}}(\zeta)$ is also $\leq_{\mathcal{J}}$-maximal among the regular $\mathcal{J}$-classes of $\overline{\Omega}_B\mathsf{V}$. The following example shows that that may not be the case.

Example 9.10. Consider the sequence of substitutions σ_n over the alphabet $A = \{\mathtt{a}, \mathtt{b}\}$ defined by

$$\sigma_n \colon \mathtt{a} \mapsto \mathtt{ab}^{\mathtt{n}},\ \mathtt{b} \mapsto \mathtt{a}.$$

Note that $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is a bounded primitive directive sequence. Let V, (Γ, ψ, x), y, x_i and y_i be as in the statement and proof of Proposition 9.9. Then, $\mathtt{ab}^{|x_i|}$ is a prefix of $\psi_{y_i}(\mathtt{a})$, so that $\mathtt{ab}^{\omega}$ is a prefix of $\psi_y(\mathtt{a})$. Hence, $\mathtt{b}^{\omega}$ is an idempotent which lies strictly $\leq_{\mathcal{J}}$-above $\psi_y(\mathtt{a})$ provided V contains Sl. Thus, for such V, the $\mathcal{J}$-class of $\psi_y(\mathtt{a})$ is not $\leq_{\mathcal{J}}$-maximal among the regular $\mathcal{J}$-classes of $\overline{\Omega}_A\mathsf{V}$.

10. Saturating directive sequences

We saw in Section 7 that when $\boldsymbol{\sigma}$ has a proper contraction directive sequence, then the V-image of $\boldsymbol{\sigma}$ is a closed subgroup of the free pro-V semigroup over the alphabet of $X(\boldsymbol{\sigma})$. It is natural to ask for necessary and sufficient conditions under which this subgroup is a *maximal* subgroup of that free pro-V semigroup. In this section, we investigate that question in a more general framework, assuming only that $\boldsymbol{\sigma}$ is primitive, not necessarily having a proper contraction. In the process, we establish a strong link with the notion of recognizable directive sequence.

This section is divided in into three subsections. In the first one, we lay the foundations for our framework by introducing the notion of V-*saturating* directive sequence (V a pseudovariety). We give a straightforward proof that primitive directive sequences consisting of pure encodings are S-saturating (Theorem 10.5), and study the case where $\boldsymbol{\sigma}$ is recurrent and consists of encodings that may not be pure (Theorem 10.7). In the second subsection, we see how the recognizability of $\boldsymbol{\sigma}$ is sufficient for $\boldsymbol{\sigma}$ to be S-saturating (Theorem 10.10), and in the last subsection we see cases where it is a necessary condition (Theorem 10.17), leading to new classes of recognizable directive sequences (Corollary 10.21 and Theorem 10.22).

10.1. The notion of S-saturating directive sequence.

In what follows, $\boldsymbol{\sigma}$ is a directive sequence $(\sigma_n)_{n\in\mathbb{N}}$ with σ_n a homomorphism from A_{n+1}^+ to A_n^+. The following definition is the cornerstone upon which this entire section is built.

Definition 10.1. Let $\boldsymbol{\sigma}$ be a primitive directive sequence and V be a pseudovariety containing N. We say that $\boldsymbol{\sigma}$ is V-*saturating* if $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ contains a maximal subgroup of $J_{\mathsf{V}}(\boldsymbol{\sigma})$.

Remark 10.2. If $\boldsymbol{\sigma}$ is primitive and has a proper contraction, then $\boldsymbol{\sigma}$ is S-saturating if and only if $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a maximal subgroup of $J_{\mathsf{V}}(\boldsymbol{\sigma})$, by Theorem 7.9.

In the next proposition we see several equivalent alternatives for Definition 10.1.

Proposition 10.3. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence and V be a pseudovariety containing N. The following conditions are equivalent:*

(i) *$\boldsymbol{\sigma}$ is V-saturating;*
(ii) *$\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ contains an $\mathcal{H}$-class of $J_{\mathsf{V}}(\boldsymbol{\sigma})$;*
(iii) *$J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ is a union of $\mathcal{H}$-classes of $J_{\mathsf{V}}(\boldsymbol{\sigma})$;*
(iv) *if p, q, r are elements of $\overline{\Omega}_{A_0}\mathsf{V}$ such that the relations $p \mathrel{\mathcal{R}} q \mathrel{\mathcal{L}} r$ hold in $\overline{\Omega}_{A_0}\mathsf{V}$, and p and r belong to $J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, then so does q.*

Proof. Let $J = J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ and recall that J is a regular $\mathcal{J}$-class of the semigroup $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, by Theorem 6.9.

The implication (i) $\Rightarrow$ (ii) holds because every maximal subgroup of a semigroup is an $\mathcal{H}$-class of that same semigroup.

For (ii) $\Rightarrow$ (iii), suppose that H is an $\mathcal{H}$-class of $J_{\mathsf{V}}(\boldsymbol{\sigma})$ contained in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. Take $h \in H$. Let $s \in J$. Since $H \subseteq J$, by Theorem 6.9 there are $u, v \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ such that $uhv = s$. By Green's Lemma (cf. [66, Lemma A.3.1]), applied to $\overline{\Omega}_{A_0}\mathsf{V}$, we deduce that uHv is the $\mathcal{H}$-class of s in $\overline{\Omega}_{A_0}\mathsf{V}$. Note that $uHv \subseteq \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, as $H \subseteq \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ and $u, v \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$. As s is an arbitrary element of J, we conclude that J is a union of $\mathcal{H}$-classes of $J_{\mathsf{V}}(\boldsymbol{\sigma})$.

We proceed to show (iii) $\Rightarrow$ (iv). For each Green's relation symbol $\mathcal{K}$, denote by $\mathcal{K}'$ the corresponding Green's relation in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, that is, $\mathcal{K}' = \mathcal{K}_{\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})}$. Take p, q, r as in (iv). Since J is a $\mathcal{J}'$-class by Theorem 6.9, there is $t \in J$ such that $p \mathrel{\mathcal{R}'} t \mathrel{\mathcal{L}'} r$. Hence, q lies in the same $\mathcal{H}$-class of $J_{\mathsf{V}}(\boldsymbol{\sigma})$ as t. Since t belongs to $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$, it follows from (iii) that q also belongs to $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$.

For (iv) $\Rightarrow$ (i), take an idempotent $e \in J$. In (iv) we may take $p = r = e$ and q an arbitrary element in the maximal subgroup H_e, and then conclude that $q \in J$. $\square$

Before proceeding, it is worth noting the following simple observation.

Proposition 10.4. *If $\boldsymbol{\sigma}$ is V-saturating and W is a pseudovariety such that $\mathsf{LSl} \subseteq \mathsf{W} \subseteq \mathsf{V}$, then $\boldsymbol{\sigma}$ is also W-saturating.*

Proof. Suppose that $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ contains a maximal subgroup H of $J_{\mathsf{V}}(\boldsymbol{\sigma})$. Then $\mathrm{Im}_{\mathsf{W}}(\boldsymbol{\sigma})$ contains $p_{\mathsf{V},\mathsf{W}}(H)$ by Proposition 6.6. Moreover, the set $p_{\mathsf{V},\mathsf{W}}(H)$ is a maximal subgroup of $J_{\mathsf{W}}(\boldsymbol{\sigma})$ by Corollary 5.10. Hence, $\boldsymbol{\sigma}$ is W-saturating. $\square$

Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a directive sequence. We say that $\boldsymbol{\sigma}$ is *pure* if σ_n is a pure encoding for all $n \in \mathbb{N}$.

Theorem 10.5. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence. If $\boldsymbol{\sigma}$ is pure, then it is S-saturating.*

Proof. There is a maximal subgroup H of $J_{\mathsf{S}}(\boldsymbol{\sigma})$ such that $H \cap \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}) \neq \emptyset$, by Theorem 6.9. Let $n \in \mathbb{N}$. In particular, we have $H \cap \mathrm{Im}(\sigma_{0,n}^{\mathsf{S}}) \neq \emptyset$. The homomorphism $\sigma_{0,n}$ is pure, as every composition of pure homomorphisms remains pure. In other words, the set $C = \sigma_{0,n}(A_n)$ is a pure code. Since $\mathrm{Im}(\sigma_{0,n}^{\mathsf{S}}) = \mathrm{Cl}_{\mathsf{S}}(C^+)$, it follows from Proposition 3.7 that $H \subseteq \mathrm{Im}(\sigma_{0,n}^{\mathsf{S}})$. As n is arbitrary, this shows that $H \subseteq \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$, thereby establishing that $\boldsymbol{\sigma}$ is S-saturating. $\square$

Example 10.6. Recall the primitive substitution over $A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ considered in Example 6.11:

$$\sigma\colon \mathtt{a} \mapsto \mathtt{ac},\ \mathtt{b} \mapsto \mathtt{bcb},\ \mathtt{c} \mapsto \mathtt{ba},$$

Set $C = \sigma(A)$. No element of C is a prefix or a suffix of some element of C, that is, C is a *bifix* code (cf. [24]). Using a GAP package [37], one may check that the syntactic semigroup of C^+ is aperiodic. Hence, σ is a pure encoding, and so the directive sequence $\boldsymbol{\sigma} = (\sigma, \sigma, \ldots)$ is S-saturating by Theorem 10.5. Denote by $\widehat{\sigma}$ the unique continuous endomorphism $\sigma^{\mathsf{S}}\colon \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ extending σ. Recall that $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}) = \mathrm{Im}(\widehat{\sigma}^{\omega})$ (cf. Example 8.7).

We may easily compute the table of first and last letters of the images of $\widehat{\sigma}^{\omega}$ (Table 1). From Proposition 10.3, it follows that $J_{\mathsf{S}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$ is the union of

ℓ	first letter of $\widehat{\sigma}^{\omega}(\ell)$	last letter of $\widehat{\sigma}^{\omega}(\ell)$
a	a	a
b	b	b
c	b	c

TABLE 1. First and last letters of the images of $\widehat{\sigma}^{\omega}$

six $\mathcal{H}$-classes of $\overline{\Omega}_A\mathsf{S}$, of which $\widehat{\sigma}^{\omega}(\mathtt{a})$, $\widehat{\sigma}^{\omega}(\mathtt{ab})$, $\widehat{\sigma}^{\omega}(\mathtt{ac})$, $\widehat{\sigma}^{\omega}(\mathtt{ba})$, $\widehat{\sigma}^{\omega}(\mathtt{b})$, $\widehat{\sigma}^{\omega}(\mathtt{ca})$ are representative elements. As seen in Example 6.11, the pseudoword $\widehat{\sigma}^{\omega}(\mathtt{a}^2)$ is not in $J_{\mathsf{S}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$, and therefore the $\mathcal{H}$-class of $\widehat{\sigma}^{\omega}(\mathtt{a})$ is not a group.

We say that a directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is *recurrent* if, seen as a right-infinite word over the alphabet $\{\sigma_n : n \in \mathbb{N}\}$, it is a recurrent right-infinite word.

Theorem 10.7. *Let $\boldsymbol{\sigma}$ be a bounded primitive directive sequence. Suppose moreover that $\boldsymbol{\sigma}$ is recurrent and encoding. If there is $k \in \mathbb{N}$ such that $\boldsymbol{\sigma}^{(k)}$ is S-saturating, then $\boldsymbol{\sigma}$ is S-saturating.*

Proof. Let k be a positive integer such that $\boldsymbol{\sigma}^{(k)}$ is S-saturating. By Corollary 6.16, we may take idempotent pseudowords $g \in J_{\mathsf{S}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$ and $h \in J_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)}) \cap \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)})$ such that $g = \sigma_{0,k}^{\mathsf{V}}(h)$. We want to show that the maximal subgroup of $\overline{\Omega}_{A_0}\mathsf{S}$ to which g belongs is contained in $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$.

Because $\boldsymbol{\sigma}$ is recurrent, Proposition 4.6 yields an S-model (Γ, ψ, e) of $\boldsymbol{\sigma}$ where e is an idempotent. Let z be the prefix of length k of e and consider the idempotent $f = e^{(k)}z = z^{-1}ez$.

Note that $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}) = \mathrm{Im}(\psi_e)$ by Corollary 9.4. Since $(\Gamma, \psi, z^{-1}e)$ is an S-model of $\boldsymbol{\sigma}^{(k)}$ by Lemma 9.5, we have

$$\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)}) = \mathrm{Im}(\psi_{z^{-1}e}) = \mathrm{Im}(\psi_f),$$

where the first equality holds by Corollary 9.4 and the second because $z^{-1}e \mathrel{\mathcal{R}} f$. On the other hand, the equalities $zf \cdot z^{-1}e = e$, $zf = ez$ yield $zf \mathrel{\mathcal{R}} e$ and so

$$\mathrm{Im}(\psi_{zf}) = \mathrm{Im}(\psi_e) = \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}).$$

As f is idempotent, the homomorphism ψ_f restricts to the identity on $\mathrm{Im}(\psi_f)$ which can be factored as in the following commutative diagram of restricted mappings, which, for simplicity, are indicated simply by adding a vertical bar:

$$\begin{tikzcd} \mathrm{Im}(\psi_f) \arrow[rr, "\psi_f|=\mathrm{id}"] \arrow[dr, "\psi_z|"'] & & \mathrm{Im}(\psi_f) \\ & \mathrm{Im}(\psi_{zf}). \arrow[ur, "\psi_{z^{-1}e}|"'] & \end{tikzcd}$$

In view of the aforementioned equalities $\mathrm{Im}(\psi_f) = \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)})$ and $\mathrm{Im}(\psi_{zf}) = \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$, it follows that ψ_z restricts to a continuous isomorphism from $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)})$ onto $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$, and that $\psi_{z^{-1}e}$ restricts to a continuous isomorphism from $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$ onto $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)})$.

In what follows, bear in mind that the equality

$$\psi_{z^{-1}e}(g) = h$$

holds: indeed, one has $g = \psi_z(h)$ as $\psi_z = \sigma_{0,k}^{\vee}$, and $h \in \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)}) = \mathrm{Im}(\psi_f)$, whence $h = \psi_f(h) = \psi_{z^{-1}e}(\psi_z(h)) = \psi_{z^{-1}e}(g)$.

Since $\boldsymbol{\sigma}^{(k)}$ is S-saturating, by Proposition 10.3 we know that $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)})$ contains the maximal subgroup H of $J_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)})$ to which the idempotent h belongs. As $\psi_{z^{-1}e}$ restricts to an isomorphism from $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$ to $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)})$, the maximal subgroup G of $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$ containing the idempotent g is such that $\psi_{z^{-1}e}(G) = H$. Let K be the maximal subgroup of $\overline{\Omega}_{A_0}\mathsf{S}$ containing g. Then, as $K \supseteq G$ and H is a maximal subgroup, we must have

$$\psi_{z^{-1}e}(G) = H = \psi_{z^{-1}e}(K).$$

Hence, to show that $\boldsymbol{\sigma}$ is S-saturating, it suffices to show that the restriction of $\psi_{z^{-1}e}$ to K is injective. The reader may wish to look at Figure 2 while checking the proof.

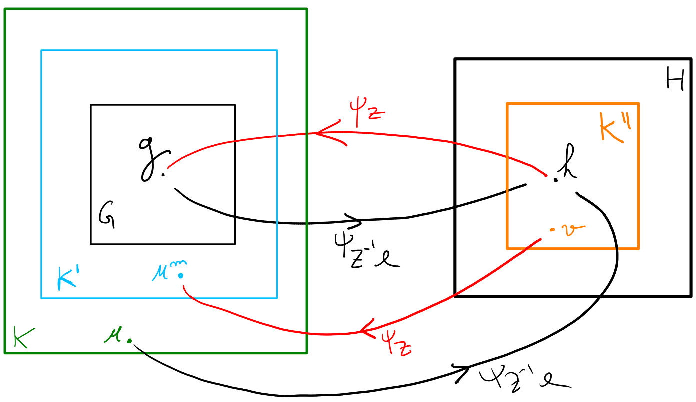

FIGURE 2. Illustration of the proof of Theorem 10.7

Let $u \in K$ be such that $\psi_{z^{-1}e}(u) = h$. Note that $\lim u^{n!} = u^{\omega} = g$. Since $g \in \mathrm{Im}(\psi_z)$, in particular we obtain

$$\lim u^{n!} \in \mathrm{Im}(\psi_z).$$

As the set $\psi_z(A_k)^+$ is a recognizable language by Kleene's theorem [53, Theorem 3.2], and the equality $\mathrm{Im}(\psi_z) = \mathrm{Cl}_{\mathsf{S}}(\psi_z(A_k)^+)$ holds by continuity of ψ_z, we know that the set $\mathrm{Im}(\psi_z)$ is clopen by Theorem 3.1. Hence, there is a positive integer m

such that $u^m \in \mathrm{Im}(\psi_z)$. Let $K' = K \cap \mathrm{Im}(\psi_z)$. Since K' is a closed subgroup, the closed subsemigroup $\psi_z^{-1}(K')$ of $\overline{\Omega}_{A_k}\mathsf{S}$ contains a closed subgroup K'' such that $\psi_z(K'') = K'$ [66, Proposition 3.1.1]. As $u^m \in K'$, we may take $v \in K''$ such that $u^m = \psi_z(v)$.

We claim that $K'' \subseteq H$. On one hand we have $\psi_z(h) = g \in K' = \psi_z(K'')$, and on the other hand, as $\boldsymbol{\sigma}$ is encoding, the homomorphism ψ_z is injective by Theorem 3.8. Therefore, we must have $h \in K''$, which establishes the claim $K'' \subseteq H$ by maximality of H.

In particular, we have $v \in H$. On the other hand, we also have

$$h = h^m = \psi_{z^{-1}e}(u^m) = \psi_{z^{-1}e}(\psi_z(v)) = \psi_f(v).$$

Since $H \subseteq \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)})$ and ψ_f restricts to the identity on $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)})$, it follows that $v = h$, thus $u^m = \psi_z(v) = \psi_z(h) = g$. But every closed subgroup of $\overline{\Omega}_{A_0}\mathsf{S}$ is torsion-free by [65, Theorem 1], and so $u = g$. This proves that the restriction of $\psi_{z^{-1}e}$ to K is injective, thereby establishing that $\boldsymbol{\sigma}$ is S-saturating. □

The next proposition and the ensuing corollary, which are not necessary for the sequel, shed additional light on Theorem 10.7.

Proposition 10.8. *Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a primitive directive sequence. Let $k, m \in \mathbb{N}$, with $k \leq m$, be such that $\sigma^{\mathsf{V}}_{k,m}$ is injective. If $\boldsymbol{\sigma}^{(k)}$ is V-saturating, then $\boldsymbol{\sigma}^{(m)}$ is V-saturating.*

Proof. The intersection $J_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)})$ is a regular $\mathcal{J}$-class of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)})$, by Theorem 6.9, and so it contains a maximal subgroup G of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)})$.

Since $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)}) = \sigma^{\mathsf{V}}_{k,m}(\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)})$ by Lemma 6.12, and $\sigma^{\mathsf{V}}_{k,m}$ is injective, we know that $\sigma^{\mathsf{V}}_{k,m}$ restricts to a continuous isomorphism $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)}) \to \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)})$. Therefore, $\sigma^{\mathsf{V}}_{k,m}(G)$ is a maximal subgroup of $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)})$. Moreover, the inclusion $\sigma^{\mathsf{V}}_{k,m}(G) \subseteq J_{\mathsf{V}}(\boldsymbol{\sigma}^{(k)})$ holds by Corollary 6.15. Since we are assuming that $\boldsymbol{\sigma}^{(k)}$ is V-saturating, the group $\sigma^{\mathsf{V}}_{k,m}(G)$ is in fact a maximal subgroup of $\overline{\Omega}_{A_k}\mathsf{V}$, by Proposition 10.3.

Let H be the maximal subgroup of $\overline{\Omega}_{A_m}\mathsf{V}$ containing G. Since $\sigma^{\mathsf{V}}_{k,m}(G)$ is a maximal subgroup of $\overline{\Omega}_{A_k}\mathsf{V}$, we necessarily have $\sigma^{\mathsf{V}}_{k,m}(G) = \sigma^{\mathsf{V}}_{k,m}(H)$, whence $G = H$ by injectivity of $\sigma^{\mathsf{V}}_{k,m}$. This shows that the maximal subgroup H of $\overline{\Omega}_{A_m}\mathsf{V}$ is contained in $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}^{(m)})$, thus establishing that $\boldsymbol{\sigma}^{(m)}$ is V-saturating. □

We say that a directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is *eventually* V*-saturating* if there is $k \in \mathbb{N}$ such that $\boldsymbol{\sigma}^{(m)}$ is V-saturating for every $m \geq k$.

Corollary 10.9. *Let $\boldsymbol{\sigma}$ be a bounded primitive encoding directive sequence. Then $\boldsymbol{\sigma}$ is eventually S-saturating if and only if there is $k \in \mathbb{N}$ such that $\boldsymbol{\sigma}^{(k)}$ is S-saturating.*

Proof. Suppose that there is $k \in \mathbb{N}$ such that $\boldsymbol{\sigma}^{(k)}$ is S-saturating. Let $m \in \mathbb{N}$ be such that $k \leq m$. Since $\boldsymbol{\sigma}$ is an encoding directive sequence, the homomorphism $\sigma_{k,n}$ is injective. Hence, $\sigma^{\mathsf{S}}_{k,n}$ is injective by Theorem 3.8. Applying Proposition 10.8, we deduce that $\boldsymbol{\sigma}^{(m)}$ is S-saturating for every integer m such that $m \geq k$. We have therefore established the "if" part of the corollary. The "only if" part is trivial. □

We do not know for which pseudovarieties V we may replace S by V in the statement of Theorem 10.7, even when $\boldsymbol{\sigma}$ is eventually S-saturating.

10.2. **Recognizable directive sequences.** In earlier work, the first two authors showed that every primitive aperiodic proper substitution $\sigma\colon A^+ \to A^+$ is such that $\mathrm{Im}((\sigma^{\mathsf{S}})^\omega)$ is a maximal subgroup of $\overline{\Omega}_A\mathsf{S}$ (cf. [11, Lemma 6.3], see also [11, Theorem 5.6]). An essential ingredient of the proof is Mossé's theorem stating that every primitive aperiodic substitution is recognizable. Therefore, the following theorem may be considered a generalization to the S-adic setting of the result of the two first authors.

Theorem 10.10. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence. If $\boldsymbol{\sigma}$ is recognizable, then it is S-saturating.*

For the proof of this theorem we need the next couple of lemmas. The first one is included in [27, Lemma 3.5] (also in [41, Proposition 6.4.16]).

Lemma 10.11. *Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a primitive directive sequence. Let $n, m \in \mathbb{N}$, with $n < m$. The substitution $\sigma_{n,m}$ is recognizable in $X(\boldsymbol{\sigma}^{(m)})$ if and only if, for every integer k such that $n \leq k \leq m$, the substitution σ_k is recognizable in $X(\boldsymbol{\sigma}^{(k)})$.*

In [18, Proposition 4.4.17] one finds a proof of the following lemma.[4]

Lemma 10.12. *Let A be a finite alphabet. Let $u, v \in \overline{\Omega}_A\mathsf{S}$. If $(w_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\overline{\Omega}_A\mathsf{S}$ such that $\lim w_n = uv$, then there are sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ of elements of $(\overline{\Omega}_A\mathsf{S})^1$ respectively converging to u and v and such that $w_n = u_n v_n$ for every $n \in \mathbb{N}$.*

We may now proceed to show Theorem 10.10.

Proof of Theorem 10.10. By Corollary 6.16, we may consider a sequence $(e_k)_{k\in\mathbb{N}}$ of idempotents such that $e_k \in J_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)}) \cap \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma}^{(k)})$ and $e_k = \sigma^{\mathsf{S}}_{k,l}(e_l)$ for every $k, l \in \mathbb{N}$ such that $k \leq l$. Set $z_k = \Bbbk(e_k)$ for each $k \in \mathbb{N}$. Note that $z_k \in X(\boldsymbol{\sigma}^{(k)})$ by Proposition 5.9. Since $\sigma^{\mathsf{S}}_{0,k}(e_k) = e_0$, we have

$$\sigma_{0,k}(z_k) = z_0.$$

Denote by H the maximal subgroup of $\overline{\Omega}_{A_0}\mathsf{S}$ containing e_0. We wish to show the inclusion $H \subseteq \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$.

Fix $s \in H$. By Proposition 5.6, we may write s as a limit

$$s = \lim_{n\to\infty} t_n, \quad t_n \in L(\boldsymbol{\sigma}).$$

Since $s = e_0 s e_0$, it follows from Lemma 10.12 that we may choose for every $n \in \mathbb{N}$ a factorization $t_n = p_n s_n q_n$ in $(A_0)^*$ such that

$$e_0 = \lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n, \quad s = \lim_{n\to\infty} s_n,$$

with the limits being taken in $(\overline{\Omega}_{A_0}\mathsf{S})^1$.

Take an arbitrary positive integer k. By Lemma 10.11, the composite $\sigma_{0,k}$ is recognizable in $X(\boldsymbol{\sigma}^{(k)})$. Since $X(\boldsymbol{\sigma}^{(k)})$ is minimal, it is generated by the sequence z_k. Hence, $\sigma_{0,k}$ is recognizable in Mossé's sense for z_k by Proposition 2.14; let ℓ_k be the corresponding constant of recognizability. Denote z_0 by z. Because $\Bbbk(H) = \Bbbk(e_0) = z$ and for every $x \in (A_0)^+$ the sets of the form $x(\overline{\Omega}_{A_0}\mathsf{S})^1$ and

[4] In [18, Proposition 4.4.17] it is used the notation $\widehat{A^*}$ instead of $(\overline{\Omega}_A\mathsf{S})^1$, and the assumption that A is finite is implicit in the statement, since it is done globally in an early point of the chapter. Indeed, $\widehat{A^*}$ denotes there the free profinite *monoid* generated by A, which is equal to $(\overline{\Omega}_A\mathsf{S})^1$, see the last paragraph in [18, Section 4.4]. See also [18, Section 4.12] for early references to this lemma.

$(\overline{\Omega}_{A_0}\mathsf{S})^1 x$ are clopen subsets of $\overline{\Omega}_{A_0}\mathsf{S}$, there is a positive integer N_k such that the following relations hold whenever $n > N_k$:

$$z[-\ell_k, 0) \geq_{\mathcal{L}} p_n, \quad z[0, \ell_k) \geq_{\mathcal{R}} s_n, \quad z[-\ell_k, 0) \geq_{\mathcal{L}} s_n, \quad z[0, \ell_k) \geq_{\mathcal{R}} q_n. \tag{9}$$

Take $n > N_k$. Since $p_n s_n q_n \in L(\boldsymbol{\sigma})$ and the minimal shift space $X(\boldsymbol{\sigma})$ is generated by z, there are integers $j_1 < j_2 < j_3 < j_4$ such that

$$z[j_1, j_2) = p_n, \quad z[j_2, j_3) = s_n, \quad z[j_3, j_4) = q_n.$$

Consider the set $C = C_{\sigma_{0,k}}(z_k)$ of $\sigma_{0,k}$-cutting points of z_k, and bear in mind the equality $\sigma_{0,k}(z_k) = z$. It follows from (9) that

$$z[j_2 - \ell_k, j_2) = z[-\ell_k, 0), \quad z[j_2, j_2 + \ell_k) = z[0, \ell_k),$$

that is, $z[j_2 - \ell_k, j_2 + \ell_k) = z[-\ell_k, \ell_k)$, and so, since $0 \in C$, by recognizability we conclude that $j_2 \in C$. Similarly, we conclude that $j_3 \in C$. Hence, as $s_n = z[j_2, j_3)$, we have $s_n \in \sigma_{0,k}(A_k^+)$. Since n is an arbitrary integer greater than N, it follows that $s \in \mathrm{Im}(\sigma_{0,k}^{\mathsf{S}})$. As k is arbitrary, this shows that $s \in \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$. We have therefore established the inclusion $H \subseteq \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$, and so $\boldsymbol{\sigma}$ is S-saturating. □

We next apply Theorem 10.10, together with other results, to deduce upper bounds on the rank of V-Schützenberger groups.

Corollary 10.13. *Let $\boldsymbol{\sigma}$ be a contraction stable, recognizable, primitive directive sequence, with finite alphabet rank n. Let V be a pseudovariety of semigroups containing LSl. Then the rank of the Schützenberger group $G_{\mathsf{V}}(\boldsymbol{\sigma})$ is at most n^2-n+1. If moreover $\boldsymbol{\sigma}$ has a proper contraction, then the rank of $G_{\mathsf{V}}(\boldsymbol{\sigma})$ is at most n.*

Proof. By Corollary 5.10, it suffices to establish the result for the case $\mathsf{V} = \mathsf{S}$.

Since $\boldsymbol{\sigma}$ is contraction stable, the profinite semigroup $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$ is contained in $J_{\mathsf{S}}(\boldsymbol{\sigma})$, by Theorems 7.1 and 7.5. As $\boldsymbol{\sigma}$ is recognizable, it follows from Theorem 10.10 that $\boldsymbol{\sigma}$ is S-saturating, and so the maximal subgroups of $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$ are actually maximal subgroups of $J_{\mathsf{S}}(\boldsymbol{\sigma})$. All maximal subgroups of $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$ have rank at most $n^2 - n + 1$ by Corollary 8.8. If $\boldsymbol{\sigma}$ moreover has a proper contraction, then $\mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$ is a profinite group by Theorem 7.9, and its rank is at most n by Corollary 8.6. □

We next see how Corollary 10.13 extends to the important class of minimal shift spaces of *finite topological rank*. A minimal shift space is said to be of finite topological rank when it can be represented by a Bratteli-Vershik diagram with a uniformly bounded number of vertices per level; and if the least such bound among all such representations is n, then it is said to have finite topological rank n; see [41, Chapter 6] for details. A minimal shift space X has finite topological rank at most n if and only if it is topologically conjugate to $X(\boldsymbol{\sigma})$ for some proper, recognizable, primitive directive sequence $\boldsymbol{\sigma}$ of alphabet rank at most n (this result is from [38], as attributed in [44, Theorem 1.1]).

Corollary 10.14. *Let X be a minimal shift space of finite topological rank n. Then, for every pseudovariety V containing LSl, the Schützenberger group $G_{\mathsf{V}}(X)$ is a profinite group of rank at most n.*

Proof. By Corollary 5.10, it suffices to establish the result for the case $\mathsf{V} = \mathsf{S}$.

By [38, Proposition 4.6], there exists a proper, recognizable, primitive directive sequence $\boldsymbol{\sigma}$ with alphabet rank at most n and such that X is topologically conjugate to $X(\boldsymbol{\sigma})$. Since the S-Schützenberger group of a minimal shift space is a topological conjugacy invariant by Theorem 5.2, we have $G_{\mathsf{S}}(X) \cong G_{\mathsf{S}}(\boldsymbol{\sigma})$. The result now follows immediately from Corollary 10.13. $\square$

Relaxing the hypothesis in Corollary 10.13, we obtain the following.

Corollary 10.15. *Let $\boldsymbol{\sigma}$ be a primitive directive sequence with finite alphabet rank n. Let V be a pseudovariety of semigroups containing LSI. If $\boldsymbol{\sigma}$ is recognizable, then the Schützenberger group $G_{\mathsf{V}}(\boldsymbol{\sigma})$ is a profinite group of rank at most n^2.*

Proof. Under the same assumptions on $\boldsymbol{\sigma}$, Donoso et al. showed that $X(\boldsymbol{\sigma})$ has topological rank at most n^2 [38, Proposition 4.7]. Combining this result with Corollary 10.14, we get that $G_{\mathsf{V}}(\boldsymbol{\sigma})$ has rank at most n^2. $\square$

We do not know if the converse of Corollary 10.14 holds:

Problem 10.16. *Let X be a minimal shift space.*

(i) *Suppose that $G_{\mathsf{S}}(X)$ is finitely generated. Does X necessarily have finite topological rank?*
(ii) *Is it true that, if H is a nontrivial pseudovariety of groups such that $G_{\overline{\mathsf{H}}}(X)$ is finitely generated, then $G_{\mathsf{S}}(X)$ is finitely generated?*

10.3. **Sufficient conditions for recognizability.** The purpose of this section is to give conditions under which saturating directive sequences are recognizable. In other words, we are proposing a partial converse to Theorem 10.10.

Theorem 10.17. *Let $\boldsymbol{\sigma}$ be an encoding directive sequence. Let V be a pseudovariety of semigroups containing LSI. Assume that $\mathrm{Im}(\sigma_{0,n})$ is V-recognizable for every $n \in \mathbb{N}$. If $\boldsymbol{\sigma}$ is V-saturating and eventually recognizable, then $\boldsymbol{\sigma}$ is recognizable.*

Proof. Using an argument of *reductio ad absurdum*, let us suppose that the hypothesis in the statement holds but $\boldsymbol{\sigma}$ is not recognizable.

Let $m \in \mathbb{N}$ be such that $\boldsymbol{\sigma}^{(m)}$ is recognizable. Take $n \in \mathbb{N}$ such that $n > m$. By the assumption that $\boldsymbol{\sigma}$ is not recognizable and by Lemma 10.11, the homomorphism $\sigma_{0,n}$ is not recognizable in $X(\boldsymbol{\sigma}^{(n)})$. Therefore, letting $A = A_0$, there is an element in $A^{\mathbb{Z}}$ with two distinct centered $\sigma_{0,n}$-representations in $X(\boldsymbol{\sigma}^{(n)})$, which means that there are $x_n, z_n \in X(\boldsymbol{\sigma}^{(n)})$ and $\ell_n \in \mathbb{N}$ such that

$$\sigma_{0,n}(x_n) = T^{\ell_n} \sigma_{0,n}(T^n(z_n)) \tag{10}$$

with $0 \leq \ell_n < |\sigma_{0,n}(z_n[n])|$ and $(0, x_n) \neq (\ell_n, T^n(z_n))$.

Despite the statement mentioning the pseudovariety V, for most of the proof we work with the pseudovariety S of all finite semigroups. By Proposition 5.9 we may take the unique idempotents e_n and f_n of $J_{\mathsf{S}}(\boldsymbol{\sigma}^{(n)})$ such that $\Bbbk(e_n) = x_n$ and $\Bbbk(f_n) = z_n$. Note that both $\sigma^{\mathsf{S}}_{0,n}(e_n)$ and $\sigma^{\mathsf{S}}_{0,n}(f_n)$ belong to $J_{\mathsf{S}}(\boldsymbol{\sigma})$ and that the following equalities hold:

$$\Bbbk(\sigma^{\mathsf{S}}_{0,n}(e_n)) = \sigma_{0,n}(x_n) \qquad \text{and} \qquad \Bbbk(\sigma^{\mathsf{S}}_{0,n}(f_n)) = \sigma_{0,n}(z_n). \tag{11}$$

Denote by r_n the prefix of length ℓ_n of $\sigma_{0,n}(z_n[n])$, cf. Figure 3. Let

$$p'_n = z_n[0,n), \ p_n = \sigma_{0,n}(p'_n) r_n.$$

Then p_n is a prefix of $\sigma_{0,n}(z_n[0,n])$ and suffix of $\sigma_{0,n}(x_n[-n',0))$ for some $n' \in \mathbb{N}$, as illustrated by Figure 3. Hence, p_n is a prefix of the idempotent $\sigma_{0,n}(f_n)$, and a suffix of the idempotent $\sigma_{0,n}(e_n)$, in view of (11). Moreover, we have

$$\begin{aligned} T^{|p_n|}(\Bbbk(\sigma^{\mathsf{S}}_{0,n}(f_n))) &= T^{|r_n|}T^{|\sigma_{0,n}(z_n[0,n))|}(\sigma_{0,n}(z_n)) \\ &= T^{\ell_n}\sigma_{0,n}(T^n(z_n)) \\ &= \sigma_{0,n}(x_n) \\ &= \Bbbk(\sigma^{\mathsf{S}}_{0,n}(e_n)), \end{aligned}$$

with the second last equality holding by (10). Therefore, the equality

$$p_n\sigma^{\mathsf{S}}_{0,n}(e_n) = \sigma^{\mathsf{S}}_{0,n}(f_n)p_n \tag{12}$$

holds by Proposition 5.13.

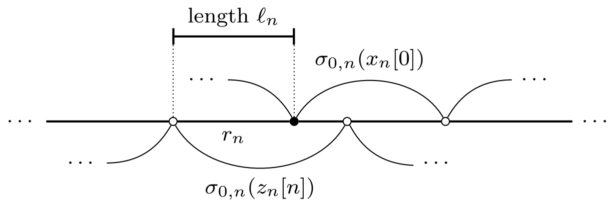

FIGURE 3. The bi-infinite word $\sigma_{0,n}(x_n) = T^{\ell_n}\sigma_{0,n}(T^n(z_n))$. The black cutting point marks the boundary of its left and right infinite parts.

Suppose that $\ell_n = 0$, that is to say $p_n = \sigma_{0,n}(p'_n)$. Then the equality (12) becomes

$$\sigma^{\mathsf{S}}_{0,n}(p'_n e_n) = \sigma^{\mathsf{S}}_{0,n}(f_n p'_n).$$

As by hypothesis $\boldsymbol{\sigma}$ is an encoding directive sequence, the homomorphism $\sigma_{0,n}$ is injective, and so $\sigma^{\mathsf{S}}_{0,n}$ is injective by Theorem 3.8. It follows that $p'_n e_n = f_n p'_n$, thus $x_n = T^n(z_n)$ by Proposition 5.13. But this contradicts $(0, x_n) \neq (\ell_n, T^n(z_n))$. Therefore, ℓ_n must be positive.

By compactness of $\overline{\Omega}_A\mathsf{S}$, the sequence $(p_n, \sigma^{\mathsf{S}}_{0,n}(e_n), \sigma^{\mathsf{S}}_{0,n}(f_n))_{n>m}$ has some subsequence converging in $(\overline{\Omega}_A\mathsf{S})^3$ to a triple (p, e, f). Bear in mind that, by Lemma 6.5, the pseudowords e, f are idempotents in $J_{\mathsf{S}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{S}}(\boldsymbol{\sigma})$, as $\sigma^{\mathsf{S}}_{0,n}(e_n)$ and $\sigma^{\mathsf{S}}_{0,n}(f_n)$ are, for every $n \in \mathbb{N}$, idempotents in the closed space $J_{\mathsf{S}}(\boldsymbol{\sigma})$. Note that $\lim |p_n| = \infty$, and so the pseudoword p has infinite length. As $p_n \in L(\boldsymbol{\sigma})$ for every $n > m$, it follows that $p \in J_{\mathsf{S}}(\boldsymbol{\sigma})$. Since p_n is a prefix of the idempotent $\sigma^{\mathsf{S}}_{0,n}(f_n)$ for each $n > m$, and the relation $\leq_{\mathcal{R}}$ is closed in $\overline{\Omega}_A\mathsf{S}$, we know that p is a prefix of f. Similarly, p is a suffix of e. By stability, we obtain $f \mathrel{\mathcal{R}} p \mathrel{\mathcal{L}} e$. It follows that $p_{\mathsf{S},\mathsf{V}}(f) \mathrel{\mathcal{R}} p_{\mathsf{S},\mathsf{V}}(p) \mathrel{\mathcal{L}} p_{\mathsf{S},\mathsf{V}}(e)$. Note that the idempotents $p_{\mathsf{S},\mathsf{V}}(f)$ and $p_{\mathsf{S},\mathsf{V}}(e)$ belong to $J_{\mathsf{V}}(\boldsymbol{\sigma}) \cap \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ by Corollary 5.5 and Proposition 6.6. Since $\boldsymbol{\sigma}$ is V-saturating, we deduce that $p_{\mathsf{S},\mathsf{V}}(p) \in \mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma})$ by Proposition 10.3.

Set $B = A_m$. Since $\mathrm{Im}_{\mathsf{V}}(\boldsymbol{\sigma}) \subseteq \mathrm{Im}_{\mathsf{V}}(\sigma^{\mathsf{V}}_{0,m})$, we have $p_{\mathsf{S},\mathsf{V}}(p) \in \mathrm{Cl}_{\mathsf{V}}(\sigma_{0,m}(B^+))$. Because $\sigma_{0,m}(B^+)$ is V-recognizable, as assumed in the statement, the set $\mathrm{Cl}_{\mathsf{V}}(\sigma_{0,m}(B^+))$ is clopen by Theorem 3.1. Note also that, since the continuous mapping $p_{\mathsf{S},\mathsf{V}}$ restricts to the identity on A^+, the pseudoword $p_{\mathsf{S},\mathsf{V}}(p)$ is a cluster point of the sequence $(p_n)_n$ in the space $\overline{\Omega}_A\mathsf{V}$. Hence, there is $k \in \mathbb{N}$ such that $k > m$ and $p_k \in \sigma_{0,m}(B^+)$.

Take $q \in B^+$ such that $p_k = \sigma_{0,m}(q)$. The equality (12) then entails

$$\sigma^{\mathsf{S}}_{0,m}\Big(q \cdot \sigma^{\mathsf{S}}_{m,k}(e_k)\Big) = \sigma^{\mathsf{S}}_{0,m}\Big(\sigma^{\mathsf{S}}_{m,k}(f_k) \cdot q\Big).$$

But $\sigma^{\mathsf{S}}_{0,m}$ is injective (by Theorem 3.8), and so the equality $q \cdot \sigma^{\mathsf{S}}_{m,k}(e_k) = \sigma^{\mathsf{S}}_{m,k}(f_k) \cdot q$ holds. Since $\mathfrak{k}(\sigma^{\mathsf{S}}_{m,k}(e_k)) = \sigma_{m,k}(x_k)$ and $\mathfrak{k}(\sigma^{\mathsf{S}}_{m,k}(f_k)) = \sigma_{m,k}(z_k)$, we then deduce from Proposition 5.13 that

$$\sigma_{m,k}(x_k) = T^{|q|}(\sigma_{m,k}(z_k))$$

and that q is a nonempty prefix of a word of the form $\sigma_{m,k}(z_k[0,l))$, with $l > 0$ (see Figure 4). Hence, we may consider the integer

$$l_0 = \min\{l \in \mathbb{N} : |\sigma_{m,k}(z_k[0,l))| \geq q\}.$$

Letting $d = |\sigma_{m,k}(z_k[0,l_0))| - q$ we see that $(d, T^{l_0-1}(z_k))$ is a centered $\sigma_{m,k}$-representation of $\sigma_{m,k}(x_k)$. Since $\boldsymbol{\sigma}^{(m)}$ is recognizable, we know that $\sigma_{m,k}$ is recognizable in $X(\boldsymbol{\sigma}^{(k)})$ by Lemma 10.11. Therefore, we must have $d = 0$, thus $q = \sigma_{m,k}(z_k[0,l_0))$.

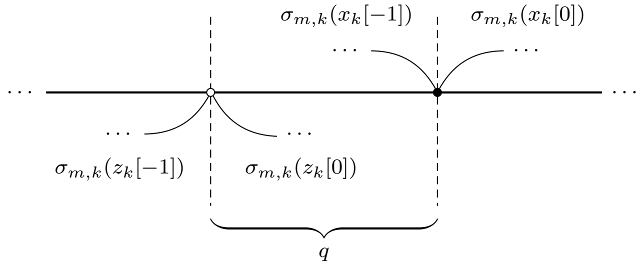

FIGURE 4. Location of q in the infinite word $\mathfrak{k}(\sigma^{\mathsf{S}}_{m,k}(e_k)) = \sigma_{m,k}(x_k)$.

We have therefore $\sigma_{0,k}(z_k[0,l_0)) = \sigma_{0,m}(q) = p_k = \sigma_{0,k}(z_k[0,k))r_k$. In particular, we must have $l_0 > k$, as $|r_k| = \ell_k \neq 0$. It follows that $\sigma_{0,k}(z_k[k])$ is a prefix of r_k. But this contradicts the fact that $|r_k| = \ell_k < |\sigma_{0,k}(z_k[k])|$ by choice of r_k and ℓ_k.

This concludes the argument by *reductio ad absurdum*, and therefore we proved that $\boldsymbol{\sigma}$ must be recognizable. ☐

In the special case where V is of the form $\overline{\mathsf{H}}$ for some extension-closed pseudovariety of groups H, the previous theorem can be specialized as follows.

Corollary 10.18. *Let $\boldsymbol{\sigma}$ be an eventually recognizable primitive directive sequence. Let H be an extension-closed pseudovariety of groups such that σ_n is an $\overline{\mathsf{H}}$-encoding for every $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) $\boldsymbol{\sigma}$ *is recognizable;*
(ii) $\boldsymbol{\sigma}$ *is S-saturating;*
(iii) $\boldsymbol{\sigma}$ *is $\overline{\mathsf{H}}$-saturating.*

The proof of the corollary requires the next lemma.

Lemma 10.19. *Let H be an extension-closed pseudovariety of groups. Let $L \subseteq A^+$ be an $\overline{\mathsf{H}}$-recognizable language and $\sigma\colon A^+ \to B^+$ be an $\overline{\mathsf{H}}$-encoding. Then $\sigma(L)$ is $\overline{\mathsf{H}}$-recognizable.*

Proof. Observe that $\sigma^{\overline{\mathsf{H}}}\colon \overline{\Omega}_A\overline{\mathsf{H}} \to \overline{\Omega}_B\overline{\mathsf{H}}$ is injective by Theorem 3.8 (ii) and its image is clopen by [19, Corollary 5.7]. By assumption, the closure of L in $\overline{\Omega}_A\overline{\mathsf{H}}$ is clopen. Hence, the closure of $\sigma(L)$ in $\overline{\Omega}_B\overline{\mathsf{H}}$, which is equal to $\sigma^{\overline{\mathsf{H}}}(L)$, is clopen and, therefore, $\sigma(L)$ is $\overline{\mathsf{H}}$-recognizable by Theorem 3.1. □

Remark 10.20. An alternative proof of this lemma is obtained by combining [62, Proposition 4.3] with [54, Theorem 3].

Proof of Corollary 10.18. The implication (i) ⇒ (ii) is Theorem 10.10, while the implication (ii) ⇒ (iii) is given by Proposition 10.4. It remains to establish the implication (iii) ⇒ (i).

Assume that (iii) holds. By Lemma 10.19, the composition of two $\overline{\mathsf{H}}$-encodings is again an $\overline{\mathsf{H}}$-encoding. Therefore, under our assumptions, the image of $\sigma_{0,n}$ is $\overline{\mathsf{H}}$-recognizable, for all $n \in \mathbb{N}$. Since $\mathsf{LSI} \subseteq \mathsf{A} \subseteq \overline{\mathsf{H}}$, we can apply Theorem 10.17 to conclude that (i) holds. □

The case $\mathsf{H} = \mathsf{I}$ in Corollary 10.18 is precisely the pure case. Since $G_{\mathsf{A}}(\boldsymbol{\sigma})$ is the trivial group, condition (iii) holds trivially in that case. We deduce the following.

Corollary 10.21. *Let $\boldsymbol{\sigma}$ be an eventually recognizable primitive directive sequence. If $\boldsymbol{\sigma}$ is pure, then $\boldsymbol{\sigma}$ is recognizable.*

The following result gives yet another sufficient condition for recognizability.

Theorem 10.22. *Let $\boldsymbol{\sigma}$ be a bounded primitive directive sequence. If $\boldsymbol{\sigma}$ is eventually recognizable, recurrent, and encoding, then it is recognizable.*

Proof. By Theorem 10.10, the assumption that $\boldsymbol{\sigma}$ is eventually recognizable entails that it is eventually S-saturating. Theorem 10.7 then yields that $\boldsymbol{\sigma}$ is S-saturating. Finally, as each σ_n is an S-encoding, Corollary 10.18 shows that $\boldsymbol{\sigma}$ is recognizable. □

Although the statements of Corollary 10.21 and Theorem 10.22 concern only symbolic dynamics, their proofs use the connection with profinite semigroups in crucial ways. For instance, the proof of Theorem 10.22 relies indirectly on the fact that closed subgroups of free profinite semigroups are torsion-free [65, Theorem 1] (needed in the proof of Theorem 10.7). This raises the question of whether or not proofs of purely dynamical and combinatorial character can be given for those results.

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