Bicategories of Lax Fractions

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Abstract

The well-known calculus of fractions of Gabriel and Zisman provides a convenient way to formally invert morphisms in a category. This was extended to bicategories by Pronk. On the other hand, the second author has developed a calculus of lax fractions for order-enriched categories that formally turns a given class of morphisms into left adjoint right inverses. We extend these constructions by presenting a calculus of lax fractions for 2-categories that formally turns a class of morphisms and pseudo-commutative squares into left adjoint right inverses and Beck–Chevalley squares.

Contents

1 Introduction

The calculus of fractions construction by Gabriel and Zisman [2] provides a universal way to formally invert a class of morphisms in a category. In [6] Pronk presents a 2-categorical generalisation allowing for freely turning morphisms from a given class into equivalences. On the other hand, in [7] the second author introduces a calculus of lax fractions for order-enriched categories in order to formally add right adjoint retractions to morphisms in a class while also controlling when the Beck–Chevalley condition holds.

In this paper we provide a synthesis of these two approaches by providing a calculus of lax fractions for 2-categories.

One application of our calculus, developed in the paper [4], is a construction of the bicategory of (strict) monoidal categories and lax monoidal functors from the 2-category of strict monoidal categories and strict monoidal functors by formally adding right adjoints to the morphisms whose underlying functors have fully faithful right adjoints.

The data for the construction involves the original 2-category $\mathcal{X}$ and a collection Σ of squares, commuting up to isomorphism, whose horizontal morphisms are to become left adjoint right inverses, and which will themselves will become Beck–Chevalley squares. This collection

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of squares must satisfy a number of axioms (Definition 2.1). In particular, the horizontal morphisms of these squares together with the squares themselves form a subcategory of the arrow category $\mathcal{X}^{\to\cong}$. The resulting bicategory of lax fractions $\mathcal{X}[\Sigma_*]$ will have the same objects as $\mathcal{X}$ and 1-cells given by cospans $A \xrightarrow{f} I \xleftarrow{s} B$ where s is a horizontal morphism from Σ. The 2-cells are certain equivalence classes of diagrams of the form

$$\begin{array}{ccccc} A & \xrightarrow{f} & I & \xleftarrow{r} & B \\ \Vert & & {\scriptstyle x_1}\downarrow & \Sigma^{\delta_1} & \Vert \\ & \overset{\alpha}{\swarrow} & X & \xleftarrow{x_3} & B \\ \Vert & & {\scriptstyle x_2}\uparrow & \Sigma^{\delta_2} & \Vert \\ A & \xrightarrow[g]{} & J & \xleftarrow[s]{} & B \end{array}$$

where Σ denotes that the squares come from the collection Σ. This bicategory is universal in the sense that, for each bicategory $\mathcal{Y}$, there is an biequivalence between the bicategory of pseudofunctors from $\mathcal{X}[\Sigma_*]$ to $\mathcal{Y}$ and a bicategory whose objects are pseudofunctors from $\mathcal{X}$ to $\mathcal{Y}$ that send the horizontal morphisms of Σ to left adjoint right inverses and the squares of Σ to Beck–Chevalley squares.

In the paper [4] we will present several special examples of bicategories of lax fractions and explore the relationship between the calculus of lax fractions, lax idempotent monads and Kan extensions.

Bourke and Garner [1] addressed to the construction of categories by freely adding a "section" to certain morphisms. Our approach here is completely different but it may be interesting to study how their work relates to our bicategory of fractions.

2 The Σ-calculus

Let $\mathcal{X}$ be a 2-category and denote by $\mathcal{X}^{\to\cong}$ the arrow category whose objects are the 1-cells of $\mathcal{X}$ and whose morphisms from $f\colon X \to Y$ to $g\colon Z \to W$ are triples $(u, v, \delta)\colon f \to g$ where $u\colon X \to Z$ and $v\colon Y \to W$ are 1-cells and $\delta\colon gu \to vf$ is an invertible 2-cell:

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ {\scriptstyle u}\downarrow & \overset{\delta}{\nearrow} & \downarrow{\scriptstyle v} \\ \bullet & \xrightarrow[g]{} & \bullet \end{array} \quad . \tag{2.1}$$

The identity morphisms are just identity 2-cells $(1, 1, \mathrm{id})\colon f \to f$; composition is vertical composition of squares, that is, for $f \xrightarrow{(u,v,\delta)} g \xrightarrow{(u',v',\delta')} h$, the composition is given by $(u', v', \delta') \cdot (u, v, \delta) = (u'u, v'v, (v' \circ \delta)(\delta' \circ u))$.

Let Σ be a subcategory of $\mathcal{X}^{\to\cong}$. In the following we will use a square

$$\begin{array}{ccc} \bullet & \xrightarrow{r} & \bullet \\ {\scriptstyle u}\downarrow & \Sigma^{\delta} & \downarrow{\scriptstyle v} \\ \bullet & \xrightarrow[s]{} & \bullet \end{array} \ ,$$

with Σ^{δ} in its center, to indicate that $(u, v, \delta)\colon r \to s$ is a morphism in Σ. If no danger of confusion exists we also use just Σ without δ. We call these squares ***Σ-squares***. Sometimes we reverse or invert them, but the horizontal arrows always refer to objects of Σ, and the vertical ones to the 1-cell part of the represented morphism.

In this way, our calculus of lax fractions becomes essentially a calculus of Σ-squares.

Definition 2.1. Left calculus of lax fractions. Given a subcategory Σ of $\mathcal{X}^{\to\cong}$, we say that it ***admits a left calculus of lax fractions*** provided that the following conditions are statisfied.

(1) **Identity.** Every identity 1-cell of $\mathcal{X}$ is an object of Σ, and for every Σ-object $s\colon X \to Y$ we have the Σ-square

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle 1_X}\downarrow & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle s} \\ X & \xrightarrow[s]{} & Y \end{array} .$$

(2) **Repletion.**

(a) **Vertical Repletion.** For every invertible 2-cell $\delta\colon r \Rightarrow s$ with $r\colon X \to Y$ in Σ, s also belongs to Σ and we have the Σ-square

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle 1_X}\downarrow & \Sigma^{\delta} & \downarrow{\scriptstyle 1_Y} \\ X & \xrightarrow[r]{} & Y \end{array} .$$

(b) **Horizontal Repletion.** For every pair of morphisms $f, g\colon X \to Y$ and every invertible 2-cell $\gamma\colon f \Rightarrow g$, we have the Σ-square

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle f}\downarrow & \Sigma^{\gamma} & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[1_Y]{} & Y \end{array} .$$

(3) **Composition.** If in the diagram

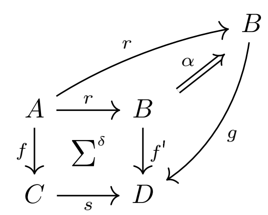
$$\begin{array}{ccccc} \bullet & \xrightarrow{r} & \bullet & \xrightarrow{s} & \bullet \\ {\scriptstyle f}\downarrow & ① & \downarrow{\scriptstyle g} & ② & \downarrow{\scriptstyle h} \\ \bullet & \xrightarrow[r']{} & \bullet & \xrightarrow[s']{} & \bullet \end{array}$$

① and ② are both Σ-squares, then the pasting diagram ①+② is also a Σ-square.

(4) **Square.** For every span $\bullet \xleftarrow{f} \bullet \xrightarrow{s} \bullet$ with $s \in \Sigma$, there is a Σ-square of the form

$$\begin{array}{ccc} \bullet & \xrightarrow{s} & \bullet \\ {\scriptstyle f}\downarrow & \Sigma & \downarrow{\scriptstyle f'} \\ \bullet & \xrightarrow[s']{} & \bullet \end{array} .$$

(5) **Equi-insertion.** For every Σ-square and every 2-cell $\alpha\colon f'r \Rightarrow gr$ as in the diagram

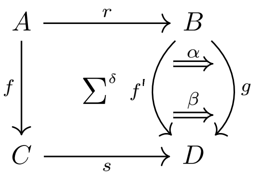


there is a 1-cell $d\colon D \to E$ and a 2-cell $\alpha'\colon df' \Rightarrow dg$ such that

$$\begin{array}{ccc} C & \xrightarrow{s} & D \\ \Vert & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle d} \\ C & \xrightarrow[ds]{} & E \end{array}$$

and $d\alpha = \alpha' r$.

(6) **Equification.** For every Σ-square and two 2-cells as in the diagram



with $\alpha r = \beta r$, there is a 1-cell $d\colon D \to E$ such that

$$\begin{array}{ccc} C & \xrightarrow{s} & D \\ \Vert & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle d} \\ C & \xrightarrow[ds]{} & E \end{array}$$

and $d\alpha = d\beta$.

Remark 2.2. Given a subcategory Σ of the arrow category $\mathcal{X}^{\to\cong}$, we say that it admits a ***right calculus of lax fractions*** if it satisfies the rules of Definition 2.1, but with all 1-cells reversed.

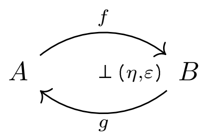
Remark 2.3. One can consider this left calculus of lax fractions in the more general context of bicategories. We opt for starting with a 2-category to simplify the exposition, and also because all the examples we know involve 2-categories.

Remark 2.4. The Composition axiom (together with Horizontal Repletion) allows to us to view Σ as a double category.

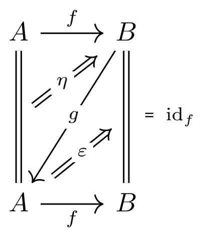
Remark 2.5. Using Vertical Repletion and Composition, we have the following property:

$$\begin{array}{ccc} B & \xrightarrow{r} & I \\ \Vert & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle x} \\ B & \xrightarrow[xr]{} & X \end{array} \quad \text{if and only if} \quad \begin{array}{ccc} B & \xrightarrow{r} & I \\ \Vert & \Sigma^{\delta} & \downarrow{\scriptstyle x} \\ B & \xrightarrow[u]{} & X \end{array} \quad \text{for some } u \text{ and } \delta.$$

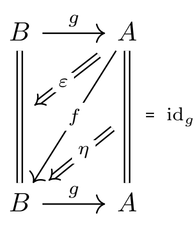
Remark 2.6. 1. A 1-cell $f\colon A \to B$ is said to be a ***lari*** (abbreviation for ***left adjoint right inverse***), if there is an adjunction



such that η is an invertible 2-cell. Thus, in this case, the conditions defining the adjunction



and

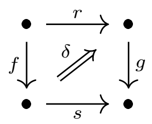


lead to

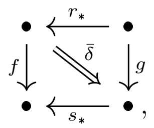
$$(f \circ \eta)^{-1} = \varepsilon \circ f \quad \text{and} \quad (\eta \circ g)^{-1} = g \circ \varepsilon.$$

In the following, we use the notation f_* to denote a right adjoint to f.

2. A diagram of the form



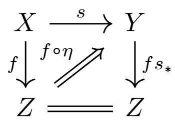
with r and s lari 1-cells and δ an invertible 2-cell has the ***Beck–Chevalley condition*** if its mate is an isomorphism — that is, in the diagram



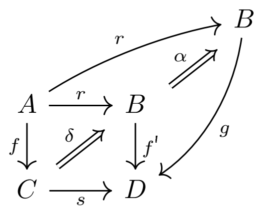
the 2-cell $\overline{\delta} = (s_* \circ g \circ \varepsilon^r) \cdot (s_* \circ \delta \circ r_*) \cdot (\eta^s \circ f \circ r_*) \colon fr_* \Rightarrow s_* g$ is invertible.

Examples 2.7. 1. **Laris.** For a 2-category $\mathcal{X}$, let Σ be the subcategory of the category $\mathcal{X}^{\to\cong}$ given by all laris and those morphisms of $\mathcal{X}^{\to\cong}$ forming Beck–Chevalley squares.

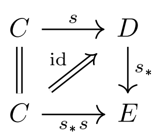
The subcategory Σ of laris admits a left calculus of lax fractions. Indeed, Identity, Repletion and Composition are clear. For Square, observe that, since s is a lari, the square



satisfies the Beck–Chevalley condition. For Equi-insertion, suppose we have



where δ is invertible and satisfying the Beck–Chevalley condition. Let $\overline{\delta}$ be the mate of δ. The Equi-insertion condition is fulfilled by putting $d = s_*$ and defining α' to be the composite $s_* f' \xrightarrow{\overline{\delta}^{-1}} fr_* \xrightarrow{\overline{\alpha\delta}} s_* g$ where $\overline{\alpha\delta}$ denotes the mate of the composite $\alpha \cdot \delta$. Note that



is indeed a Beck–Chevalley square. To see that $s_*\alpha = \alpha' r$, it suffices to show $s_*\alpha \cdot \overline{\delta} r \cdot f\eta^r = \alpha' r \cdot \overline{\delta} r \cdot f\eta^r$, since $\overline{\delta}$ and η^r are invertible. But $\alpha' r \cdot \overline{\delta} r \cdot f\eta^r = \overline{\alpha\delta} r \cdot f\eta^r$. Thus, we can just show that $s_*\alpha \cdot \overline{\delta} r \cdot f\eta^r = \overline{\alpha\delta} r \cdot f\eta^r$. Expanding the definition of the mates we have

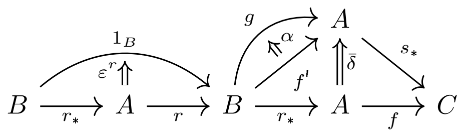
$$
\begin{aligned}
& s_*\alpha \cdot s_* f' \varepsilon^r r \cdot s_* \delta r_* r \cdot \eta^s f r_* r \cdot f\eta^r \\
={}& s_*\alpha \cdot s_* f' \varepsilon^r r \cdot s_* \delta r_* r \cdot s_* s f \eta^r \cdot \eta^s f \\
={}& s_*\alpha \cdot s_* f' \varepsilon^r r \cdot s_* f' r \eta^r \cdot s_* \delta \cdot \eta^s f \\
={}& s_*\alpha \cdot s_* \delta \cdot \eta^s f
\end{aligned}
$$

on the left-hand side and

$$\begin{aligned} & s_* g\varepsilon^r r \cdot s_*\alpha r_* r \cdot s_*\delta r_* r \cdot \eta^s f r_* r \cdot f\eta^r \\ = \ & s_* g\varepsilon^r r \cdot s_*\alpha r_* r \cdot s_* f' r\eta^r \cdot s_*\delta \cdot \eta^s f \\ = \ & s_* g\varepsilon^r r \cdot s_* g r\eta^r \cdot s_*\alpha \cdot s_*\delta \cdot \eta^s f \\ = \ & s_*\alpha \cdot s_*\delta \cdot \eta^s f \end{aligned}$$

on the right-hand side. Thus, they are indeed equal.

For Equification, we show that again we may take $d = s_*$. Let $\bar{\delta} : fr_* \Rightarrow s_* f'$ be the mate of δ. Composing α and β with $\varepsilon^r : rr_* \Rightarrow 1_B$ and $\bar{\delta}$,



as $s_* \circ \alpha \circ r = s_* \circ \beta \circ r$, we have $s_* \circ \alpha \circ \varepsilon^r = s_* \circ \beta \circ \varepsilon^r$, hence we obtain that $(s_* \circ \alpha) \cdot (f \circ r_* \circ \varepsilon^r) \cdot (\bar{\delta} \circ r \circ r_*) = (s_* \circ \beta) \cdot (f \circ r_* \circ \varepsilon^r) \cdot (\bar{\delta} \circ r \circ r_*)$. Since $(f \circ r_* \circ \varepsilon^r) \cdot (\bar{\delta} \circ r \circ r_*)$ is invertible, $s_* \circ \alpha = s_* \circ \beta$.

2. **Ordinary left calculus of fractions.** Given an ordinary category $\mathcal{X}$ and a class Σ of morphisms of $\mathcal{X}$, let us look at $\mathcal{X}$ as a 2-category with trivial 2-cells (in particular, laris are just isomorphisms) and at Σ as a full subcategory of the arrow category $\mathcal{X}^{\rightarrow}$. Then, for Σ, to admit a left calculus of lax fractions just means to admit a left calculus of fractions in the classical sense ([2]).

3. **Pronk's calculus.** In [6], Dorette Pronk introduced a *right bicalculus of fractions* for a class Σ of 1-cells generalising the classical calculus for bicategories (see also [5]). With this calculus, the localization process yields a bicategory where morphisms in Σ become equivalences. Here we show that in a 2-category $\mathcal{X}$, a class Σ of 1-cells admits a left bicalculus of fractions, in the sense of Pronk, if and only if Σ, viewed as a full subcategory of $\mathcal{X}^{\rightarrow\cong}$, admits a left calculus of lax fractions.

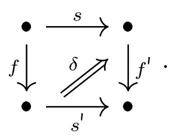
   First, we observe that, if Σ is a full subcategory of $\mathcal{X}^{\rightarrow\cong}$, we can look at Σ as a class of 1-cells of $\mathcal{X}$, and our *left calculus of lax fractions* becomes one with the following rules:

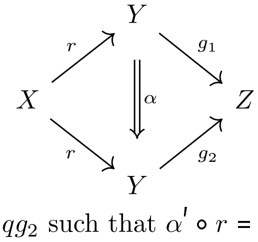
   (Id) All identity 1-cells belong to Σ.

   (Rep) For every invertible 2-cell $\delta : r \Rightarrow s$ with $r \in \Sigma$, also s belongs to Σ.

   (Comp) Σ is closed under composition.

   (Sq) For every span $X \xleftarrow{f} Y \xrightarrow{s} Z$ with $s \in \Sigma$, there are 1-cells $s' : Z \to W$ and $f' : Y \to W$, with s' in Σ, and an invertible 2-cell



   (Eq1) For every 2-cell  with $r \in \Sigma$, there is $q : Z \to W$ in Σ and a 2-cell $\alpha' : qg_1 \Rightarrow qg_2$ such that $\alpha' \circ r = q \circ \alpha$.

(Eq2) For every diagram $X \xrightarrow{r} Y \overset{g_1}{\underset{g_2}{\rightrightarrows}} Z$ (with 2-cells $\alpha, \beta\colon g_1 \Rightarrow g_2$) with $r \in \Sigma$, there is $q\colon Z \to W$ in Σ such that $q \circ \alpha = q \circ \beta$.

For obtaining (Eq1) from Equi-insertion, observe that now $\begin{array}{ccc} \cdot & \xrightarrow{r} & \cdot \\ {\scriptstyle g_1 r}\downarrow & = & \downarrow{\scriptstyle g_1} \\ \cdot & = & \cdot \end{array}$ is a Σ-square. Conversely, from (Eq1) and (Comp) we obtain Equi-insertion. A similar analysis works for (Eq2).

Comparing with the *left bicalculus of fractions* of Pronk, we see that rules (Id), (Rep), (Comp) and (Sq) are common to Pronk's calculus (except that in (Id) we just impose identities to belong to Σ instead of all equivalences). The remaining rule of the left bicalculus of fractions states that:

(PR) Given $X \overset{r}{\to} Y$ in Σ, 1-cells $g_1, g_2\colon Y \to Z$ and a 2-cell $\alpha\colon g_1 \circ r \Rightarrow g_2 \circ r$, we have that:
(i) There is $q\colon Z \to Q$ in Σ and $\alpha'\colon q \circ g_1 \Rightarrow q \circ g_2$ with $\alpha' \circ r = q \circ \alpha$.
(ii) If we have other q' and α'' as q and α' in (i), that is, $q'\colon Z \to Q'$ belongs to Σ and $\alpha''\colon q' \circ g_1 \Rightarrow q' \circ g_2$ with $\alpha'' \circ r = q' \circ \alpha$, then there are 1-cells u and u' and an invertible 2-cell $\varepsilon\colon u \circ q \Rightarrow u' \circ q'$ such that $(\varepsilon \circ g_2) \cdot (u \circ \alpha') = (u' \circ \alpha'') \cdot (\varepsilon \circ g_1)$.

In [6], there is another part of (PR), namely,

(PR)(iii) If, in (i), α is invertible, then α' may be chosen invertible too.

But this condition is not needed, since it follows from the others, as it was shown in [5].

(PR)(i) is just (Eq1). Thus, in order to show that Pronk's bicalculus is equivalent to our calculus of lax fractions, we only need to prove that, in the presence of the rules (Id), (Rep), (Comp), (Sq) and (Eq1), (PR)(ii) $\Longleftrightarrow$ (Eq2).

(PR)(ii) $\Longrightarrow$ (Eq2). Given 2-cells $\alpha, \beta\colon g_1 \circ r \Rightarrow g_2 \circ r$ as in (Eq2), both $(1_Z, \alpha)$ and $(1_Z, \beta)$ play the same role as (q, α') in (Eq1). Then, by (PR)(ii), there are $u, u'\colon Z \to W$ in Σ and an invertible $\varepsilon\colon u \to u'$ such that $\varepsilon \circ \alpha = \varepsilon \circ \beta$, thus $u \circ \alpha = u \circ \beta$.

(Eq2) $\Longrightarrow$ (PR)(ii). Let (q, α') be as in (Eq1) and let another pair (q', α'') play the same role. Apply (Sq) to q' and q obtaining $\begin{array}{ccc} \cdot & \xrightarrow{q'} & \cdot \\ {\scriptstyle q}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle p'} \\ \cdot & \xrightarrow[p]{} & \cdot \end{array}$. Then,

$$(\theta \circ g_2) \cdot (p \circ \alpha') \circ r = \theta \circ \alpha = (p' \circ \alpha'') \cdot (\theta \circ g_1) \circ r.$$

By (Eq2) there is a 1-cell d belonging to Σ such that $d\circ(\theta\circ g_2)\cdot(p\circ\alpha') = d\circ(p'\circ\alpha'')\cdot(\theta\circ g_1)$. Hence, $u = dp$, $u' = dp'$ and $\varepsilon = d \circ \theta$ fulfill the conditions of (PR)(ii).

4. **Order-enriched categories.** For an order-enriched category $\mathcal{X}$, that is, a 2-category where all $\mathcal{X}(A, B)$, $A, B \in \mathcal{X}$, are just posets, we can remove the rule Equification because it trivially always holds. This way, our calculus of lax fractions becomes the calculus introduced in [7] by the second author, except that Horizontal Repletion, stating that identity squares of the form $\begin{array}{ccc} \cdot & = & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \cdot & = & \cdot \end{array}$ are Σ-squares, was not used there. The following examples in order-enriched categories can be found in [7].

   (a) **Embbedings in Pos.** Let D be the contravariant endofunctor on **Pos** taking each poset X into the poset of lower sets of X, and every monotone map $f\colon X \to Y$

to the preimage map $Df\colon DY \to DX$. Let Σ consist of all embbeddings of **Pos** and commutative squares $\begin{array}{ccc} X & \xrightarrow{m} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ Z & \xrightarrow{n} & W \end{array}$ such that $(Du)^* \cdot Dm = Dn \cdot (Dv)^*$, where $(-)^*$ stands for the left adjoint. Equivalently, these squares are those such that, for every $y \in Y$ and $z \in Z$, if $n(z) \leq v(y)$ then there is some $x \in X$ with $z \leq u(x)$ and $m(x) \leq y$. Then Σ admits a left calculus of lax fractions.

(b) **Embeddings in Loc.** Let **Loc** be the category of locales (i.e., frames) and localic maps, i.e., maps preserving all infima and whose left adjoint f^* preserves finite meets. Recall that embeddings in **Loc** are precisely the localic maps h made split monomorphisms by its left adjoint: $h^* h = id$.

Let Σ consist of all embeddings and commutative squares

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ Z & \xrightarrow{n} & W \end{array}$$

satisfying the Beck–Chevalley condition $v^* n = m u^*$. Then Σ admits a left calculus of lax fractions.

(c) **Flat embeddings in Loc.** In the following two cases we have a subcategory of $\mathbf{Loc}^{\to}$ which admits a left calculus of lax fractions:

- All dense embeddings and squares as above.
- All flat embeddings and squares as above.

5. **Lax epimorphisms.** Recall that in a 2-category $\mathcal{X}$ a 1-cell $f\colon X \to Y$ is said to be a *lax epimorphism* (or *co-fully faithful*) if, for every object Z, the functor $\mathcal{X}(Y,Z) \xrightarrow{(-)\circ f} \mathcal{X}(X,Z)$ is fully faithful, i.e., every 2-cell $\alpha\colon g_1 f \Rightarrow g_2 f$ factors uniquely through f. Lax epimorphisms are stable under bi-pushouts, that is, a bi-pushout of a lax epimorphism along any morphism is a lax epimorphism (see [3]).

   Let Σ be the full subcategory of $\mathcal{X}^{\to\cong}$ of all lax epimorphisms. Then, Σ admits a left calculus of lax fractions, that is, it fulfills rules (Id), (Rep), (Comp), (Sq), (Eq1) and (Eq2) of Example 3 above. All rules are obvious, (Sq) is obtained with a bi-pushout.

   Analogously, if we take all lax epimorphisms and just all squares obtained by a finite vertical and horizontal composition of identity squares of the form $\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle r} \\ X & \xrightarrow{r} & Y \end{array}$ with r a lax epimorphism and bi-pushout squares, we obtain a left calculus of lax fractions.

6. **Fully faithful functors.** Let **Cat** be the 2-category of small categories. Let Σ consist of all fully faithful functors and squares

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{M} & \mathbb{B} \\ {\scriptstyle F}\downarrow & \overset{\delta}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{C} & \xrightarrow{N} & \mathbb{D} \end{array}$$

   (with M and N fully faithful and δ invertible) such that if $(\bar{Y}, \kappa)$ is the a Kan extension of the Yoneda embedding $Y\colon \mathbb{C} \to [\mathbb{C}^{op}, \mathbf{Set}]$ along N then $(\bar{Y}G, (\bar{Y} \circ \delta) \cdot (\kappa \circ F))$ is a left Kan extension of YF along M.

7. **Strict monoidal functors.** Let $\mathbf{Cat}(\mathbf{Mon})$ be the 2-category of categories internal to the category **Mon** of monoids. A category in **Mon** is the same thing as a strict monoidal category, while an internal functor is a strict monoidal functor. The class of strict monoidal functors whose underlying functors have fully faithful right adjoints and the pseudo-commutative squares whose underlying functors form Beck–Chevalley squares admits a right calculus of lax fractions.

The last two examples, as well as other examples and the corresponding bicategories of fractions, will be studied in detail in the paper [4], where we explore the relation of the calculus of lax fractions with lax-idempotent monads and Kan extensions.

In the next proposition, from the rules (1)–(6) of the left calculus of lax fractions we obtain new rules which will be very useful in what follows.

Proposition 2.8. *Let Σ be a subcategory of $\mathcal{X}^{\to\cong}$ admitting a left calculus of lax fractions. Then it satisfies the following rules:*

Rule 1. *Every square obtained as a finite (horizontal and vertical) composition of Σ-squares is a Σ-square.*

Rule 2. *For composable $r, s \in \Sigma$, we have*

$$\begin{array}{ccc} \bullet & \xrightarrow{r} & \bullet \\ \Vert & \Sigma^{\mathrm{id}} & \big\downarrow s \\ \bullet & \xrightarrow[sr]{} & \bullet \end{array}$$

Rule 3. *If* $\begin{array}{ccc} \bullet & \xrightarrow{r} & \bullet \\ s\big\downarrow & \Sigma^{\delta} & \big\downarrow u \\ \bullet & \xrightarrow[t]{} & \bullet \end{array}$ *and* $s \in \Sigma$ *then* $\begin{array}{ccc} \bullet & \xrightarrow{r} & \bullet \\ \Vert & \Sigma^{\delta} & \big\downarrow u \\ \bullet & \xrightarrow[ts]{} & \bullet \end{array}$ *and* $\begin{array}{ccc} \bullet & \xrightarrow{s} & \bullet \\ \Vert & \Sigma^{\mathrm{id}} & \big\downarrow t \\ \bullet & \xrightarrow[ts]{} & \bullet \end{array}$.

Rule 4. *If we have two diagrams of the form*

$$\begin{array}{ccc} B_1 & \xrightarrow{r_1} & I_1 \\ b_1\big\downarrow & \Sigma^{\delta_1} & \big\downarrow x_1 \\ A & \xrightarrow{x_3} & X \\ b_2\big\uparrow & \Sigma^{\delta_2} & \big\uparrow x_2 \\ B_2 & \xrightarrow[r_2]{} & I_2 \end{array} \qquad\qquad \begin{array}{ccc} B_1 & \xrightarrow{r_1} & I \\ b_1\big\downarrow & \Sigma^{\varepsilon_1} & \big\downarrow y_1 \\ A & \xrightarrow{y_3} & Y \\ b_2\big\uparrow & \Sigma^{\varepsilon_2} & \big\uparrow y_2 \\ B_2 & \xrightarrow[r_2]{} & I_2 \end{array}$$

then there are $X \xrightarrow{d_x} D \xleftarrow{d_y} Y$ *and invertible 2-cells* $\gamma_i\colon d_x x_i \Rightarrow d_y y_i$, $i = 1, 2$, *such that we have the following Σ-squares formed with d_x and d_y and the equality of pasting diagrams:*

$$\begin{array}{ccc} B_1 & \xrightarrow{r_1} & I_1 \\ b_1\big\downarrow & \Sigma^{\delta_1} & \big\downarrow x_1 \\ A & \xrightarrow{x_3} & X \overset{\gamma_1}{\Rightarrow} \; d_y y_1 \\ \Vert & \Sigma^{\varphi} & \big\downarrow d_x \\ A & \xrightarrow{u} & D \\ \Vert & \Sigma^{\varphi} & \big\uparrow d_x \\ A & \xrightarrow{x_3} & X \overset{\gamma_2}{\Rightarrow} \; d_y y_2 \\ b_2\big\uparrow & \Sigma^{\delta_2} & \big\uparrow x_2 \\ B_2 & \xrightarrow[r_2]{} & I_2 \end{array} \quad = \quad \begin{array}{ccc} B_1 & \xrightarrow{r_1} & I_1 \\ b_1\big\downarrow & \Sigma^{\varepsilon_1} & \big\downarrow y_1 \\ A & \xrightarrow{y_3} & Y \\ \Vert & \Sigma^{\chi} & \big\downarrow d_y \\ A & \xrightarrow{u} & D \\ \Vert & \Sigma^{\chi} & \big\uparrow d_y \\ A & \xrightarrow{y_3} & D \\ b_2\big\uparrow & \Sigma^{\varepsilon_2} & \big\uparrow y_2 \\ B_2 & \xrightarrow[r_2]{} & I_2 \end{array}$$

Rule 4'. *If we have Σ-squares as the two ones on the top of the diagrams*

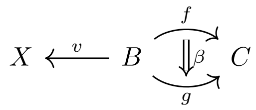
$$\begin{array}{ccc} B & \xrightarrow{r} & I \\ b\big\downarrow & \Sigma^{\delta} & \big\downarrow x \\ A & \longrightarrow & X \overset{\gamma}{\Rightarrow} \; d_y y \\ \Vert & \Sigma^{\varphi} & \big\downarrow d_x \\ A & \xrightarrow{u} & D \end{array} \quad = \quad \begin{array}{ccc} B & \xrightarrow{r} & I \\ b\big\downarrow & \Sigma^{\varepsilon} & \big\downarrow y \\ A & \longrightarrow & Y \\ \Vert & \Sigma^{\chi} & \big\downarrow d_y \\ A & \xrightarrow{u} & D \end{array}$$

then there are 1-cells $X \xrightarrow{d_x} D \xleftarrow{d_y} Y$ *forming* Σ*-squares as in the bottom of the diagrams and an invertible 2-cell* $\gamma : d_x x \Rightarrow d_y y$ *forming the above equality of pasting diagrams.*

Rule 5. *For every two spans* $X \xleftarrow{v} B \xrightarrow{f} C$ *and* $X \xleftarrow{v} B \xrightarrow{g} C$ *with* $v \in \Sigma$*, there are* $w\colon C \to D$ *and* Σ*-squares of the form*

$$\begin{array}{ccc} B & \xrightarrow{v} & X \\ {\scriptstyle f}\downarrow & \Sigma^{\delta} & \downarrow{\scriptstyle f'} \\ C & \xrightarrow[w]{} & D \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} B & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & \Sigma^{\varepsilon} & \downarrow{\scriptstyle g'} \\ C & \xrightarrow[w]{} & D \end{array}.$$

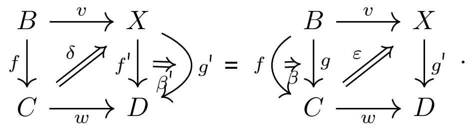
Rule 6. *Given a diagram*



with $v \in \Sigma$*, there are a 1-cell* $w\colon C \to D$*,* Σ*-diagrams of the form*

$$\begin{array}{ccc} B & \xrightarrow{v} & X \\ {\scriptstyle f}\downarrow & \Sigma^{\delta} & \downarrow{\scriptstyle f'} \\ C & \xrightarrow[w]{} & D \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} B & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & \Sigma^{\varepsilon} & \downarrow{\scriptstyle g'} \\ C & \xrightarrow[w]{} & D \end{array}$$

and a 2-cell $\beta'\colon f' \Rightarrow g'$ *performing the following equality of pasting diagrams:*



Proof. 1. Horizontal composition of Σ-squares is given by Composition, the vertical one is the composition in the subcategory Σ.

2. It is obtained by using Vertical Repletion, Identity and Composition:

$$\begin{array}{ccccc} \bullet & \xrightarrow{r} & \bullet & = & \bullet \\ \| & \Sigma^{\mathrm{id}} & \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle s} \\ \bullet & \xrightarrow[r]{} & \bullet & \xrightarrow[s]{} & \bullet \end{array} \;=\; \begin{array}{ccc} \bullet & \xrightarrow{r} & \bullet \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle s} \\ \bullet & \xrightarrow[sr]{} & \bullet \end{array}$$

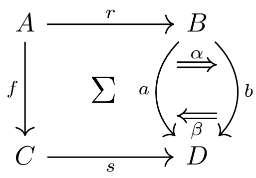
3. Observe that

$$\begin{array}{ccccc} \bullet & = & \bullet & \xrightarrow{r} & \bullet \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle s} & \Sigma^{\delta} & \downarrow{\scriptstyle u} \\ \bullet & \xrightarrow[s]{} & \bullet & \xrightarrow[t]{} & \bullet \end{array} \;=\; \begin{array}{ccc} \bullet & \xrightarrow{r} & \bullet \\ \| & \Sigma^{\delta} & \downarrow{\scriptstyle u} \\ \bullet & \xrightarrow[ts]{} & \bullet \end{array}.$$

The other Σ-square is obtained by Rule 2, since $t \in \Sigma$.

4. A. First, we prove two auxiliary rules, namely 4a and 4b, as follows:

4a. If we have a diagram of the form



with $\alpha \circ r = (\beta \circ r)^{-1}$, then there is a 1-cell $d\colon D \to E$ such that

$$\begin{array}{ccc} C & \xrightarrow{s} & D \\ \| & \Sigma & \big\downarrow {\scriptstyle d} \\ C & \xrightarrow[ds]{} & E \end{array}$$

and $d \circ \alpha = (d \circ \beta)^{-1}$.

4b. Given Σ-squares

$$\begin{array}{ccc} A & \xrightarrow{r} & B \\ {\scriptstyle f}\big\downarrow & \Sigma^{\delta} & \big\downarrow {\scriptstyle a} \\ C & \xrightarrow[s]{} & D \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{r} & B \\ {\scriptstyle f}\big\downarrow & \Sigma^{\varepsilon} & \big\downarrow {\scriptstyle b} \\ C & \xrightarrow[s]{} & D \end{array}$$

there is a 1-cell $d\colon D \to E$ and an invertible 2-cell $\gamma\colon da \Rightarrow db$ such that $\begin{array}{ccc} C & \xrightarrow{s} & D \\ \| & \Sigma^{\mathrm{id}} & \big\downarrow {\scriptstyle d} \\ C & \xrightarrow[ds]{} & E \end{array}$

and $(\gamma \circ r) \cdot (d \circ \delta) = d \circ \varepsilon$.

Proof of 4a. Since we have $(\beta \cdot \alpha) \circ r = \mathrm{id}_a \circ r$ and $(\alpha \cdot \beta) \circ r = \mathrm{id}_b \circ r$, by Equification twice we obtain, successively, 1-cells $d_1\colon D \to D_1$ and $d_2\colon D_1 \to D_2$ such that

$$\begin{array}{ccc} C & \xrightarrow{s} & D \\ \| & \Sigma & \big\downarrow {\scriptstyle d_1} \\ C & \xrightarrow[d_1 s]{} & D_1 \end{array} \qquad \qquad \begin{array}{ccc} C & \xrightarrow{d_1 s} & D_1 \\ \| & \Sigma & \big\downarrow {\scriptstyle d_2} \\ C & \xrightarrow[d_2 d_1 s]{} & D_2 \end{array}$$

and, also, first $d_1 \circ (\beta \cdot \alpha) = d_1 \circ \mathrm{id}_a$ and, secondly, $d_2 \circ d_1 \circ (\alpha \cdot \beta) = d_2 \circ d_1 \circ \mathrm{id}_b$. Thus, the 1-cell $d = d_2 d_1$ is as desired.

Proof of 4b. We have the 2-cell $ar \overset{\delta^{-1}}{\Longrightarrow} sf \overset{\varepsilon}{\Longrightarrow} br$. Then, using Equi-insertion, there is $d_1\colon D \to D_1$ and $\gamma_1\colon d_1 a \Rightarrow d_1 b$ such that $\begin{array}{ccc} C & \xrightarrow{s} & D \\ \| & \Sigma^{\mathrm{id}} & \big\downarrow {\scriptstyle d_1} \\ C & \xrightarrow[d_1 s]{} & D_2 \end{array}$ and $\gamma_1 \circ r = d_1 \circ (\varepsilon \cdot \delta^{-1})$.

Analogously, departing from $d_1 br \overset{d_1 \varepsilon^{-1}}{\Longrightarrow} d_1 sf \overset{d_1 \delta}{\Longrightarrow} d_1 ar$, we obtain $d_2\colon D_1 \to D_2$ and a 2-cell $\gamma_2\colon d_2 d_1 b \Rightarrow d_2 d_1 a$ such that $\begin{array}{ccc} C & \xrightarrow{d_1 s} & D \\ \| & \Sigma^{\mathrm{id}} & \big\downarrow {\scriptstyle d_2} \\ C & \xrightarrow[d_2 d_1 s]{} & D_2 \end{array}$ and $\gamma_2 \circ r = d_2 \circ ((d_1 \circ \delta) \cdot (d_1 \circ \varepsilon^{-1}))$.

This way, we have the Σ-square

$$\begin{array}{ccc} C & \xrightarrow{s} & D \\ \| & \Sigma^{\mathrm{id}} & \big\downarrow {\scriptstyle d_2 d_1} \\ C & \xrightarrow[d_2 d_1 s]{} & D_2 \end{array}$$

and the 2-cell composition $d_2 d_1 a \overset{d_2 \gamma_1}{\Longrightarrow} d_2 d_1 b \overset{\gamma_2}{\Longrightarrow} d_2 d_1 a$ such that $(\gamma_2 \circ r)^{-1} = d_2 \circ \gamma_1 \circ r$. Using the rule 4a, we conclude that there is $d_3\colon D_2 \to D_3$ such that $\begin{array}{ccc} C & \xrightarrow{d_2 d_1 s} & D \\ \| & \Sigma^{\mathrm{id}} & \big\downarrow {\scriptstyle d_3} \\ C & \xrightarrow[d_3 d_2 d_1 s]{} & D_2 \end{array}$. and $(d_3 \circ \gamma_2)^{-1} = d_3 d_2 \gamma_1$. Hence, the 1-cell $d = d_3 d_2 d_1$ and the invertible 2-cell $\gamma = d_3 d_2 \gamma_1\colon da \Rightarrow db$ are as desired.

B. Now, we prove Rule 4.

We obtain successively:

(i)
$$\begin{array}{ccc} A & \xrightarrow{x_3} & X \\ {\scriptstyle y_3}\downarrow & \Sigma^{\varphi} & \downarrow{\scriptstyle a'} \\ Y & \xrightarrow[a]{} & Z \end{array}$$
by Square

(ii)
$$\begin{array}{ccc} A & \xrightarrow{x_3} & X \\ \| & \Sigma^{\varphi} & \downarrow{\scriptstyle a'} \\ A & \xrightarrow[ay_3]{} & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{y_3} & Y \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle a} \\ A & \xrightarrow[ay_3]{} & Z \end{array}$$
by (i) and Rule 3

(iii)
$$\begin{array}{ccc} B_i & \xrightarrow{r_i} & I_i \\ {\scriptstyle b_i}\downarrow & \Sigma^{\delta_i} & \downarrow{\scriptstyle x_i} \\ A & \xrightarrow[x_3]{} & X \\ \| & \Sigma^{\varphi} & \downarrow{\scriptstyle a'} \\ A & \xrightarrow[ay_3]{} & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} B_i & \xrightarrow{r_i} & I_i \\ {\scriptstyle b_i}\downarrow & \Sigma^{\varepsilon_i} & \downarrow{\scriptstyle y_i} \\ A & \xrightarrow[y_3]{} & Y \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle a} \\ A & \xrightarrow[ay_3]{} & Z \end{array} \quad , \quad i = 1, 2$$
using the initial data, (ii) and Rule 1

(iv)
$$\begin{array}{ccc} B_1 & \xrightarrow{r_1} & I_1 \\ {\scriptstyle b_1}\downarrow & \Sigma^{\delta_1} & \downarrow{\scriptstyle x_1} \\ A & \xrightarrow{x_3} & X \\ \| & \Sigma^{\varphi} & {\scriptstyle a'}\downarrow \\ A & \xrightarrow{ay_3} & Z \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle q} \\ A & \xrightarrow[v_1]{} & Q \end{array} \overset{\gamma}{\Rightarrow} qay_1 = \begin{array}{ccc} B_1 & \xrightarrow{r_1} & I_1 \\ {\scriptstyle b_1}\downarrow & \Sigma^{\varepsilon_1} & \downarrow{\scriptstyle y_1} \\ A & \xrightarrow{y_3} & Y \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle a} \\ A & \xrightarrow{ay_3} & Z \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle q} \\ A & \xrightarrow[v_1]{} & Q \end{array}$$
with γ invertible

by (iii) and rule 4b.

(v)
$$\begin{array}{ccc} B_2 & \xrightarrow{r_2} & I_2 \\ {\scriptstyle b_2}\downarrow & \Sigma & \downarrow{\scriptstyle qa'x_2} \\ A & \xrightarrow[v_1]{} & Q \end{array} \quad \text{and} \quad \begin{array}{ccc} B_2 & \xrightarrow{r_2} & I_2 \\ {\scriptstyle b_2}\downarrow & \Sigma & \downarrow{\scriptstyle qay_2} \\ A & \xrightarrow[v_1]{} & Q \end{array}$$
using Rule 1 applied to Σ-squares of (iii) and (iv)

(vi)
$$\begin{array}{ccc} B_2 & \xrightarrow{r_2} & I_2 \\ {\scriptstyle b_2}\downarrow & \Sigma & \downarrow{\scriptstyle qa'x_2} \\ A & \xrightarrow[v_1]{} & Q \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle q'} \\ A & \xrightarrow[q'v_1]{} & D \end{array} \overset{\gamma_2}{\Rightarrow} q'qay_2 = \begin{array}{ccc} B_2 & \xrightarrow{r_2} & I_2 \\ {\scriptstyle b_2}\downarrow & \Sigma & \downarrow{\scriptstyle qay_2} \\ A & \xrightarrow[v_1]{} & Q \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle q'} \\ A & \xrightarrow[q'v_1]{} & D \end{array}$$
with γ_2 invertible

by (v) and rule 4b.

Thus, setting $d_x = q'qa'$, $d_y = q'qa$ and $\gamma_1 = q'\gamma$, we obtain the desired result.

4'. This is immediate from Rule 4. Indeed unfolding symmetrically each one of the Σ-squares, we get a particular case of Rule 4.

5. Use Square to obtain successively

$$\begin{array}{ccc} B & \xrightarrow{v} & X \\ {\scriptstyle f}\downarrow & \Sigma^{\delta} & \downarrow{\scriptstyle \tilde f} \\ C & \xrightarrow[w_1]{} & D_1 \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & \Sigma^{\varepsilon} & \downarrow{\scriptstyle \tilde g} \\ C & \xrightarrow[w_2]{} & D_2 \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{w_1} & D_1 \\ {\scriptstyle w_2}\downarrow & \Sigma^{\varphi} & \downarrow{\scriptstyle d_1} \\ D_2 & \xrightarrow[d_2]{} & D \end{array}$$

Now, using Rule 3 and Rule 1,we have:

$$\begin{array}{ccc} B & \xrightarrow{v} & X \\ {\scriptstyle f}\downarrow & \Sigma^{\delta} & \downarrow{\scriptstyle \tilde f} \\ C & \xrightarrow[w_1]{} & D_1 \\ \| & \Sigma^{\varphi} & \downarrow{\scriptstyle d_1} \\ C & \xrightarrow[d_2 w_2]{} & D \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & \Sigma^{\varepsilon} & \downarrow{\scriptstyle \tilde g} \\ C & \xrightarrow[w_2]{} & D_2 \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle d_2} \\ C & \xrightarrow[d_2 w_2]{} & D \end{array} .$$

6. First, departing from v, f and g, obtain the Σ-squares as in 5. with $\bar\delta$, $\bar\varepsilon$, $\bar D$, $\bar w$, $\bar f$ and $\bar g$ instead of δ, ε, D, w, f' and g', respectively. Then we have a 2-cell $\mu = \big(\bar f v \overset{\bar\delta^{-1}}{\Longrightarrow} \bar w f \overset{\bar w\beta}{\Longrightarrow} \bar w g \overset{\bar\varepsilon}{\Rightarrow} \bar g v\big)$. By Equi-insertion we get $d : \bar D \to D$, forming a Σ-square with $\bar w$, and a 2-cell $d\bar f \overset{\mu'}{\Longrightarrow} d\bar g$ such that $\mu' \circ v = d \circ \mu$. The desired Σ-squares Σ^{δ} and Σ^{ε}, the 1-cell w and the 2-cell β' are then given by $\delta = d \circ \bar\delta$, $\varepsilon = d \circ \bar\varepsilon$, $w = d\bar w$ and $\beta' = \mu'$. $\square$

Notation 2.9. For composable Σ-squares

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \Sigma^{\alpha} & \downarrow \\ \bullet & \longrightarrow & \bullet \\ \downarrow & \Sigma^{\beta} & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} \qquad \begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & \Sigma^{\alpha} & \downarrow & \Sigma^{\gamma} & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

we sometimes refer to the Σ-squares which result by vertical and horizontal composition as $\Sigma^{\beta\odot\alpha}$ and $\Sigma^{\alpha\oplus\gamma}$, respectively.

3 The bicategory of lax fractions

Let $\mathcal{X}$ be a 2-category and let Σ be a subcategory of $\mathcal{X}^{\to\cong}$ admitting a left calculus of lax fractions. This section is devoted to the description of a ***bicategory of lax fractions*** $\mathcal{X}[\Sigma_*]$.

The classical calculus of fractions with respect to a class of morphisms Σ in a category $\mathcal{X}$, introduced by Gabriel and Zisman [2] provides a nice description of a category $\mathcal{X}[\Sigma^{-1}]$ and a functor $P_\Sigma\colon \mathcal{X} \to \mathcal{X}[\Sigma^{-1}]$ such that the morphisms of $P_\Sigma[\Sigma]$ are all invertible and P_Σ is universal with respect to this property. Our definition of $\mathcal{X}[\Sigma_*]$ gives a generalization of the classical case. In the next section, we define a pseudofunctor $P_\Sigma\colon \mathcal{X} \to \mathcal{X}[\Sigma_*]$ which freely adds to each $P_\Sigma(s)$ with $s \in \mathrm{ob}(\Sigma)$ a right adjoint making $P_\Sigma(s)$ a lari in $\mathcal{X}[\Sigma_*]$ and sends Σ-squares to Beck-Chevalley squares. Moreover, $P_\Sigma(s)$ is universal with respect to these properties. The bicategory $\mathcal{X}[\Sigma_*]$ has strict units, that is, the unitors are just identities. Analogously the pseudofunctor P_Σ is strictly unitary, that is, it strictly preserves units.

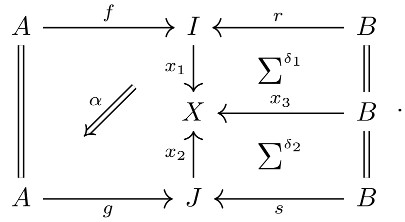
The proofs will consist essentially on a convenient calculus of Σ-squares based on the rules of Definition 2.1.

3.1 The categories $\mathcal{X}[\Sigma_*](A,B)$

The objects of $\mathcal{X}[\Sigma_*]$ are just those of $\mathcal{X}$, the 1-cells are the Σ-cospans (see Definition 3.1) and the 2-cells are $\approx$-equivalence classes of 2-morphisms (see Definition 3.1 and Definition 3.2). In this subsection we describe the categories $\mathcal{X}[\Sigma_*](A,B)$ for every pair of objects A and B.

Definition 3.1. A cospan $A \xrightarrow{f} I \xleftarrow{r} B$ with $r \in \Sigma$ is said to be a Σ-***cospan*** from A to B, and is written (f, I, r) or simply (f, r).

Given two Σ-cospans (f,r) and (g,s), both from A to B, a ***2-morphism*** from (f,r) to (g,s) consists of two Σ-squares and a 2-cell $\alpha\colon x_1 f \Rightarrow x_2 g$ as in the following diagram:


(3.1)

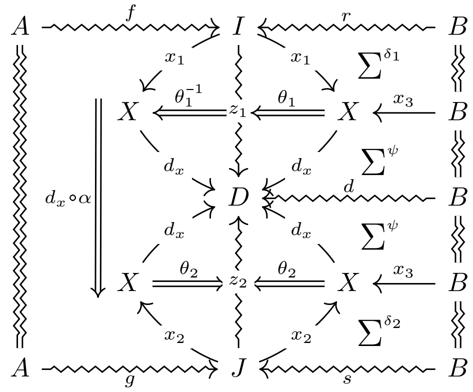
We denote it by

$$(\alpha, x_1, x_2, x_3, \delta_1, \delta_2)\colon (f,r) \Rightarrow (g,s)$$

or just $(\alpha, x_1, x_2)\colon (f,r) \Rightarrow (g,s)$.

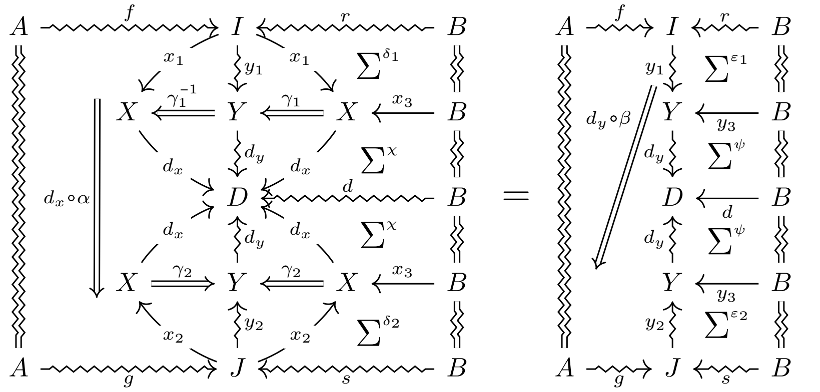
Our bicategory $\mathcal{X}[\Sigma_*]$ has the same objects as $\mathcal{X}$ and has Σ-cospans as 1-cells; the 2-cells are $\approx$-equivalence classes of 2-morphisms for a convenient $\approx$-relation, which we describe next.

Definition 3.2. 1. A Σ-***extension*** of a 2-morphism as in (3.1) above is any 2-morphism of the form $((\theta_2 \circ g)\cdot(d_x \circ \alpha)\cdot(\theta_1^{-1} \circ f), z_1, z_2)$ indicated by the wavy line part of the following diagram, where the 2-cells $\theta_i\colon d_x x_i \Rightarrow z_i$, $i = 1,2$, are invertible:



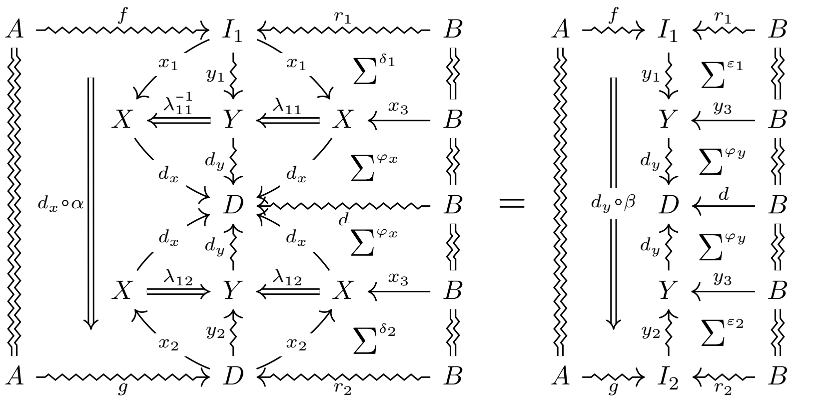
2. We say that two 2-morphisms with common domain and codomain are $\approx$***-related*** if they have a common Σ-extension. This is clearly equivalent to say that two 2-morphisms $(\alpha, x_1, x_2, x_3, \delta_1, \delta_2)$ and $(\beta, y_1, y_2, y_3, \varepsilon_1, \varepsilon_2)$ from (f,r) to (g,s) are $\approx$-related if there are Σ-squares $\begin{array}{ccc} B & \xrightarrow{x_3} & X \\ \| & \Sigma^{\chi} & \downarrow{\scriptstyle d_x} \\ B & \xrightarrow[d]{} & D \end{array}$ and $\begin{array}{ccc} B & \xrightarrow{y_3} & X \\ \| & \Sigma^{\psi} & \downarrow{\scriptstyle d_y} \\ B & \xrightarrow[d]{} & D \end{array}$ and invertible 2-cells $\gamma_i : d_x x_i \Rightarrow d_y y_i$ such that the 2-morphisms given by the wavy lines in the next diagram

are equal.

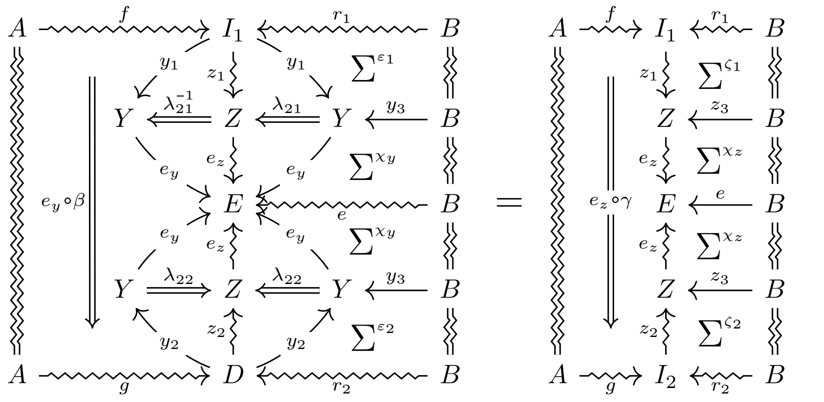


Lemma 3.3. *The relation ≈ is an equivalence relation.*

Proof. Reflexivity and symmetry are obvious. To show transitivity, consider three 2-morphisms $\bar{\alpha} = (\alpha, x_1, x_2, x_3, \delta_1, \delta_2)$, $\bar{\beta} = (\beta, y_1, y_2, y_3, \varepsilon_1, \varepsilon_2)$ and $\bar{\gamma} = (\gamma, z_1, z_2, z_3, \zeta_1, \zeta_2)$ from (f, I_1, r_1) to (g, I_2, r_2), with $\bar{\alpha} \approx \bar{\beta}$ and $\bar{\beta} \approx \bar{\gamma}$. Let the ≈-relation between $\bar{\alpha}$ and $\bar{\beta}$, and the one between $\bar{\beta}$ and $\bar{\gamma}$, be given by the data represented by (3.2) and (3.3) below, respectively.

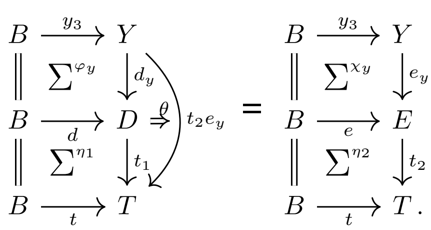


(3.2)

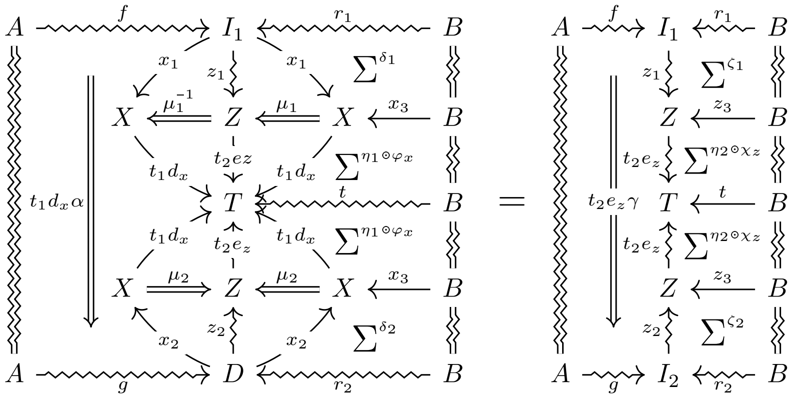


(3.3)

Using the Σ-squares Σ^{φ_y} and Σ^{χ_y}, combined with Rule 4' of Proposition 2.8, we obtain Σ-squares and an invertible 2-cell θ such that



Consequently, we obtain a common Σ-extension of $\bar{\alpha}$ and $\bar{\gamma}$. Namely, for $\mu_i = (t_2 \circ \lambda_{2i})(\theta \circ y_i)(t_1 \circ \lambda_{1i})$, we get the equality

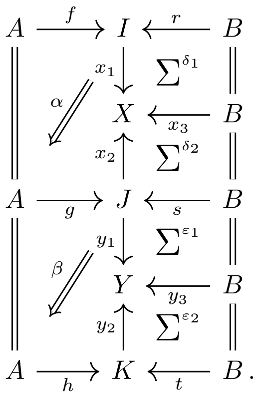


showing that $\bar{\alpha} \approx \bar{\gamma}$. □

Notation 3.4. A ***2-cell*** between Σ-cospans is an $\approx$-equivalence class of 2-morphisms between them. Given a 2-morphism $\bar{\alpha} = (\alpha, x_1, x_2, x_3, \delta_1, \delta_2)$, we use the notation $[\bar{\alpha}]$ for indicating the $\approx$-equivalence class of $\bar{\alpha}$. Sometimes we use $[\alpha, x_1, x_2, x_3, \delta_1, \delta_2]$ or simply $[\alpha, x_1, x_2]$.

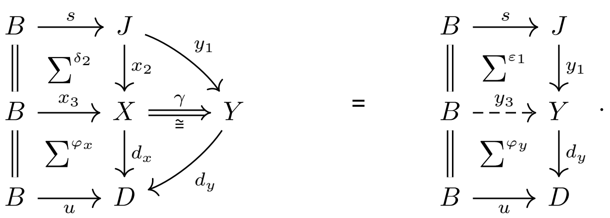
We present now the vertical composition of 2-cells between Σ-cospans. For that we define first the vertical composition between 2-morphisms and then take the corresponding $\approx$-equivalence classes.

Definition 3.5. ***Vertical composition***. Let $\bar{\alpha} = (\alpha, x_1, x_2, x_3, \delta_1, \delta_2)\colon (f, r) \Rightarrow (g, s)$ and $\bar{\beta} = (\beta, y_1, y_2, y_3, \varepsilon_1, \varepsilon_2)\colon (g, s) \Rightarrow (h, t)$ be 2-morphisms, as illustrated in the diagram



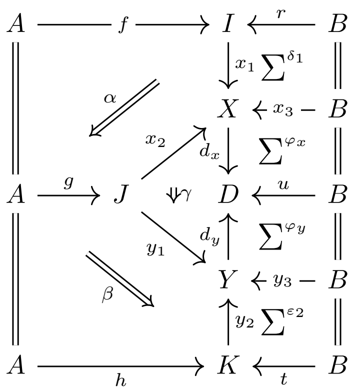
(3.4)

Using the Σ-squares Σ^{δ_2} and Σ^{ε_1} and Rule 4', we obtain new Σ-squares and an invertible 2-cell γ performing the following equality between pasting diagrams:



(3.5)

The vertical composition of the two 2-morphisms is any resulting 2-morphism $\bar{\beta} \circ \bar{\alpha}$ represented by the following diagram:



(3.6)

The corresponding vertical composition of the two 2-cells is given by

$$[\bar{\beta}] \cdot [\bar{\alpha}] = [\bar{\beta} \cdot \bar{\alpha}].$$

We now show that it is well-defined.

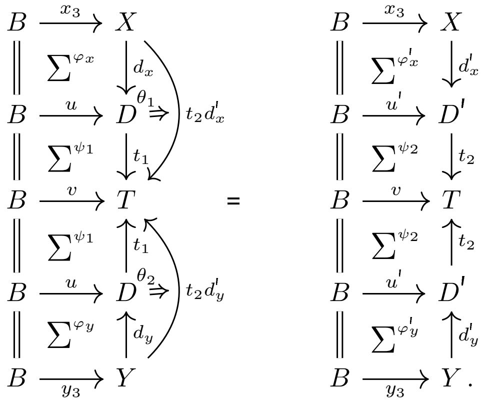
Proposition 3.6. *The vertical composition between 2-cells is well-defined.*

Proof. Consider 2-morphisms $\bar{\alpha} = (\alpha, x_1, x_2, x_3, \delta_1, \delta_2) \colon (f, r) \Rightarrow (g, s)$ and $\bar{\beta} = (\beta, y_1, y_2, y_3, \varepsilon_1, \varepsilon_2) \colon (g, s) \Rightarrow (h, t)$ as above in Equation (3.4).

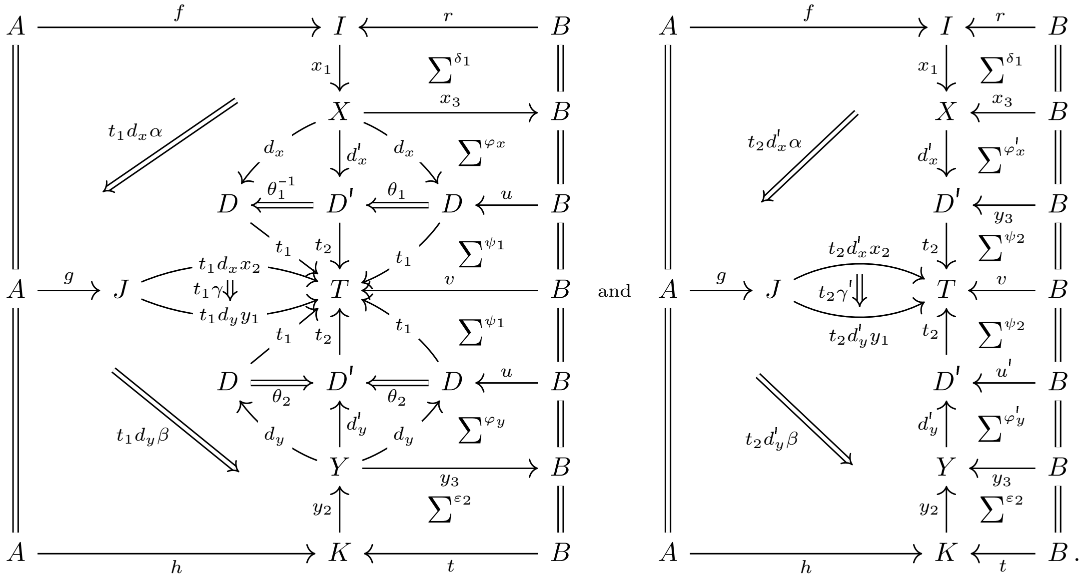
(1) We first show that the composition does not depend on the the choice of the Σ-squares and the isomorphisms $\gamma \colon d_x x_2 \Rightarrow d_y y_1$ in Equation (3.5). Indeed, suppose we have two different choices of this data with $\gamma \colon d_x x_2 \Rightarrow d_y y_1$ and $\gamma' \colon d'_x x_2 \Rightarrow d'_y y_1$ as in the diagrams

$$\begin{array}{ccc} B & \xrightarrow{x_3} & X \\ \| & \Sigma^{\varphi_x} & \downarrow{\scriptstyle d_x} \\ B & \xrightarrow{u} & D \\ \| & \Sigma^{\varphi_y} & \uparrow{\scriptstyle d_y} \\ B & \xrightarrow[y_3]{} & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} B & \xrightarrow{x_3} & X \\ \| & \Sigma^{\varphi'_x} & \downarrow{\scriptstyle d'_x} \\ B & \xrightarrow{u'} & D' \\ \| & \Sigma^{\varphi'_y} & \uparrow{\scriptstyle d'_y} \\ B & \xrightarrow[y_3]{} & Y\,. \end{array}$$

By Rule 4 of Proposition 2.8, we then obtain Σ-squares and invertible 2-cells θ_i, $i = 1, 2$, such that



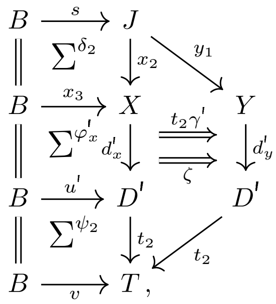
These can be used to form Σ-extensions each of the two compositions:



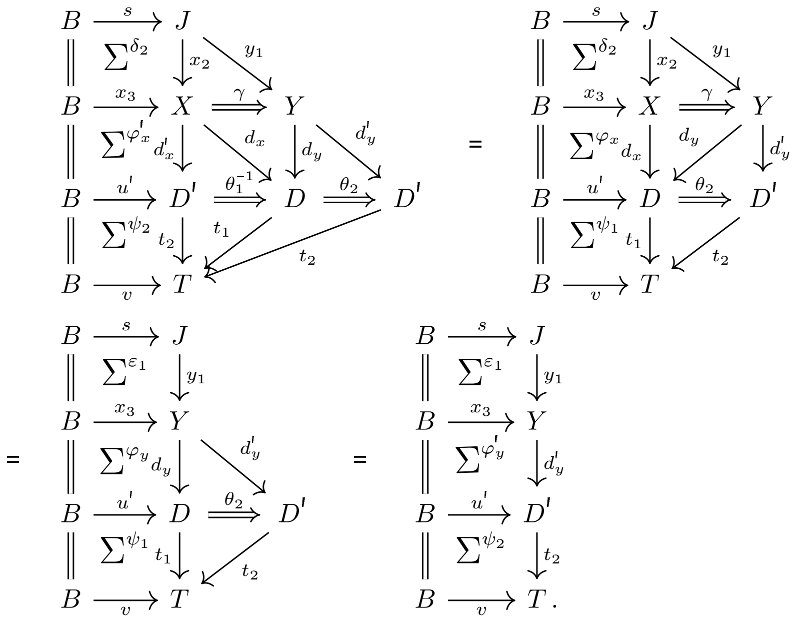
For these two Σ-extensions to coincide we would need the 2-cells on the left-hand-side of each diagram to be equal. That is, we want the outer rectangle of the following diagram to commute.

$$\begin{array}{ccccccc} t_1d_xx_1f & \xRightarrow{t_1d_x\alpha} & t_1d_xx_2g & \xRightarrow{t_1\gamma g} & t_1d_yy_1g & \xRightarrow{t_1d_y\beta} & t_1d_yy_2h \\ \Uparrow{\scriptstyle \theta_1^{-1}x_1f} & & \Uparrow{\scriptstyle \theta_1^{-1}x_2g} & & \Downarrow{\scriptstyle \theta_2y_1g} & & \Downarrow{\scriptstyle \theta_2y_2h} \\ t_2d'_xx_1f & \xRightarrow[t_2d'_x\alpha]{} & t_2d'_xx_2g & \xRightarrow[t_2\gamma' g]{} & t_2d'_yy_1g & \xRightarrow[t_2d'_y\beta]{} & t_2d'_yy_2h \end{array}$$

The left and right squares commute by naturality and so it suffices to show the central square commutes. Actually, this square might not commute, but we can force it by passing to a further Σ-extension. We want to force $\theta_2y_1 \cdot t_1\gamma \cdot \theta_1^{-1}x_2 = t_2\gamma'$ using Equification. Thus, we consider the diagram



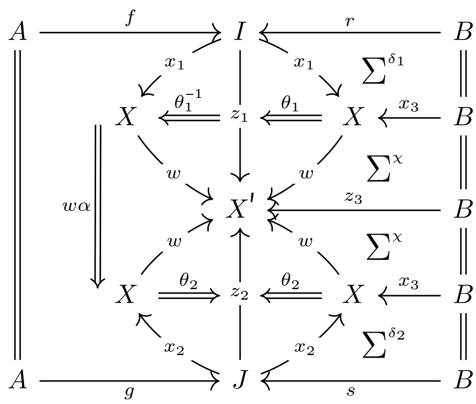
where $\zeta = \theta_2y_1 \cdot t_1\gamma \cdot \theta_1^{-1}x_2$. To apply Equification we must first show $t_2\gamma's = \zeta s$. Note that the composite of the above squares with the map $t_2\gamma's$ is equal to $\sum^{\psi_2\odot\varphi'_y\odot\varepsilon_1}$ by the assumption on γ'. On the other hand, the composite of the squares with ζs is



So the composites are the same, and since the 2-cells in the squares are invertible, we indeed have $t_2\gamma' s = \zeta s$. So by Equification, we obtain a map $q\colon T \to Q$ and associated Σ-square that allows us to pass to a common Σ-extension of the two 2-morphisms above, as required.

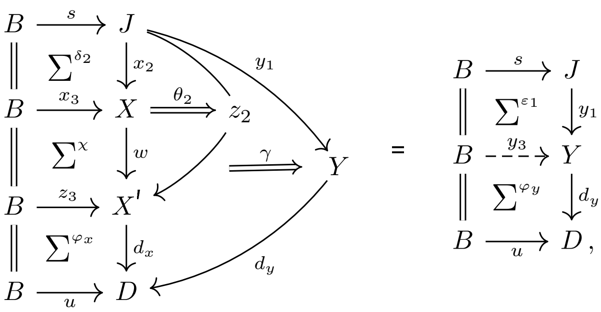
(2) We now show that vertical composition of 2-morphisms respects $\approx$ — that is, if $\overline{\alpha_1} \approx \overline{\alpha_2}$ and $\overline{\beta_1} \approx \overline{\beta_2}$ then $\overline{\beta_1} \cdot \overline{\alpha_1} \approx \overline{\beta_2} \cdot \overline{\alpha_2}$. By transitivity and symmetry, we may assume $\overline{\alpha_2}$ is a Σ-extension of $\overline{\alpha_1} = \overline{\alpha}$ and $\overline{\beta_2}$ is a Σ-extension of $\overline{\beta_1} = \overline{\beta}$ without loss of generality. Moreover, again by transitivity, we may assume α and β are replaced in turn. We will consider the case α, but the case of β is entirely analogous.

Suppose $\overline{\alpha_2}$ is given by the diagram

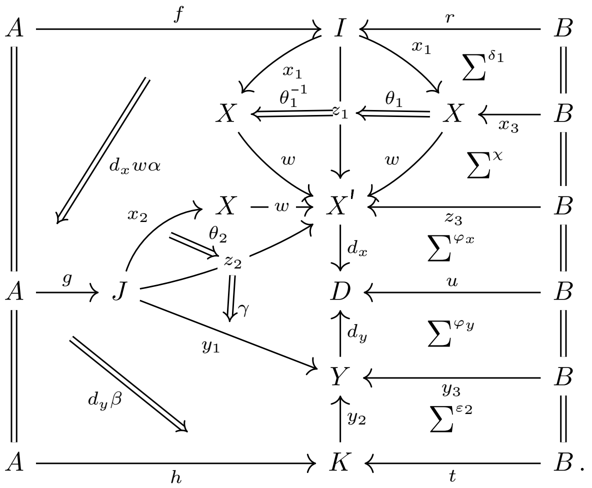


and suppose the composite $\overline{\beta} \cdot \overline{\alpha_2}$ is defined using the data $d_x\colon X' \to D, d_y\colon Y \to D, u\colon B \to$

D, φ_x, φ_y and γ. We then have



yielding the composite

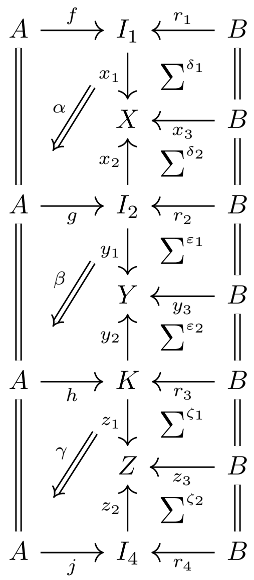


We now observe that this is a Σ-extension (using $d_x\theta_1$ and $\mathrm{id}_{d_y y_2}$) of the composite for $\overline{\beta} \cdot \overline{\alpha}$ given by the data $d_x w, d_y, u, \varphi_x \odot \chi$, φ_y and $\gamma \cdot d_x\theta$, and hence the two composites are equivalent. □

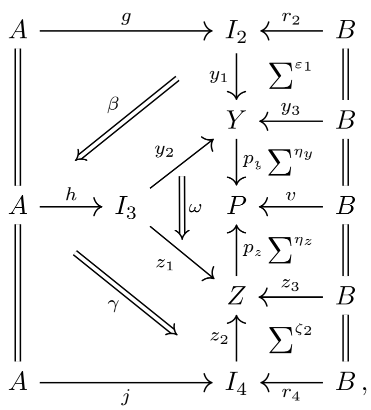
Remark 3.7. ***Identity 2-cells.*** The identity 2-cell on the Σ-cospan (f, I, r) is $[\mathrm{id}_f, 1_I, 1_I, r, \mathrm{id}_r, \mathrm{id}_r]$. It is represented by any 2-morphism of the form $[\mathrm{id}_{df}, d, d, u, \delta, \delta]$.

Proposition 3.8. *The vertical composition between 2-cells is associative and the identity 2-cells act as identities.*

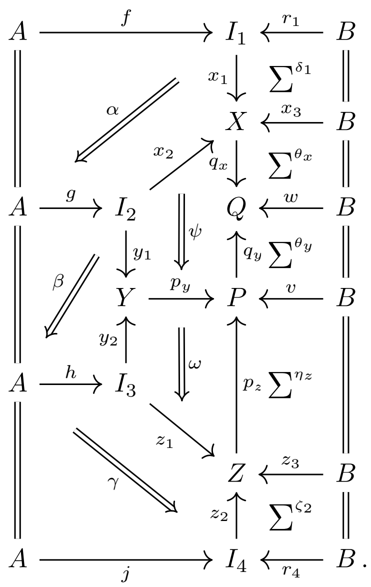
Proof. Consider the following triple composite.



We must show that the two ways of forming this composite, $(\overline{\gamma} \cdot \overline{\beta}) \cdot \overline{\alpha}$ and $\overline{\gamma} \cdot (\overline{\beta} \cdot \overline{\alpha})$, give equivalent results. For the former of these we find $\overline{\gamma} \cdot \overline{\beta}$ is given by

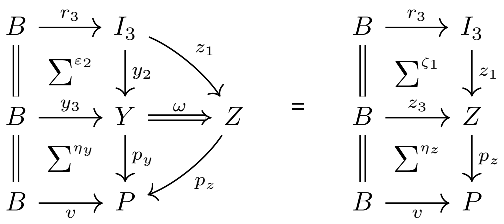


and hence $(\overline{\gamma} \cdot \overline{\beta}) \cdot \overline{\alpha}$ is given by



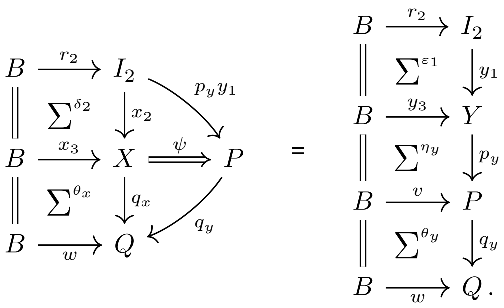
(3.7)

The data used for these compositions satisfy



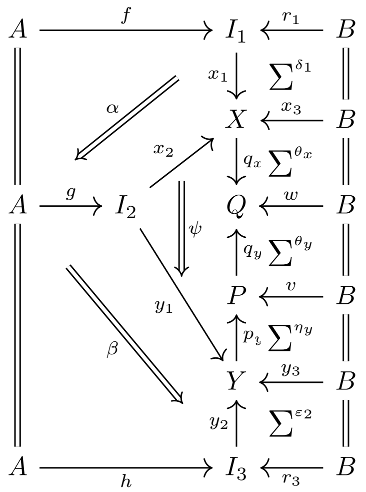
(3.8)

and

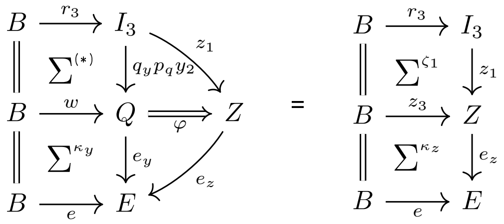


(3.9)

As for the other composite, we can form a composite of $\overline{\beta} \cdot \overline{\alpha}$ using the Σ-squares Σ^{θ_x} and $\Sigma^{\theta_y \odot \eta_y}$ from Equation (3.9) above.

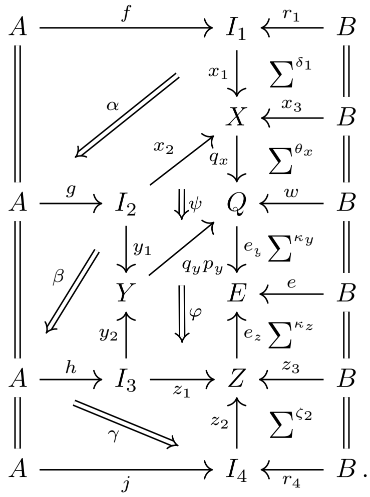


We then find $\overline{\gamma} \cdot (\overline{\beta} \cdot \overline{\alpha})$ using the data



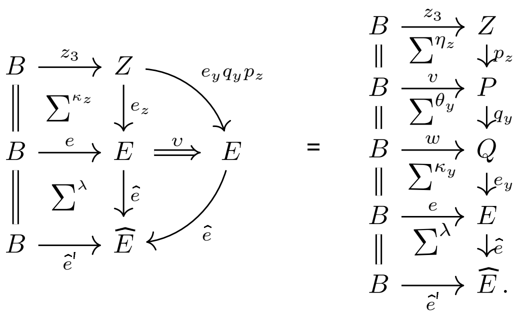
(3.10)

(where $\Sigma^{(*)}$ denotes $\Sigma^{\theta_y \odot \eta_y \odot \varepsilon_2}$) to be



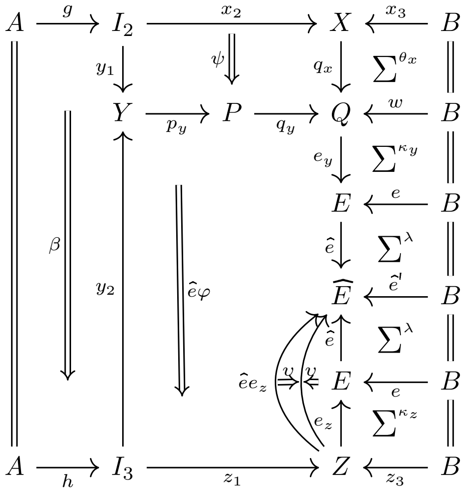
(3.11)

We want to show that this is an equivalent 2-morphism to the composite (3.11) above. By Rule 4' of Proposition 2.8 applied to Σ^{κ_z} and $\Sigma^{\kappa_y \odot \theta_y \odot \eta_z}$ we obtain

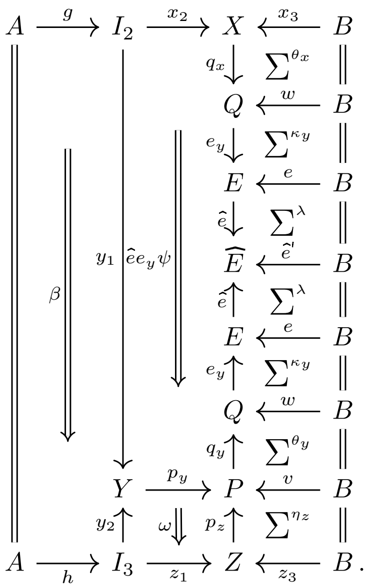


(3.12)

Using $\widehat{e}$ and υ to form a Σ-extension of $\overline{\gamma} \cdot (\overline{\beta} \cdot \overline{\alpha})$ and $\widehat{e}e_y$ to form an extension of $(\overline{\gamma} \cdot \overline{\beta}) \cdot \overline{\alpha}$, the relevant parts of the 2-morphism diagrams become:

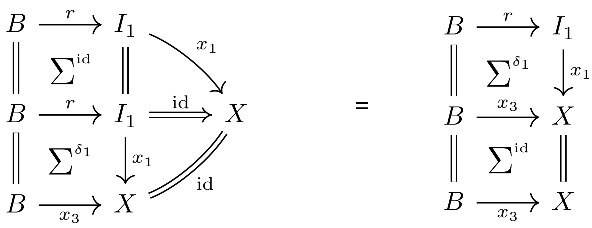


and



Note that the Σ-squares on the right-hand side of each diagram agree by Equation (3.12). For the remaining 2-cells, we have the composites $\upsilon z_1 h \cdot \widehat{e}\varphi h \cdot \widehat{e}e_y q_y p_y \beta \cdot \widehat{e}e_y \psi g$ and $\widehat{e}e_y q_y \omega h \cdot \widehat{e}e_y q_y p_y \beta \cdot \widehat{e}e_y \psi g$. These will agree if $\upsilon z_1 \cdot \widehat{e}\varphi = \widehat{e}e_y q_y \omega$. Composing $\Sigma^{\lambda \odot \kappa_y \odot \theta_y \odot \eta_y \odot \varepsilon_2}$ with $\widehat{e}\varphi$ and υz_1 and using Equation (3.10) and Equation (3.12) obtain $\Sigma^{\lambda \odot \kappa_y \odot \theta_y \odot \eta_z \odot \zeta_1}$. Composing the same Σ-square with $\widehat{e}e_y q_y \omega$ and using Equation (3.8) we obtain the same result. Since the 2-cells in the Σ-squares are invertible, this implies $\upsilon z_1 r_3 \cdot \widehat{e}\varphi r_3 = \widehat{e}e_y q_y \omega r_3$ and hence we can apply Equification to obtain an extension where the desired equality indeed holds. Thus, we have proved associativity.

We now show that the identity 2-cells are indeed identities with respect to vertical composition. To form the composite $\overline{\alpha} \circ \overline{\mathrm{id}}_{(f,I_1,r_1)}$ we use the data



and easily compute the composite to be equal to $\overline{\alpha}$. The composite $\overline{\mathrm{id}}_{(g,I_2,r_2)} \circ \overline{\alpha}$ can be shown to also be equal to $\overline{\alpha}$ in a similar way. □

3.2 Σ-schemes and Ω 2-cells

In order to define the horizontal composition and the associator, and prove the due properties on it, we will make use of a special kind of 2-cells between Σ-cospans that we present in this subsection. Here we define Σ-schemes, Σ-paths and Ω 2-cells, and state the properties which will have a role in the following. Most of the proofs are provided in Appendix A.

1. The basic Ω 2-cells. Departing from $I \xleftarrow{r} B \xrightarrow{g} J$, with $r \in \Sigma$, let us have two Σ-squares

Σ^{α} and $\Sigma^{\alpha'}$ as below, and apply Rule 4' to obtain the equality

$$\begin{array}{ccccc} B & \xrightarrow{r} & I & & \\ g\big\downarrow & \Sigma^{\alpha} & \big\downarrow g_1 & \searrow^{g_2} & \\ J & \xrightarrow{r_1} & I_1 & \overset{\theta}{\Longrightarrow} & I_2 \\ \Big\| & \Sigma^{\delta_1} & \big\downarrow d_1 & \swarrow_{d_2} & \\ J & \xrightarrow[d]{} & D & & \end{array} \quad = \quad \begin{array}{ccc} B & \xrightarrow{r} & I \\ g\big\downarrow & \Sigma^{\alpha'} & \big\downarrow g_2 \\ J & \xrightarrow{r_2} & I_2 \\ \Big\| & \Sigma^{\delta_2} & \big\downarrow d_2 \\ J & \xrightarrow[d]{} & D \end{array} \tag{3.13}$$

where θ is invertible. This way we get a 2-morphism as follows:

$$\begin{array}{ccccc} I & \xrightarrow{g_1} & I_1 & \xleftarrow{r_1} & J \\ \Bigg\| & & d_1\big\downarrow & \Sigma^{\delta_1} & \Big\| \\ & \overset{\theta}{\swarrow\!\!\swarrow} & D & \xleftarrow{d} & J \\ \Bigg\| & & d_2\big\uparrow & \Sigma^{\delta_2} & \Big\| \\ I & \xrightarrow[g_2]{} & I_2 & \xleftarrow[r_2]{} & J\,. \end{array}$$

It is easy to see that this 2-morphism is $\approx$-independent of the data used in the application of Rule 4'. (Indeed, given e_1, e_2, e, ε_1, ε_2 and θ', instead of d_1, d_2, d, δ_1, δ_2 and θ, in (3.13), apply Rule 4 to $\begin{array}{ccc} & \xrightarrow{r_1} & \\ \| & \Sigma^{\delta_1} & \downarrow d_1 \\ & \xrightarrow{d} & \\ \| & \Sigma^{\delta_2} & \uparrow d_2 \\ & \xrightarrow[r_2]{} & \end{array}$ and $\begin{array}{ccc} & \xrightarrow{r_1} & \\ \| & \Sigma^{\varepsilon_1} & \downarrow e_1 \\ & \xrightarrow{e} & \\ \| & \Sigma^{\varepsilon_2} & \uparrow e_2 \\ & \xrightarrow[r_2]{} & \end{array}$. This leads to a commom Σ-extension of $[\theta, d_1, d_2, d, \delta_1, \delta_2]$ and $[\theta', e_1, e_2, e, \varepsilon_1, \varepsilon_2]$.)

We say that this 2-morphism and the corresponding 2-cell of $\mathcal{X}[\Sigma_*]$ are of ***basic*** Ω ***type***. We denote this 2-cell by

$$\Omega_{\alpha,\alpha'}\colon (g_1, r_1) \Rightarrow (g_2, r_2)$$

or just by Ω. If we have any $f\colon A \to I$ and $s\colon C \to J$ in Σ, we use also the notation

$$\Omega_{\alpha,\alpha'}\colon (g_1 f, r_1 s) \Rightarrow (g_2 f, r_2 s)$$

for the 2-cell obtained composing with f and s on the left and on the right, respectively. This 2-cell is also said to be of ***basic*** Ω ***type***.

We have the following:

Lemma 3.9. *For Σ-squares* $\begin{array}{ccc} B & \xrightarrow{r} & I \\ g\big\downarrow & \Sigma^{\alpha_i} & \big\downarrow g_i \\ J & \xrightarrow[r_i]{} & B_i \end{array}$, $i = 1, 2, 3$,

(1) $\Omega_{\alpha_1,\alpha_2}$ is an invertible 2-cell between Σ-cospans and $\Omega^{-1}_{\alpha_1,\alpha_2} = \Omega_{\alpha_2,\alpha_1}$;

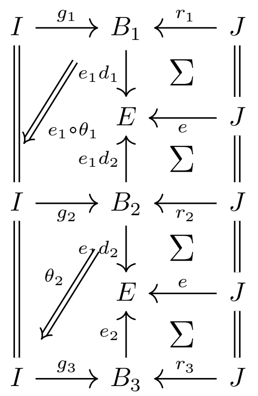
(2) $\Omega_{\alpha_2,\alpha_3} \cdot \Omega_{\alpha_1,\alpha_2} = \Omega_{\alpha_1,\alpha_3}$.

Proof. (1) is clear.

(2) Successively, consider invertible 2-cells in $\mathcal{X}$, θ_1 and θ_2, such that

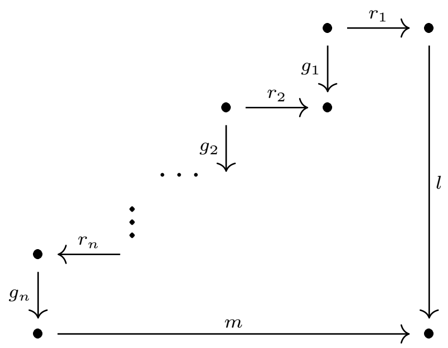
$$\begin{array}{ccccc} B & \xrightarrow{r} & I & & \\ g\big\downarrow & \Sigma^{\alpha_1} & \big\downarrow g_1 & \searrow^{g_2} & \\ J & \xrightarrow[r_1]{} & B_1 & \overset{\theta_1}{\Longrightarrow} & B_2 \\ \Big\| & \Sigma & \big\downarrow d_1 & \swarrow_{d_2} & \\ J & \xrightarrow[d]{} & D & & \end{array} = \begin{array}{ccc} B & \xrightarrow{r} & I \\ g\big\downarrow & \Sigma^{\alpha_2} & \big\downarrow g_2 \\ J & \xrightarrow[r_2]{} & B_2 \\ \Big\| & \Sigma & \big\downarrow d_2 \\ J & \xrightarrow[d]{} & D \end{array} \quad \text{and} \quad \begin{array}{ccccc} B & \xrightarrow{r} & I & & \\ g\big\downarrow & \Sigma^{\alpha_2} & \big\downarrow g_2 & \searrow^{g_3} & \\ J & \xrightarrow[r_2]{} & B_1 & & \\ \Big\| & \Sigma & \big\downarrow d_2 & \overset{\theta_2}{\nearrow\!\!\nearrow} & \\ J & \xrightarrow[d]{} & D & & B_3 \\ \Big\| & \Sigma & \big\downarrow e_1 & \swarrow_{e_2} & \\ J & \xrightarrow[e]{} & E & & \end{array} = \begin{array}{ccc} B & \xrightarrow{r} & I \\ g\big\downarrow & \Sigma^{\alpha_3} & \big\downarrow g_3 \\ J & \xrightarrow[r_3]{} & B_3 \\ \Big\| & \Sigma & \big\downarrow e_2 \\ J & \xrightarrow[e]{} & E \end{array}.$$

We obtain the diagram

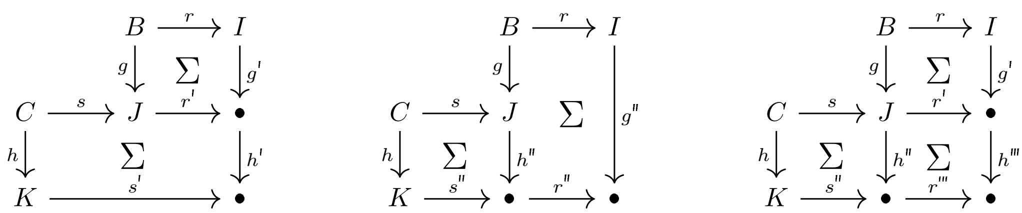


Vertically, this diagram is the juxtaposition of two 2-morphisms which represent $\Omega_{\alpha_1,\alpha_2}$ and $\Omega_{\alpha_2,\alpha_3}$. Clearly, $(\theta_2 \cdot (e_1 \circ \theta_1), e_1 d_1, e_2)$ is a representative of the vertical composition $\Omega_{\alpha_2,\alpha_3} \cdot \Omega_{\alpha_1,\alpha_2}$, and also a representative of $\Omega_{\alpha_1,\alpha_3}$. □

2. Σ-schemes and Σ-paths. A Σ-scheme is any diagram of the form

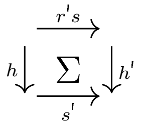
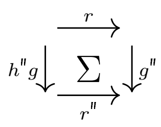


obtained by means of vertical and horizontal composition of Σ-squares. For instance, the following three diagrams

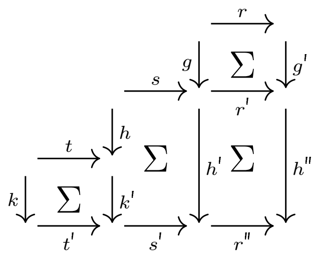


(1) (2) (3)

are Σ-schemes. We say that $(r_1, g_1, r_2, g_2, \ldots, r_n, g_n)$ is the ***left border*** of the Σ-scheme and (l, m) is the ***right border***. A Σ-scheme with left border $(r_1, \ldots, g_n)$ is said to be of ***level*** n. A Σ-scheme of level 1 is just a Σ-square.

Let S be a Σ-scheme. Any Σ-square used in the formation of S is said to be a sub-Σ-square of S. Between all sub-Σ-squares of S, we are interested in those whose lower right vertex coincides with the lower right vertex of S, let us call them ***replaceable Σ-squares*** in S. For instance, in (1) above,  is replaceable; in (2)  is replaceable. In the

Σ-scheme of level 3



(3.14)

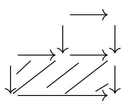
the Σ-squares $\begin{array}{ccc} & \xrightarrow{r's} & \\ k'h\downarrow & \Sigma & \downarrow h'' \\ & \xrightarrow[r''s']{} & \end{array}$ and $\begin{array}{ccc} & \xrightarrow{r} & \\ h'g\downarrow & \Sigma & \downarrow h''g' \\ & \xrightarrow[r'']{} & \end{array}$ are replaceable.

A Σ-***step*** from a Σ-scheme S_1 to a Σ-scheme S_2 with the same left border is a transformation of S_1 into S_2 which substitutes a replaceable Σ-square of S_1 by a Σ-square with the same left border, obtaining S_2. We are going to indicate a Σ-step from S_1 to S_2 by a wavy arrow

$$S_1 \rightsquigarrow S_2 .$$

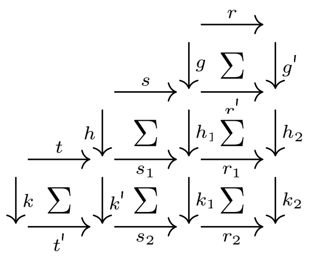
For instance, the Σ-step from (1) to (3) above which replaces $\begin{array}{ccc} & \xrightarrow{r's} & \\ h\downarrow & \Sigma & \downarrow h' \\ & \xrightarrow[s']{} & \end{array}$ by $\begin{array}{ccc} & \xrightarrow{r's} & \\ h\downarrow & \Sigma & \downarrow h''' \\ & \xrightarrow[r'''s'']{} & \end{array}$

takes a Σ-scheme of the type



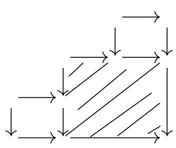
into another one of the same type, by replacing the Σ-square corresponding to the shaded area. Observe that, as required, the two Σ-squares involved, when seen as Σ-schemes, have the same left border, namely $(r's, h)$.

Another example: Consider the Σ-scheme of level 3



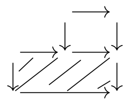
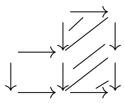
(3.15)

The Σ-step from the Σ-scheme (3.14) to the Σ-scheme (3.15) which replaces the Σ-square $\begin{array}{ccc} & \xrightarrow{r's} & \\ k'h\downarrow & \Sigma & \downarrow h'' \\ & \xrightarrow[r''s']{} & \end{array}$ by $\begin{array}{ccc} & \xrightarrow{r's} & \\ k'h\downarrow & \Sigma & \downarrow k_2h_2 \\ & \xrightarrow[r_2s_2]{} & \end{array}$ takes a Σ-scheme of the form



into another one of the same type by replacing the Σ-square corresponding to the shaded area.

A Σ-***path*** between Σ-schemes with the same left border is a finite sequence of Σ-steps.

Concerning Σ-paths between Σ-schemes of level 2, we use the notations d and u to indicate the type of Σ-steps involved, where d (down) stands by the type  and u (upper) stands by the type . We may for instance consider the following Σ-path from (1) to (2):

$$(1) \overset{d}{\rightsquigarrow} (3) \overset{u}{\rightsquigarrow} (2) \quad . \tag{3.16}$$

3. Ω 2-cells. Associated to each Σ-step from a Σ-scheme S_1 with right border (l_1, m_1) to another Σ-scheme S_2 with right border (l_2, m_2), we have a basic Ω 2-cell from the cospan (l_1, m_1) to the cospan (l_2, m_2) as described next.

Assume that the Σ-step replaces a Σ-square R_1 with right border (k_1, n_1) with a Σ-square R_2 with right border (k_2, n_2) such that $l_i = k_i l_o$ and $m_i = n_i m_0$, $i = 1, 2$, as illustrated below.

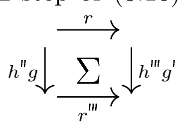
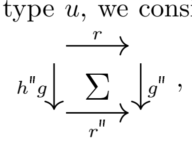
Consider the basic 2-cell from (k_1, n_1) to (k_2, n_2) determined by the passage from R_1 to R_2. Composing on the left and on the right with l_0 and m_0, respectively, we obtain a basic Ω 2-cell from (l_1, m_1) to (l_2, m_2).

For instance, consider the first Σ-step, of type d, in (3.16). First we take the basic Ω 2-cell

$$\Omega_1 \colon (h', s') \Rightarrow (h''', r''' s'').$$

The part remaining unchanged in the right border of the Σ-scheme is just g'. Then we compose with g', obtaining a (well-determined) basic Ω 2-cell

$$\Omega_1 \colon (h' g', s') \Rightarrow (h''' g', r''' s'').$$

Analogously, for the second Σ-step of (3.16), of type u, we consider the basic Ω 2-cell determined by the passage from  to , obtaining

$$\Omega_2 \colon (h''' g', r''') \Rightarrow (g'', r'')\,;$$

then, composing with s'', we have the resulting basic Ω 2-cell

$$\Omega_2 \colon (h''' g', r''' s'') \Rightarrow (g'', r'' s'')\,.$$

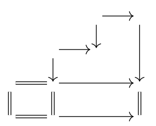
A Ω ***2-cell*** is a finite vertical composition of basic Ω 2-cells, determined by Σ-steps between Σ- schemes with the same left border. For instance, the vertical composition $\Omega_2 \cdot \Omega_1$ of the two basic Ω 2-cells above gives an Ω 2-cell from $(h'g', s')$ to $(g'', r''s'')$.

This way, to each Σ-path corresponds a Ω 2-cell given by the vertical composition of the basic Ω 2-cells corresponding to the Σ-steps of the Σ-path.

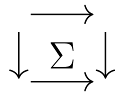
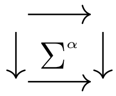
We say that two Σ-paths are ***equivalent*** if they give rise to the same Ω 2-cell.

4. Properties of Σ-paths. The Σ-schemes of level 3 will be of special interest in what follows.

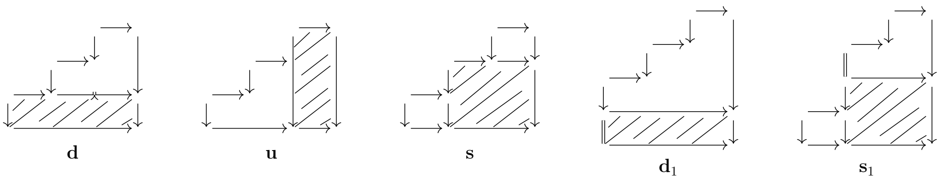
Observe that a Σ-scheme of level 2 can be looked as a Σ-scheme of level 3 by adding a bottom row made of identities:



Therefore, all that we are going to conclude on Σ-schemes of level 3 has obvious consequences for Σ-schemes of level 2.

Remark 3.10. In what follows, we frequently write  instead of  omitting the name of the invertible 2-cell of the Σ-square. Sometimes we will also use just numbered squares, as for instance in the proof of Lemma 3.12, to refer to Σ-squares.

Notation 3.11. Consider the following five types of Σ-schemes of level 3 with the same left border, that we identify by the letters below, namely, **d** (down), **u** (upper), **s** (square), $\mathbf{d}_1$ and $\mathbf{s}_1$:



Between two Σ-schemes of type **d**, which agree on the non-shaded part and on the left border of the shaded part, we may perform a Σ-step by replacing just the shaded part with its left border. We then say that this Σ-step is also of type **d**. Analogously, we use the same terminology for the remaining four cases.

Lemma 3.12. *We have the following properties for Σ-paths between Σ-schemes of level 3:*

(1) A Σ-path of two Σ-steps of the same type is equivalent to the Σ-path consisting of just a Σ-step of that type.

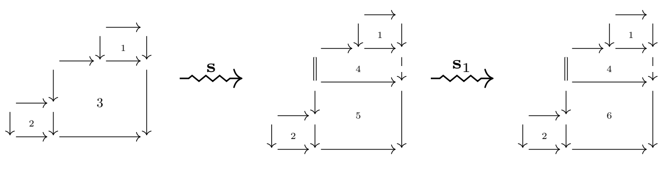
(2) A Σ-path consisting of two Σ-steps, one of the form **s***, the other of the form* $\mathbf{s}_1$*, is equivalent to a Σ-step of the form* **s***.*

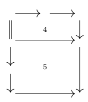
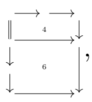
(3) A Σ-path consisting of two Σ-steps, one of the form **d***, the other of the form* $\mathbf{d}_1$*, is equivalent to a Σ-step of the form* **d***.*

(4) Every two possible Σ-steps between two given Σ-schemes are equivalent. The basic Ω 2-cell corresponding to a Σ-step of any type from a Σ-scheme to itself is the identity 2-cell.

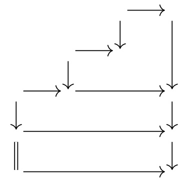
Proof. (1) is clear from Lemma 3.9.

Concerning (2), given a Σ-path $S_1 \overset{\mathbf{s}}{\rightsquigarrow} S_3 \overset{\mathbf{s}_1}{\rightsquigarrow} S_2$, necessarily we have that S_1 is of the form **s**, S_2 is of the form $\mathbf{s}_1$, and S_3 is of both forms; that is, the Σ-path is of the following form, where we use numbers to indicate the various Σ-squares envolved:



But the application of Rule 4' to the squares  and , which leads to the basic Ω 2-cell corresponding to $\mathbf{s}_1$, may be obtained by applying Rule 4' to the squares number 5 and 6 and then composing with square number 4. Consequently, under the present circumstances, $\mathbf{s}_1$ is equivalent to **s**, and the entire Σ-path is equivalent to the Σ-step of type **s**.

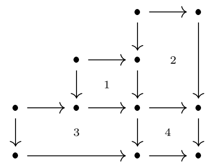
(3) Given a Σ-path of the form $S_1 \overset{\mathbf{d}}{\rightsquigarrow} S_3 \overset{\mathbf{d}_1}{\rightsquigarrow} S_2$ the Σ-scheme S_3 has to be simultaneously of type $\mathbf{d}$ and $\mathbf{d}_1$. Thus, it has the form



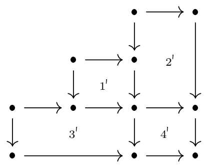
(3.17)

Arguing similarly to (2), we conclude that the second Σ-step is also of type $\mathbf{d}$, thus the entire Σ-path is equivalent to just a Σ-step of type $\mathbf{d}$.

Concerning (4), let, for instance, have two Σ-steps between two Σ-schemes, one of type $\mathbf{d}$, the other one of type $\mathbf{u}$. Then the two Σ-schemes must be simultaneously of type $\mathbf{d}$ and $\mathbf{u}$, say



and

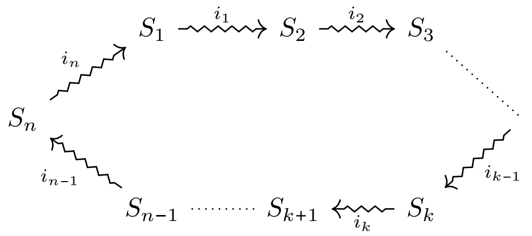


.

respectively. The Σ-step of type $\mathbf{d}$ determines that $1 = 1'$ and $2 = 2'$; the Σ-step of type $\mathbf{u}$ determines that $1 = 1'$ and $3 = 3'$. Applying Rule 4' to the Σ-squares 4 and $4'$, and composing with the right side of 2 and with the bottom side of 3, we obtain simultaneously the Ω 2-cell corresponding to $\mathbf{d}$ and also to $\mathbf{u}$. For the remaining cases the argument is similar. □

As we have just seen, all Σ-paths of length 1 between two given Σ-schemes are equivalent. We show in Proposition 3.14 that this is also true for Σ-paths of length 2. That is, every two Σ-paths of the form $S_1 \overset{i}{\rightsquigarrow} S_3 \overset{j}{\rightsquigarrow} S_2$ and $S_1 \overset{k}{\rightsquigarrow} S_4 \overset{l}{\rightsquigarrow} S_2$, where $i, j, k, l \in \{\mathbf{d}, \mathbf{u}, \mathbf{s}, \mathbf{d}_1, \mathbf{s}_1\}$, are equivalent. For Σ-paths of length greater than 2, see Remark 3.17 and Proposition 3.20.

Remark 3.13. Observe that, given a cycle of Σ-steps,

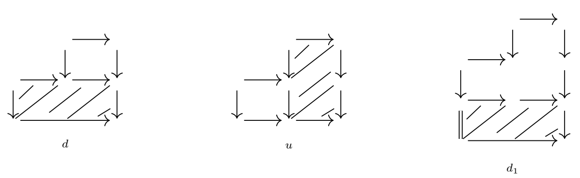


the corresponding Σ-path from S_1 to S_1 is equivalent to the identity Σ-step if, and only if, for some k with $1 \leq k \leq n$, the Σ-paths $S_1 \overset{i_1}{\rightsquigarrow} S_2 \cdots\cdots S_{k-1} \overset{i_{k-1}}{\rightsquigarrow} S_k$ and $S_1 \overset{i_n}{\rightsquigarrow} S_n \cdots\cdots S_{k+1} \overset{i_k}{\rightsquigarrow} S_k$ are equivalent.

Proposition 3.14. *Every two Σ-paths of length 2 between two given Σ-schemes of level 3 with the same left border are equivalent. Equivalently, any cycle as in Remark 3.13 of length 4 corresponds to an identity 2-cell in $\mathcal{X}[\Sigma_*]$.*

Proof. See Proposition A.3 and Corollary A.4 in Appendix A. □

Notation 3.15. For Σ-steps between Σ-schemes of level 2, apart from the letters d and u, already mentioned, we use also the letter d_1 in the indication of the type of the Σ-step. To summarize, we use the following types:

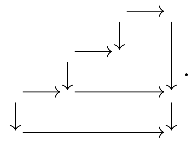


where, in each case, the shaded rectangle is the Σ-square replaced in the corresponding Σ-step.

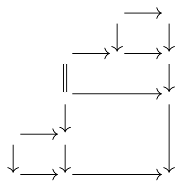
Corollary 3.16. *For Σ-schemes of level 2, S_1, S_2, S_3 and S_4, we have that for $i, j, k, l \in \{d, u, d_1\}$, any two Σ-paths of the form $S_1 \overset{i}{\rightsquigarrow} S_3 \overset{j}{\rightsquigarrow} S_2$ and $S_1 \overset{k}{\rightsquigarrow} S_4 \overset{l}{\rightsquigarrow} S_2$ are equivalent.*

Remark 3.17. We believe that the property stated in 3.14 is true for Σ-paths of any finite length. Equivalently, any cycle as in Remark 3.13 of finite length corresponds to an identity 2-cell in $\mathcal{X}[\Sigma_*]$. Although we do not have a complete proof involving all possible Σ-paths, we have a proof of the property for Σ-paths that we call *of interest* and that we are going to define next. This is stated in Proposition 3.20. This result combined with Proposition 3.14 encompass all Σ-paths with a role along the paper.

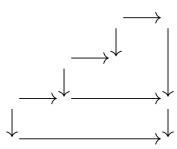
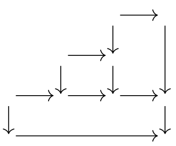
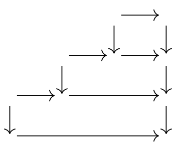
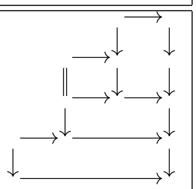
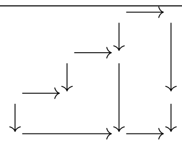
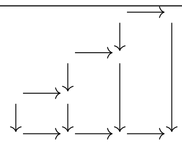
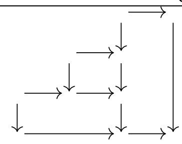
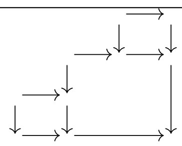
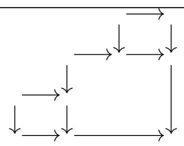
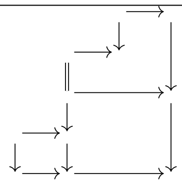
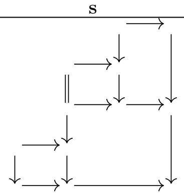
Next we define Σ-paths of interest. A Σ-scheme S in which a Σ-step of type **d** may start (or end) is necessarily of type **d** (see Notation 3.11), that is, of the form



Analogously for the types **u**, **s**, $\mathbf{d}_1$ and $\mathbf{s}_1$. Of course the Σ-schemes of one of these types may have different configurations. In the next table we consider some special configurations which are going to be of interest, each one with a name. For **s** and $\mathbf{s}_1$ we consider just one configuration of each one. Observe that the configuration of type $\mathbf{s}_1$ given by



is also of type **s**, thus we do not consider it in the row of $\mathbf{s}_1$.

Type	General	Configurations of interest		
d		 **da**	 **db**	 **dc**
u		 **ua**	 **ub**	
s		 **s**		
$\mathbf{s}_1$		 $\mathbf{s}_1$		

Definition 3.18. A Σ-scheme of level 3 has a ***configuration of interest*** if it has one of the seven configurations indicated in the right-side column of the above table. If a Σ-scheme has a configuration of interest we say that it is a Σ***-scheme of interest***. A **Σ-path of interest** is a Σ-path consisting only of Σ-schemes of interest.

Remark 3.19. Of course, in a Σ-path of interest, every Σ-step is incident to two Σ-schemes with a same configuration which starts with the letter representing the type of the Σ-step.

Proposition 3.20. *Every two Σ-paths of interest starting and ending at the same Σ-schemes are equivalent.*

Proof. See Appendix A. $\square$

The following property has also a role in what follows and is proven in Appendix A.

Proposition 3.21. *Let $A \xrightarrow{\bar f} B \xrightarrow{\bar g} C \xrightarrow{\bar h} D \xrightarrow{\bar k} E$ be Σ-cospans, where $\bar f = (f, r)$, $\bar g = (g, s)$, $\bar h = (h, t)$ and $\bar k = (k, u)$.*

(1) Let $\Omega\colon (l_1 g, m_1 u) \Rightarrow (l_2 g, m_2 u)$ be a basic Ω 2-cell determined by a Σ-step of level 2 of the type **d** *or* **u** *between two Σ-schemes of left border (s, h, t, k) and right border (l_i, m_i), respectively. Then the 2-cell $\Omega \circ 1_{\bar f}$ is an Ω 2-cell corresponding to a Σ-path of interest of Σ-schemes of level 3 and left border (r, g, s, h, t, k).*

(2) Let $\Omega\colon (l_1 f, m_1 t) \to (l_2 f, m_2 t)$ be a basic Ω 2-cell determined by a Σ-step of level 2 of the type d or u between two Σ-schemes of left border (r, g, s, h) and right border (l_i, m_i), respectively. Then the 2-cell $1_{\bar k} \circ \Omega$ is an Ω 2-cell corresponding to a Σ-path of interest of Σ-schemes of level 3 and left border (r, g, s, h, t, k).

Proof. See Appendix A. $\square$

3.3 The horizontal composition and the associator

In the composition of Σ-cospans we are going to use prefixed Σ-squares, called canonical Σ-squares.

Definition 3.22. ***Canonical Σ-squares.*** Since Σ admits a left calculus of lax fractions, by Square, we know that for each span $I \xleftarrow{r} B \xrightarrow{g} J$ with $r \in \Sigma$, there is some Σ-cospan forming a Σ-square with it. In order to define the composition, we assume a pregiven map which assigns to each such a pair (r, g) a fixed Σ-square

$$\begin{array}{ccc} B & \xrightarrow{r} & I \\ {\scriptstyle g}\downarrow & \Sigma^{\dot\alpha} & \downarrow{\scriptstyle \dot g} \\ J & \xrightarrow[\dot r]{} & \dot B \end{array}$$

called a ***canonical*** Σ***-square***. For $r\colon A \to B$ in Σ and any $f\colon A \to B$, we assume that we have

$$\begin{array}{ccc} A & \xrightarrow{r} & B \\ {\scriptstyle 1_A}\downarrow & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow{r} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle f}\downarrow & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle f} \\ B & \xrightarrow{1_A} & B \end{array}$$

as canonical Σ-squares. Sometimes the canonical Σ-squares will be indicated with just the symbol $\dot\Sigma$.

The ***canonical*** Σ***-scheme*** of level 3 of a given left border, which sometimes we will denote by Can, is given by

$$\begin{array}{ccccccc}
 & & & & & \longrightarrow & \\
 & & & \longrightarrow & \downarrow & \underset{\longrightarrow}{\dot\Sigma} & \downarrow \\
 & \longrightarrow & \downarrow & \underset{\longrightarrow}{\dot\Sigma} & \downarrow & \underset{\longrightarrow}{\dot\Sigma} & \downarrow \\
\downarrow & \underset{\longrightarrow}{\dot\Sigma} & \downarrow & \underset{\longrightarrow}{\dot\Sigma} & \downarrow & \underset{\longrightarrow}{\dot\Sigma} & \downarrow
\end{array}$$

and likewise for Σ-schemes of level 2.

Definition 3.23. ***(Horizontal) composition between Σ-cospans.*** Given Σ-cospans $(f, I, r)\colon A \to B$ and $(g, J, s)\colon B \to C$, its composition is the Σ-cospan $(\dot g f, \dot B, \dot r s)\colon A \to C$, obtained by taking the canonical Σ-square of r and g:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & I & \xleftarrow{r} & B & & \\ & & {\scriptstyle \dot g}\big\downarrow & \Sigma^{\dot\alpha} & \big\downarrow{\scriptstyle g} & & \\ & & \dot B & \xleftarrow[\dot r]{} & J & \xleftarrow[s]{} & C \end{array}$$

Definition 3.24. ***Horizontal composition of 2-cells.*** Given two horizontally composable 2-morphisms as in the diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{f_1} & I_1 & \xleftarrow{r_1} & B & \xrightarrow{g_1} & J_1 & \xleftarrow{s_1} & C \\ \Big\| & & \big\downarrow{\scriptstyle x_1}\ \Sigma^{\delta_1} & & \Big\| & & \big\downarrow{\scriptstyle y_1}\ \Sigma^{\varepsilon_1} & & \Big\| \\ \Big\| & \overset{\alpha}{\Swarrow} & X & \xleftarrow[x_3]{} & B & \overset{\beta}{\Swarrow} & Y & \xleftarrow{y_3} & C \\ \Big\| & & \big\uparrow{\scriptstyle x_2}\ \Sigma^{\delta_2} & & \Big\| & & \big\uparrow{\scriptstyle y_2}\ \Sigma^{\varepsilon_2} & & \Big\| \\ A & \xrightarrow[f_2]{} & I_2 & \xleftarrow[r_2]{} & B & \xrightarrow[g_2]{} & J_2 & \xleftarrow[s_2]{} & C \end{array} \tag{3.18}$$

use Rule 6 to obtain the following equality of pasting diagrams:

$$\begin{array}{ccc} B & \xrightarrow{x_3} & X \\ {\scriptstyle g_1}\big\downarrow & & \\ J_1 & \Sigma^{\xi_1} & \big\downarrow{\scriptstyle y_1'} \overset{\beta'}{\Rightarrow} \curvearrowright{\scriptstyle y_2'} \\ {\scriptstyle y_1}\big\downarrow & & \\ Y & \xrightarrow{v} & V \end{array} \quad = \quad \begin{array}{cccc} & B & \xrightarrow{x_3} & X \\ \swarrow{\scriptstyle g_1} & \big\downarrow{\scriptstyle g_2} & & \\ J_1 & \overset{\beta}{\Longrightarrow} J_2 & \Sigma^{\xi_2} & \big\downarrow{\scriptstyle y_2'} \\ \searrow{\scriptstyle y_1} & \big\downarrow{\scriptstyle y_2} & & \\ & Y & \xrightarrow{v} & V \end{array} \tag{3.19}$$

In this way, we get the following 2-morphism:

$$\begin{array}{ccccccccccc} A & \xrightarrow{f_1} & I_1 & \xrightarrow{x_1} & X & \xrightarrow{y_1'} & V & \xleftarrow{v} & Y & \xleftarrow{y_3} & C \\ \Big\| & & & & \Big\| & & \Big\| & & \Sigma^{\mathrm{id}} & & \Big\| \\ \Big\| & & \overset{\alpha}{\Swarrow} & & \Big\| & \overset{\beta'}{\Swarrow} & V & \xleftarrow[v]{} & Y & \xleftarrow[y_3]{} & C \\ \Big\| & & & & \Big\| & & \Big\| & & \Sigma^{\mathrm{id}} & & \Big\| \\ A & \xrightarrow[f_2]{} & I_2 & \xrightarrow[x_2]{} & X & \xrightarrow[y_2']{} & V & \xleftarrow[v]{} & Y & \xleftarrow[y_3]{} & C \end{array} \tag{3.20}$$

Consider the following Σ-path between Σ-schemes of level 2, where d and u refer to the type of the Σ-steps (see Notation 3.15):

$$\begin{array}{ccccc} & & B & \xrightarrow{r_i} & I_i \\ & & {\scriptstyle g_i}\big\downarrow & \dot\Sigma & \big\downarrow{\scriptstyle \dot g_i} \\ C & \xrightarrow{s_i} & J_i & \xrightarrow[\dot r_i]{} & \dot B_i \\ \Big\| & \dot\Sigma & \Big\| & \dot\Sigma & \Big\| \\ C & \xrightarrow[s_i]{} & J_i & \xrightarrow[\dot r_i]{} & \dot B_i \end{array} \overset{d}{\rightsquigarrow} \begin{array}{ccccc} & & B & \xrightarrow{r_i} & I_i \\ & & {\scriptstyle g_i}\big\downarrow & \dot\Sigma & \big\downarrow{\scriptstyle \dot g_i} \\ C & \xrightarrow[s_i]{} & J_i & \xrightarrow[\dot r_i]{} & \dot B_i \\ \Big\| & \Sigma^{\varepsilon_i} & \big\downarrow{\scriptstyle y_i} & \dot\Sigma & \big\downarrow \\ C & \xrightarrow[y_3]{} & Y & \longrightarrow & \end{array} \overset{u}{\rightsquigarrow} \begin{array}{ccccc} & & B & \xrightarrow{r_i} & I_i \\ & & {\scriptstyle g_i}\big\downarrow & & \Big| \\ C & \xrightarrow{s_i} & J_i & \Sigma^{\xi_i \circ \delta_i} & \Big|{\scriptstyle y_i' x_i} \\ \Big\| & \Sigma^{\varepsilon_i} & \big\downarrow{\scriptstyle y_i} & & \big\downarrow \\ C & \xrightarrow[y_3]{} & Y & \xrightarrow[v]{} & V \end{array} . \tag{3.21}$$

Let $\Omega_i\colon (g_i, s_i) \circ (f_i, r_i) = (\dot g_i f_i, \dot r_i s_i) \implies (y_i' x_i f_i, v y_3)$ be the Ω 2-cell determined by that Σ-path, and obtain three vertically composable 2-cells

$$(g_1, s_1) \circ (f_1, r_1) \xRightarrow{\Omega_1} (y_1' x_1 f_1, v y_3) \xRightarrow{[\beta' \circ \alpha, 1_V, 1_V]} (y_2' x_2 f_2, v y_3) \xRightarrow{\Omega_2^{-1}} (g_2, s_2) \circ (f_2, r_2) \ .$$

The horizontal composition $[\beta, y_1, y_2] \circ [\alpha, x_1, x_2]$ is given by the vertical composition

$$\Omega_2^{-1} \cdot [\beta' \circ \alpha, 1_V, 1_V] \cdot \Omega_1 \,.$$

In Proposition 3.28 we show that the horizontal composition is well-defined — that is, Definition 3.24 does not depend on the choice of the Σ-squares and β' in (3.19), and it respects the ≈-relation.

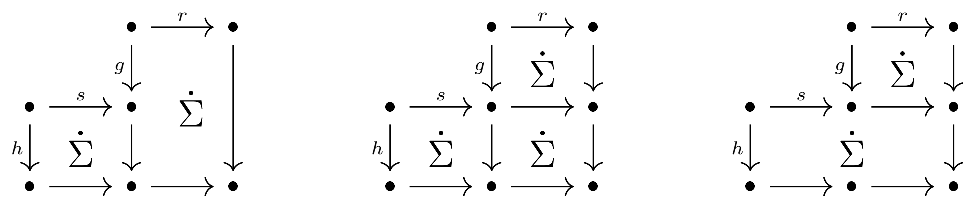
Definition 3.25. ***Composition.*** For every triple of objects A, B and C the map comp_{ABC} assigns to two Σ-cospans $A \xrightarrow{(f,r)} B \xrightarrow{(g,s)} C$ its composition $(g,s) \circ (f,r)$, and to two 2-cells $\bar{\alpha}\colon (f,r) \Rightarrow (f',r')$ and $\bar{\beta}\colon (g,s) \Rightarrow (g',s')$ its composition $\bar{\beta} \circ \bar{\alpha}\colon (g,s) \circ (f,r) \Rightarrow (g',s') \circ (f',r')$.

In Proposition 3.30 we will prove that comp_{ABC} is indeed a functor.

Definition 3.26. ***Associator.*** Let us be given three composable Σ-cospans:

$$A \xrightarrow{\bar{f}} B \xrightarrow{\bar{g}} C \xrightarrow{\bar{f}} D$$

with $\bar{f} = (f, I, r)$, $\bar{g} = (g, J, s)$ and $\bar{h} = (h, K, t)$. We consider the Σ-schemes of level 2 as below, where all Σ-squares are canonical and the middle Σ-scheme is the canonical one.



(1) (2) (3)

The right borders of the Σ-schemes (1) and (3), after performing the composition with f on the left and with t on the right, give the compositions $((h,t) \circ (g,s)) \circ (f,r)$ and $(h,t) \circ ((g,s) \circ (f,r))$, respectively.

The associator, given componentwise by

$$\mathrm{Assoc}_{\bar{f},\bar{g},\bar{h}}\colon ((h,t) \circ (g,s)) \circ (f,r) \Rightarrow (h,t) \circ ((g,s) \circ (f,r)) ,$$

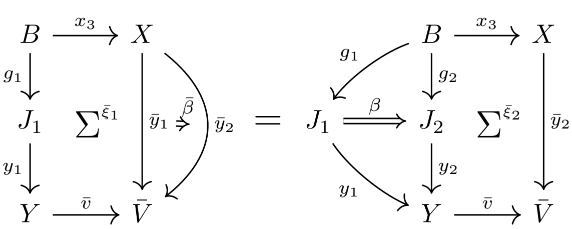
is the Ω 2-cell corresponding to the Σ-path $(1) \overset{\mathbf{u}}{\rightsquigarrow} (2) \overset{\mathbf{d}}{\rightsquigarrow} (3)$.

In Proposition 3.31 we will prove that the associator, as defined above, form indeed a natural isomorphism from $- \circ (- \circ -)$ to $(- \circ -) \circ -$.

Definition 3.27. ***Unitors.*** Taking into account Definition 3.23 and the Identity rule of the calculus of lax fractions (see Definition 2.1), the identity 1-cells act indeed as identities concerning the horizontal composition. Thus, we define the ***left unitor*** and the ***right unitor*** to be given simply by identity 2-cells.

Proposition 3.28. *The horizontal composition is well-defined.*

Proof. (1) First we observe that, up to the ≈-relation, the horizontal composition does not depend on the choice of the Σ-squares and the 2-cell β' of (3.19). Indeed, let us have the equality of pasting diagrams



(3.22)

leading to the 2-morphism $(\bar{\beta}\circ\alpha, 1_{\bar{V}}, 1_{\bar{V}})\colon(\bar{y}_1x_1f_1, \bar{v}y_3)\Rightarrow(\bar{y}_2x_2f_2, \bar{v}y_3)$. Using Rule 4 of Proposition 2.8, there are $Q \xrightarrow{p} T \xleftarrow{\bar{p}} \bar{Q}$, Σ-squares Σ^{χ} and $\Sigma^{\bar{\chi}}$, and invertible 2-cells θ_i, such that

$$\begin{array}{ccccc} B & \xrightarrow{x_3} & X & \overset{\bar{y}_1}{\searrow} & \\ {\scriptstyle y_1g_1}\downarrow & \Sigma^{\xi_1} & \downarrow{\scriptstyle y_1'} & & \\ Y & \overset{v}{\cdots\to} & V & \overset{\theta_1}{\Longrightarrow} & \bar{V} \\ \Vert & \Sigma^{\chi} & \downarrow{\scriptstyle p} & \overset{\bar{p}}{\swarrow} & \\ Y & \xrightarrow{t} & T & & \\ \Vert & \Sigma^{\chi} & \uparrow{\scriptstyle p} & \overset{\bar{p}}{\nwarrow} & \\ Y & \overset{v}{\cdots\to} & V & \overset{\theta_2}{\Longrightarrow} & \bar{V} \\ {\scriptstyle y_2g_2}\uparrow & \Sigma^{\xi_2} & \uparrow{\scriptstyle y_2'} & & \\ B & \xrightarrow{x_3} & X & \underset{\bar{y}_2}{\nearrow} & \end{array} \quad = \quad \begin{array}{ccc} B & \xrightarrow{x_3} & X \\ {\scriptstyle y_1g_1}\downarrow & \Sigma^{\bar{\xi}_1} & \downarrow{\scriptstyle \bar{y}_1} \\ Y & \overset{\bar{v}}{\cdots\to} & \bar{V} \\ \Vert & \Sigma^{\bar{\chi}} & \downarrow{\scriptstyle \bar{p}} \\ Y & \xrightarrow{t} & T \\ \Vert & \Sigma^{\bar{\chi}} & \uparrow{\scriptstyle \bar{p}} \\ Y & \overset{\bar{v}}{\cdots\to} & \bar{V} \\ {\scriptstyle y_2g_2}\uparrow & \Sigma^{\bar{\xi}_2} & \uparrow{\scriptstyle \bar{y}_2} \\ B & \xrightarrow{x_3} & X \end{array} \tag{3.23}$$

Using the equalities (3.19), (3.22) and (3.23), we see that the pasting of the Σ-square

$$\begin{array}{ccc} B & \xrightarrow{x_3} & X \\ {\scriptstyle y_1g_1}\downarrow & \Sigma^{\bar{\chi}\circ\bar{\xi}_1} & \downarrow{\scriptstyle \bar{p}\bar{y}_1} \\ Y & \xrightarrow{t} & T \end{array}$$

with $\theta_2\cdot(p\circ\beta')\cdot\theta_1^{-1}$ is equal to its pasting with $\bar{p}\circ\bar{\beta}$. Using Equification, we obtain a Σ-square

$$\begin{array}{ccc} Y & \xrightarrow{t} & T \\ \Vert & \Sigma^{\psi} & \downarrow{\scriptstyle q} \\ Y & \xrightarrow[u]{} & Q \end{array}$$

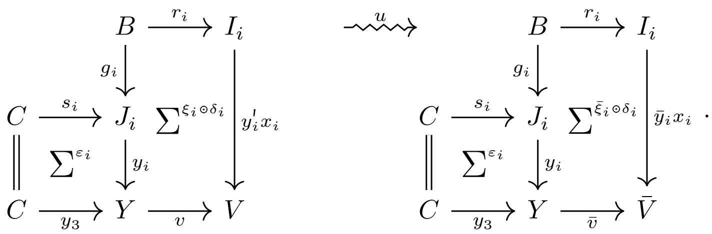
such that $q\circ\big(\theta_2\cdot(p\circ\beta')\cdot\theta_1^{-1}\big) = q\circ\bar{p}\circ\bar{\beta}$. Consider the following diagram representing three vertically composable 2-morphisms:

$$\begin{array}{ccccccccccc} A & \xrightarrow{f_1} & I_1 & \xrightarrow{x_1} & X & \xrightarrow{\bar{y}_1} & \bar{V} & \xleftarrow{\bar{v}} & Y & \xleftarrow{y_3} & C \\ \Vert & & \Vert & & \Vert & \Swarrow & {\scriptstyle q\bar{p}}\downarrow & \Sigma^{\psi\circ\bar{\chi}} & \Vert & \Sigma^{\mathrm{id}} & \Vert \\ & & & & & q\theta_1^{-1} & Q & \xleftarrow{u} & Y & \xleftarrow{y_3} & C \\ & & & & & \Swarrow & {\scriptstyle qp}\uparrow & \Sigma^{\psi\circ\chi} & \Vert & \Sigma^{\mathrm{id}} & \Vert \\ A & \xrightarrow{f_1} & I_1 & \xrightarrow{x_1} & X & \xrightarrow{y_1'} & V & \xleftarrow{v} & Y & \xleftarrow{y_3} & C \\ \Vert & & \overset{\alpha}{\Swarrow} & & \Vert & \overset{\beta'}{\Swarrow} & \Vert & & \Sigma^{\mathrm{id}} & & \Vert \\ & & & & & & V & \xleftarrow[v]{} & Y & \xleftarrow[y_3]{} & C \\ & & & & & & \Vert & & \Sigma^{\mathrm{id}} & & \Vert \\ A & \xrightarrow[f_2]{} & I_2 & \xrightarrow[x_2]{} & X & \xrightarrow[y_2']{} & V & \xleftarrow[v]{} & Y & \xleftarrow[y_3]{} & C \\ \Vert & & \Vert & & \Vert & \Swarrow & {\scriptstyle qp}\downarrow & \Sigma^{\psi\circ\chi} & \Vert & \Sigma^{\mathrm{id}} & \Vert \\ & & & & & q\theta_2 & Q & \xleftarrow[u]{} & Y & \xleftarrow[y_3]{} & C \\ & & & & & \Swarrow & {\scriptstyle q\bar{p}}\uparrow & \Sigma^{\psi\circ\bar{\chi}} & \Vert & \Sigma^{\mathrm{id}} & \Vert \\ A & \xrightarrow[f_2]{} & I_2 & \xrightarrow[x_2]{} & X & \xrightarrow[\bar{y}_2]{} & \bar{V} & \xleftarrow[\bar{v}]{} & Y & \xleftarrow[y_3]{} & C\,. \end{array}$$

The top and the bottom 2-morphisms of this diagram are of type Ω; more precisely,

$$\tilde{\Omega}_i = [q\theta_i x_i f_i, qp, q\bar{p}]\colon(\bar{y}_i x_i f_i, \bar{v}y_3) \Longrightarrow (y_i' x_i f_i, v y_3)$$

corresponds to the Σ-step



Adding this Σ-step to the Σ-path (3.21), and taking into account Lemma 3.12(1), we obtain a Σ-path of the form $\bullet \overset{d}{\rightsquigarrow} \bullet \overset{u}{\rightsquigarrow} \bullet$ which determines, for $i = 1, 2$, the Ω 2-cells

$$\bar{\Omega}_i \colon (g_i, s_i) \circ (f_i, r_i) \Longrightarrow (\bar{y}_i x_i f_i, \bar{v} y_3)$$

required in Definition 3.24.

Then,

$$\begin{aligned}
\bar{\Omega}_2^{-1} \cdot [\bar{\beta} \circ \alpha, \mathrm{id}, \mathrm{id}] \cdot \bar{\Omega}_1 &= \bar{\Omega}_2^{-1} \cdot [(q \circ \theta_2 \circ x_2 f_2) \cdot (qp \circ \beta' \circ \alpha) \cdot (q \circ \theta_1^{-1} \circ x_1 f_1), q\bar{p}, q\bar{p}] \cdot \bar{\Omega}_1 \\
&= \bar{\Omega}_2^{-1} \cdot [q \circ \theta_2 \circ x_2 f_2, qp, q\bar{p}] \cdot [qp \circ \beta' \circ \alpha, qp, qp] \cdot [((q \circ \theta_1^{-1} \circ x_1 f_1), q\bar{p}, qp] \cdot \bar{\Omega}_1 \\
&= \bar{\Omega}_2^{-1} \cdot \tilde{\Omega}_2^{-1} \cdot [\beta' \cdot \alpha, 1_V, 1_V] \cdot \tilde{\Omega}_1 \cdot \bar{\Omega}_1 \\
&= \Omega_2^{-1} \cdot [\beta' \circ \alpha, 1_V, 1_V] \cdot \Omega_1
\end{aligned}$$

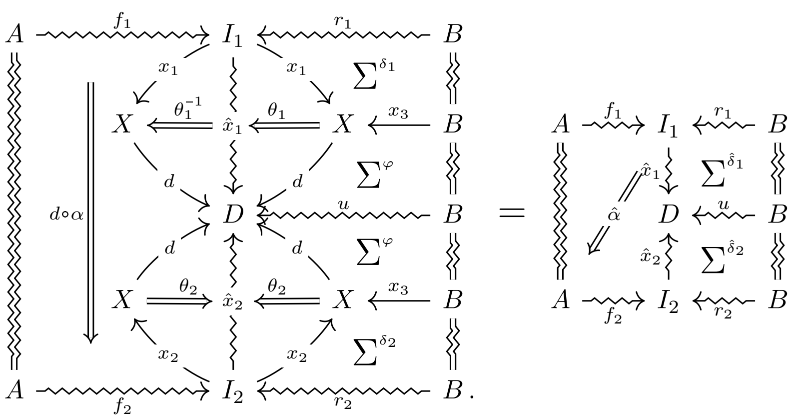
as desired.

(2) Secondly, we show that this composition respects the ≈-relation. That is, given $(\hat{\alpha}, \hat{x}_1, \hat{x}_2) \approx (\alpha, x_1, x_2) \colon (f_1, r_1) \Rightarrow (f_2, r_2) \colon A \to B$ and $(\hat{\beta}, \hat{y}_1, \hat{y}_2) \approx (\beta, y_1, y_2) \colon (g_1, s_1) \Rightarrow (g_2, s_2) \colon B \to C$, it holds $(\hat{\beta}, \hat{y}_1, \hat{y}_2) \circ (\hat{\alpha}, \hat{x}_1, \hat{x}_2) \approx (\beta, y_1, y_2) \circ (\alpha, x_1, x_2)$. It is enough to consider the following two special cases:

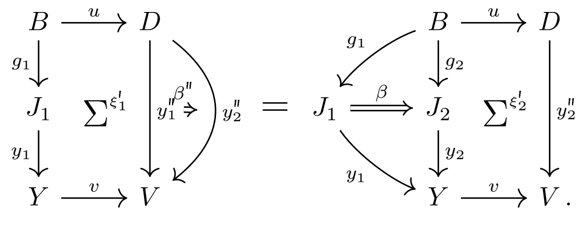
(2a) $(\hat{\alpha}, \hat{x}_1, \hat{x}_2)$ is a Σ-extension of (α, x_1, x_2) and $(\beta, y_1, y_2) = (\hat{\beta}, \hat{y}_1, \hat{y}_2)$;

(2b) $(\hat{\beta}, \hat{y}_1, \hat{y}_2)$ is a Σ-extension of (β, y_1, y_2) and $(\hat{\alpha}, \hat{x}_1, \hat{x}_2) = (\alpha, x_1, x_2)$.

(2a) In the first case, let $(\hat{\alpha}, \hat{x}_1, \hat{x}_2)$ be a Σ-extension of (α, x_1, x_2) by means of the following equality:



In order to get $(\beta, y_1, y_2) \circ (\hat{\alpha}, \hat{x}_1, \hat{x}_2)$, we apply Rule 6:



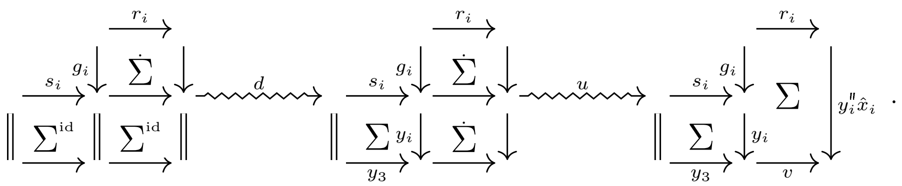
By considering

$$y_i' = y_i'' d,\ \xi_i = \xi_i' \odot \varphi\,(i = 1, 2),\ \text{and}\ \beta' = \beta'' \circ d$$

we obtain the data to define $[\beta, y_1, y_2] \circ [\alpha, x_1, x_2]$, see (3.19). Let

$$\begin{aligned} &\Omega_i\colon (g_i, s_i) \circ (f_i, r_i) \Rightarrow (y_i' x_i f_i, v y_3), \text{ and} \\ &\hat{\Omega}_i\colon (g_i, s_i) \circ (f_i, r_i) \Rightarrow (y_i'' \hat{x}_i f_i, v y_3), \end{aligned}$$

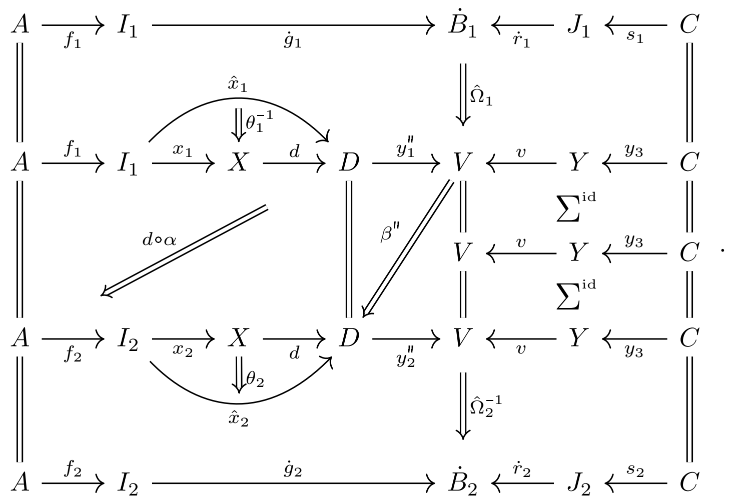
be the Ω 2-cells obtained as in Definition 3.24 to arrive at $[\beta, y_1, y_2] \circ [\hat{\alpha}, \hat{x}_1, \hat{x}_2]$ and $[\beta, y_1, y_2] \circ [\alpha, x_1, x_2]$, respectively; in particular, $\hat{\Omega}_i$ is the Ω 2-cell corresponding to the Σ-path



(3.24)

Observe that $\tilde{\Omega}_i = [y_i'' \circ \theta_i \circ f_i, 1, 1]$ is a basic Ω 2-cell produced by a Σ-step of type u. Hence, using Lemma 3.12(1), we see that $\tilde{\Omega}_i \cdot \hat{\Omega}_i = \Omega_i$.

Consider the diagram



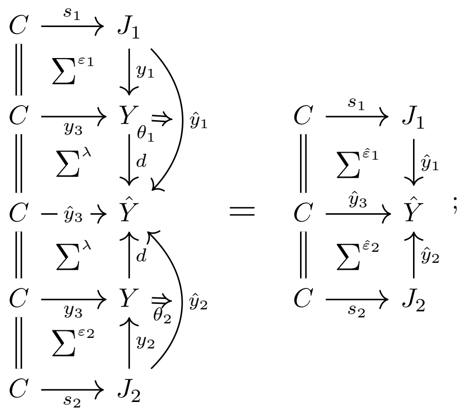
.

We then obtain:

$$\begin{aligned} [(\beta, y_1, y_2) \circ (\hat{\alpha}, \hat{x}_1, \hat{x}_2)] &= \hat{\Omega}_2^{-1} \cdot [\beta'' \circ \hat{\alpha}, 1_V, 1_V] \cdot \hat{\Omega}_1 \\ &= \hat{\Omega}_2^{-1} \cdot [\beta'' \circ \big((\theta_2 \circ f_2) \cdot (d \circ \alpha) \cdot (\theta_1^{-1} \circ f_1)\big), 1_V, 1_V]) \cdot \hat{\Omega}_1 \\ &= \hat{\Omega}_2^{-1} \cdot \tilde{\Omega}_2^{-1} \cdot [\beta' \circ \alpha, 1_V, 1_V] \cdot \tilde{\Omega}_1 \cdot \hat{\Omega}_1 \\ &= \Omega_2^{-1} \cdot [\beta' \circ \alpha, 1_V, 1_V] \cdot \Omega_1 \\ &= [(\beta, y_1, y_2) \circ (\alpha, x_1, x_2)] \end{aligned}$$

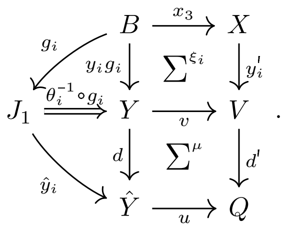
as desired.

(2b) For the second case, let $(\hat{\beta}, \hat{y}_1, \hat{y}_2)$ be a Σ-extension of (β, y_1, y_2) through the following equalities, where the 2-cells θ_i are invertible:

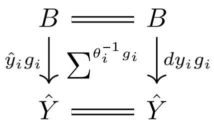


$$; \ \hat{\beta} = (\theta_2 \circ g_2) \cdot (d \circ \beta) \cdot (\theta_1^{-1} \circ g_1). \tag{3.25}$$

Consider the Σ-squares Σ^{ξ_i} of (3.19) of the definition of horizontal composition of (α, x_1, x_2) with (β, y_1, y_2), and use Square and the 2-cells θ_i to get



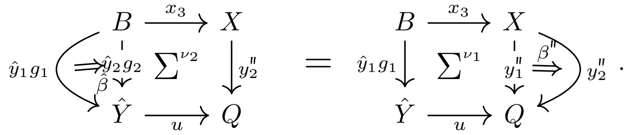
By Horizontal Repletion, we have the Σ-square



Put

$$\nu_i = (\theta_i^{-1} g_i) \oplus (\mu \odot \xi_i), \ y_i'' = d' y_i' \text{ and } \beta'' = d' \circ \beta'.$$

It is easy to see that



Therefore, we may take the Σ-squares Σ^{ν_i} and the 2-cell β'' to perform the conditions corresponding to (3.19) in the process of obtaining the horizontal composition of (α, x_1, x_2) with $(\hat{\beta}, \hat{x}_1, \hat{x}_2)$. In this way, we get the 2-morphism in the middle of diagram (3.27) below. This diagram illustrates the desired $\approx$-equivalence. Indeed, as in Definition 3.24, consider

$$\Omega_i \colon (g_i, s_i) \circ (f_i, r_i) \Longrightarrow (y_i' x_i f_i, v y_3)$$

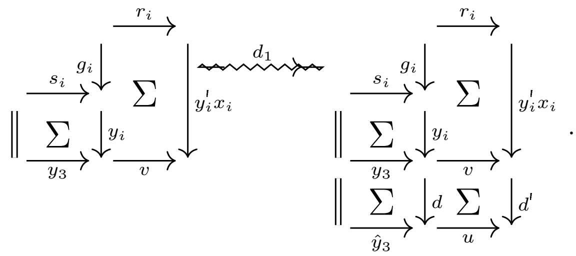
and

$$\hat{\Omega}_i \colon (g_i, s_i) \circ (f_i, r_i) \Longrightarrow (d' y_i' x_i f_i, u \hat{y}_3)$$

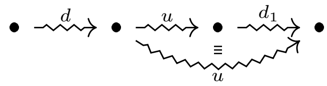
corresponding to the compositions $[\beta, y_1, y_2] \circ [\alpha, x_1, x_2]$ and $[\hat{\beta}, \hat{y}_1, \hat{y}_2] \circ [\alpha, x_1, x_2]$, respectively. Moreover, let

$$\tilde{\Omega}_i \colon (y_i' x_i f_i, v y_3) \Longrightarrow (d' y_i' x_i f_i, u \hat{y}_3)$$

be the basic Ω 2-cell corresponding to the following Σ-step of type d_1 (see Notation 3.15):

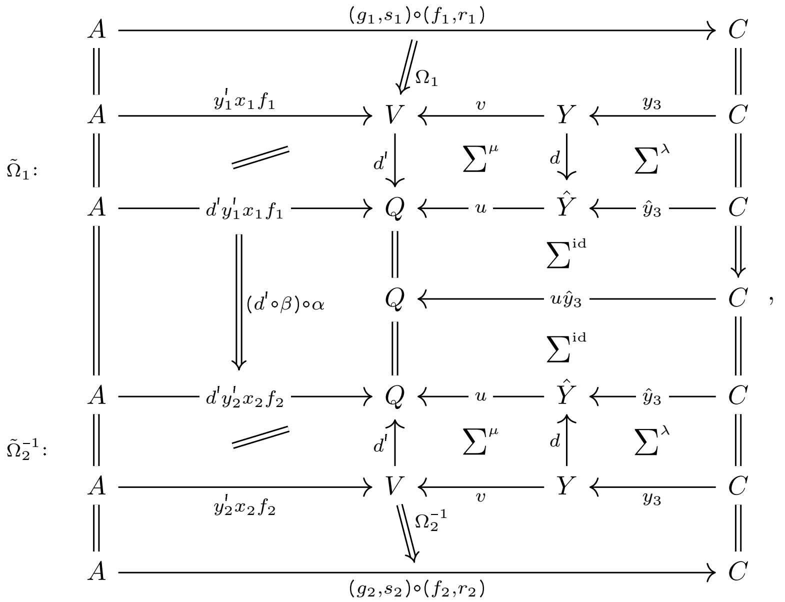


Recall that Ω_i is given by the Σ-step (3.21) as in Definition 3.24. Comparing $\tilde{\Omega}_i \circ \Omega_i$ and $\hat{\Omega}_i$, which have the same domain and codomain, we see that, on one hand, $\tilde{\Omega}_i \circ \Omega_i$ corresponds to a Σ-path of the form $\bullet \overset{d}{\rightsquigarrow} \bullet \overset{u}{\rightsquigarrow} \bullet \overset{d_1}{\rightsquigarrow} \bullet$, as in the top line of diagram (3.26) below, and, on the other hand, $\hat{\Omega}_i$ is obtained by a Σ-path of the form $\bullet \overset{d}{\rightsquigarrow} \bullet \overset{u}{\rightsquigarrow} \bullet$. See (3.24), as in the bottom of diagram (3.26), starting with the same Σ-step $\bullet \overset{d}{\rightsquigarrow} \bullet$ as the one for $\tilde{\Omega}_i \circ \Omega_i$.



(3.26)

By Lemma 3.12, we have the equivalence of Σ-paths indicated in the diagram. Thus, $\tilde{\Omega}_i \circ \Omega_i = \hat{\Omega}_i$.



(3.27)

We obtain:

$[(\beta, y_1, y_2) \circ (\alpha, x_1, x_2)] = \Omega_2^{-1} \cdot [\beta' \circ \alpha, 1_V, 1_V] \cdot \Omega_1,$ see Definition 3.24

$= \Omega_2^{-1} \cdot (\tilde{\Omega}_2^{-1} \cdot [(d' \circ \beta') \circ \alpha, 1_Q, 1_Q] \cdot \tilde{\Omega}_1) \cdot \Omega_1),$

$= (\Omega_2^{-1} \cdot \tilde{\Omega}_2^{-1}) \cdot [(d' \circ \beta') \circ \alpha, 1_Q, 1_Q] \cdot (\tilde{\Omega}_1 \cdot \Omega_1),$ by the associativity of the vertical composition

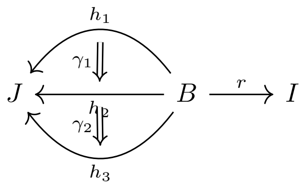
$= \hat{\Omega}_2^{-1} \cdot [(d' \circ \beta') \circ \alpha, 1_Q, 1_Q] \cdot \hat{\Omega}_1,$

$= [(\hat{\beta}, \hat{y}_1, \hat{y}_2) \circ (\alpha, x_1, x_2)].$

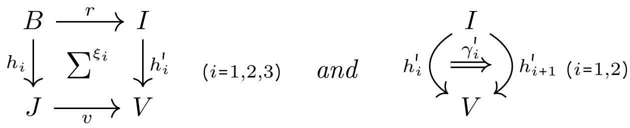
□

The following remark will be useful in the proof of the next proposition.

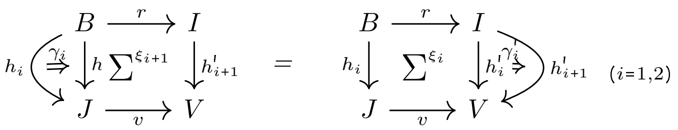
Lemma 3.29. *Assume we are given the data*



with $r \in \Sigma$. *Then there are*

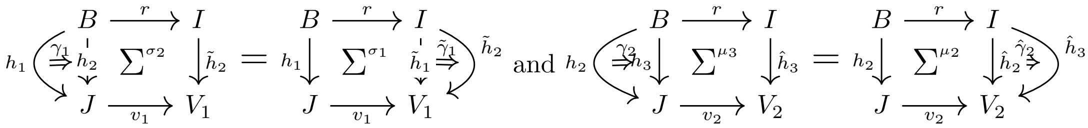


such that

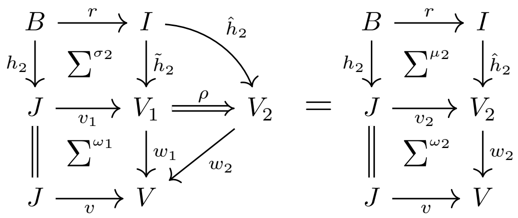


(3.28)

Proof. Rule 6 of Proposition 2.8 tells us that this is true for just one 2-cell γ_1. Thus, we have equalities of the form



Using Rule 4', we obtain the equality



where ρ is invertible.

Put:

$h'_1 = w_1\tilde{h}_1$; $h'_2 = w_2\hat{h}_2$; $h'_3 = w_2\hat{h}_3$;

$\gamma'_1 = \rho \cdot (w_1\tilde{\gamma}_1)$; $\gamma'_2 = w_2\hat{\gamma}_2$;

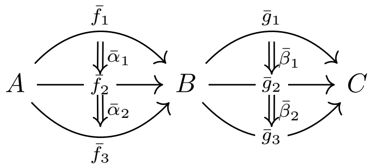
$\Sigma^{\xi_1} = \Sigma^{\omega_1 \odot \sigma_1}$; $\Sigma^{\xi_2} = \Sigma^{(\omega_1 \odot \sigma_2) \oplus \rho}$; $\Sigma^{\xi_3} = \Sigma^{\omega_2 \odot \mu_3}$.

It is easy to see that this way we obtain the desired result. □

Proposition 3.30. $\mathrm{comp}_{ABC} \colon \mathcal{X}[\Sigma_*](A,B) \times \mathcal{X}[\Sigma_*](B,C) \to \mathcal{X}[\Sigma_*](A,C)$ *is a functor.*

Proof. The equality $\mathrm{id}_g \circ \mathrm{id}_f = \mathrm{id}_{g\circ f}$ is clear: observe that in the application of (3.19), for $\beta = \mathrm{id}_g$, we may put $\beta' = \mathrm{id}_{\dot{g}}$ using the canonical Σ-square of r and g.

Now, given 1-cells $\bar{f}_i = (f_i, r_i)$ and $\bar{g}_i = (g_i, s_i)$, and 2-cells



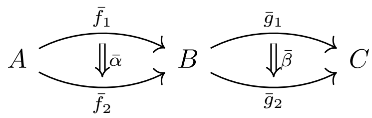
we want to show that

$$(\bar\beta_2 \cdot \bar\beta_1) \circ (\bar\alpha_2 \cdot \bar\alpha_1) = (\bar\beta_2 \circ \bar\alpha_2) \cdot (\bar\beta_1 \circ \bar\alpha_1)\,. \tag{3.29}$$

First we show that, for $A \xrightarrow{\bar f} B \xrightarrow{\bar g} C$, we always have

$$(\bar g \circ \bar\alpha_2) \cdot (\bar g \circ \bar\alpha_1) = \bar g \circ (\bar\alpha_2 \cdot \bar\alpha_1) \text{ and } (\bar\beta_2 \circ \bar f) \cdot (\bar\beta_1 \circ \bar f) = (\bar\beta_2 \cdot \bar\beta_1) \circ \bar f\,. \tag{3.30}$$

After that, to obtain (3.29), it is enough to prove the whiskering law — that is, that, given two 2-cells $\bar\alpha$ and $\bar\beta$ of the form



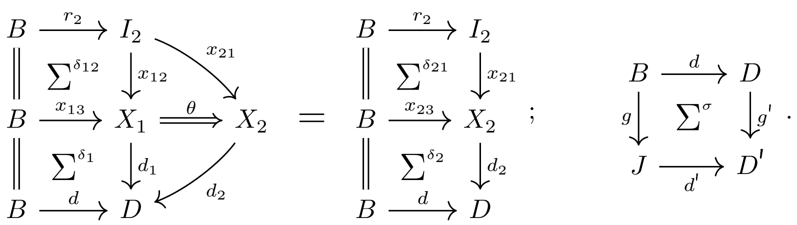
the equalities

$$(\bar\beta \circ \bar f_2) \cdot (\bar g_1 \circ \bar\alpha) = \bar\beta \circ \bar\alpha = (\bar g_2 \circ \bar\alpha) \cdot (\bar\beta \circ \bar f_1) \tag{3.31}$$

hold.

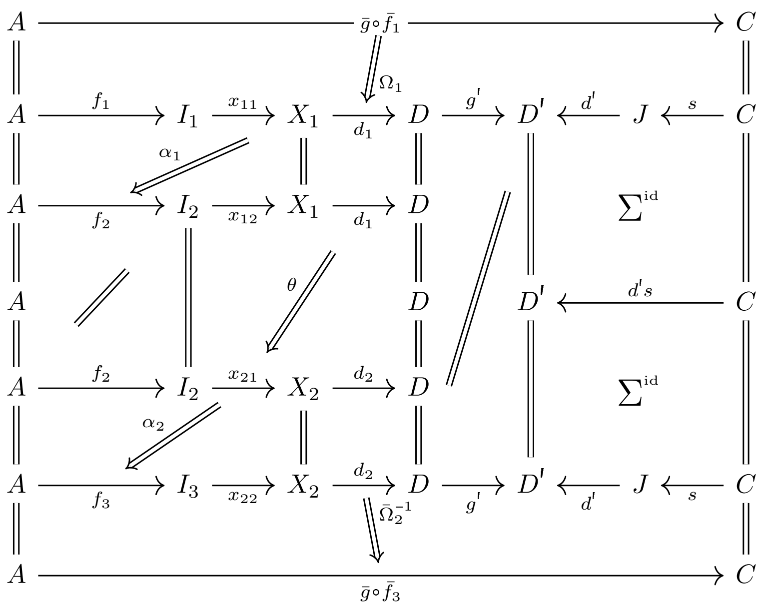
(1) In order to show the first equality of (3.30), suppose we have $\bar f_i = (f_i, I_i, r_i)\colon A \to B$, $i = 1,2,3$, $\bar\alpha_i = [\alpha_i, x_{i1}, x_{i2}, x_{i3}, \delta_{i1}, \delta_{i2}]\colon \bar f_i \Rightarrow \bar f_{i+1}$, and $\bar g = (g, J, s)$.

Consider the following existing data:



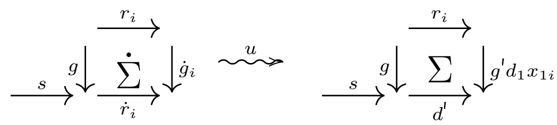
(3.32)

Then $\bar\alpha_2 \cdot \bar\alpha_1 = [(d_2 \circ \alpha_2) \cdot (\theta \circ f_2) \cdot (d_1 \circ \alpha_1),\ d_1 x_{11},\ d_2 x_{22},\ d,\ \delta_1 \odot \delta_{11},\ \delta_2 \odot \delta_{22}]$, and $\bar g \circ (\bar\alpha_2 \cdot \bar\alpha_1)$ is represented by the diagram

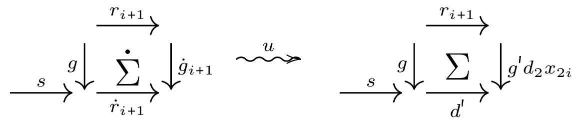


(3.33)

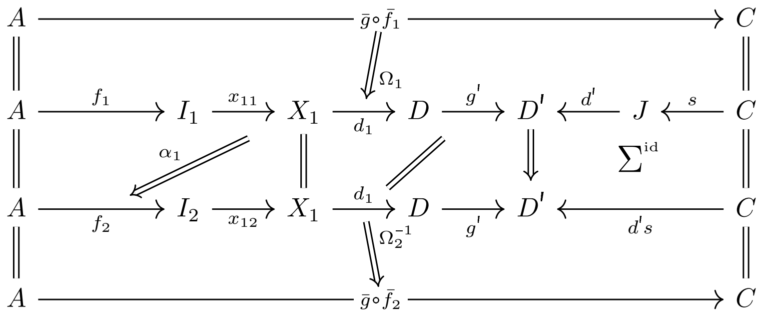
where $\Omega_i\colon \bar{g}\circ\bar{f}_i \Rightarrow (g'd_1x_{1i}f_i, d's)$ is the Ω 2-cell corresponding just to the Σ-path



and, similarly, $\bar{\Omega}_i\colon \bar{g}\circ\bar{f}_{i+1} \Rightarrow (g'd_2x_{2i}f_{i+1}, d's)$ corresponds to

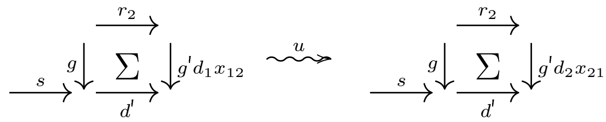


Using again (3.32), the horizontal compositions $\bar{g}\circ\bar{\alpha}_1$ is represented by



(3.34)

An analogous diagram represents the composition $\bar{g}\circ\bar{\alpha}_2$. Moreover the 2-cell $[g'\circ\theta\circ f_2, 1_{D'}, 1_{D'}]$ is just the Ω 2-cell corresponding to the Σ-path

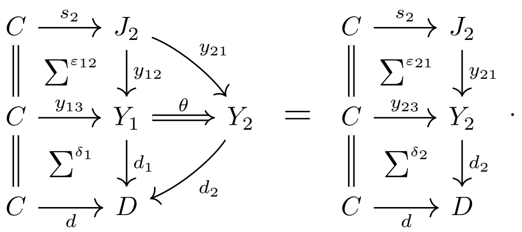


Then, by Lemma 3.12, $[g'\circ\theta\circ f_2,\, 1_{D'}, 1_{D'}]\circ\Omega_2 = \bar{\Omega}_1$.

Thus, we have:

$$
\begin{aligned}
\bar{g}\circ(\bar{\alpha}_2\cdot\bar{\alpha}_1) &= \bar{\Omega}_2^{-1}\cdot[g'\circ((d_2\circ\alpha_2)\cdot(\theta\circ f_2)\cdot(d_1\circ\alpha_1)), 1_{D'}, 1_{D'}]\cdot\Omega_1\\
&= \bar{\Omega}_2^{-1}\cdot[g'd_2\circ\alpha_2, 1_{D'}, 1_{D'}]\cdot[g'\circ\theta\circ f_2, 1_{D'}, 1_{D'}]\cdot[g'd_1\circ\alpha_1, 1_{D'}, 1_{D'}]\cdot\Omega_1\\
&= \bar{\Omega}_2^{-1}\cdot[g'd_2\circ\alpha_2, 1_{D'}, 1_{D'}]\cdot([g'\circ\theta\circ f_2, 1_{D'}, 1_{D'}]\cdot\Omega_2)\cdot(\Omega_2^{-1}\cdot[g'd_1\circ\alpha_1, 1_{D'}, 1_{D'}]\cdot\Omega_1)\\
&= (\bar{\Omega}_2^{-1}\cdot[g'd_2\circ\alpha_2, 1_{D'}, 1_{D'}]\cdot\bar{\Omega}_1)\cdot(\Omega_2^{-1}\cdot[g'd_1\circ\alpha_1, 1_{D'}, 1_{D'}]\cdot\Omega_1)\\
&= (\bar{g}\circ\bar{\alpha}_2)\cdot(\bar{g}\circ\bar{\alpha}_1)
\end{aligned}
$$

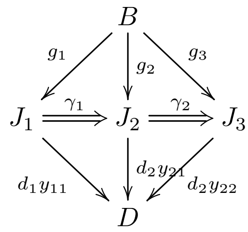
(2) In order to show the second equality of (3.30), let us have $\bar{f} = (f, I, r)\colon A\to B$, $\bar{g}_i = (g_i, J_i, s_i)\colon B\to C$ and $\bar{\beta}_i = [\beta_i, y_{i1}, y_{i2}, y_{i3}, \varepsilon_{i1}, \varepsilon_{i2}]\colon \bar{g}_i \Rightarrow \bar{g}_{i+1}$. Use Rule 4' to form the equality



We can use this data to obtain Σ-extensions of $(\beta_1, y_{11}, y_{12})$ and $(\beta_2, y_{21}, y_{22})$; more precisely, we have that

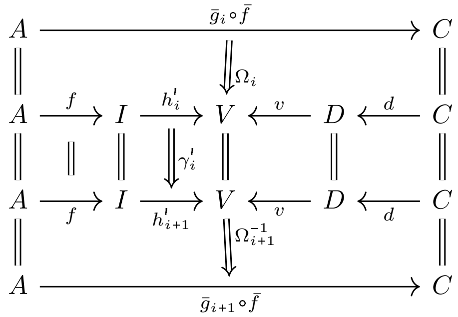
$$\bar{\beta}_1 = [(\theta\circ g_2)\cdot(d_1\circ\beta_1), d_1y_{11}, d_2y_{21}, d, \delta_1\odot\varepsilon_{11}, \delta_2\odot\varepsilon_{21}] \quad\text{and}\quad \bar{\beta}_2 = [d_2\circ\beta_2, d_2y_{21}, d_2y_{22}, d, \delta_2\odot\varepsilon_{21}, \delta_2\odot\varepsilon_{22}].$$

We form $\bar{\beta}_i \circ \bar{f}$, for $i = 1, 2$, using these last $\approx$-representatives. For that, apply the result of Lemma 3.29, with $\gamma_1 = (\theta \circ g_2) \cdot (d_1 \circ \beta_1)$ and $\gamma_2 = d_2 \circ \beta_2$,

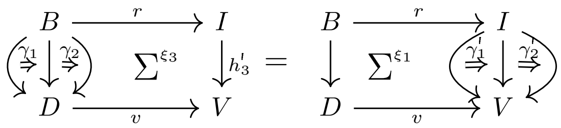


to obtain a situation as in Equation (3.28), with $h_1 = d_1 y_{11} g_1$, $h_2 = d_2 y_{21} g_2$ and $h_3 = d_2 y_{22} g_3$,

Then, we may represent the 2-cells $(\bar{\beta}_i \circ \bar{f})$ of $\mathcal{X}[\Sigma_*]$ by the following diagram, where Ω_i and Ω_{i+1} are the due 2-cells of type Ω, according to Definition 3.24:



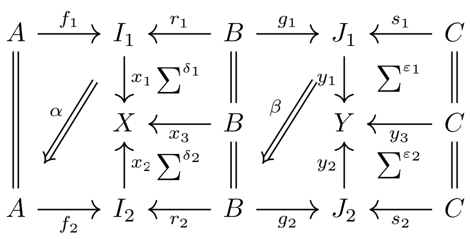
Observing that $\bar{\beta}_2 \cdot \bar{\beta}_1 = [\gamma_2 \cdot \gamma_1, d_1 y_{11}, d_2 y_{22}]$, and that we have



we obtain:

$$
\begin{aligned}
(\bar{\beta}_2 \circ \bar{f}) \cdot (\bar{\beta}_1 \circ \bar{f}) &= (\Omega_3^{-1} \cdot [\gamma_2' \circ f, 1_V, 1_V] \cdot \Omega_2) \cdot (\Omega_2^{-1} \cdot [\gamma_1' \circ f, 1_V, 1_V] \cdot \Omega_1) \\
&= \Omega_3^{-1} \cdot [\gamma_2' \circ f, 1_V, 1_V] \cdot [\gamma_1' \circ f, 1_V, 1_V] \cdot \Omega_1 \\
&= \Omega_3^{-1} \cdot [(\gamma_2' \cdot \gamma_1') \circ f, 1_V, 1_V] \cdot \Omega_1 \\
&= (\bar{\beta}_2 \cdot \bar{\beta}_1) \circ \bar{f}.
\end{aligned}
$$

(3) Concerning (3.31), let us consider two 2-cells $\overline{\alpha} = [\alpha, x_1, x_2]$ and $\overline{\beta} = [\beta, y_1, y_2]$ represented in the following diagram:

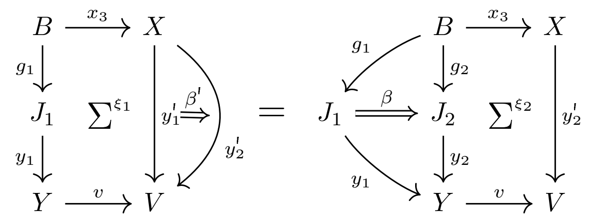


(3.35)

We want to show that

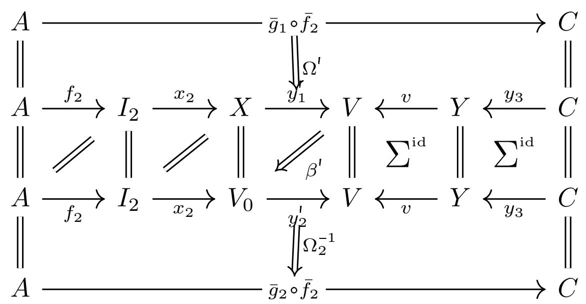
$$(\bar{\beta} \circ \bar{f}_2) \cdot (\bar{g}_1 \circ \bar{\alpha}) = \bar{\beta} \circ \bar{\alpha} = (\bar{g}_2 \circ \bar{\alpha}) \cdot (\bar{\beta} \circ \bar{f}_1). \tag{3.36}$$

Consider the following data, used in the composition $\bar{\beta} \circ \bar{\alpha}$ (see (3.19)).



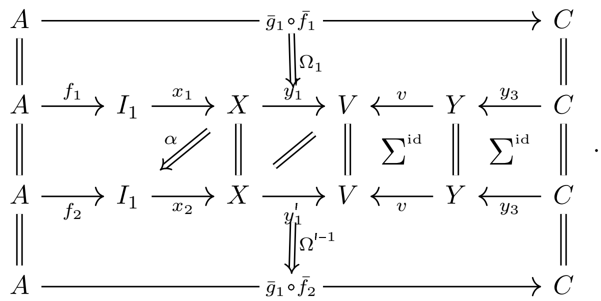
(3.37)

We prove the first equality of (3.36). First observe that $\bar{\beta} \circ \bar{f}_2 = \Omega_2^{-1} \cdot [\beta' \circ x_2 f_2, 1, 1] \cdot \Omega'$, where $\Omega_i \colon \bar{g}_i \circ \bar{f}_i \Rightarrow (y_i' x_i f_i, v y_3)$ and $\Omega' \colon \bar{g}_1 \circ \bar{f}_2 \Rightarrow (y_1' x_2 f_2, v y_3)$ are the due 2-cells of type Ω, see diagram (3.38) below.



(3.38)

Moreover, taking into account that $(\mathrm{id}_{g_1}, 1_{J_1}, 1_{J_1}, s_1, \mathrm{id}, \mathrm{id}) \approx (\mathrm{id}_{y_1 g_1}, y_1, y_1, y_3, \varepsilon_1, \varepsilon_1)$, the composition $\bar{g}_1 \circ \bar{\alpha}$ may be represented by



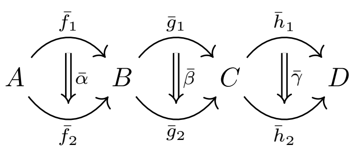
. (3.39)

Composing vertically (3.39) with (3.38), we immediately see that the resulting 2-morphism is indeed a representative of the composition $\bar{\beta} \circ \bar{\alpha}$, that is, $(\bar{\beta} \circ \bar{f}_2) \cdot (\bar{g}_1 \circ \bar{\alpha}) \approx \bar{\beta} \circ \bar{\alpha}$.

The proof of the second equality of (3.36) is analogous. ☐

Proposition 3.31. *The isomorphisms of Definition 3.26 indeed form a natural transformation from* $- \circ (- \circ -)$ *to* $(- \circ -) \circ -$.

Proof. We want to show that, for 2-cells

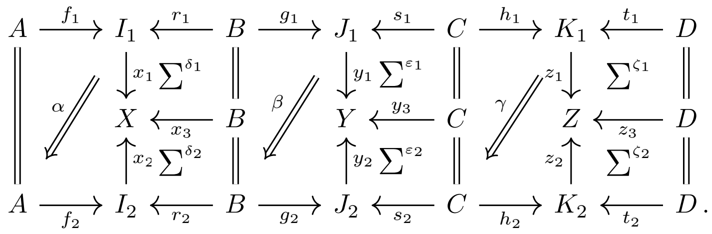


in $\mathcal{X}[\Sigma_*]$ the following diagram

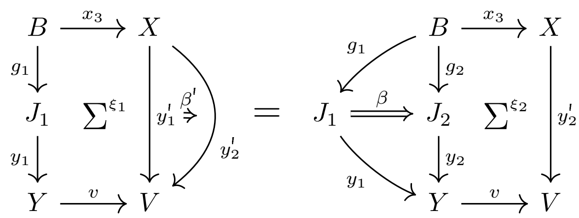
$$\begin{array}{ccc} (\bar{h}_1 \circ \bar{g}_1) \circ \bar{f}_1 & \xRightarrow{\mathrm{Assoc}_{\bar{f}_1, \bar{g}_1, \bar{h}_1}} & \bar{h}_1 \circ (\bar{g}_1 \circ \bar{f}_1) \\ {\scriptstyle (\bar{\gamma} \circ \bar{\beta}) \circ \bar{\alpha}} \Downarrow & & \Downarrow {\scriptstyle \bar{\gamma} \circ (\bar{\beta} \circ \bar{\alpha})} \\ (\bar{h}_2 \circ \bar{g}_2) \circ \bar{f}_2 & \xRightarrow{\mathrm{Assoc}_{\bar{f}_2, \bar{g}_2, \bar{h}_2}} & \bar{h}_2 \circ (\bar{g}_2 \circ \bar{f}_2) \end{array} \tag{3.40}$$

is commutative in $\mathcal{X}[\Sigma_*](A, D)$.

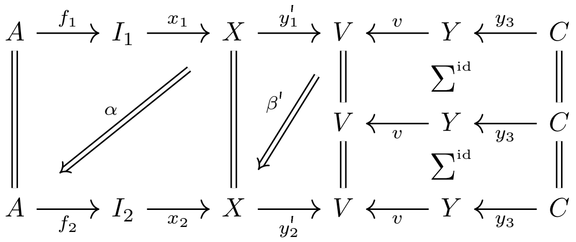
We are given



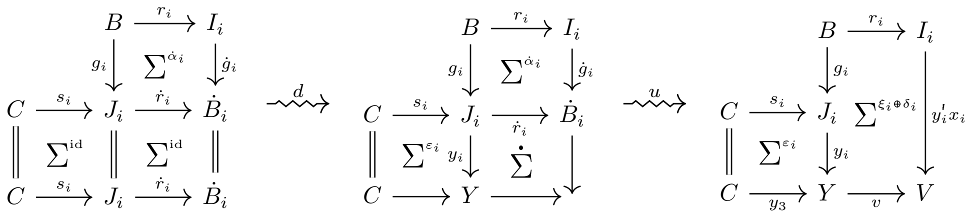
(a) Formation of $\bar{\beta} \circ \bar{\alpha}$:
We apply Rule 6 to obtain



(3.41)

and we get the following 2-morphism  .

Now, consider the Σ-path

(Ω_i) 

(3.42)

and let

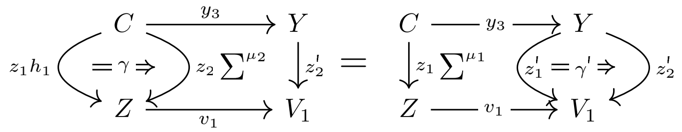
$$\Omega_i \colon (\dot{g}_i f_i, \dot{r}_i s_i) \Rightarrow (y_i' x_i f_i, v y_3) \qquad (i = 1, 2)$$

be the corresponding Ω 2-cell. Thus, according to Definition 3.24, $\bar{\beta} \circ \bar{\alpha}$ is given by

$$\bar{g}_1 \circ \bar{f}_1 = (\dot{g}_1 f_1, \dot{r}_1 s_1) \xRightarrow{\Omega_1} (y_1' x_1 f_1, v y_3) \xRightarrow{[\beta' \circ \alpha, 1_V, 1_V]} (y_2' x_2 f_2, v y_3) \xRightarrow{\Omega_2^{-1}} (\dot{g}_2 f_2, \dot{r}_2 s_2) = \bar{g}_2 \circ \bar{f}_2 .$$

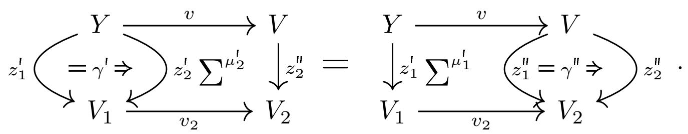
(b) Formation of $\bar{\gamma} \circ (\bar{\beta} \circ \bar{\alpha})$ and $(\bar{\gamma} \circ \bar{\beta}) \circ \bar{\alpha}$:

First use Rule 6 to find



(3.43)

and, subsequently, also

 .

(3.44)

These data will be used in both compositions.

(b1) Observe that, using the interchange law, already proven to be valid, we have that

$$\bar{\gamma} \circ (\bar{\beta} \circ \bar{\alpha}) = (\mathrm{id}_{\bar{h}_2} \circ \Omega_2^{-1}) \cdot (\bar{\gamma} \circ [\beta' \circ \alpha, 1, 1]) \cdot (\mathrm{id}_{\bar{h}_1} \circ \Omega_1)\,.$$

Concerning $\bar{\gamma} \circ [\beta' \circ \alpha, 1, 1]$, from (3.43) and (3.44), we have that

$$\bar{\gamma} \circ [\beta' \circ \alpha, 1, 1] = \Omega_{12}^{-1} \cdot [\gamma'' \circ \beta' \circ \alpha, 1_{V_2}, 1_{V_2}] \cdot \Omega_{11}$$

where

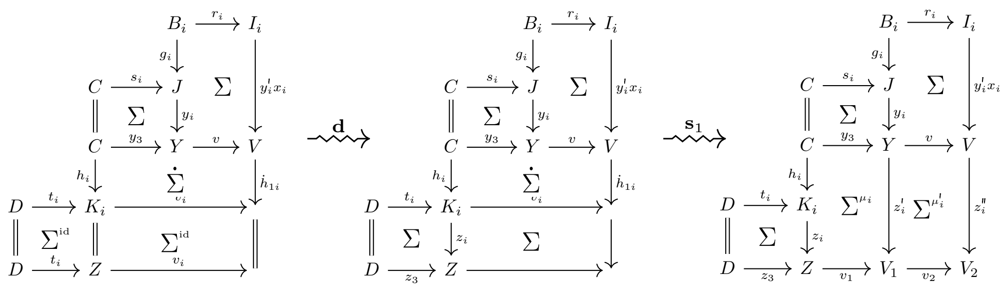
$$\Omega_{1i} \colon \bar{h}_i \circ (y_i' x_i f_i, v y_3) = (\dot{h}_{1i} y_i' x_i f_i, \dot{v} t_1) \Rightarrow (z_i'' y_1' x_1 f_1, v_2 v_1 z_3) \tag{3.45}$$

are the obvious Ω 2-cells, in accordance with Definition 3.24. Hence,

$$\bar{\gamma} \circ (\bar{\beta} \circ \bar{\alpha}) = (\mathrm{id}_{\bar{h}_2} \circ \Omega_2^{-1}) \cdot \Omega_{12}^{-1} \cdot [\gamma'' \circ \beta' \circ \alpha, 1_{V_2}, 1_{V_2}] \cdot \Omega_{11} \cdot (\mathrm{id}_{\bar{h}_1} \circ \Omega_1)\,. \tag{3.46}$$

Note that, combining the formation of $(y_i' x_i, v y_3)$ with the definition of composition of Σ-cospans, we see that the 2-cell Ω_{1i} is determined by the following Σ-path between Σ-schemes of level 3, where $\dot{\Sigma}$ indicates a canonical Σ-square, and Equation (3.43) and Equation (3.44) are used:

(Ω_{1i})



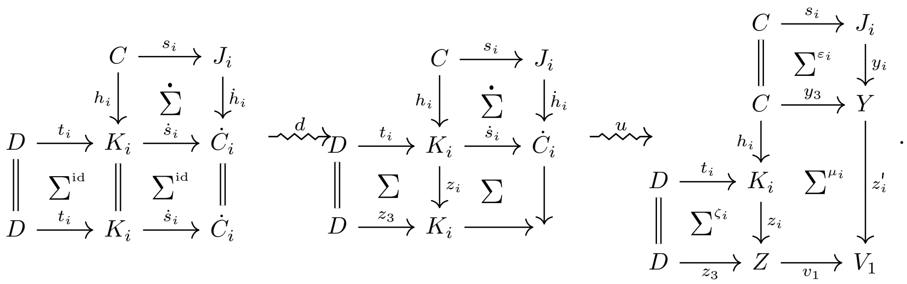
(3.47)

(b2) Concerning $\bar{\gamma} \circ \bar{\beta}$, let

$$\Omega_{2i} \colon \bar{h}_i \circ \bar{g}_i = (\dot{h}_i g_i, \dot{s}_i t_i) \Rightarrow (z_i' y_i g_i, v_1 z_3)$$

be the 2-cell of Ω type obtained via the following Σ-path between Σ-schemes of level 2:

(Ω_{2i})



(3.48)

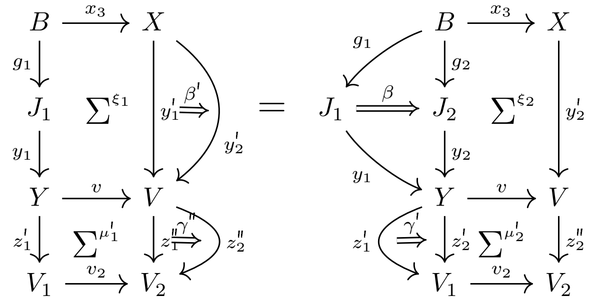
Then $\bar{\gamma} \circ \bar{\beta} = \Omega_{22}^{-1} \cdot [\gamma' \circ \beta, 1_{V_1}, 1_{V_1}] \cdot \Omega_{21}$ where $[\gamma' \circ \beta, 1_{V_1}, 1_{V_1}]$ is represented by

$$\begin{array}{ccccccccccc} B & \xrightarrow{g_1} & J_1 & \xrightarrow{y_1} & Y & \xrightarrow{z_1'} & V_1 & \xleftarrow{v_1} & Z & \xleftarrow{z_3} & D \\ \| & & \Downarrow \beta & & \| & \Downarrow \gamma' & \| & \Sigma & \| & \Sigma & \| \\ B & \xrightarrow{g_2} & J_2 & \xrightarrow{y_2} & Y & \xrightarrow{z_2'} & V_1 & \xleftarrow{v_1} & Z & \xleftarrow{z_3} & D \end{array}$$

Using again the interchange law, we have that

$$(\bar{\gamma} \circ \bar{\beta}) \circ \bar{\alpha} = (\Omega_{22}^{-1} \circ \mathrm{id}_{\bar{f}_2}) \cdot ([\gamma' \circ \beta, 1_{V_1}, 1_{V_1}] \circ \bar{\alpha}) \cdot (\Omega_{21} \circ \mathrm{id}_{\bar{f}_1}). \tag{3.49}$$

In order to obtain $[\gamma' \circ \beta, 1_{V_1}, 1_{V_1}] \circ \bar{\alpha}$, we are going to use Equation (3.41) and Equation (3.43), more precisely, the fact that we have:



Indeed, we obtain again the 2-morphism $(\gamma'' \circ \beta' \circ \alpha, 1_{V_2}, 1_{V_2})$, already appearing in Equation (3.46), more precisely, we have that

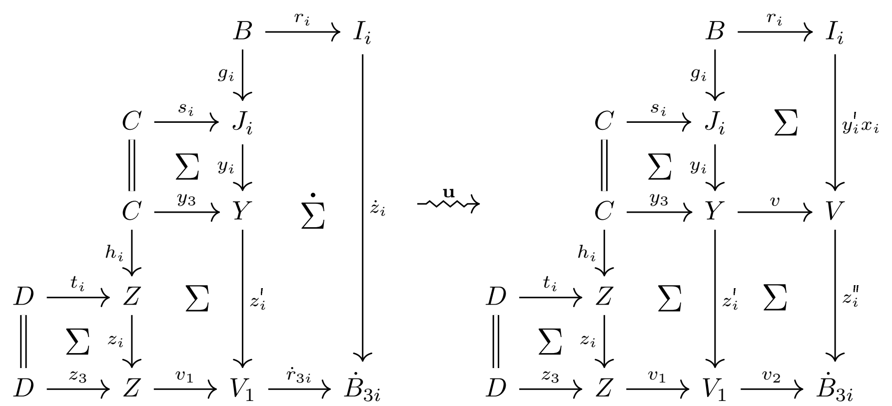
$$[\gamma' \circ \beta, 1_{V_1}, 1_{V_1}] \circ \bar{\alpha} = \Omega_{32}^{-1} \cdot [\gamma'' \circ \beta' \circ \alpha, 1_{V_2}, 1_{V_2}] \cdot \Omega_{31}$$

where

$$\Omega_{3i} \colon (z_i' y_i g_i, v_1 z_3) \circ \bar{f}_i = (\dot{z}_i f_i, \dot{r}_{3i} v_1 z_3) \Rightarrow (z_i'' y_i' x_i f_i, v_2 v_1 z_3)$$

is the basic Ω 2-cell determined by the Σ-step below:

(Ω_{3i})



(3.50)

Thus, using (3.49),

$$(\bar{\gamma} \circ \bar{\beta}) \circ \bar{\alpha} = (\Omega_{22}^{-1} \circ \mathrm{id}_{\bar{f}_2}) \cdot \Omega_{32}^{-1} \cdot [\gamma'' \circ \beta' \circ \alpha, 1_{V_2}, 1_{V_2}] \cdot \Omega_{31} \cdot (\Omega_{21} \circ \mathrm{id}_{\bar{f}_1}) . \qquad (3.51)$$

Put

$$\hat{\Omega}_i = \Omega_{3i} \cdot (\Omega_{2i} \circ \mathrm{id}_{\bar{f}_i})$$

and

$$\tilde{\Omega}_i = \Omega_{1i} \cdot (\mathrm{id}_{\bar{h}_i} \circ \Omega_i) .$$

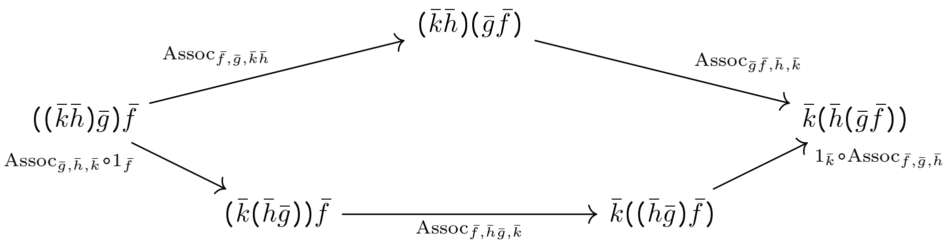
Thus, using Equation (3.46) and Equation (3.51), we see that, in order to prove (3.40), we just need to show that $\mathrm{Assoc}_{\bar{f}_2, \bar{g}_2, \bar{h}_2} \cdot \hat{\Omega}_2^{-1} = \tilde{\Omega}_2^{-1}$ and $\hat{\Omega}_1 = \tilde{\Omega}_1 \cdot \mathrm{Assoc}_{\bar{f}_1, \bar{g}_1, \bar{h}_1}$. That is, we need to prove that

$$\mathrm{Assoc}_{\bar{f}_i, \bar{g}_i, \bar{h}_i} = \tilde{\Omega}_i^{-1} \cdot \hat{\Omega}_i \qquad (i = 1, 2). \qquad (3.52)$$

We know that the first member of this equality is an Ω 2-cell. The 2-cells Ω_{1i} and Ω_{3i} correspond to Σ-paths of interest, namely (3.47) and (3.50). Indeed the first one have a Σ-step of type **d** between Σ-schemes of configuration **dc** and a Σ-step of type $\mathbf{s}_1$ between Σ-schemes of configuration $\mathbf{s}_1$; the second one is just a Σ-step of type **u** between Σ-schemes of configuration **ua**. The 2-cells Ω_i and Ω_{2i} correspond to Σ-steps between Σ-schemes of level 2, each one made of Σ-steps of type d and u, see ((3.42)) and ((3.48)); then, by Proposition A.5 (see also Corollary A.6), $\Omega_{2i} \circ \mathrm{id}_{\bar{f}}$ and $\mathrm{id}_{\bar{h}} \circ \Omega_i$ are Ω 2-cells corresponding to Σ-paths of interest. Since both members of (3.52) correspond to Σ-paths of interest between Σ-schemes of left border $(r_i, g_i, s_i, h_i, t_i, 1)$, they coincide. □

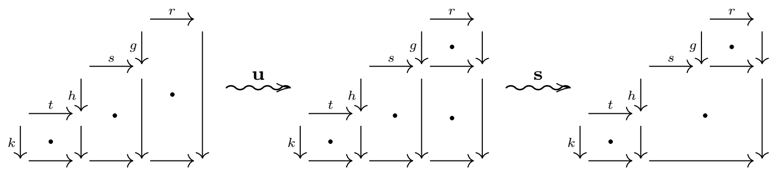
Proposition 3.32. *The isomorphisms of Definition 3.26 fulfil the Pentagon Axiom.*

Proof. Let $A \xrightarrow{\bar f} B \xrightarrow{\bar g} C \xrightarrow{\bar h} D \xrightarrow{\bar k} E$ be composable 1-cells in $\mathcal{X}[\Sigma_*]$, where $\bar f = (f, r)$, $\bar g = (g, s)$, $\bar h = (h, t)$ and $\bar k = (k, u)$. We want to prove the commutativity of the diagram

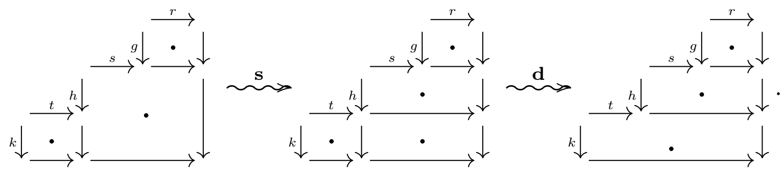


in $\mathcal{X}[\Sigma_*](A, E)$.

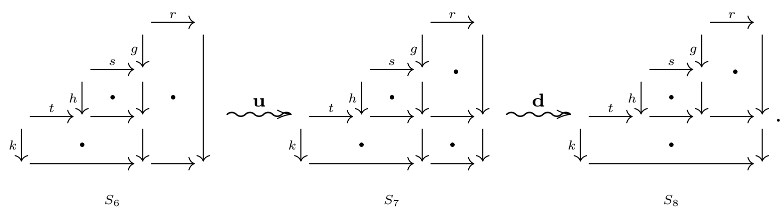
Concerning the top of the pentagon diagram, using the definitions of composition of Σ-cospans and of associator, we see that $\mathrm{Assoc}_{\bar f, \bar g, \bar k \bar h}$ is an Ω 2-cell corresponding to a Σ-path of interest of the form



and $\mathrm{Assoc}_{\bar g \bar f, \bar h, \bar k}$ is an Ω 2-cell corresponding to a Σ-path of interest of the form

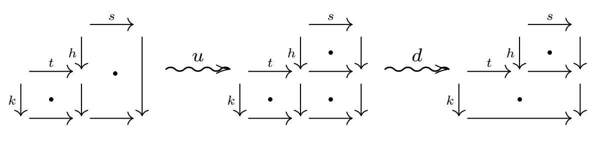


The middle map in the bottom line of the pentagon, $\mathrm{Assoc}_{\bar f, \bar h \bar g, \bar k}$, is the Ω 2-cell obtained via the Σ-path of interest

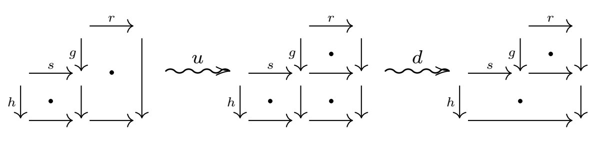


(3.53)

The 2-cells $\mathrm{Assoc}_{\bar g, \bar h, \bar k}$ and $\mathrm{Assoc}_{\bar f, \bar g, \bar h}$ correspond, respectively, to the Σ-paths of level 2



and



Hence, by Proposition A.5, $\mathrm{Assoc}_{\bar g, \bar h, \bar k} \circ 1_{\bar f}$ and $1_{\bar k} \circ \mathrm{Assoc}_{\bar f, \bar g, \bar h}$ are Ω 2-cells corresponding to Σ-paths of interest between Σ-schemes of left border (r, g, s, t, k).

Since the top and the bottom of the pentagon correspond to two Σ-paths of interest with the same starting and ending, those Σ-paths are equivalent — that is, the pentagon is commutative in the category $\mathcal{X}[\Sigma_*](A, E)$. □

Since the unitors are just identities, this finishes the proof of the main result, which was proved in the present section.

Theorem 3.33. *$\mathcal{X}[\Sigma_*]$ is a bicategory.*

4 The universal property

We want to show that $\mathcal{X}[\Sigma_*]$ is the universal bicategory which turns Σ-morphisms into laris and Σ-squares to Beck–Chevalley squares. Let us start by showing that $\mathcal{X}[\Sigma_*]$ at least does do this.

Definition 4.1. We define a pseudofunctor $P_\Sigma \colon \mathcal{X} \to \mathcal{X}[\Sigma_*]$ such that:

- on objects, $P_\Sigma(X) = X$,
- on morphisms, $P_\Sigma(f) = (f, 1)$,
- on 2-morphisms, $P_\Sigma(\alpha) = [\alpha, 1, 1, 1, \mathrm{id}, \mathrm{id}]$,
- the unitors $\iota^P_X \colon 1_{P_\Sigma(X)} \to P_\Sigma(1_X)$ are identities,
- the compositors $\gamma^P_{g,f} \colon P_\Sigma(g) \circ P_\Sigma(f) \to P_\Sigma(gf)$ are also identities.

Lemma 4.2. *P_Σ as defined above is indeed a pseudofunctor.*

Proof. It is clear that P_Σ preserves identity 2-cells. For vertical composition of 2-cells, note that to compute the composite $P_\Sigma(\beta) \cdot P_\Sigma(\alpha)$ as in Definition 3.5 we may take the Σ-squares φ_x, φ_y to be vertical identity squares and the isomorphism γ to be an identity 2-cell. Then it is easy to see that $P_\Sigma(\beta) \cdot P_\Sigma(\alpha) = P_\Sigma(\beta \cdot \alpha)$ and so P_Σ preserves vertical composition.

It is also clear that P_Σ preserves identity 1-cells and so the unitors are well-defined. For morphisms $g \colon A \to B$ and $f \colon B \to C$ in $\mathcal{X}$, the composite $P_\Sigma(f) \circ P_\Sigma(g)$ is found using the canonical Σ-square

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xleftarrow{1_B} & B \\ & & {\scriptstyle g}\downarrow & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle g} \\ & & C & \xleftarrow[1_C]{} & C & \xleftarrow[1_C]{} & C, \end{array}$$

and so give $P_\Sigma(gf)$ ensuring the compositors are well-defined.

The naturality condition for the compositors requires that $P_\Sigma(\beta) \circ P_\Sigma(\alpha) = P_\Sigma(\beta \circ \alpha)$. Computing the left-hand side as in Definition 3.24 we may take the necessary Σ-squares in Equation (3.19) to be Σ^{id} and $\beta' = \beta$. The desired equality follows.

Finally, we must show the coherence conditions. Coherence for the compositor (the 'associativity' condition) reduces to requiring that the Ω 2-cell

$$\mathrm{assoc}_{P_\Sigma(f), P_\Sigma(g), P_\Sigma(h)} \colon (P_\Sigma(h) \circ P_\Sigma(g)) \circ P_\Sigma(f) \Rightarrow P_\Sigma(h) \circ (P_\Sigma(g) \circ P_\Sigma(f))$$

is the identity. This is immediate, since, by replacing the morphisms r and s with identities in the diagrams (1), (2) and (3) of Definition 3.26, the Ω 2-cell which corresponds to the Σ-path $(1) \overset{\mathbf{u}}{\rightsquigarrow} (2) \overset{\mathbf{d}}{\rightsquigarrow} (3)$ have coincident domain and codomain and is clearly the identity. The left and right unit conditions are automatic since all the maps in the diagram are identities. □

Proposition 4.3. *The pseudofunctor P_Σ sends 1-cells in Σ to laris and Σ-squares to Beck–Chevalley squares.*

Proof. Consider $s\colon A \to B$ in Σ. We first show $P_\Sigma(s) = (s, 1_B)$ is a lari. We claim its right adjoint is $(1_B, s)$. Note that $(1_B, s) \circ (s, 1_B) = (s, s)$. Put

$$\bar{\eta} = [\mathrm{id}_s, s, 1_B, s, \mathrm{id}, \mathrm{id}].$$

This is indeed the basic Ω 2-cell given by the Σ-path $\begin{array}{ccc} \cdot & \xrightarrow{1} & \cdot \\ {\scriptstyle 1}\downarrow & \dot{\Sigma} & \downarrow{\scriptstyle 1} \\ \cdot & \xrightarrow[1]{} & \cdot \end{array} \rightsquigarrow \begin{array}{ccc} \cdot & \xrightarrow{1} & \cdot \\ {\scriptstyle 1}\downarrow & \Sigma & \downarrow{\scriptstyle s} \\ \cdot & \xrightarrow[s]{} & \cdot \end{array}$. In particular, note that $\bar{\eta}$ is invertible.

On the other hand, we find $(s, 1_B) \circ (1_B, s)$ using

$$\begin{array}{ccccc} B & \xrightarrow{1_B} & B & \xleftarrow{s} & A \\ & & {\scriptstyle \dot{s}_1}\downarrow & \Sigma^{\dot{\alpha}} & \downarrow{\scriptstyle s} \\ & & \dot{A} & \xleftarrow[\dot{s}_2]{} & B & \xleftarrow[1_B]{} & B, \end{array}$$

to give $(\dot{s}_1, \dot{s}_2)$. Applying Equi-insertion to the above Σ-square and the 2-cell $\dot{\alpha}^{-1}\colon \dot{s}_1 s \to \dot{s}_2 s$ we obtain a morphism $q\colon \dot{A} \to Q$ and a 2-cell $\varepsilon\colon q\dot{s}_1 \to q\dot{s}_2$ such that $q\dot{\alpha}^{-1} = \varepsilon s$ and the following is a Σ-square.

$$\begin{array}{ccc} B & \xrightarrow{\dot{s}_2} & \dot{A} \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle q} \\ B & \xrightarrow[q\dot{s}_2]{} & Q \end{array}$$

We can now form a 2-morphism $\bar{\varepsilon}\colon (\dot{s_1}, \dot{s}_2) \to (\mathrm{id}_B, \mathrm{id}_B)$ (using Identity for the lower Σ-square).

$$\begin{array}{ccccc} B & \xrightarrow{\dot{s_1}} & \dot{A} & \xleftarrow{\dot{s_2}} & B \\ \| & \overset{\varepsilon}{\swarrow} & {\scriptstyle q}\downarrow & \Sigma^{\mathrm{id}} & \| \\ \| & & Q & \xleftarrow{q\dot{s_2}} & B \\ \| & & {\scriptstyle q\dot{s_2}}\uparrow & \Sigma^{\mathrm{id}} & \| \\ B & \xrightarrow[1_B]{} & B & \xleftarrow[1_B]{} & B \end{array} \tag{4.1}$$

Using the notations $\bar{s} = (s, 1)$ and $s_* = (1, s)$, we now want to show the triangle identites: $\mathrm{id}_{s_*} = (s_* \circ \bar{\varepsilon}) \cdot (\bar{\eta} \circ s_*)$ and $\mathrm{id}_s = (\bar{\varepsilon} \circ \bar{s}) \cdot (\bar{s} \circ \bar{\eta})$. Since $\bar{\eta}$ is an Ω 2-cell and using the coherence theorems for Ω 2-cells, it suffices to show that $s_* \circ \bar{\varepsilon}$ and $\bar{\varepsilon} \circ s$ are (equivalent to) Ω 2-cells. Moreover, we can ignore the Ω 2-cells in the definition of horizontal composition and just consider the central 2-cell in Definition 3.24.

For $(1, s) \circ \bar{\varepsilon}$ we arrive at the 2-cell

$$\begin{array}{ccccccccccc} B & \xrightarrow{\dot{s}_1} & \dot{A} & \xrightarrow{q} & Q & \xleftarrow{q} & \dot{A} & \xleftarrow{\dot{s}_2} & B & \xleftarrow{s} & A \\ \| & & \overset{\varepsilon}{\swarrow} & & \| & & & \Sigma^{\mathrm{id}} & & & \| \\ \| & & & & Q & & & \xleftarrow{q\dot{s}_2 s} & & & A \\ \| & & & & \| & & & \Sigma^{\mathrm{id}} & & & \| \\ B & \xrightarrow[\dot{s}_2]{} & \dot{A} & \xrightarrow[q]{} & Q & \xleftarrow[q]{} & \dot{A} & \xleftarrow[\dot{s}_2]{} & B & \xleftarrow[s]{} & A \end{array} \tag{4.2}$$

from the equality

$$\begin{array}{ccc} B & \xrightarrow{q\dot{s}_2} & Q \\ \| & & \| \\ B & \Sigma^{\mathrm{id}} & \| \overset{\mathrm{id}}{\Rightarrow} \mathrm{id}_B \\ \| & & \| \\ B & \xrightarrow[q\dot{s}_2]{} & Q \end{array} = \begin{array}{ccc} B & \xrightarrow{q\dot{s}_2} & Q \\ \| & & \| \\ \mathrm{id}_B \overset{\mathrm{id}}{\Rightarrow} B & \Sigma^{\mathrm{id}} & \| \\ \| & & \| \\ B & \xrightarrow[q\dot{s}_2]{} & Q. \end{array}$$

For $\bar{\varepsilon} \circ (s, 1)$ we have the 2-cell

$$\begin{array}{ccccccccc}
A & \xrightarrow{s} & B & \xrightarrow{q\dot{s}_1} & Q & \xleftarrow{q} & \dot{A} & \xleftarrow{\dot{s}_2} & A \\
\| & & \| & & \| & & \Sigma^{\text{id}} & & \| \\
\| & \overset{\text{id}}{\swarrow} & \| & \overset{\varepsilon}{\swarrow} & Q & & \xleftarrow{q\dot{s}_2} & & A \\
\| & & \| & & \| & & \Sigma^{\text{id}} & & \| \\
A & \xrightarrow[s]{} & B & \xrightarrow[q\dot{s}_2]{} & Q & \xleftarrow[q]{} & \dot{A} & \xleftarrow[\dot{s}_2]{} & A
\end{array} \tag{4.3}$$

from the equality

$$\begin{array}{ccc}
B & = & B \\
{\scriptstyle \dot{s}_1}\downarrow & & \\
\dot{A} & \Sigma^{\text{id}} & \downarrow q\dot{s}_1 \overset{\varepsilon}{\Rightarrow} q\dot{s}_2 \\
{\scriptstyle q}\downarrow & & \\
Q & = & Q
\end{array}
\quad = \quad
\begin{array}{cccc}
 & B & = & B \\
\overset{\dot{s}_1}{\swarrow} & \| & & \\
\dot{A} \overset{\varepsilon}{\Longrightarrow} & B & \Sigma^{\text{id}} & \downarrow q\dot{s}_2 \\
\underset{q}{\searrow} & \downarrow q\dot{s}_2 & & \\
 & Q & = & Q.
\end{array}$$

Now consider the Σ-squares

$$\begin{array}{ccc}
A & \xrightarrow{s} & B \\
{\scriptstyle s}\downarrow & \Sigma^{\dot{\alpha}} & \downarrow{\scriptstyle \dot{s}_1} \\
B & \xrightarrow{\dot{s}_2} & \dot{A} \\
\| & \Sigma^{\text{id}} & \downarrow{\scriptstyle q} \\
B & \xrightarrow[q\dot{s}_2]{} & Q
\end{array} \tag{4.4}$$

and apply both parts of rule 3 from Proposition 2.8 to obtain the Σ-squares

$$\begin{array}{ccc}
A & \xrightarrow{s} & B \\
\| & \Sigma^{q\dot{\alpha}} & \downarrow{\scriptstyle q\dot{s}_1} \\
A & \xrightarrow[q\dot{s}_2 s]{} & Q
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A & \xrightarrow{s} & B \\
\| & \Sigma^{\text{id}} & \downarrow{\scriptstyle q\dot{s}_2} \\
A & \xrightarrow[q\dot{s}_2 s]{} & Q.
\end{array}$$

To obtain an Ω 2-cell from these squares we use the equality

$$\begin{array}{ccc}
A & \xrightarrow{s} & B \\
\| & \Sigma^{q\dot{\alpha}} & \downarrow{\scriptstyle q\dot{s}_1} \\
A & \xrightarrow[q\dot{s}_2 s]{} & Q \overset{\varepsilon}{\Rightarrow} q\dot{s}_2 \\
\| & \Sigma^{\text{id}} & \| \\
A & \xrightarrow[q\dot{s}_2 s]{} & Q
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{s} & B \\
\| & \Sigma^{\text{id}} & \downarrow{\scriptstyle q\dot{s}_2} \\
A & \xrightarrow[q\dot{s}_2 s]{} & Q \\
\| & \Sigma^{\text{id}} & \| \\
A & \xrightarrow[q\dot{s}_2 s]{} & Q.
\end{array}$$

The Ω 2-cell so obtained is then precisely 2-cell (4.2) above for $(1, s) \circ \bar{\varepsilon}$.

For $\bar{\varepsilon} \circ (s, 1)$, a similar approach works. We first compose Σ-square (4.4) with the Σ-square from the Identity axiom, giving

$$\begin{array}{ccc}
A & = & A \\
\| & \Sigma^{\text{id}} & \downarrow{\scriptstyle s} \\
A & \xrightarrow{s} & B \\
{\scriptstyle s}\downarrow & \Sigma^{\dot{\alpha}} & \downarrow{\scriptstyle \dot{s}_1} \\
B & \xrightarrow{\dot{s}_2} & \dot{A} \\
\| & \Sigma^{\text{id}} & \downarrow{\scriptstyle q} \\
B & \xrightarrow[q\dot{s}_2]{} & Q.
\end{array}$$

We also compose Σ-squares from the Identity and Horizontal Repletion axioms to give

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle s}\downarrow & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle s} \\ B & = & B \\ \| & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle q\dot{s}_2} \\ B & \xrightarrow[q\dot{s}_2]{} & Q. \end{array}$$

Using the equality

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle s}\downarrow & \Sigma^{q\dot{\alpha}} & \downarrow{\scriptstyle q\dot{s}_1 s} \\ B & \xrightarrow[q\dot{s}_2]{} & Q \overset{\varepsilon^s}{\Rightarrow} \\ \| & \Sigma^{\mathrm{id}} & \| \\ B & \xrightarrow[q\dot{s}_2]{} & Q \end{array} \; {\scriptstyle q\dot{s}_2 s} \quad = \quad \begin{array}{ccc} A & = & A \\ {\scriptstyle s}\downarrow & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle q\dot{s}_2 s} \\ B & \xrightarrow[q\dot{s}_2]{} & Q \\ \| & \Sigma^{\mathrm{id}} & \| \\ B & \xrightarrow[q\dot{s}_2]{} & Q \end{array}$$

we find an Ω 2-cell that coincides with 2-cell (4.3). Thus, the triangle identities hold and P_Σ indeed sends 1-cells in Σ to laris.

We now show P_Σ sends Σ-squares to Beck–Chevalley squares. Suppose we have a Σ-square

$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ {\scriptstyle f}\downarrow & \Sigma^{\delta} & \downarrow{\scriptstyle g} \\ C & \xrightarrow[t]{} & D. \end{array}$$

Then applying P_Σ we have $P_\Sigma(\delta)\colon (t,1)\circ(f,1) \to (g,1)\circ(s,1)$. The mate of this is a morphism $(f,1)\circ(1,s) \to (1,t)\circ(1,g)$ given by the composite:

$$\begin{aligned} &((g,t)\circ\varepsilon^s)\cdot \mathrm{Assoc}_{(g,t),(s,1),(1,s)} \cdot (\mathrm{Assoc}^{-1}_{(1,t),(g,1),(1,s)} \circ (1,s)) \\ &\cdot (((1,t)\circ P_\Sigma(\delta))\circ(1,s)) \cdot (\mathrm{Assoc}_{(1,t),(t,1),(f,1)} \circ (1,s)) \cdot ((\eta^t \circ f)\circ(1,s)). \end{aligned}$$

The Beck–Chevalley condition says that this composite is an isomorphism. All of the factors except $(g,t)\circ\varepsilon^s$ are always isomorphisms. We now show that $(g,t)\circ\varepsilon^s$ is also an isomorphism.

From Σ^δ and the canonical Σ-square

$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ {\scriptstyle f}\downarrow & \Sigma^{\ddot{\alpha}} & \downarrow{\scriptstyle \dot{f}} \\ B & \xrightarrow[\dot{s}]{} & \ddot{A} \end{array}$$

we have an Ω 2-cell between (g,t) and $(\dot{f},\dot{s})$. Note that $(\dot{f},\dot{s})$ is the composite $(f,1)\circ(1,s)$. Hence $(g,t)\cong(f,1)\circ(1,s)$ and it suffices to show $(f,1)\circ((1,s)\circ\overline{\varepsilon})$ is an isomorphism. But $(1,s)\circ\overline{\varepsilon}$ is an isomorphism by one of the triangle identities and so we are done. □

Theorem 4.4. *Let $\mathcal{X}$ be a 2-category admitting a left calculus of lax fractions for Σ. The pseudofunctor $P_\Sigma\colon \mathcal{X}\to\mathcal{X}[\Sigma_*]$ is the universal (strictly unital) pseudofunctor that satisfies the conditions of Proposition 4.3. More precisely, we have*

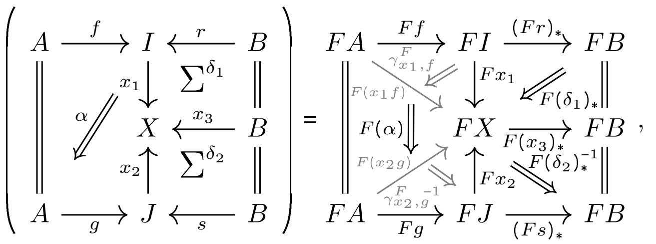
a) *If $F\colon\mathcal{X}\to\mathcal{C}$ is a pseudofunctor sending Σ-morphisms to laris and Σ-squares to BC squares, then there is a pseudofunctor $H\colon\mathcal{X}[\Sigma_*]\to\mathcal{C}$ such that $F\simeq H\circ P_\Sigma$.*

b) *If $H,H'\colon\mathcal{X}[\Sigma_*]\to\mathcal{C}$ are pseudofunctors and $\xi\colon H\circ P_\Sigma\to H'\circ P_\Sigma$ is a pseudonatural transformation for which the pseudo-naturality squares for $r\colon B\to I$ in $\mathcal{X}$ are BC squares whenever $r\in\Sigma$, then there is a pseudonatural transformation $\upsilon\colon H\to H'$ such that $\xi\cong\upsilon\circ P_\Sigma$.*

c) *If* $H, H' \colon \mathcal{X}[\Sigma_*] \to \mathcal{C}$ *are pseudofunctors,* $\upsilon, \upsilon' \colon H \to H'$ *are pseudonatural transformations, and* $\aleph \colon \upsilon \circ P_\Sigma \to \upsilon' \circ P_\Sigma$ *is a modification, then there is a unique modification* $\beth \colon \upsilon \to \upsilon'$ *such that* $\aleph = \beth \circ P_\Sigma$.

Remark 4.5. We also remark that if $H \colon \mathcal{X}[\Sigma_*] \to \mathcal{C}$ is a pseudofunctor, then $H \circ P_\Sigma$ is a pseudofunctor which sends Σ-morphisms to laris and Σ-squares to BC squares (since being a lari or a BC square is preserved by pseudofunctors). Moroever, if $H, H' \colon \mathcal{X}[\Sigma_*] \to \mathcal{C}$ are pseudofunctors and $\upsilon \colon H \to H'$ is a pseudonatural transformation, then the pseudonaturality squares for $r \in \Sigma$ in $\upsilon \circ P_\Sigma$ are BC squares (since for any pseudonatural transformation υ, the component at the adjoint of a 1-morphism is the inverse of the mate of the component at the 1-morphism itself).

Proof of Theorem 4.4. Let $\mathcal{C}$ be a bicategory and let $F \colon \mathcal{X} \to \mathcal{C}$ be a pseudofunctor that sends 1-cells in Σ to laris in $\mathcal{C}$ and Σ-squares to BC squares. We may assume the unitors of $\mathcal{C}$ are identities and that F is strictly unitary. For each lari f of $\mathcal{C}$ we choose a right adjoint f_* such that $(1_X)_* = 1_X$ for every identity 1-cell. There is a pseudofunctor $H \colon \mathcal{X}[\Sigma_*] \to \mathcal{C}$ defined as follows.

- On objects, $H(X) = F(X)$,
- On morphisms, $H((f, r)) = (Fr)_*(Ff)$,
- On 2-morphisms, H 

  where $F(\delta_1)_*$ and $F(\delta_2)_*$ denote the mate of $F(\delta_1)$ and $F(\delta_2)$, respectively.
- The unitors $\iota^H_X \colon 1_{H(X)} \to H(1_X)$ are identites,
- The compositors $\gamma^H_{(g,s),(f,r)} \colon H((g, s)) \circ H((f, r)) \to H((g, s) \circ (f, r))$ are given by the composite

$$
\begin{aligned}
&(F(s)_*F(g)) \circ (F(r)_*F(f)) \xrightarrow{\alpha^{\mathcal{C}\,-1}_{F(s)_*F(g),F(r)_*,F(f)}} ((F(s)_*F(g)) \circ F(r)_*) \circ F(f) \xrightarrow{\alpha^{\mathcal{C}}_{F(s)_*,F(g),F(r)_*} \circ F(f)} \\
&(F(s)_* \circ (F(g)F(r)_*)) \circ F(f) \xrightarrow{(F(s)_* \circ F(\dot\beta)_*) \circ F(f)} (F(s)_* \circ (F(\dot r)_*F(\dot g))) \circ F(f) \xrightarrow{\alpha^{\mathcal{C}\,-1}_{F(s)_*,F(\dot r),F(\dot g)_*} \circ F(f)} \\
&((F(s)_*F(\dot r)_*) \circ F(\dot g))) \circ F(f) \xrightarrow{\alpha^{\mathcal{C}}_{F(s)_*F(\dot r),F(\dot g)_*,F(f)}} (F(s)_*F(\dot r)_*) \circ (F(\dot g)F(f)) \xrightarrow{(F(s)_*F(\dot r)_*) \circ \gamma^F_{\dot g,f}} \\
&(F(s)_*F(\dot r)_*) \circ F(\dot g f) \xrightarrow{\sigma \circ F(\dot g f)} (F(\dot r)F(s))_* \circ F(\dot g f) \xrightarrow{(\gamma^{F\,-1}_{\dot r,s})_* \circ F(\dot g f)} F(\dot r s)_*F(\dot g f)
\end{aligned}
$$

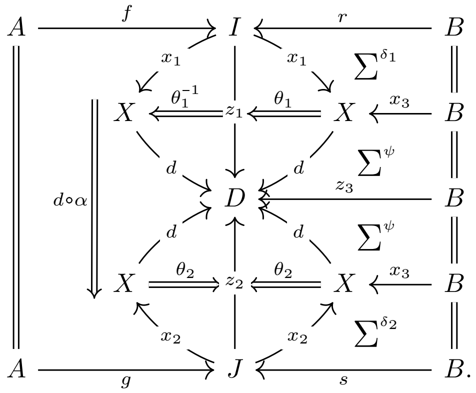
  where $\dot g$, $\dot r$ and $\dot\beta$ are given by the composite $(g, s) \circ (f, r) = (\dot g f, \dot r s)$ from

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & I & \xleftarrow{r} & B & & \\
 & & {\scriptstyle \dot g}\downarrow & \Sigma^{\dot\beta} & \downarrow{\scriptstyle g} & & \\
 & & \dot B & \xleftarrow[\dot r]{} & J & \xleftarrow[s]{} & C,
\end{array}
$$

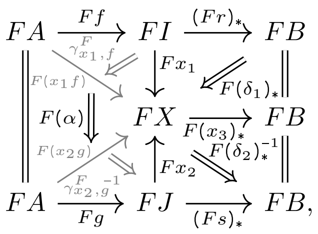
  and where $\alpha^{\mathcal{C}}_{\bullet,\bullet,\bullet}$ denotes the associator for $\mathcal{C}$, $\gamma^F_{\bullet,\bullet}$ denotes the compositor for F, $\sigma \colon F(s)_*F(\dot r)_* \to (F(\dot r)F(s))_*$ is the canonical isomorphism given by composition of adjoints and $(\gamma^{F\,-1}_{\dot r,s})_* \colon (F(\dot r)F(s))_* \to F(\dot r s)_*$ is the mate of the inverse of the compositor.

We show this indeed is a pseudofunctor. Here it will be useful to make use of string diagrams.

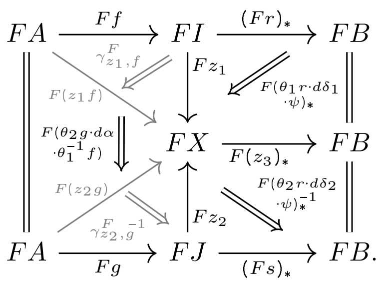
First we must show that H is well-defined on 2-morphisms. Let us consider a Σ-extension of the 2-morphism $(\alpha, x_1, x_2, x_3, \delta_1, \delta_2)$:



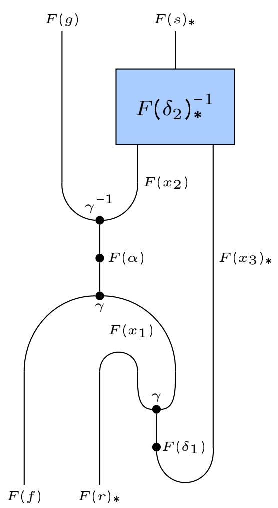
Applying H to $(\alpha, x_1, x_2, x_3, \delta_1, \delta_2)$ gives



while applying H to the above Σ-extension gives

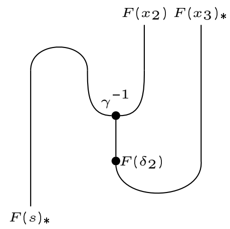


To show these are equal we express both in terms of string diagrams. The former gives



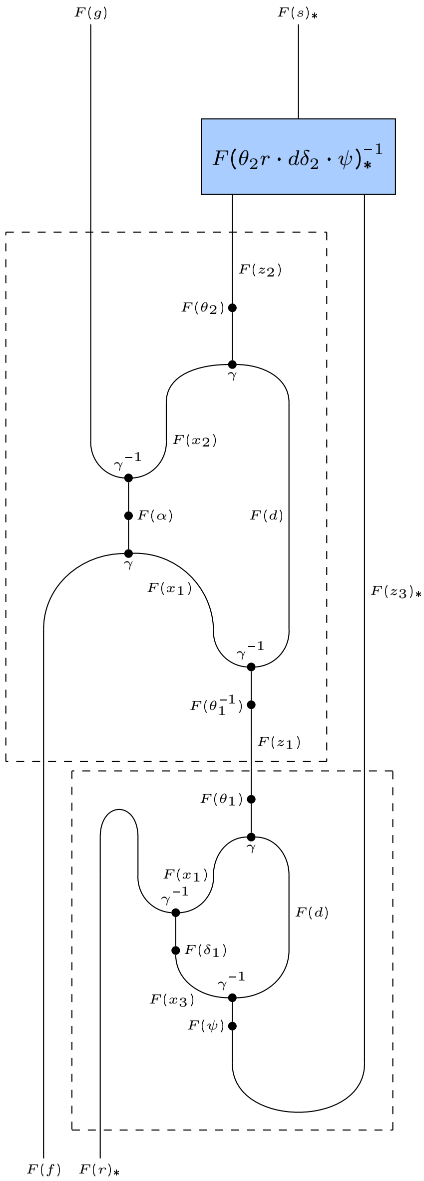
(4.5)

where the blue box depicts the inverse of the 2-morphism represented the following diagram.



(4.6)

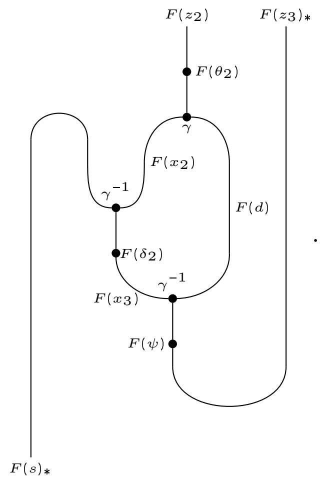
The latter gives



(4.7)

(Here the three boxes represent the three rectangles in the diagram for H applied to the

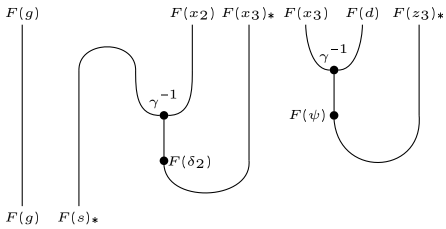
Σ-extension.) Also, $F(\theta_2 r \cdot d\delta_2 \cdot \psi)_*$ is represented by



. (4.8)

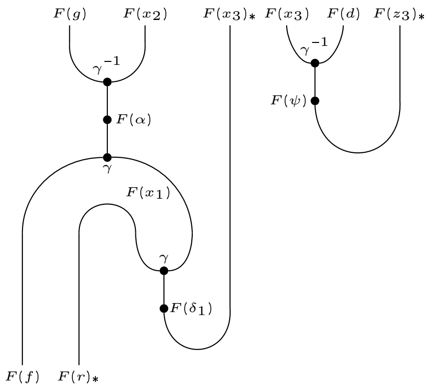
In the string diagram 4.7, $F(\theta_1)$ cancels with $F(\theta_1^{-1})$, and $F(\theta_2)$ (together with its nearby compositor) cancels with the $F(\theta_2)^{-1}$ (and its compositor) from the inverse of $F(\theta_2 r \cdot d\delta_2 \cdot \psi)_*$.

Now we compose Equation (4.7) and Equation (4.5) on the top with

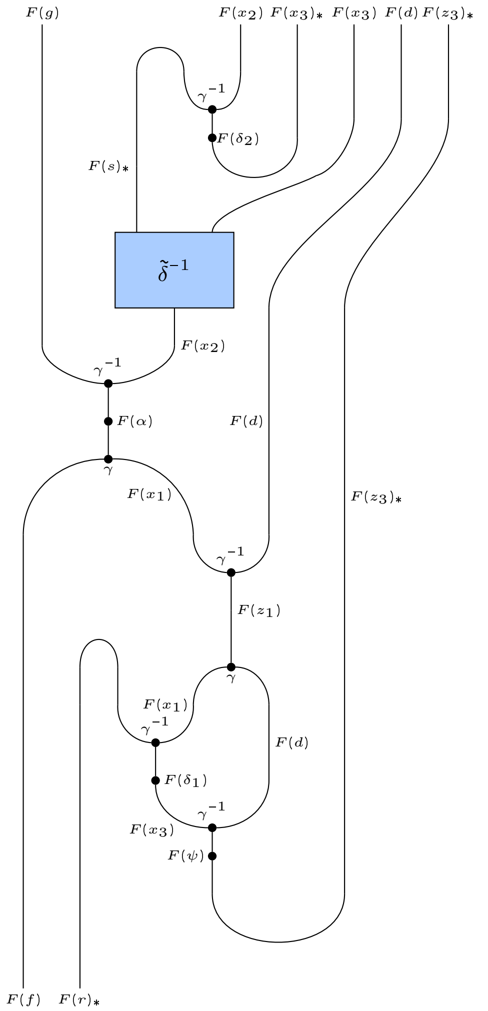


(4.9)

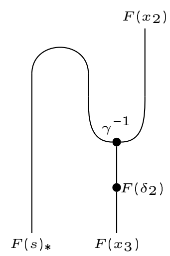
(which is an isomorphism since $\eta^{F(z_3)}$ is and $F(\delta_2)$ forms a BC square). Part of this cancels with the blue box in Equation (4.5) and another part partially cancels with the blue box in Equation (4.7). So what was Equation (4.5) becomes



and what was Equation (4.7) yields



where $\tilde{\delta}^{-1}$ denotes the inverse of



Observe that if we compose both of these with $\varepsilon^{F(x_3)}$ (whiskered with the appropriate morphisms) on the top and apply one of the triangle identities we arrive at the same result in each case. While $\varepsilon^{F(x_3)}$ is not an isomorphism, its resuling composition with Equation (4.9) is one by the BC condition for $F(\delta_2)$ and the form of the resulting inverse of its mate. So we have proved the equality of the two original 2-cells, and hence H is well-defined on 2-cells.

One can then show that H preserves vertical composition of 2-cells and that the coherence conditions hold. It is then easy to see that $F = H \circ P_\Sigma$. We omit these proofs from the current version of this document.

Now we show the '2-dimensional' universality condition (b). Assume $H, H' \colon \mathcal{X}[\Sigma_*] \to \mathcal{C}$ and that $\xi \colon H \circ P_\Sigma \to H' \circ P_\Sigma$ is a pseudonatural transformation. We define $\upsilon \colon H \to H'$ by

- $\upsilon_X = \xi_X$ for objects $X \in \mathcal{X}$,

- $\upsilon_{(f,r)}$ for $(f,r) \in \mathcal{X}[\Sigma_*]$ is given by the composite

$$\begin{array}{ccccc} H(A) & \xrightarrow{H(f)} & H(I) & \xrightarrow{H(r)_*} & H(B) \\ {\scriptstyle \xi_A}\downarrow & \nearrow\!\!\!\nearrow {\scriptstyle \xi_f} & \downarrow{\scriptstyle \xi_I} & \nearrow\!\!\!\nearrow {\scriptstyle (\xi_r)_*^{-1}} & \downarrow{\scriptstyle \xi_B} \\ H'(A) & \xrightarrow[H'(f)]{} & H'(I) & \xrightarrow[H'(r)_*]{} & H'(B) \end{array}$$

  where we write $H(f)$ for $H((f,\mathrm{id})) = H(P_\Sigma(f))$ and where $(\xi_r)_*^{-1}$ denotes the inverse of the mate of ξ_r.

As before we omit from the current version the proofs that this is indeed a pseudonatural transformation and that $\xi = \upsilon \circ P_\Sigma$.

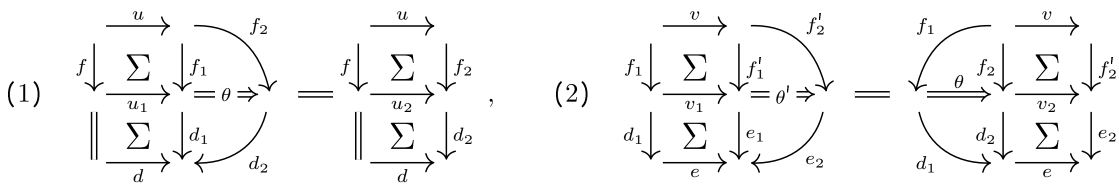
Finally, for the '3-dimensional' universality condition (c). For data as in the statement of the theorem we simply set $\beth_X = \aleph_X$. Yet again, we omit from this version the (easy) proof that this is a modifcation. It is also clearly the only modifcation $\beth$ such that $\aleph = \beth \circ P_\Sigma$.
$\square$

A On Σ-paths and Ω 2-cells

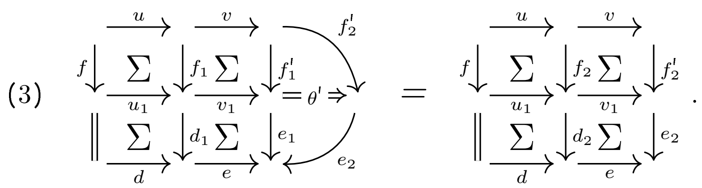
This appendix completes Section 3.2 by presenting the proofs of Proposition 3.14 (here Corollary A.4), Proposition 3.21 (here included in Proposition A.5), and Proposition 3.20 (here Proposition A.7).

Lemma A.1 and Remark A.2 are going to have a role in the proof of Proposition A.3, which states a fundamental property needed for the rest of this section.

Lemma A.1. *Given* $\begin{array}{ccccc} \cdot & \xrightarrow{u} & \cdot & \xrightarrow{v} & \cdot \\ {\scriptstyle f}\downarrow & \Sigma & \downarrow{\scriptstyle f_i} & \Sigma & \downarrow{\scriptstyle f_i'} \\ \cdot & \xrightarrow[u_i]{} & \cdot & \xrightarrow[v_i]{} & \cdot \end{array}$, $i = 1, 2$, *there are* Σ*-squares and invertible 2-cells* θ *and* θ' *such that*

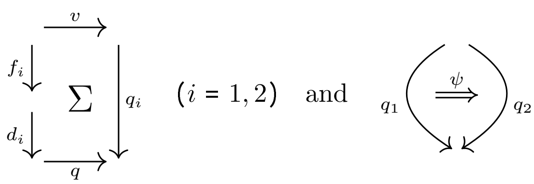


and, consequently, also

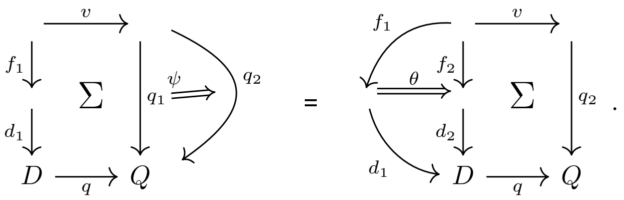


Proof. (1) It is just Rule 4' of Proposition 2.8.

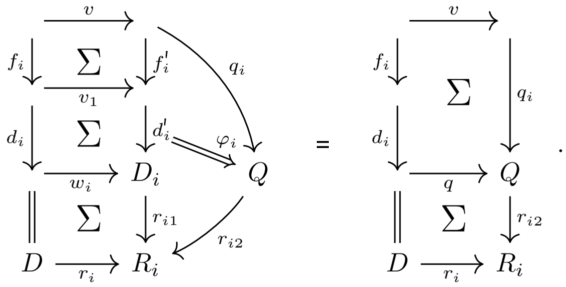
(2) By Rule 6 of the same proposition, we obtain



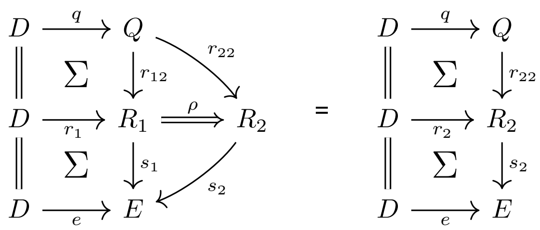
such that



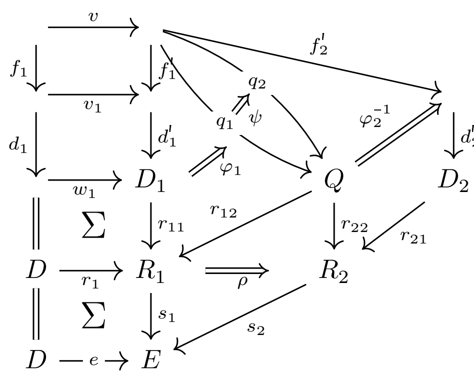
On the other hand, for every $i = 1, 2$, by applying Square followed by Rule 4', we get



Using again Rule 4', we obtain

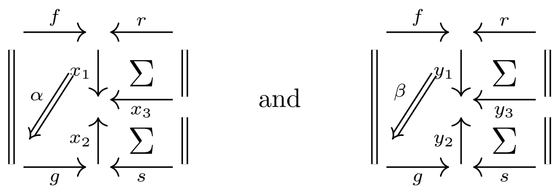


with ρ invertible. Thus we have the following pasting diagram:



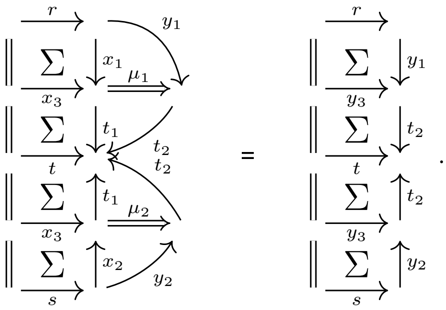
Put $e_i = s_i r_{i1} d_i'$ and $\theta' = (s_2 \circ \varphi_2^{-1}) \cdot (\rho \circ \psi) \cdot (s_1 \circ \varphi_1)$. It is easy to see that in this way we obtain the desired situation (2). □

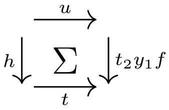
Remark A.2. Given two 2-morphisms

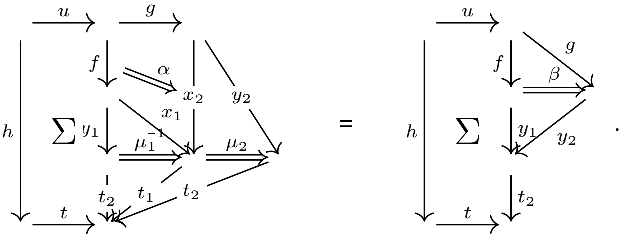


(A.1)

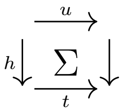
apply Rule 4 to obtain μ_1 and μ_2 invertible such that



Assume that there is a Σ-square  such that



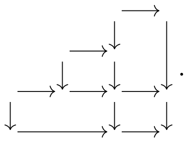
Then the 2-morphisms of (A.1) are $\approx$-equivalent.

Indeed, applying Equification to the Σ-square  together with the 2-cells $(\mu_2 \circ g) \cdot (t_1 \circ \alpha) \cdot (\mu_1^{-1} \circ f)$ and $t_2 \circ \beta$, we obtain a 2-morphism which is a Σ-extension of both the two 2-morphisms of (A.1).

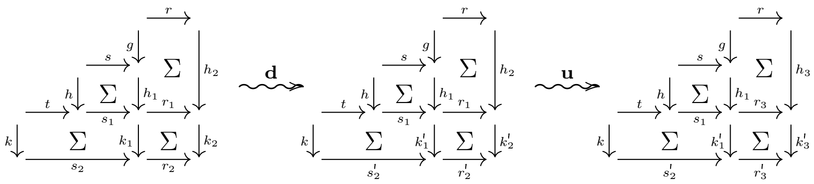
Proposition A.3. *For Σ-schemes S_1, S_2, S_3 and S_4 of level 3 with common left border, every two Σ-paths of the form $S_1 \overset{i}{\rightsquigarrow} S_3 \overset{j}{\rightsquigarrow} S_2$ and $S_1 \overset{j}{\rightsquigarrow} S_4 \overset{i}{\rightsquigarrow} S_2$, where $i, j \in \{\mathbf{d}, \mathbf{u}, \mathbf{s}, \mathbf{d}_1, \mathbf{s}_1\}$, are equivalent.*

Proof. The cases $i = j$, $\{i, j\} = \{\mathbf{d}, \mathbf{d}_1\}$ and $\{i, j\} = \{\mathbf{s}, \mathbf{s}_1\}$ follow from Lemma 3.12.

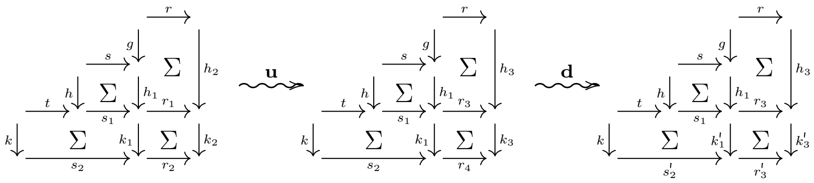
Case $i = \mathbf{d}$ and $j = \mathbf{u}$. We have $S_1 \overset{\mathbf{d}}{\rightsquigarrow} S_3 \overset{\mathbf{u}}{\rightsquigarrow} S_4$ and $S_1 \overset{\mathbf{u}}{\rightsquigarrow} S_4 \overset{\mathbf{d}}{\rightsquigarrow} S_2$. The combination of the Σ-steps of type $\mathbf{d}$ with the ones of type $\mathbf{u}$ oblige the four Σ-schemes to have the configuration



Hence, our Σ-paths look as follows:

$(P1)$ 

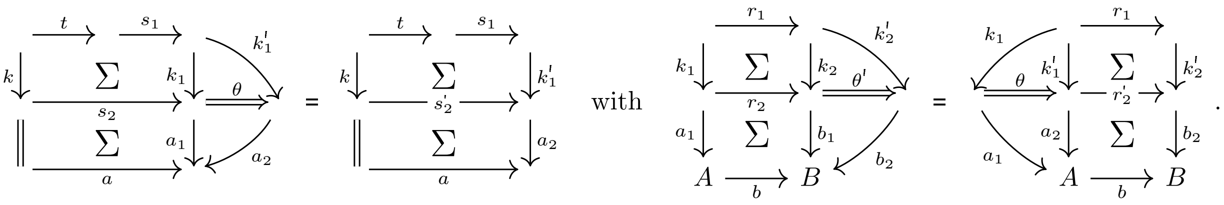
(P2)



We want to show that they give rise to the same Ω 2-cell.

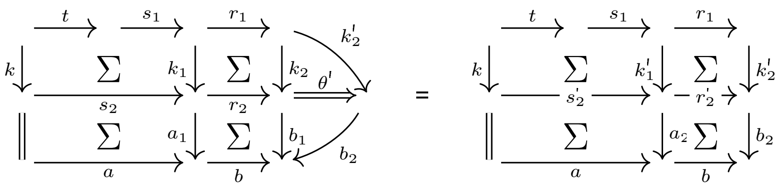
A. We describe a 2-morphism representing the Ω 2-cell corresponding to (P1).

Observe that, using Lemma A.1, we obtain Σ-squares and invertible 2-cells θ and θ' such that



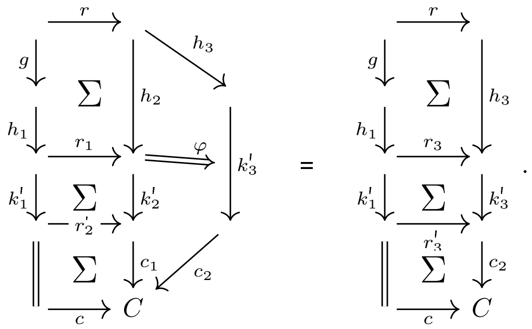
(A.2)

Thus, as in Lemma A.1, we have



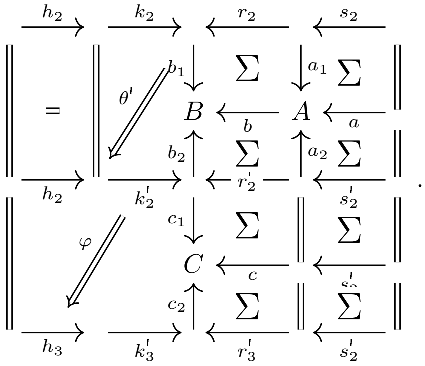
(A.3)

Moreover, by Rule 4', we have Σ-squares and φ invertible such that



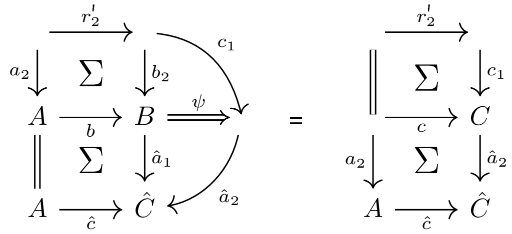
(A.4)

Hence, (A.3) and (A.4) give rise to the following vertical juxtaposition of the two basic Ω 2-morphisms corresponding to (P1):

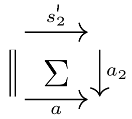


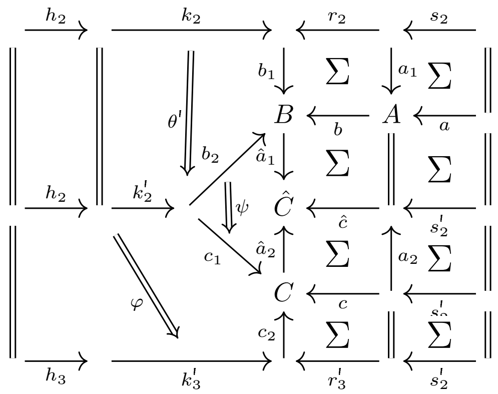
(A.5)

In order to obtain the vertical composition of these two 2-morphisms, first observe that with the help of Rule 4', we have Σ-squares and an invertible 2-cell ψ such that



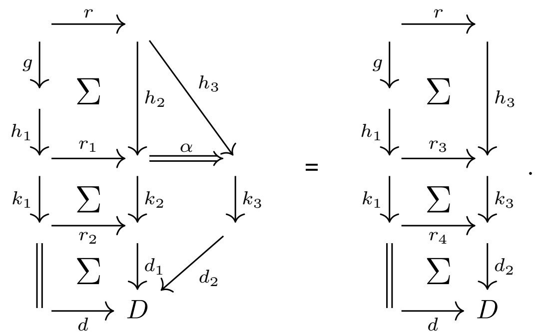
(A.6)

By composing both members on the left with  , we see that then the vertical composition of the 2-morphisms of (A.7) is given by the 2-morphism

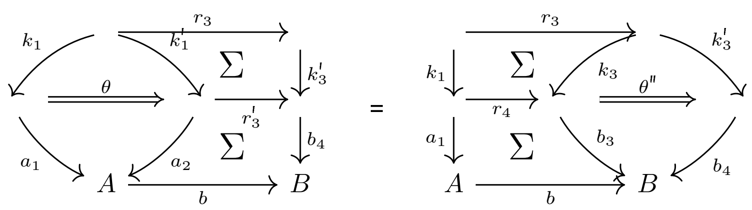


(A.7)

B. In order to form the vertical composition corresponding to (P2), observe that, for the first Σ-step, we have a basic Ω 2-cell determined by the following data:



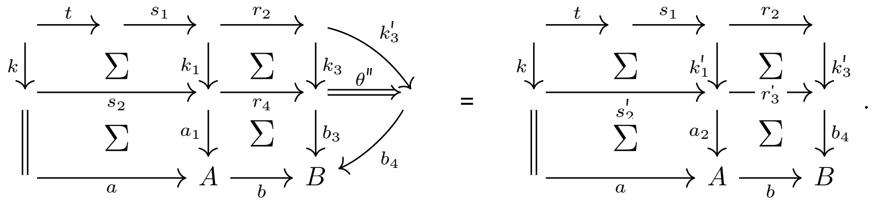
For the second Σ-step, use Lemma A.1 to first obtain, θ exactly as the first equality in (A.2), then the equality



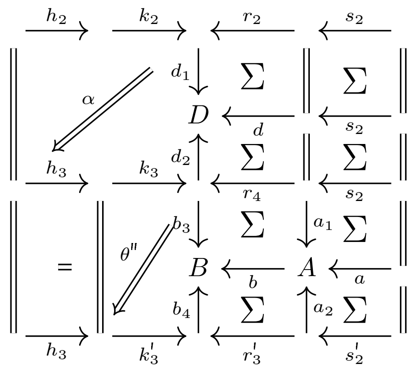
(A.8)

Observe that, without loss of generality, we may indeed use the same morphism $A \xrightarrow{b} B$ in (A.4) and (A.8): if in (A.8) we have $A \xrightarrow{b'} B'$, instead of $A \xrightarrow{b} B$, just take Square of b along b', and then apply Rule 3 of Proposition 2.8.

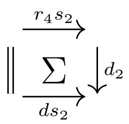
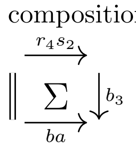
Thus, as described in Lemma A.1, we have that

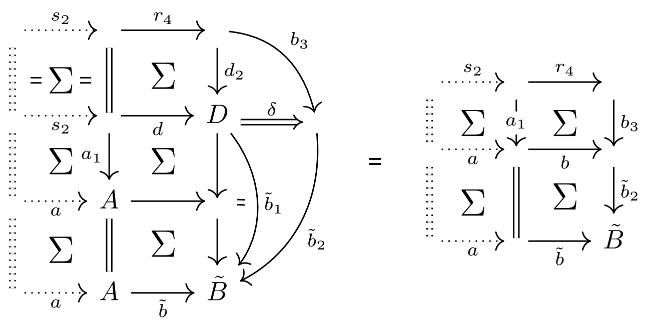


Combining the two Σ-steps of (P2), we obtain:

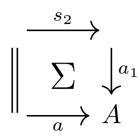


(A.9)

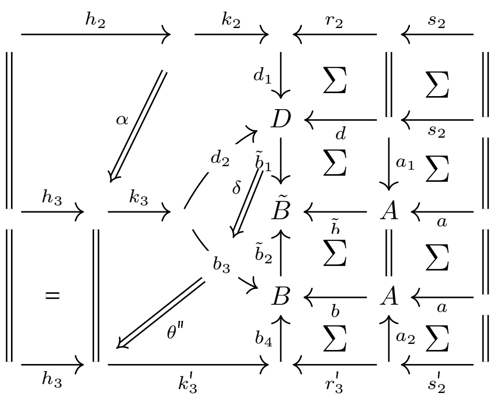
Now, in order to obtain the vertical composition of the two 2-morphisms of (A.9), we need to apply Rule 4' to  and  . For that, use Square applied to d and a_1, followed by Rule 4', to obtain the equality of the part drawn in solid lines in diagram (A.10) below.



(A.10)

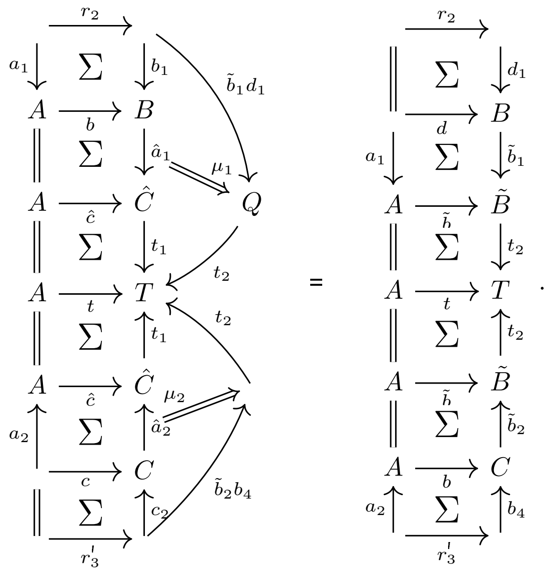
Then compose both members on the left of the solid-line part of the diagram with  .

The resulting equality leads to the vertical composition of the two 2-morphisms of (A.9):



(A.11)

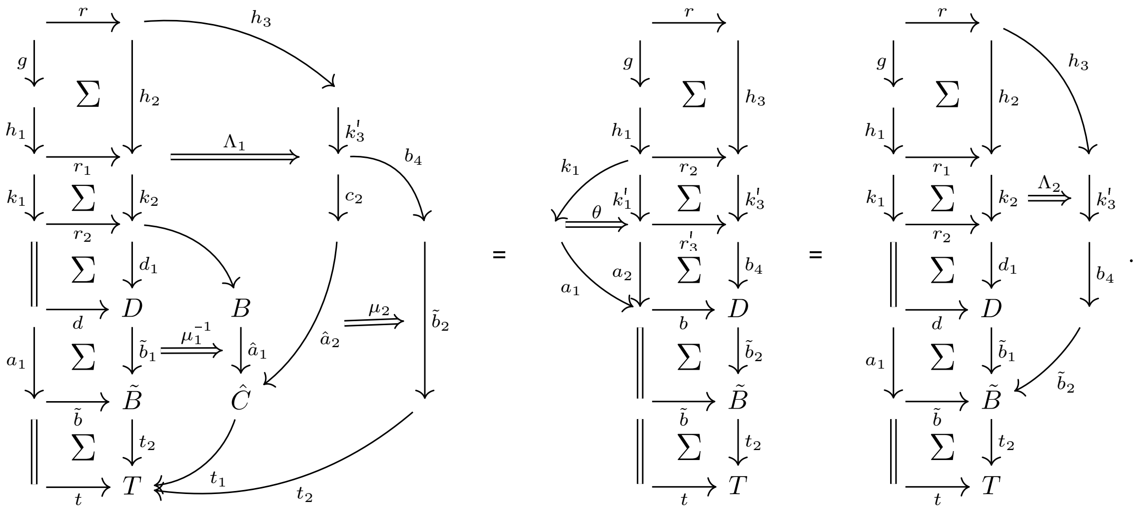
C. Now, in order to compare the 2-morphism (A.7) with the 2-morphism (A.11), observe first that the column of Σ-squares on the right-side is equal in both of them. Thus, in the following we ignore that column and take care only of the remaining part of the two 2-morphisms. For concluding the $\approx$-equivalence between them we use the property stated in Remark A.2. Accordingly, apply Rule 4 to obtain



With respect to the left-side part of (A.7) and (A.11), put

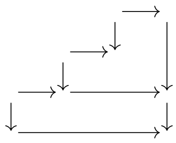
$$\Lambda_1 = (\hat{a}_2 \circ \varphi) \cdot (\psi \circ k_2' h_2) \cdot (\hat{a}_1 \circ \theta' \circ h_2) \colon \hat{a}_1 b_1 k_2 h_2 \Rightarrow \hat{a}_2 c_2 k_3' h_3;$$
$$\Lambda_2 = (\tilde{b}_2 \circ \theta'' \circ h_3) \cdot (\delta \circ k_3 h_3) \cdot (\tilde{b}_1 \circ \alpha) \colon \tilde{b}_1 d_1 k_2 h_2 \Rightarrow \tilde{b}_2 b_4 k_3' h_3.$$

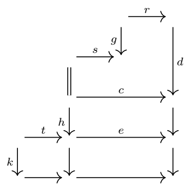
Using the previous equalities, we easily obtain the equalities

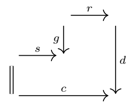
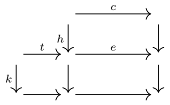
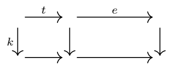


Consequently, by Remark A.2, the 2-morphisms represented in (A.7) and (A.11) are $\approx$-equivalent.

Case $i = \mathbf{d}$ and $j = \mathbf{s}_1$ (or $j = \mathbf{s}$). We consider $j = \mathbf{s}_1$, for $j = \mathbf{s}$ the procedure is the same.

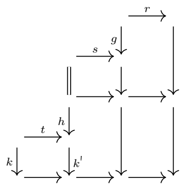
We have two Σ-paths of the form $S_1 \overset{\mathbf{d}}{\rightsquigarrow} S_3 \overset{\mathbf{s}_1}{\rightsquigarrow} S_2$ and $S_1 \overset{\mathbf{s}_1}{\rightsquigarrow} S_4 \overset{\mathbf{d}}{\rightsquigarrow} S_2$. The Σ-steps $S_1 \overset{\mathbf{d}}{\rightsquigarrow} S_3$ and $S_4 \overset{\mathbf{d}}{\rightsquigarrow} S_2$ determine that the four Σ-schemes are of the form . The combination of this with the existence of the Σ-steps $S_3 \overset{\mathbf{s_1}}{\rightsquigarrow} S_2$ and $S_1 \overset{\mathbf{s_1}}{\rightsquigarrow} S_4$ implies that all Σ-schemes are of the form

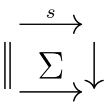
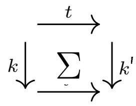
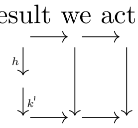


Observe that along the two Σ-paths,  remains unchanged. Thus, to obtain the basic Ω 2-cells, and the subsequent two Ω 2-cells corresponding to the two Σ-paths, we just need to work with the part . Now the procedure is analogous to the previous case: we make use of Lemma A.1 applied to the parts of the form  in the passages of type **d** and, at the end, we use Remark A.1 to conclude that the two Σ-paths give rise to ≈-equivalent 2-morphisms.

Case $i = \mathbf{u}$ and $j = \mathbf{s}_1$ (or $j = \mathbf{s}$). We consider $j = s_1$, for $j = s$, the procedure is the same.

We have two Σ-paths of the form $S_1 \overset{\mathbf{u}}{\rightsquigarrow} S_3 \overset{\mathbf{s_1}}{\rightsquigarrow} S_2$ and $S_1 \overset{\mathbf{s_1}}{\rightsquigarrow} S_4 \overset{\mathbf{u}}{\rightsquigarrow} S_2$. Thus, the four Σ-schemes are of the form

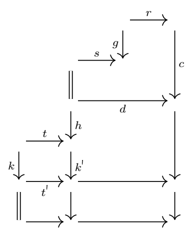


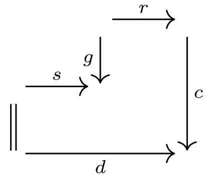
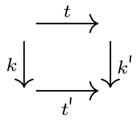
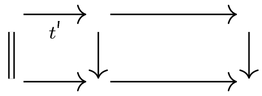
In each Σ-step above, the Σ-squares  and  remain unchanged, then they do not interfere on the Σ-steps. To obtain the desired result we act as for $i = d$ and $j = u$: Apply Lemma A.1 to the parts of the Σ-schemes given by  in the passages of type **s**, and, at the end, use Remark A.2.

Case $i = \mathbf{u}$ and $j = \mathbf{d}_1$. It is similar to $i = \mathbf{u}$ and $j = \mathbf{d}$.

Case $i = \mathbf{s}$ and $j = \mathbf{d}_1$. It is similar to $i = \mathbf{d}$ and $j = \mathbf{s}$.

Case $i = \mathbf{d_1}$ and $j = \mathbf{s}_1$. We easily see that all the Σ-schemes have the configuration



where the parts  and  are unchanged. We proceed as for the first case, by applying Lemma A.1 to the part  when obtaining the basic Ω 2-cells corresponding to the passages d_1 and use Remark A.2. ☐

Corollary A.4. *Every two Σ-paths of length equal or less than 2 between Σ-schemes of level 3 are equivalent. Equivalently, every cycle of 4 or less Σ-steps between Σ-schemes of level 3 are equivalent to an identity Σ-step.*

Proof. For cycles of length 1 or 2, it was already seen in Lemma 3.12. Concerning cycles of length 3, it suffices to consider the cases where we have three consecutive Σ-steps all of different types. Indeed in case we have two repeated types, then, since a Σ-path of two Σ-steps of the same type are equivalent to one Σ-path of just one Σ-step of that type, we fall in the case of length 1 or 2.

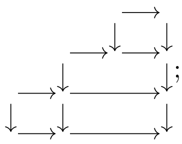
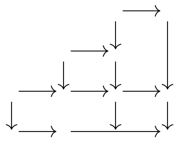
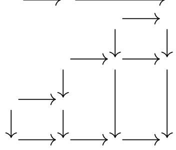
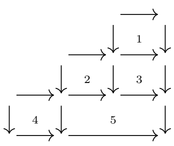
It remains to consider all cycles of length 3 or 4 not yet encompassed by Proposition A.3. Indeed, all of them follow from Lemma 3.12 and that proposition. Moreover, most of the cases follow also from Proposition 3.20, where the property is stated for cycles of all finite lengths when we restrict to certain configurations of Σ-schemes.

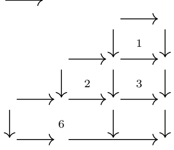
Cycles of length 3. If the cycle contains the Σ-steps $\mathbf{s}$ and $\mathbf{s}_1$, or the Σ-steps $\mathbf{d}$ and $\mathbf{d}_1$, then the result follows immediately from Lemma 3.12. Since every Σ-step may be seen as an undirected edge (because basic Ω 2-cells are invertible and the inverse corresponds to a Σ-step of the same type, just reversed), the only cases to study are $\mathbf{dus}$, $\mathbf{dus}_1$, $\mathbf{d}_1\mathbf{us}$ and $\mathbf{d}_1\mathbf{us}_1$. We analyse $\mathbf{dus}$, the remaining cases are similar.

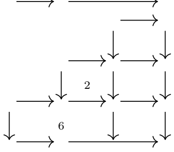
Case $\mathbf{dus}$. Let us consider

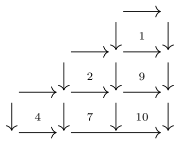
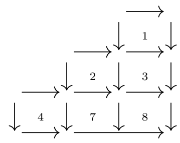
$$S_1 \overset{\mathbf{d}}{\rightsquigarrow} S_2 \overset{\mathbf{u}}{\rightsquigarrow} S_3 \overset{\mathbf{s}}{\rightsquigarrow} S_1 \quad .$$

Observe that, taking into account the type of the Σ-steps in turn, we conclude as follows, where the necessary equal Σ-squares will be indicated with the same number:

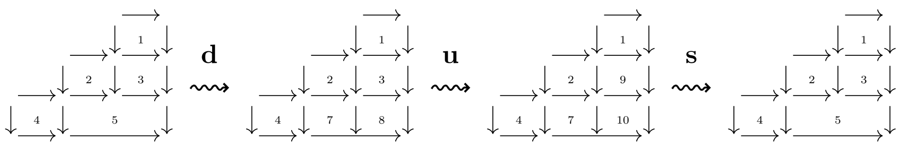
- S_1 is simultaneously of the forms $\mathbf{d}$ and $\mathbf{s}$, hence S_1 = ;
- S_2 is simultaneously of the forms $\mathbf{d}$ and $\mathbf{u}$, hence S_2 = ;
- S_3 is simultaneously of the forms $\mathbf{u}$ and $\mathbf{s}$, hence S_3 = ;
- $\mathbf{d}$: $S_1 \rightsquigarrow S_2$ obliges us to have S_1 = 

and S_2 = ;

- $\mathbf{u}$: $S_2 \rightsquigarrow S_3$ obliges us to have S_3 = ;

- **s**: $S_3 \rightsquigarrow S_1$ requires $S_3 =$  and $S_2 =$ .

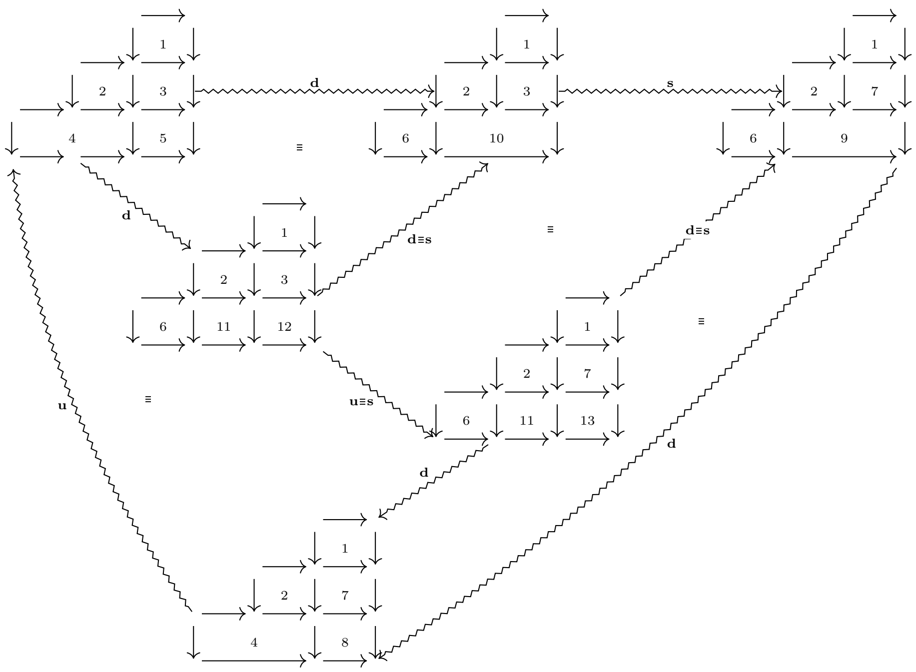
Consequently, we have:



Observe that the basic Ω 2-cell corresponding to **d** may be obtained by applying Rule 4' just to the Σ-squares 5 and $7 \oplus 8$ and composing with the Σ-square 4. Then, it coincides with the basic Ω 2-cell obtained by the Σ-step of type **s**. Analogously, the Ω 2-cell corresponding to **u** may be obtained by applying Rule 4' to the Σ-squares $3 \odot 8$ and $9 \odot 10$ and composing with the Σ-square 1. Thus, this Ω 2-cell also coincides with the one obtaining by a Σ-step of type **s**. Therefore, we conclude that the Σ-path is equivalent to a Σ-path of length 1 made of a Σ-step of type **s**, and hence it is just the identity.

Cycles of length 4. We need to analyse the cases not reducible to a cycle of length ≤ 3 and not encompassed by Proposition A.3.

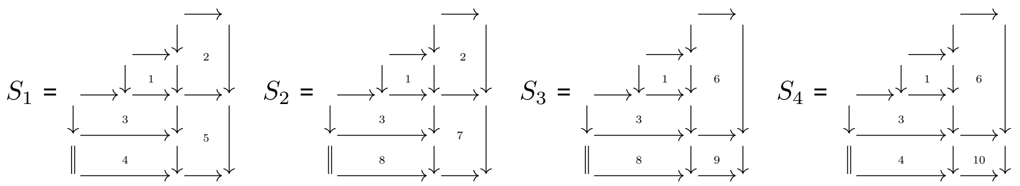
Case **duds**. Arguing in a way similar to the one of case **dsu**, we see that the Σ-schemes of a cycle of Σ-steps **duds** must be of the form represented by the outside Σ-path of the diagram below. The equivalences of Σ-paths and Σ-steps indicated by the symbol ≡ are easily seen — namely, we use the fact that the juxtaposition of Σ-steps of the same type is equivalent to a Σ-step of that type, and the case $i = \mathbf{d}$ and $j = \mathbf{u}$ studied in Proposition A.3. Consequently, we see that the Ω 2-cell corresponding to the Σ-cycle is the identity.



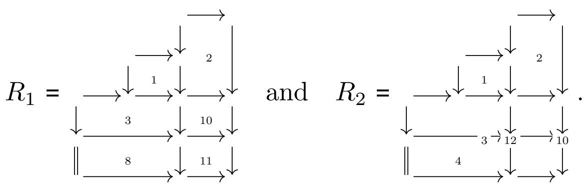
Case $\mathbf{dud_1u}$. Arguing in a similar way as above we see that our cycle is of the form

$$S_1 \overset{\mathbf{d}}{\rightsquigarrow} S_2 \overset{\mathbf{u}}{\rightsquigarrow} S_3 \overset{\mathbf{d_1}}{\rightsquigarrow} S_4 \overset{\mathbf{u}}{\rightsquigarrow} S_1$$

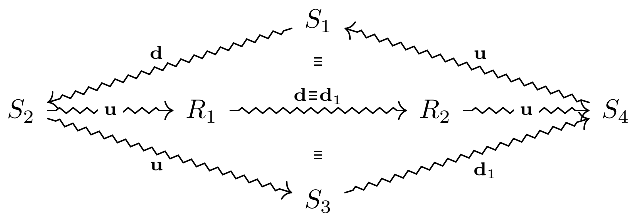
with



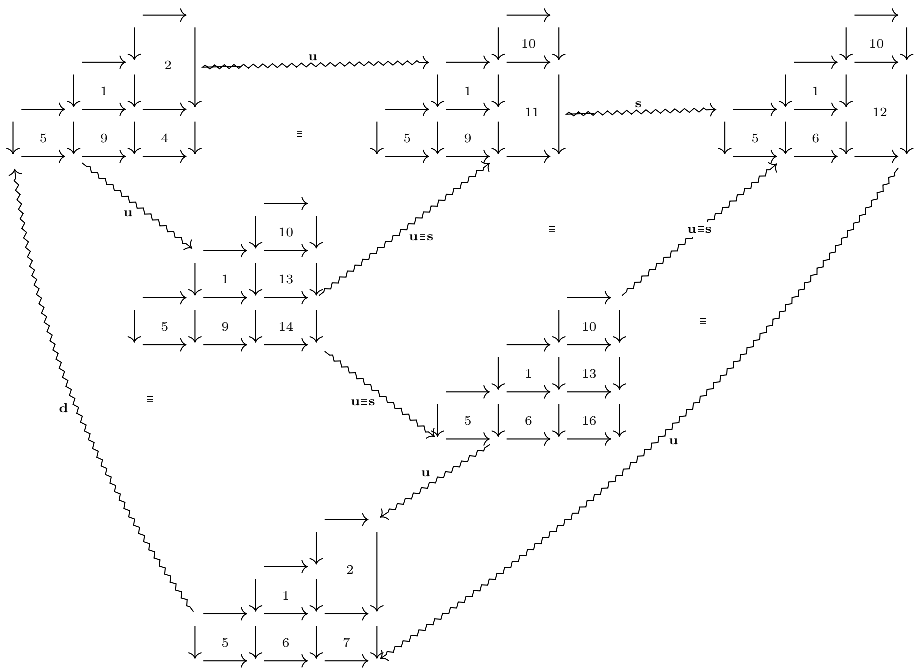
Let



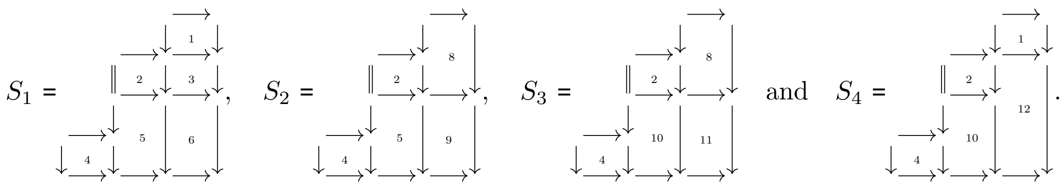
Then, using Lemma 3.12 and Proposition 3.14, we see the indicated equivalences of Σ-paths:



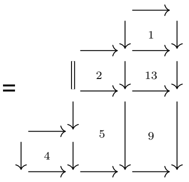
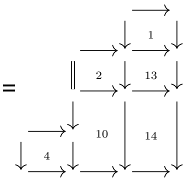
Case **udus**. Analogously, we conclude as desired seeing that the cycle **udus** must be as described in the outside part of the diagram below and observing that the indicated equivalences between Σ-paths are verified.



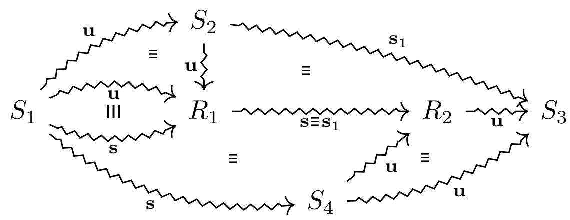
Case $\mathbf{us}_1\mathbf{us}$. In order to have the sequence of Σ-steps $S_1 \overset{\mathbf{u}}{\rightsquigarrow} S_2 \overset{\mathbf{s}_1}{\rightsquigarrow} S_3 \overset{\mathbf{u}}{\rightsquigarrow} S_4 \overset{\mathbf{s}}{\rightsquigarrow} S_1$, the Σ-schemes are of the form



Let

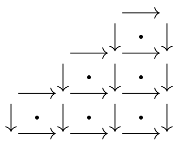
R_1 =  and R_2 =  .

The following diagram shows that the given cycle is indeed equivalent to the identity Σ-path.



The remaining cases can be obtained by arguing in a similar way. □

Recall the notion of ***configuration of interest*** and ***Σ-path of interest*** given in Definition 3.18. Recall also that the canonical Σ-scheme of level 3 and of a given left border is made of canonical Σ-squares, and is represented by

Can =  .

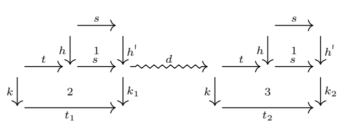
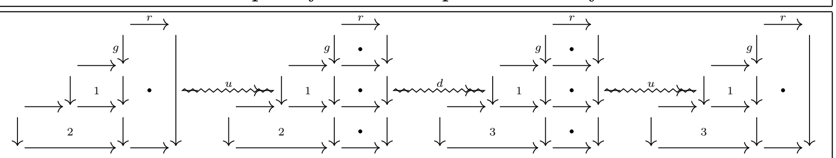
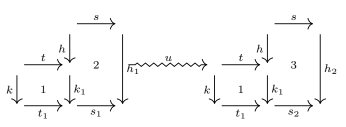
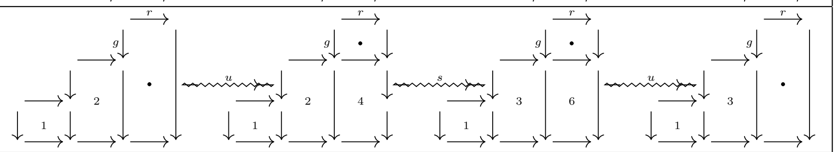
The next proposition, which is just Proposition 3.21 given in more detail, states that the horizontal composition of an identity Ω 2-cell with an Ω 2-cell is also an Ω 2-cell.

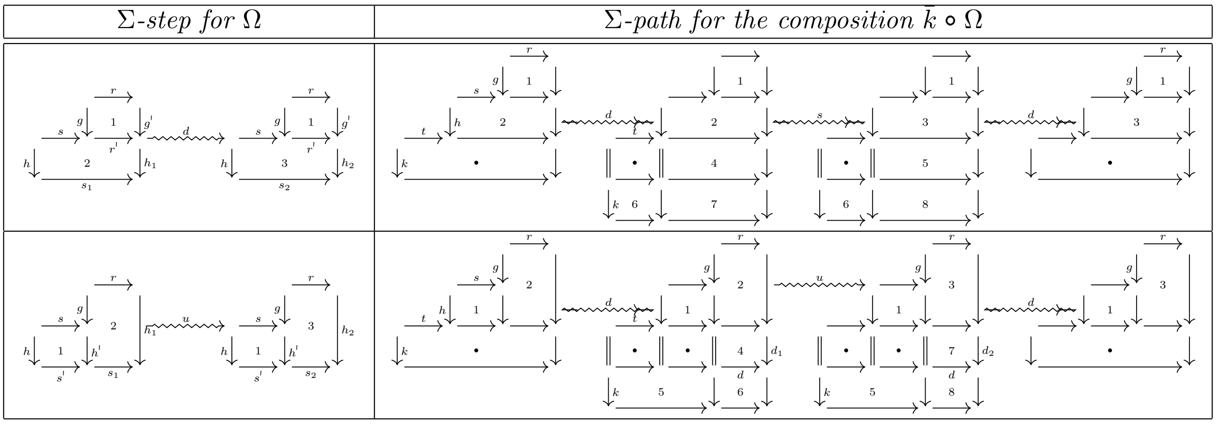
Proposition A.5. *Let $A \xrightarrow{\bar f} B \xrightarrow{\bar g} C \xrightarrow{\bar h} D \xrightarrow{\bar k} E$ be Σ-cospans, where $\bar f = (f, r)$, $\bar g = (g, s)$, $\bar h = (h, t)$ and $\bar k = (k, u)$.*

(1) Let $\Omega\colon (l_1 g, m_1 u) \to (l_2 g, m_2 u)$ be a basic Ω 2-cell determined by a Σ-step of level 2 of the type d or u between two Σ-schemes of left border (s, h, t, k) and right border (l_i, m_i), respectively. Then the 2-cell $\Omega \circ 1_{\bar f}$ is an Ω 2-cell corresponding to a Σ-path of interest of Σ-schemes of level 3 and left border (r, g, s, h, t, k).

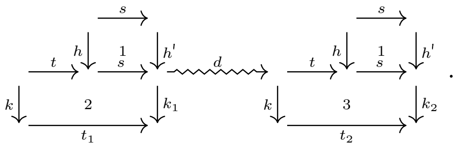
(2) Let $\Omega\colon (l_1 f, m_1 t) \to (l_2 f, m_2 t)$ be a basic Ω 2-cell determined by a Σ-step of level 2 of the type d or u between two Σ-schemes of left border (r, g, s, h) and right border (l_i, m_i), respectively. Then the 2-cell $1_{\bar k} \circ \Omega$ is an Ω 2-cell corresponding to a Σ-path of interest of Σ-schemes of level 3 and left border (r, g, s, h, t, k).

Moreover, $\Omega \circ 1_{\bar f}$ and $1_{\bar k} \circ \Omega$ are given by the following table, where the canonical Σ-squares are indicated by a bullet, and the others Σ-squares are indicated by numbers.

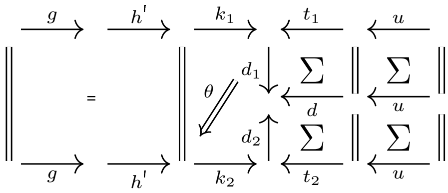
Σ-step for Ω	*Σ-path for the composition $\Omega \circ \bar f$*
	
	



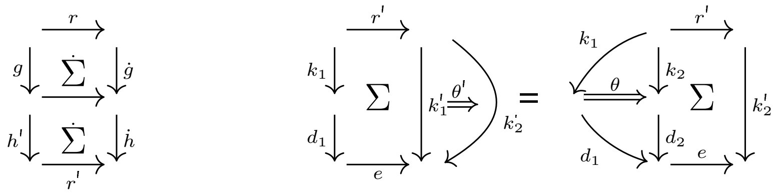
Proof. (1) Let Ω be the Ω 2-cell corresponding to the Σ-step



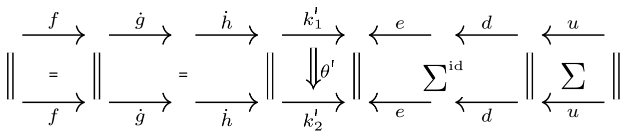
Thus, here $(l_i, m_i) = (k_i h', t_i)$. Let it be represented by the 2-morphism



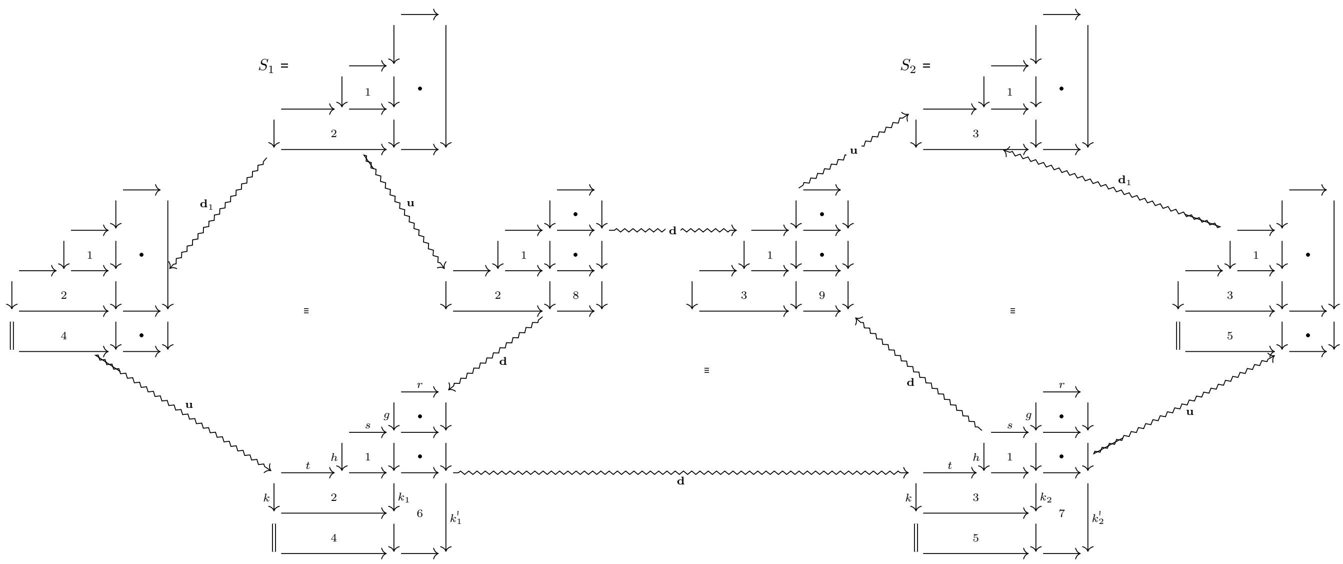
with θ invertible. Now, in order to compose with $\bar{f} = (f, r)$, consider the following vertical composition of Σ-squares and an invertible 2-cell θ' as follows:



Thus, we can use $\theta' \circ \dot{h} \circ \dot{g}$ to obtain the middle 2-morphism of the desired horizontal composition (see Definition 3.24):

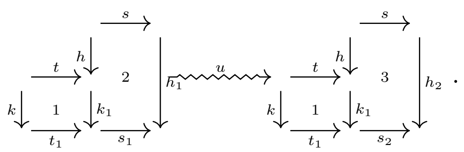


This 2-morphism is an Ω 2-morphism. Indeed it corresponds to a Σ-step of type **d** between Σ-schemes of level 3, as illustrated by the Σ-step **d** of the bottom of the next diagram. Combining this with the definition of horizontal composition as described in Definition 3.24, we see that the composition $\Omega \circ \bar{f}$ is given by the vertical composition of basic Ω 2-cells corresponding to the Σ-path $\overset{\mathbf{d}_1}{\rightsquigarrow} \overset{\mathbf{u}}{\rightsquigarrow} \overset{\mathbf{d}}{\rightsquigarrow} \overset{\mathbf{u}}{\rightsquigarrow} \overset{\mathbf{d}_1}{\rightsquigarrow}$ formed by the bottom part from S_1 to S_2 in the following diagram:

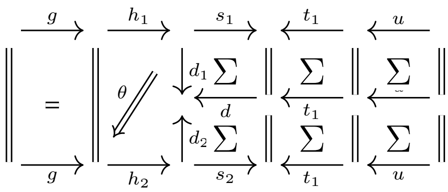


The equivalences of Σ-paths indicated in this diagram are clear: recall from Corollary A.4 that two Σ-paths of the form $S_1 \overset{\mathbf{d}_1}{\rightsquigarrow} S_3 \overset{\mathbf{u}}{\rightsquigarrow} S_2$ and $S_1 \overset{\mathbf{u}}{\rightsquigarrow} S_4 \overset{\mathbf{d}}{\rightsquigarrow} S_2$ are equivalent. Therefore, we conclude that $\Omega \circ \bar{f}$ is given by a Σ-path of the form indicated in the table.

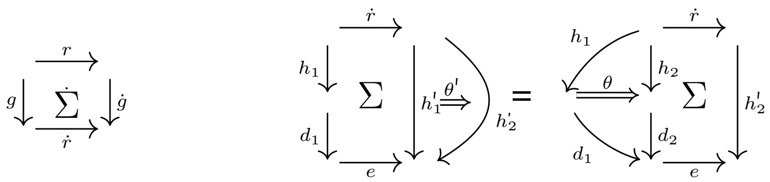
Let now Ω be the Ω 2-cell corresponding to the Σ-step



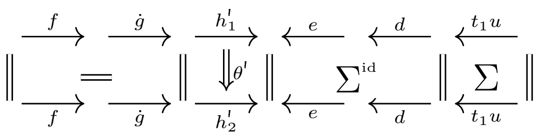
Thus, here $(l_i, m_i) = (h_i, s_i t_i)$. Let it be represented by the 2-morphism



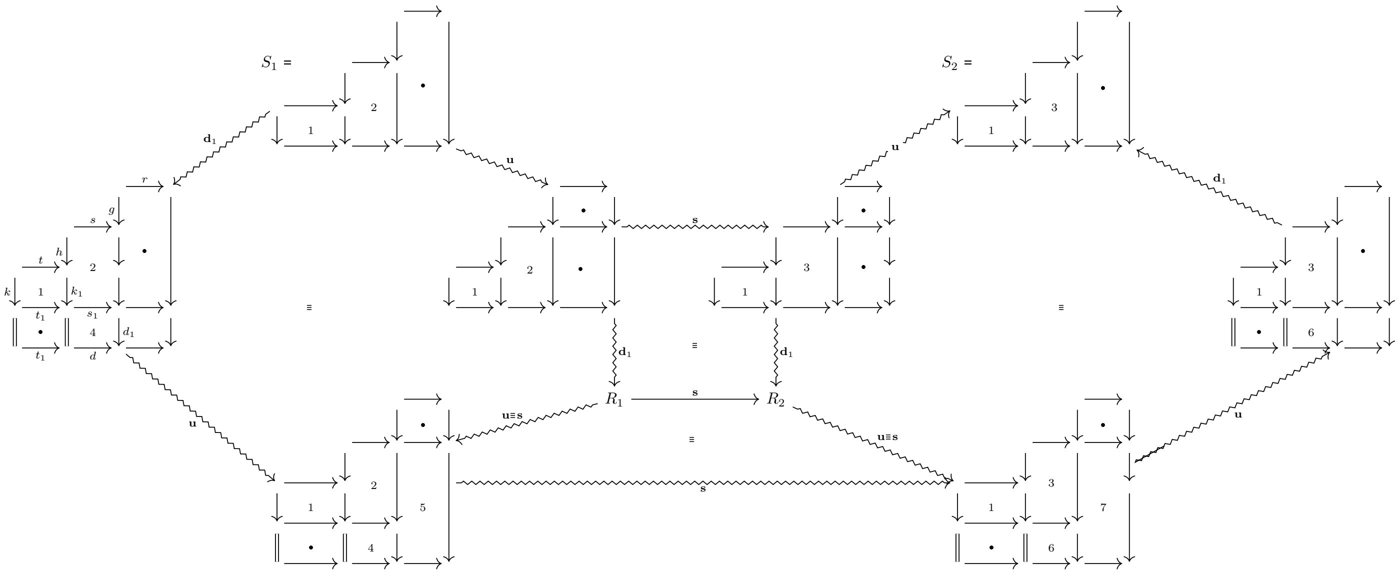
with θ invertible. Now, in order to compose it with $\bar{f} = (f, r)$, consider the canonical Σ-square of r along g and an invertible 2-cell θ' as follows:



The 2-morphism of the middle of the representative of $\theta \circ 1_{\bar{f}}$ is given by the diagram



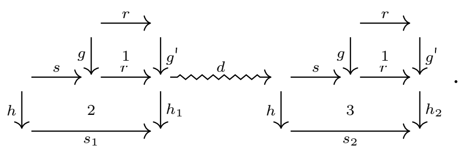
This 2-morphism is an Ω 2-morphism. Indeed it corresponds to a Σ-step of type **s** between Σ-schemes of level 3, as illustrated by the Σ-step **s** of the bottom of the next diagram. Moreover, taking into account Definition 3.24, and the original Σ-step of type u between Σ-schemes of level 2, we see that $\Omega \circ 1_{\bar{f}}$ corresponds to the Σ-path $S_1 \overset{\mathbf{d_1}}{\rightsquigarrow} \overset{\mathbf{u}}{\rightsquigarrow} \overset{\mathbf{s}}{\rightsquigarrow} \overset{\mathbf{u}}{\rightsquigarrow} \overset{\mathbf{d_1}}{\rightsquigarrow} S_2$ formed by the bottom part of the diagram



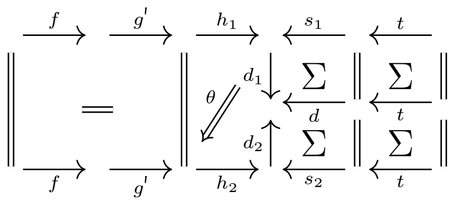
where R_1 and R_2 are obvious Σ-schemes.

The equivalences of Σ-paths indicated in this diagram are clear, from Lemma 3.12 and Proposition A.3. Therefore, we conclude that $\Omega \circ f$ is given by a Σ-path of the form indicated in the table.

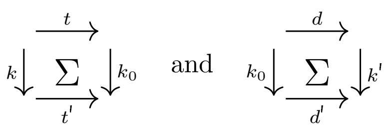
(2) Let Ω be the Ω 2-cell corresponding to the Σ-step



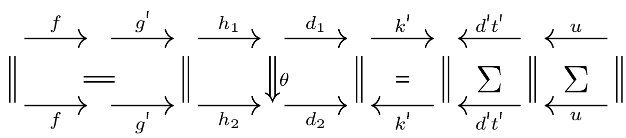
Thus, here $(l_i, m_i) = (h_i g', s_i)$. Let it be represented by the 2-morphism



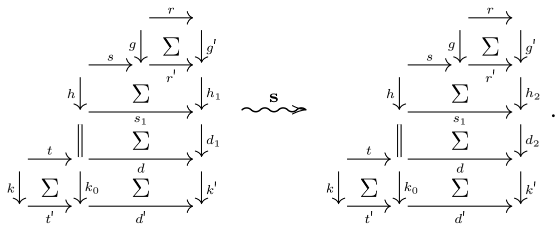
with θ invertible. Now, in order to compose with $\bar{k} = (k, u)$, just consider the Σ-squares



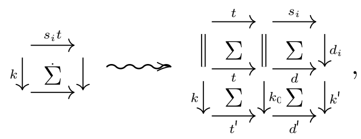
and their horizontal composition. Following Definition 3.24, we obtain the middle 2-morphism of the desired composition $\bar{k} \circ \Omega$:



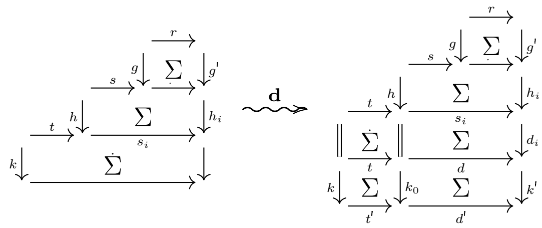
This 2-morphism is an Ω 2-morphism. Indeed, it corresponds to the following Σ-step of type **s**



Observe that, in this case, the Ω 2-cells Ω_i, $i = 1, 2$, as in Definition 3.24, just reduce to the basic Ω 2-cell determined by the Σ-path of level 1



equivalently, determined by the following Σ-step of level 3:

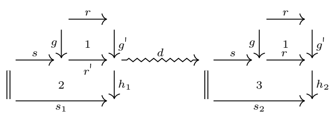


In conclusion, we obtain the Σ-path as in the first row of the second table.

The case of the second row of the second table, where Ω corresponds to the Σ-step of type u, works similarly.

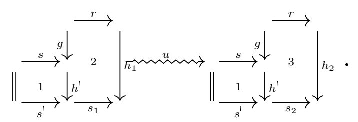
Finally, it is clear that every of the four Σ-paths given in the above tables leading to $\Omega \circ 1_{\bar{f}}$ and $1_{\bar{k}} \circ \Omega$ are Σ-paths of interest. □

Corollary A.6. *(1) Let* $\Omega\colon (h_1 g' f, s_1) \to (h_2 g' f, s_2)$ *be the* Ω *2-cell (with the indicated domain and codomain) corresponding to the* Σ*-step of left border* $(r, g, s, 1)$



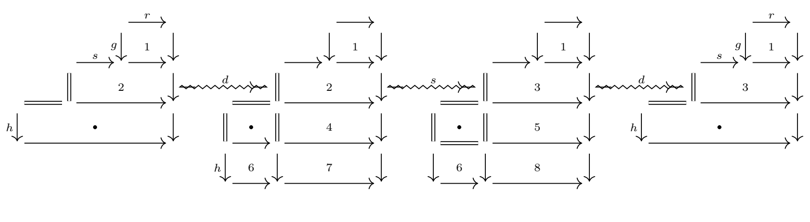
and let $\bar{h} = (h, t)$ *be horizontally composable with* Ω*. Then the* Ω *2-cell* $\bar{h} \circ \Omega$ *corresponds to a* Σ*-path of interest with left border* $(r, g, s, h, t, 1)$*.*

(2) Let $\Omega\colon (h_1, s_1 s') \to (h_2, s_2 s')$ *be the* Ω *2-cell (with the indicated domain and codomain) corresponding the* Σ*-step*

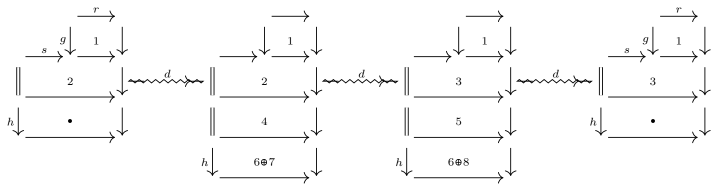


and let $\bar{h} = (h, t)$ *be horizontaly composable with* Ω*. Then, the 2-cell* $\bar{h} \circ \Omega$ *corresponds to a* Σ*-path of interest with left border* $(r, g, s, h, t, 1)$*.*

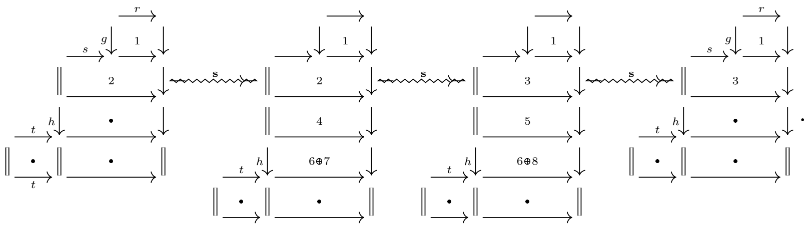
Proof. (1) From Proposition A.5, we know that the 2-cell $\bar{h} \circ \Omega$ is the one corresponding to the Σ-path



Observe that, equivalently, $\bar{h} \circ \Omega$ corresponds to the Σ path of Σ-schemes of level 2 given by

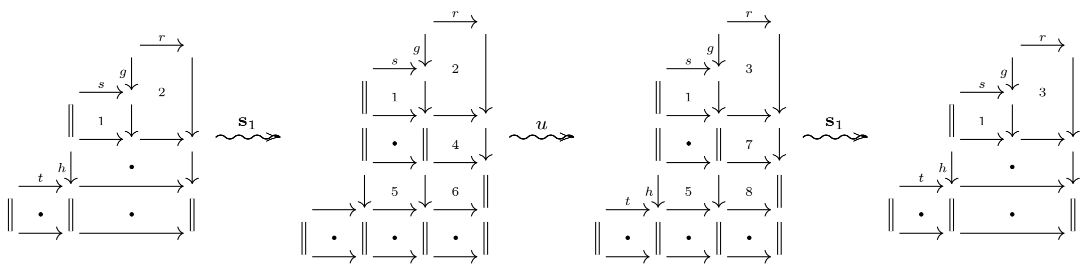


Moreover, adding just an identity row below each Σ-scheme, we obtain a Σ-path between Σ-schemes of level 3 and left border $(r, g, s, h, t, 1)$:



The 2-cell $\bar{h} \circ \Omega$ also corresponds to this Σ-path, which indeed is equivalent to just a Σ-step of type $\mathbf{s}$. Hence, a Σ-path of interest.

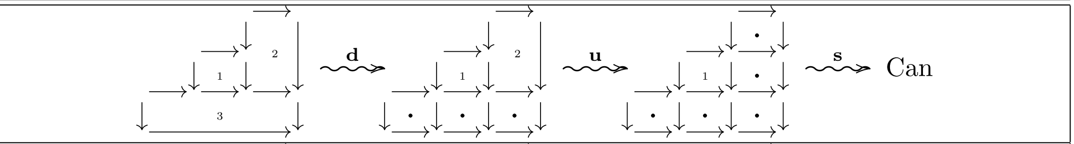
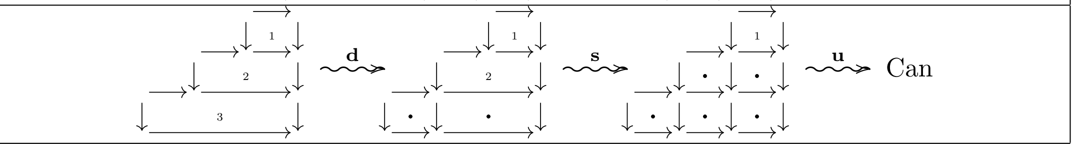
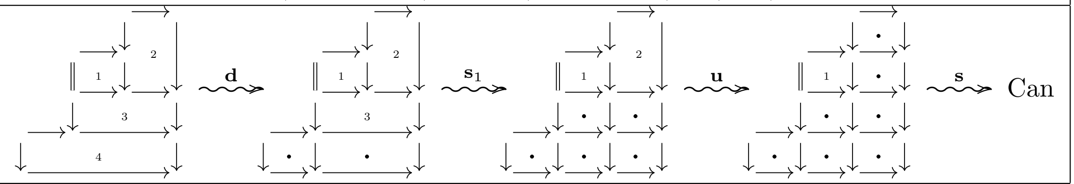
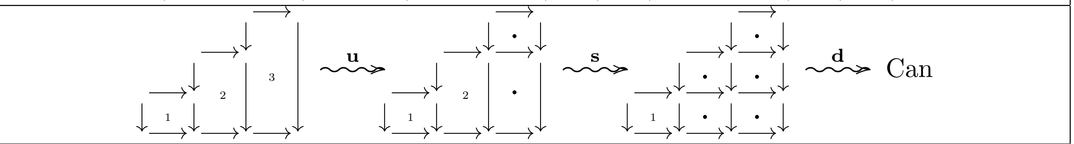
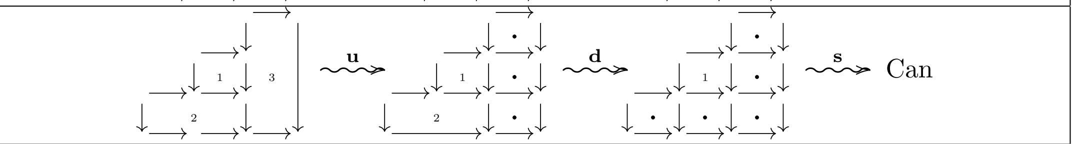
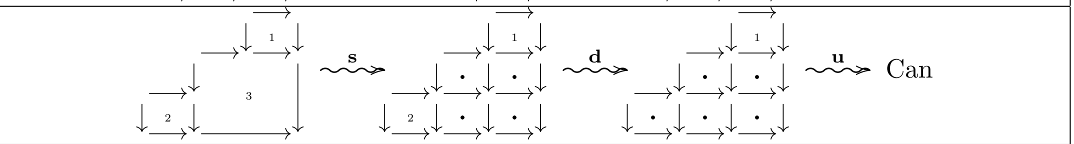
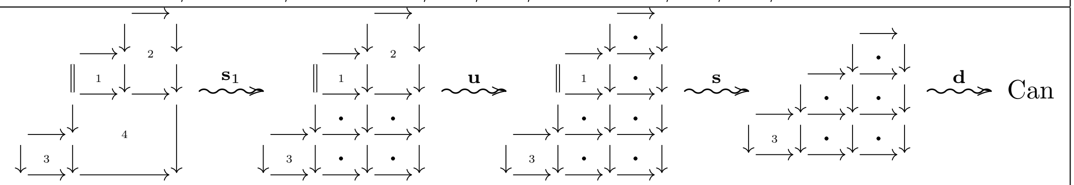
(2) Acting in a similar way for the Σ-step u, we obtain that the Ω 2-cell $\bar{h} \circ \Omega$ corresponds to the following Σ-path



which is of interest. □

The last proposition of this section states that every two Σ-paths of interest, of any finite length, between two Σ-schemes of level 3, are equivalent. As an auxiliary tool we are going to consider certain Σ-paths into the canonical Σ-scheme.

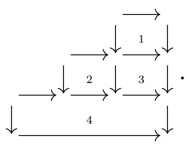
For a Σ-scheme of a given configuration of interest x (see Definition 3.18) there may exist several Σ-paths from it into the canonical Σ-scheme. In the next table we choose one of that Σ-paths to be the x***-canonical*** Σ***-path***.

x	x-canonical Σ-path
da	
db	
dc	
ua	
ub	
s	
$\mathbf{s}_1$	

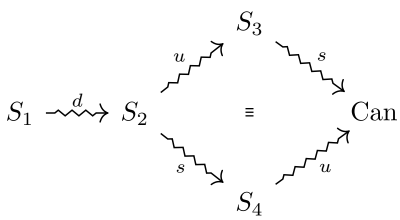
Proposition A.7. *Every two Σ-paths of interest starting and ending at the same Σ-schemes are equivalent.*

Proof. **A.** First, we show that for every Σ-scheme of interest exhibiting two different kinds of configurations of interest, the corresponding two canonical Σ-paths are equivalent.

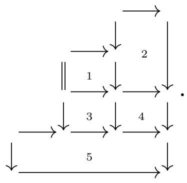
1) **da** and **db**. A Σ-scheme exhibiting the both configurations looks as follows:

$S_1 =$  .

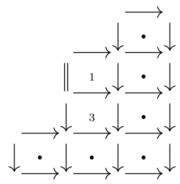
Taking the **da**-canonical Σ-path and the **db**-canonical Σ-path we see that we have the following diagram, where the top and bottom lines are the canonical Σ-paths and the equivalence $\equiv$ is given by Proposition A.3. Hence, the two canonical Σ-paths are equivalent.



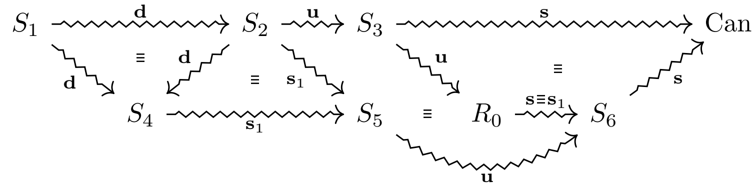
2) **da** and **dc**. A Σ-scheme exhibiting the both configurations looks as follows:

S_1 =  .

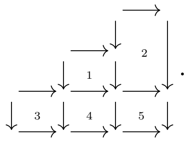
Let

R_0 = 

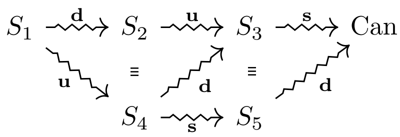
Putting the **da**-canonical Σ-path and the **dc**-canonical Σ-path on the top and on the bottom, respectively, we obtain the following equivalences of Σ-paths, showing that the two canonical Σ-paths are equivalent:



3) **da** and **ua**. A Σ-scheme exhibiting the both configurations looks as follows:

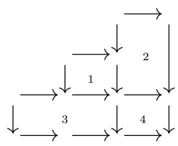
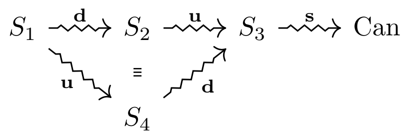
S_1 =  .

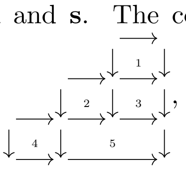
Let the **da**-canonical and **ua**-canonical Σ-paths be, respectively, the top line and the bottom line of the following diagram. There is a Σ-step of type **d** from S_4 to S_3.

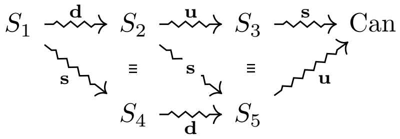


Hence, by Proposition A.3, the two canonical Σpaths are equivalent.

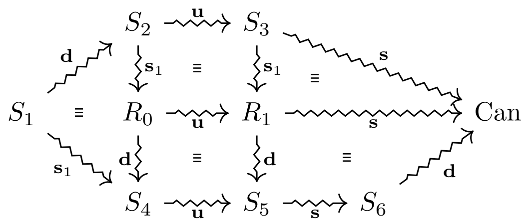
The remaining cases are shown in an analogous way. Next we just exhibit the scheme of the diagram with the two corresponding canonical Σ-paths and the equivalences leading to the desired conclusion.

4) **da** and **ub**. S_1 =  and we have  .

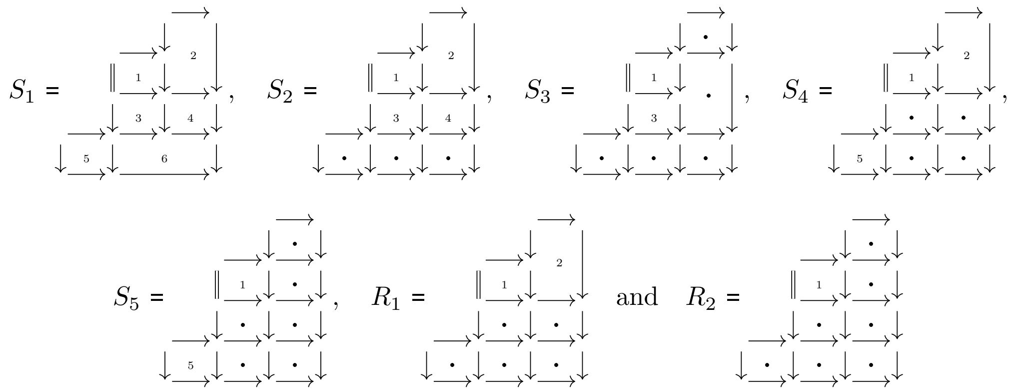
5) **da** and **s**. The combination of the two configurations give a Σ-scheme of the form S_1 = , and the equivalence of the two canonical Σ-paths is shown by the diagram

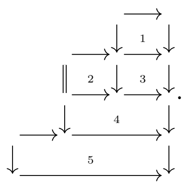
 .

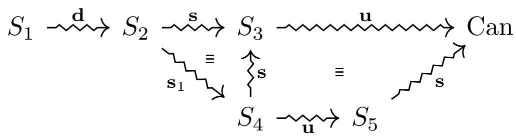
6) **da** and $\mathbf{s}_1$. The equivalence is shown by the following diagram



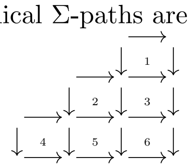
where

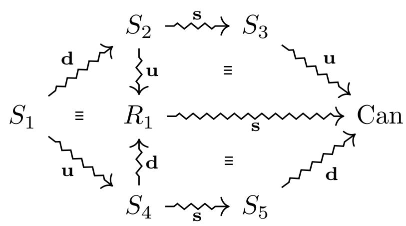


7) **db** and **dc**. Put S_1 = . We can construct a diagram of the form

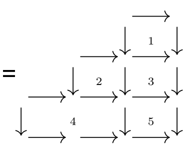
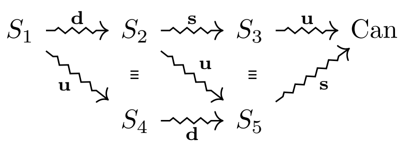


showing that the two canonical Σ-paths are equivalent.

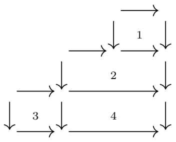
8) **db** and **ua**. Put S_1 =  and use the diagram

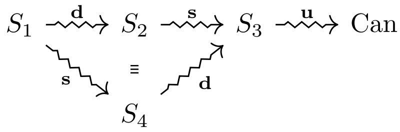


where R_1 is obvious.

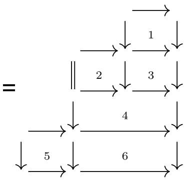
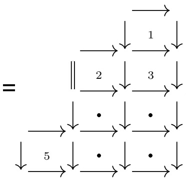
9) **db** and **ub**. Put S_1 =  and use the diagram 

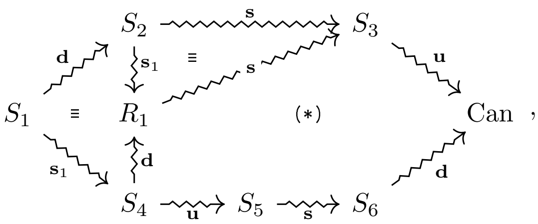
to conclude that the two canonical Σ-steps are equivalent.

10) **db** and **s**. Put S_1 =  and use the following diagram

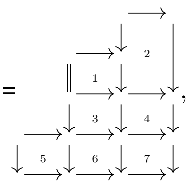
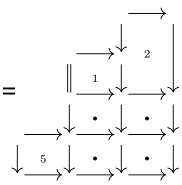


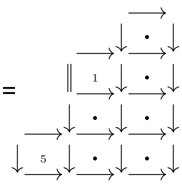
to conclude the equivalence of the two canonical Σ-paths.

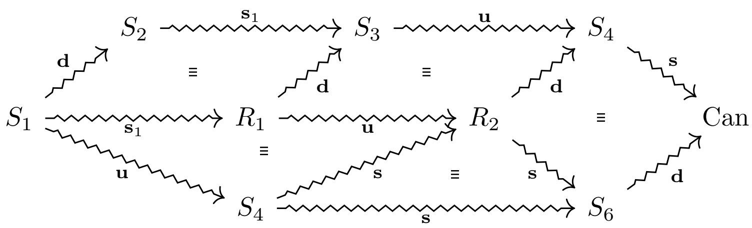
11) **db** and $\mathbf{s}_1$. Put S_1 =  and consider R_1 = . Use the following diagram



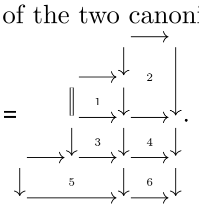
where the equivalence in (∗) is obtained as for the case **db** and **ua**, to conclude the equivalence of the two canonical Σ-paths.

12) **dc** and **ua**. Put S_1 = , and consider R_1 =  and

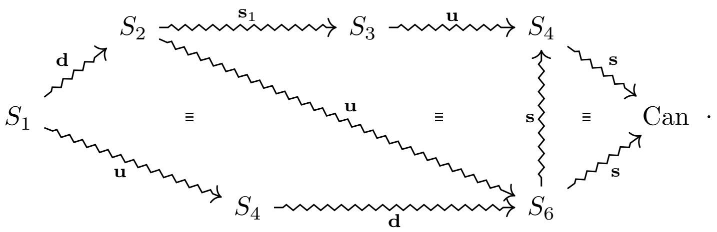
R_2 = . Use the following diagram

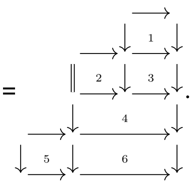


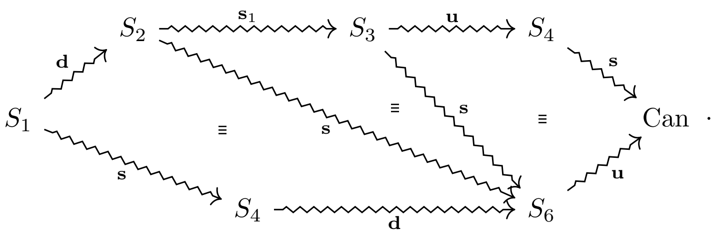
to conclude the equivalence of the two canonical Σ-paths.

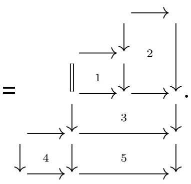
13) **dc** and **ub**. Put S_1 = . We obtain a diagram with the two canonical

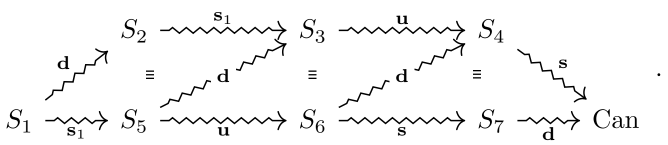
Σ-paths which are equivalent as illustrated by the diagram



14) **dc** and **s**. Put S_1 = . We obtain a diagram with the two canonical Σ-paths showing that they are equivalent:



15) **dc** and $\mathbf{s}_1$. Put S_1 = . We obtain a diagram with the two canonical Σ-paths showing that they are equivalent:

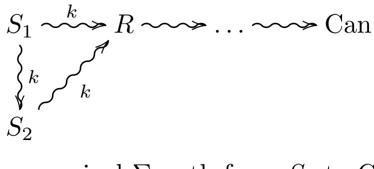


The remaining six cases are proved with similar easy arguments.

B. Observe that given a Σ-step of interest of a certain type k, say

$$S_1 \overset{k}{\rightsquigarrow} S_2 \ ,$$

then S_1 and S_2 have a common configuration y where the string y starts with the character k. Moreover, the y-canonical Σ-path from S_1 to Can factors through $S_1 \overset{k}{\rightsquigarrow} S_2$; more precisely, we have



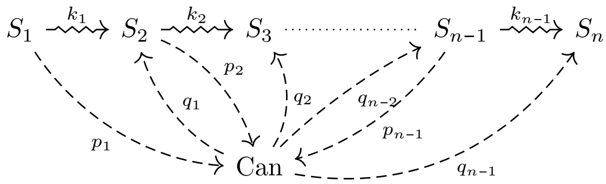
(A.12)

where the top Σ-path is the y-canonical Σ-path from S_1 to Can and the bottom line is the y-canonical Σ-path from S_2 to Can.

Now suppose we have a Σ-path of interest

$$S_1 \overset{k_1}{\rightsquigarrow} S_2 \overset{k_2}{\rightsquigarrow} \dots \overset{k_{n-1}}{\rightsquigarrow} S_n \ .$$

In the following diagram, let p_i be the y_i-canonical Σ-path from S_i to Can, and let q_i be the reverse of the y_i-canonical Σ-path from S_{i+1} to Can.



For every $i = 1, \ldots, n-2$, the Ω 2-cell corresponding to $\text{Can} \overset{q_i}{\rightsquigarrow} S_2 \overset{p_{i+1}}{\rightsquigarrow} \text{Can}$ is the identity, since p_{i+1} is a canonical Σ-path, q_i is the reverse of a canonical Σ-path, and canonical Σ-paths departing from the same Σ-scheme are equivalent. Consequently, we conclude that the given Σ-path is indeed equivalent to

$$S_1 \overset{p_1}{\dashrightarrow} \text{Can} \overset{q_{n-1}}{\dashrightarrow} S_n \quad . \tag{A.13}$$

Since all canonical Σ-paths from a given Σ-scheme to Can are equivalent, we conclude that all Σ-paths of interest from S_1 to S_n are equivalent to (A.13). □

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