DARBOUX-LIE DERIVATIVES

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ABSTRACT. We introduce the Darboux-Lie derivative along a vector field of fiber bundle maps from natural bundles to associated fiber bundles and study its properties.

1. INTRODUCTION

This article is an accompanying paper to the forthcoming series of articles by the same authors on the theory of G-structures. In the course of reformulating the basic notions pertaining to G-structures on manifolds in terms of gauge equivalence classes of soldering forms, we stumbled upon the absence of a properly developed calculus of derivatives for such forms and for gauge transformations. A *soldering form* β in this context refers to an isomorphism of vector bundles $\beta\colon TM \to P \times_G V$, where P is a principal G-bundle and V is a G-module (cf. [1]) and a *gauge transformation* is a section of the bundle $P \times_G {}_cG$.

In this article we introduce and study properties of Darboux-Lie and covariant Darboux-Lie derivatives of fiber bundle morphisms from $F(M)$ to $P \times_G N$, where M is a manifold of dimension n, F is a natural bundle on n-dimensional manifolds, P is a principal G-bundle over M, and N is a manifold equipped with a smooth left G-action. The case of soldering forms is covered by $F(M) = TM$ and $N = V$ and the case of gauge transformations corresponds to $F(M) = M$ and $N = {}_cG$. We choose the generality of an arbitrary natural bundle with a shot for future generalizations to high-order G-structures.

For a fixed G-module W and G-equivariant 1-form, the Darboux-Lie derivative $\pounds^\omega_{\widetilde{X}}\beta\colon F(M) \to P \times_G W$ of the bundle map $\beta\colon F(M) \to P \times_G N$ is taken along a G-equivariant vector field $\widetilde{X}$ on P. The complete definition, given in Section 4, is slightly too technical to be fully stated in the introduction.

If P is equipped with a G-principal connection, then the horizontal lift X^H of a vector field X on M is G-equivariant. We define the covariant Darboux-Lie derivative of β along X as $\pounds^\omega_{X^H}\beta$.

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We show in the article that various classically known derivatives (covariant, Lie, Darboux) can be considered as special cases of (covariant) Darboux-Lie derivatives. We also prove various properties, including a characterization of (covariant) Darboux-Lie derivatives in terms of flows and a suitable Leibniz rule. We conclude with the proof of the Cartan magic formula for covariant Darboux-Lie derivatives.

The paper is organized as follows. Section 2 contains some preliminary material. In particular, we fix the notation for principal and associated bundles, and recall the definition and main properties of natural bundles.

In Section 3, we discuss a metamathematical concept of derivative. This discussion culminates in the definition of an α-derivative. We show that this notion incorporates Lie derivatives of Janyška-Kolář and of Godina-Matteucci. The (covariant) Darboux-Lie derivatives are defined in Section 4 as a special case of α-derivative.

Section 5 demonstrates how classical Lie derivatives of tensor fields and standard covariant derivatives of sections of vector bundles can be seen as particular instances of the Darboux-Lie derivative.

In Section 6, we calculate the Darboux-Lie derivative along vertical G-invariant vector fields on P. This permits to compare covariant Darboux-Lie derivatives calculated with respect to two G-principal connections on P.

Section 7 is dedicated to establishing various Leibniz rules for the (covariant) Darboux-Lie derivative with respect to operations like fiber products, tensor products, and compositions of fiber bundle maps.

In Section 8, we derive a Cartan-type magic formula for covariant Darboux-Lie derivative when applied to differential forms with values in an associated bundle.

In Section 9, we announce some results related to G-structures.

2. Preliminaries

2.1. **Principal bundles.** Let G be a Lie group, M a smooth manifold and P a principal G-bundle. Then there is a unique smooth map

$$\begin{aligned} \backslash \colon P \times_M P &\to G \\ (p, p') &\mapsto p\backslash p', \end{aligned}$$

such that $p(p\backslash p') = p'$ for every pair p, p' in the same fiber of P. This map will be handy to explicitly write various maps involving principal bundles.

Clearly, $p\backslash p = e_G$. Moreover, for all $g \in G$ and $(p, p') \in P \times_M P$, we have

$$(pg)\backslash p' = g^{-1}(p\backslash p'), \quad p\backslash(p'g) = (p\backslash p')g, \quad p'\backslash p = (p\backslash p')^{-1}. \tag{1}$$

2.2. **Associated bundles.** Let N be a manifold equipped with a smooth left G-action. The *associated fiber bundle* $P \times_G N$ over M is defined as the quotient $(P \times N)/G$, where the group G acts on $P \times N$ by $(p, y)g = (pg, g^{-1}y)$ for $(p, y) \in P \times N$ and $g \in G$. We write $[p, y]$ for the G-orbit of (p, y) in $P \times N$. The projection from $P \times_G N$ to M is defined by sending the orbit $[p, y] = \{ (pg, g^{-1}y) \mid g \in G\}$ to the common base point of the elements pg, $g \in G$.

One can verify that the following map is both well-defined and smooth:

$$\begin{aligned} \backslash\colon P \times_M (P \times_G N) &\to N \\ (p, [p', y]) &\mapsto p\backslash[p', y] := (p\backslash p')y \end{aligned}$$

Notice that $p\backslash[p, y] = y$. This map is particularly useful for explicit construction of maps between associated bundles.

2.3. **Vector fields and their flows.** Given a vector field X on M, we write γ_x^X for the maximal integral curve of X that passes through $x \in M$ at time 0. We denote its interval of definition by $J_x^X \subset \mathbb{R}$. The union

$$D(X) := \bigcup_{x \in M} J_x^X \times \{x\} \subset \mathbb{R} \times M$$

is the *domain of the flow of* X and the *flow of* X is the map

$$\begin{aligned} \varphi_X\colon D(X) &\to M \\ (t, x) &\mapsto \gamma_x^X(t). \end{aligned}$$

For each $t \in \mathbb{R}$, we define the open set

$$U_t^X := \{x \in M \mid (t, x) \in D(X)\},$$

consisting of points whose integral curves are defined at time t. For a fixed $t \in \mathbb{R}$, the *flow operator* is the map

$$\Phi_X^t\colon U_t^X \to U_{-t}^X, \quad x \mapsto \varphi_X(t, x),$$

which is a diffeomorphism. These operators satisfy the usual composition rule:

$$\Phi_X^t \circ \Phi_X^s = \Phi_X^{t+s} \quad \text{on } U_s^X \cap U_{s+t}^X.$$

2.4. **Natural bundles.** A systematic treatment of natural bundles can be found in [6, Ch. IX]. Here we will just recall the definition of natural bundles and their properties that will be needed later. It is important to mention that tensor bundles are instances of natural bundles.

Write $\mathcal{M}f$ for the category of smooth manifolds with smooth maps between them and $\mathcal{M}f_n$ for its subcategory of manifolds of dimension n with local diffeomorphisms between them as morphisms. Denote by $\mathcal{E}_n$ the embedding of $\mathcal{M}f_n$ into $\mathcal{M}f$. One of the possible equivalent definitions of a natural bundle is to say that a *natural bundle* is a pair (F, p), where $F\colon \mathcal{M}f_n \to \mathcal{M}f$ is a functor and $p\colon F \to \mathcal{E}_n$ is a natural transformation such that

(a) for each $M \in \mathcal{M}f_n$ the map $p_M\colon F(M) \to M$ turns $F(M)$ into a fiber bundle over M;

(b) for each $f\colon M \to N$ in $\mathcal{M}f_n$ the map $F(f)\colon F(M) \to F(N)$ is a fiberwise diffeomorphism that covers $f\colon M \to N$, i.e. the commutative diagram

$$\begin{tikzcd} F(M) \arrow[r, "F(f)"] \arrow[d, "p_M"'] & F(N) \arrow[d, "p_N"] \\ M \arrow[r, "f"] & N \end{tikzcd} \tag{2}$$

is a pull-back square[1].

A *natural map* from a natural bundle (F,p) and to a natural bundle (F',p') is a natural transformation of functors $\eta\colon F \to F'$ such that $p = p' \circ \eta$. It is customary to refer to a natural bundle just by its functor part F.

One defines a *natural vector bundle* as a natural bundle (F,p) such that each $p_M\colon F(M) \to M$ is a vector bundle and for each $f\colon M \to N$ in $\mathcal{M}f_n$ the map $F(f)\colon F(M) \to F(N)$ is a morphism of vector bundles. Similarly, one defines *natural principal bundles.*

One of the key features of natural bundles is the existence of canonical lifts of vector fields. However, a clear statement and proof of the existence of canonical lifts for non-complete vector fields are difficult to find in the literature. For the sake of rigor, we include the following proposition.

Proposition 2.1. *Let (F,p) be a natural bundle on $\mathcal{M}f_n$. For every $M \in \mathcal{M}f_n$ there is a unique well-defined map*

$$\mathcal{F}\colon \mathfrak{X}(M) \to \mathfrak{X}(F(M))$$
$$X \mapsto \mathcal{F}(X)$$

such that for each $X \in \mathfrak{X}(M)$

(i) the vector field $\mathcal{F}(X)$ is a lift of X (called the canonical lift of X*);*

(ii) for all $x \in M$ and $y \in F(M)_x$ the intervals J^X_x and $J^{\mathcal{F}(X)}_y$ coincide;

(iii) for each $t \in \mathbb{R}$, the image of the embedding $F(U^X_t \hookrightarrow M)\colon F(U^X_t) \to F(M)$ coincides with $U^{\mathcal{F}(X)}_t$ and the diagram

$$\begin{tikzcd} F(U^X_t) \arrow[r, "F(\Phi^t_X)"] \arrow[d, "F(U^X_t \hookrightarrow M)"'] & F(U^X_{-t}) \arrow[d, "F(U^X_{-t} \hookrightarrow M)"] \\ U^{\mathcal{F}(X)}_t \arrow[r, "\Phi^t_{\mathcal{F}(X)}"] & U^{\mathcal{F}(X)}_{-t}. \end{tikzcd} \tag{3}$$

commutes.

Proof. Let $M \in \mathcal{M}f_n$ and $X \in \mathfrak{X}(M)$. The tricky point is that the family of diffeomorphisms $F(\Phi^t_X)$ is not a 1-parameter group on $F(M)$. Indeed, for a fixed

[1] Natural transformations with this property are known as *Cartesian natural transformations* in Category Theory.

t, the map $F(\Phi_X^t)$ is a diffeomorphism between $F(U_t^X)$ and $F(U_{-t}^X)$, but $F(U_t^X)$ and $F(U_{-t}^X)$ are not open subsets of $F(M)$ for a general natural bundle F. To overcome this peculiarity we use the fact that every natural bundle is isomorphic to an associated bundle of a suitable jet bundle.

For $M \in \mathcal{M}f_n$ and $k \geq 1$, write $\mathrm{inv}J_0^k(\mathbb{R}^n, M)$ for the space of invertible k-jets at 0 of smooth maps from $\mathbb{R}^n$ to M. Denote by G_n^k the subspace of $\mathrm{inv}J_0^k(\mathbb{R}^n, \mathbb{R}^n)$ consisting of jets that have value 0 at 0. Then G_n^k is a Lie group under composition of jets and $\mathrm{inv}J_0^k(\mathbb{R}^n, M)$ is a principal G_n^k-bundle.

Given a smooth map $\psi\colon \mathbb{R}^n \to M$ write $j^k(\psi)$ for the k-th jet of ψ at 0. Given a local diffeomorphism $f\colon M \to N$, we define

$$\begin{aligned}\mathrm{inv}J_0^k(\mathbb{R}^n, f)\colon \mathrm{inv}J_0^k(\mathbb{R}^n, M) &\to \mathrm{inv}J_0^k(\mathbb{R}^n, N)\\ j_0^k(\psi) &\mapsto j^k(f \circ \psi).\end{aligned}$$

This turns the functor $\mathrm{inv}J_0^k(\mathbb{R}^n, -)$ into a natural principal G_n^k-bundle on $\mathcal{M}f_n$.

One can check that for every manifold N equipped with a smooth G_n^k-action the functor $\mathrm{inv}J_0^k(\mathbb{R}^n, -)\times_{G_n^k} N$ is a natural bundle. It was proved in [2] that every natural bundle is isomorphic to a natural bundle of the form $\mathrm{inv}J_0^k(\mathbb{R}^n, -)\times_{G_n^k} N$.

Now, let k be a natural number and N a manifold equipped with a smooth left G_n^k-action such that F is isomorphic to $H = \mathrm{inv}J_0^k(\mathbb{R}^n, -)\times_{G_n^k} N$. We will show that there is $\mathcal{H}\colon \mathfrak{X}(M) \to \mathfrak{X}(H(M))$ that satisfies the properties stated in the proposition with F and $\mathcal{F}$ replaced by H and $\mathcal{H}$, respectively. Then $\mathcal{F}(X)$ is obtained from $\mathcal{H}(X)$ via a natural isomorphism between $F(M)$ and $H(M)$. The stated properties for $\mathcal{F}$ follow from the corresponding properties for $\mathcal{H}$, since the isomorphisms are natural.

To get a vector field $\mathcal{H}(X)$ on $H(M)$, we construct its flow on $H(M)$. Write σ_t for the embedding of U_t^X into M. Define the open subsets $\widetilde{U}_t$ of $H(M)$ by

$$\widetilde{U}_t = \left\{\, [j^k(\psi), y] \in H(M) \,\middle|\, \mathrm{Im}(\psi) \subset U_t^X \right\}$$

and the diffeomorphisms $\widetilde{\Phi}^t\colon \widetilde{U}_t \to \widetilde{U}_{-t}$ by

$$\widetilde{\Phi}^t([j^k(\psi), y]) = [j^k(\Phi_X^t \circ \psi), y].$$

It is easy to check that $\widetilde{\Phi}^t$ is a local flow. We define $\mathcal{H}(X)$ as the unique vector field on $H(M)$ such that $\Phi_{\mathcal{H}(X)}^t(z) = \widetilde{\Phi}^t(z)$ for all $z \in \widetilde{U}_t$. Write p for the canonical projection from $H(M)$ to M. Thus $p([j^k(\psi), y]) = \psi(0)$ for all $[j^k(\psi), y] \in H(M)$. Then for all $x \in M$ and $[j^k(\psi), y] \in H(M)_x \cap \widetilde{U}_t$, we get

$$p(\Phi_{\mathcal{H}(X)}^t([j^k(\psi), y])) = p([j^k(\Phi_X^t \circ \psi), y]) = (\Phi_X^t \circ \psi)(0) = \Phi_X^t(x).$$

Thus $\mathcal{H}(X)$ is a lift of X. Moreover, $\widetilde{U}_t \subset U_t^{\mathcal{H}(X)}$, which implies that $\bigcup_{t\in\mathbb{R}}\{t\}\times\widetilde{U}_t \subset D(\mathcal{H}(X))$. Thus, for every $x \in M$ and $z \in H(M)_x$ the interval $J_z^{\mathcal{H}(X)}$ contains the interval J_x^X. But, since $\mathcal{H}(X)$ is a lift of X, also $J_z^{\mathcal{H}(X)} \subset J_x^X$. Thus $J_z^{\mathcal{H}(X)} = J_x^X$ as claimed. We also conclude that $U_t^{\mathcal{H}(X)} = \widetilde{U}_t$ for all $t \in \mathbb{R}$.

It is clear that if $\psi' \colon \mathbb{R}^n \to U_t^X$ then $\sigma_t \circ \psi' \colon \mathbb{R}^n \to M$ has its image in U_t^X. Also for every $\psi \colon \mathbb{R}^n \to M$ with $\mathrm{Im}(\psi) \subset U_t^X$ there is a unique $\psi' \colon \mathbb{R}^n \to U_t^X$ such that $\psi = \sigma_t \circ \psi'$. This shows that $\widetilde{U}_t$, and hence also $U_t^{\mathcal{H}(X)}$, is the image of $H(\sigma_t)$. The commutativity of the diagram (3) follows from a straightforward computation. ☐

Let $\eta \colon F' \to F$ be a natural map between natural bundles on $\mathcal{M}f_n$ and $M \in \mathcal{M}f_n$. Then for every $X \in \mathfrak{X}(M)$, the vector fields $\mathcal{F}'(X)$ and $\mathcal{F}(X)$ are η_M-related. To see this, let $x \in M$ and $y \in F'(M)_x$. We have

$$
\begin{aligned}
T\eta_M(\mathcal{F}'(X)_y) &= \left.\frac{d}{dt}\right|_{t=0} \eta_M\left(\Phi^t_{\mathcal{F}'(X)} y\right) \\
\mathcal{F}(X)_{\eta_M(y)} &= \left.\frac{d}{dt}\right|_{t=0} \Phi^t_{\mathcal{F}(X)}(\eta_M(y)).
\end{aligned} \tag{4}
$$

For $t \in \mathbb{R}$, write σ_t for the embedding $U_t^X \hookrightarrow M$. Then $F'(\sigma_t)$ is a diffeomorphism between $F'(U_t^X)$ and $U_t^{\mathcal{F}'(X)} \subset F'(M)$. Denote by y_t the unique element of $F'(U_t)$ such that $F'(\sigma_t)(y_t) = y$. Using the commutativity of (3) and the naturality of η_M, we get

$$
\begin{aligned}
\eta_M(\Phi^t_{\mathcal{F}'(X)} y) &= \eta_M(\Phi^t_{\mathcal{F}'(X)} \circ F'(\sigma_t)(y_t)) = \eta_M(F'(\sigma_{-t}) \circ F'(\Phi^t_X)(y_t)) \\
&= F(\sigma_{-t}) \circ F(\Phi^t_X)(\eta_{U_t}(y_t)) = \Phi^t_{\mathcal{F}(X)} \circ F(\sigma_t)(\eta_{U_t}(y_t)) \\
&= \Phi^t_{\mathcal{F}(X)}(\eta_M(F'(\sigma_t)(y_t))) = \Phi^t_{\mathcal{F}(X)}(\eta_M(y)).
\end{aligned}
$$

Thus (4) implies that

$$
T\eta_M(\mathcal{F}'(X)_y) = \mathcal{F}(X)_{\eta_M(y)}, \tag{5}
$$

i.e. the vector fields $\mathcal{F}'(X)$ and $\mathcal{F}(X)$ are η_M-related.

3. Derivatives

We start with a discussion of what one might expect from a notion of derivative. Given a map h between two smooth manifolds, the tangent map Th captures the first-order behavior of h. The drawback of Th is that it retains h inside it. A metamathematical idea of a derivative of h would be Th with h stripped out and only the first-order information retained. It is evident how to do this in the case of $\mathbb{R}$-valued smooth functions. Namely, given a smooth function $f \colon M \to \mathbb{R}$ the derivative $df \colon TM \to \mathbb{R}$ is determined by

$$
(T_x f)(X) = df(X) \left.\frac{d}{dt}\right|_{t=f(x)}.
$$

In other words $df = dt \circ Tf$. The key point of the definition is the existence of the canonical nowhere zero 1-form dt on $\mathbb{R}$. More generally, let N be an n-dimensional manifold and $\omega\colon TN \to \mathbb{R}^n$ a 1-form on N such that $\omega_y\colon T_yN \to \mathbb{R}^n$ is an isomorphism for every $y \in N$. Then for a smooth map $h\colon M \to N$, one can define its derivative as $\omega \circ Th$. Notice that a form ω with the above property exists if and only if N is a parallelizable manifold. For this reason we call such a form *a parallelization form.*

Of course, $\omega \circ Th$ depends on the choice of the parallelization form ω. For example, in the case N is a Lie group H, there are two natural choices for ω: either left-invariant or right-invariant Maurer-Cartan form. In this case the resulting derivative is called *left (right) Darboux derivative*, respectively.

Another standard example of a derivative is the *covariant derivative* of a section $s\colon M \to P \times_G V$, where P is a principal G-bundle with a fixed principal connection and V is a G-module. In [5], Janyška and Kolář developed a general theory of Lie derivatives that includes as special cases the usual Lie derivatives and covariant derivatives.

Below, we introduce the notion of α-derivatives that captures the idea of derivative explained above. For this, we use the following auxiliary construction.

Let M_1 and M_2 be smooth manifolds and $h\colon M_1 \to M_2$ a smooth map. For a pair of vector fields $X_1 \in \mathfrak{X}(M_1)$, $X_2 \in \mathfrak{X}(M_2)$, we call the map

$$\begin{aligned} \tilde{\mathcal{L}}_{(X_1,X_2)}h\colon M_1 &\to TM_2 \\ x &\mapsto (T_xh)(X_1) - (X_2)_{h(x)}. \end{aligned}$$

the Trautman lift of h. Alternatively (cf. [6, Lemma 47.2]), we have

$$(\tilde{\mathcal{L}}_{(X_1,X_2)}h)_x = \left.\frac{d}{dt}\right|_{t=0} \Phi^{-t}_{X_2} \circ h \circ \Phi^t_{X_1}(x). \tag{6}$$

Remark 3.1. The map $\tilde{\mathcal{L}}_{(X_1,X_2)}h$ was initially introduced by Trautman in [7] without a specific name. It was later referred to as the "generalized Lie derivative" in a series of articles by Kolář and his coauthors, with the main results summarized in [6, Ch. XI]. However, we have chosen to deviate from this terminology because "generalized Lie derivatives" do not align with our concept of a derivative and never specialize to Lie derivatives. Instead, they serve as an intermediate step in the construction of Lie derivatives.

The Trautman lift is particularly useful when $h : \mathbf{F}_1 \to \mathbf{F}_2$ is a map of fiber bundles over a manifold M and X_1, X_2 are lifts of the same vector field on M. In this case, the values of $\tilde{\mathcal{L}}_{(X_1,X_2)}h$ lie in the vertical subbundle $\mathcal{V}\mathbf{F}_2$ of the tangent bundle $T\mathbf{F}_2$.

Definition 3.2. Let $h\colon \mathbf{F}_1 \to \mathbf{F}_2$ be a bundle morphism over M and (X_1, X_2) a pair of lifts of the same vector field on M to $\mathbf{F}_1$ and to $\mathbf{F}_2$, respectively. For a vector bundle E over M and a vector bundle map $\alpha\colon \mathcal{V}\mathbf{F}_2 \to E$ that covers the projection $\mathbf{F}_2 \to M$, we denote the composite

$$\alpha \circ \tilde{\mathcal{L}}_{(X_1,X_2)}h\colon \mathbf{F}_1 \to E$$

by $\mathcal{L}^{\alpha}_{(X_1,X_2)}h$ and call it the *α-derivative of h along* (X_1, X_2).

Remark 3.3. The α-derivative carries all the first-order information about h provided that, for each base point $x \in M$ and each $y \in \mathbf{F}_{2,x}$, the map

$$\alpha_y : \mathcal{V}_y\mathbf{F}_2 \to E_x \tag{7}$$

is injective.

A potential issue arises if the images of α_y vary with the choice of y inside the same fiber $\mathbf{F}_{2,x}$. In that case, the α-derivative may contain *more* than just differential information: it might even allow one to recover the actual values of h in the fiber.

To avoid this, one can assume that, for each $x \in M$, the image of α_y is the same for all $y \in \mathbf{F}_{2,x}$. Imposing injectivity and this independence of the image ensures that $\operatorname{Im}\alpha$ forms a vector subbundle of E. Thus, the notion of α-derivative fully aligns with the intended philosophy of a derivative if and only if α induces a fiberwise isomorphism onto a subbundle of E. Of course, by replacing E with this subbundle, nothing is lost.

In all instances we are aware of where the α-derivative specializes to a previously studied derivative (e.g., Lie or covariant derivative), the corresponding choice of α is a fiberwise isomorphism onto E. We have nevertheless not imposed this assumption in the definition, since doing so would prevent a clean formulation of the Leibniz rule for α-derivatives.

A family of suitable choices of α can be obtained using vertical splittings. A *vertical splitting* for a fiber bundle $\mathbf{F}_2$ over M is an isomorphism of vector bundles $\beta\colon \mathcal{V}\mathbf{F}_2 \to \mathbf{F}_2 \times_M E$ over $\mathbf{F}_2$ for some vector bundle E over M. Then $\alpha = \mathrm{pr}_2 \circ \beta$ is a vector bundle map from $\mathcal{V}\mathbf{F}_2$ to E that covers the projection $\mathbf{F}_2 \to M$ such that the maps α_y are isomorphisms.

If $\mathbf{F}_2$ is a vector bundle on its own, then the inverse of the *vertical lift*

$$\begin{aligned} \mathrm{vl}_{\mathbf{F}_2}\colon \mathbf{F}_2 \times_M \mathbf{F}_2 &\to \mathcal{V}\mathbf{F}_2 \\ (u, v) &\mapsto \left.\frac{d}{dt}\right|_{t=0} (u + vt) \end{aligned}$$

is an example of vertical splitting. The composite $\mathrm{pr}_2 \circ \mathrm{vl}^{-1}_{\mathbf{F}_2}$ is called the *vertical projection* and is denoted by $\mathrm{vpr}_{\mathbf{F}_2}$. The $\mathrm{vpr}_{\mathbf{F}_2}$-derivative $\mathcal{L}^{\mathrm{vpr}_{\mathbf{F}_2}}_{(X_1,X_2)}h$ is called the Lie derivative of h with respect to X_1 and X_2 in [6, Section 47.7]. This derivative was introduced by Janyška and Kolář in [5].

Now, we return to the context of a general fiber bundle $\mathbf{F}_2$, but assume that $\mathbf{F}_1 = M$ with the projection map id_M. In this setup $h\colon M \to \mathbf{F}_2$ is just a section of $\mathbf{F}_2$. If $X_2 \in \mathfrak{X}(\mathbf{F}_2)$ is a projectable vector field lifting $X \in \mathfrak{X}(M)$ and $\beta\colon \mathcal{V}\mathbf{F}_2 \to \mathbf{F}_2 \times_M E$ is a vertical splitting, then the $(\mathrm{pr}_2 \circ \beta)$-derivative of h along (X, X_2) was called the (restricted) Lie derivative of h with respect to X_2 by Godina and Matteucci in [3, 4].

The Lie derivatives of Janyška-Kolář and of Godina-Matteucci can be further specialized to get classical Lie and covariant derivatives.

Example 3.4 (*Lie derivative*). Suppose F is a natural bundle and E a natural vector bundle on $\mathcal{M}f_n$. Given $M \in \mathcal{M}f_n$, a vector field $X \in \mathfrak{X}(M)$ and a smooth map $h\colon F(M) \to E(M)$, *the Lie derivative* $\mathcal{L}_X h$ *of* h *along* X is the $\mathrm{vpr}_{E(M)}$-derivative of h along $(\mathcal{F}(X), \mathcal{E}(X))$. It coincides with the Lie derivative defined in textbooks in the case both $F(M)$ and $E(M)$ are tensor bundles over M.

Example 3.5 (*Covariant derivative*). If E is a vector bundle equipped with a linear connection then the covariant derivative $\nabla_X s$ of a section $s\colon M \to E$ is the vpr_E-derivative of s along (X, X^∇), where X^∇ is the horizontal lift of $X \in \mathfrak{X}(M)$ (see [6, Section 47.5]).

Despite the generality of α-derivatives, it is still possible to obtain some results for them. Namely, we will show that there is a version of Leibniz rule for an α-derivative of a composition.

We start with the Leibniz rule for the Trautman lift of a composition. The following proposition follows directly from the definition of the Trautman lift.

Proposition 3.6. *Let M_1, M_2, M_3 be manifolds and $X_i \in \mathfrak{X}(M_i)$, $i = 1, 2, 3$. For any smooth maps $h_1\colon M_1 \to M_2$, $h_2\colon M_2 \to M_3$, we have*

$$\tilde{\mathcal{L}}_{(X_1,X_3)}(h_2 \circ h_1) = Th_2 \circ \tilde{\mathcal{L}}_{(X_1,X_2)} h_1 + (\tilde{\mathcal{L}}_{(X_2,X_3)} h_2) \circ h_1. \tag{8}$$

Notice that the vector field X_2 is not present on the left-hand side of (8) and thus can be chosen arbitrarily on the right-hand side.

The above proposition immediately implies the following result.

Corollary 3.7. *Let $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ be fiber bundles over a manifold M and $h_1\colon \mathbf{F}_1 \to \mathbf{F}_2$, $h_2\colon \mathbf{F}_2 \to \mathbf{F}_3$ fiber bundle maps. Suppose $\alpha\colon \mathcal{V}\mathbf{F}_3 \to E$ is a vector bundle map covering the projection from $\mathbf{F}_3$ to M. Write $h_2^*\alpha$ for the composite of α with the restriction of Th_2 to $\mathcal{V}\mathbf{F}_2$. It is a vector bundle map covering the projection from $\mathbf{F}_2$ to M. For every triple $(X_1, X_2, X_3) \in \mathfrak{X}(\mathbf{F}_1) \times \mathfrak{X}(\mathbf{F}_2) \times \mathfrak{X}(\mathbf{F}_3)$ of projectable vector fields lifting the same vector field on M, we get*

$$\mathcal{L}^{\alpha}_{(X_1,X_3)}(h_2 \circ h_1) = \mathcal{L}^{h_2^*\alpha}_{(X_1,X_2)} h_1 + (\mathcal{L}^{\alpha}_{(X_2,X_3)} h_2) \circ h_1. \tag{9}$$

4. Darboux-Lie derivatives

The peculiarity of α-derivative is that it is defined along pairs of vector fields on fiber bundles. This is a striking contrast with classical derivatives that are defined along vector fields on the base manifold. The utility of α-derivative, as demonstrated by Examples 3.4 and 3.5, lies in the fact that usual derivatives can be expressed in terms of α-derivative. Thus α-derivatives provide a framework to introduce new types of derivatives. By specializing the notion of α-derivative, we introduce the notion of Darboux-Lie derivative taken along invariant vector fields on a principal bundle over the base manifold. We accompany this notion by the notion of covariant Darboux-Lie derivative taken along vector fields on the base manifold, but additionally depending on the choice of a principal connection.

Let G be a Lie group, P a principal G-bundle and N a manifold equipped with a smooth left G-action. We will identify $\mathcal{V}(P \times_G N)$ with $P \times_G TN$ via the isomorphism

$$\begin{aligned} \nu \colon \mathcal{V}(P \times_G N) &\to P \times_G TN \\ \left.\frac{d}{dt}\right|_{t=0} [\gamma_P(t), \gamma_N(t)] &\mapsto \left[\gamma_P(0), \left.\frac{d}{dt}\right|_{t=0} (\gamma_P(0)\backslash\gamma_P(t))\gamma_N(t)\right], \end{aligned} \tag{10}$$

where γ_P is a vertical curve in P and γ_N is an arbitrary curve in N, both defined on the same open interval containing 0.

Suppose $\widetilde{X}$ is a G-invariant vector field on the principal G-bundle P. Write $\widetilde{X}_N$ for the vector field on $P \times_G N$, whose flow is given by $\Phi^t_{\widetilde{X}_N}[p, y] = [\Phi^t_{\widetilde{X}} p, y]$. Notice that the vector fields $\widetilde{X}$ and $\widetilde{X}_N$ are both projectable and correspond to the same vector field X on M.

Let F be a natural bundle on $\mathcal{M}f_n$. Recall that $\mathcal{F}(X)$ is the canonical lift of X to $F(M)$, whose existence is proved in Proposition 2.1. For a fiber bundle map $h \colon F(M) \to P \times_G N$, we denote the Trautman lift $\tilde{\mathcal{L}}_{(\mathcal{F}(X), \widetilde{X}_N)} h$ by $\tilde{\pounds}_{\widetilde{X}} h$.

Let V be a G-module. For a G-equivariant 1-form $\omega \colon TN \to V$ define

$$\begin{aligned} \mathrm{id} \times_G \omega \colon P \times_G TN &\to P \times_G V \\ [p, Y] &\mapsto [p, \omega(Y)]. \end{aligned}$$

It is clear that $\mathrm{id} \times_G \omega$ is a vector bundle map covering the projection $P \times_G N \to M$.

Definition 4.1. Let F be a natural bundle on $\mathcal{M}f_n$, M an n-dimensional smooth manifold, P a principal G-bundle, and N a smooth manifold equipped with a G-action. Suppose $\omega \colon TN \to V$ is a G-equivariant 1-form and $\widetilde{X} \in \mathfrak{X}(P)^G$. For a bundle map $h \colon F(M) \to P \times_G N$, the *Darboux-Lie derivative*

$$\pounds^{\omega}_{\widetilde{X}} h \colon F(M) \to P \times_G V$$

of h along $\widetilde{X}$ with respect to ω is the α-derivative $\mathcal{L}^{\alpha}_{(\mathcal{F}(X), \widetilde{X}_N)} h$ with $\alpha = (\mathrm{id} \times_G \omega) \circ \nu$, i.e. $\pounds^{\omega}_{\widetilde{X}} h = (\mathrm{id} \times_G \omega) \circ \nu \circ \tilde{\pounds}_{\widetilde{X}} h$.

Definition 4.2. If P is additionally equipped with a G-principal connection, we define the *covariant Darboux-Lie derivative* $\mathcal{L}^{\omega,\nabla}_X h$ of h along $X \in \mathfrak{X}(M)$ with respect to ω to be $\pounds^{\omega}_{X^H} h$, where X^H is the horizontal lift of X to P.

If N is a G-module V, then there is the canonical G-equivariant parallelization form $\mathrm{vpr}_V := \mathrm{pr}_2 \circ \mathrm{vl}_V^{-1}$ on V, where

$$\begin{aligned}\mathrm{vl}_V \colon V \oplus V &\to TV \\ (v,w) &\mapsto \left.\frac{d}{dt}\right|_{t=0} (v+wt).\end{aligned}$$

Remark 4.3. If γ is a curve in V, then $\mathrm{vpr}_V(\dot\gamma(0))$ coincides with the derivative of $\gamma(t)$ at 0 as it is usually defined in standard Calculus courses.

Another interesting case is when N coincides with G considered with the conjugation action. In this case, we can take ω to be the left-invariant Maurer-Cartan form on G, defined by

$$\begin{aligned}\omega_{MC} \colon TG &\to \mathfrak{g} = T_eG \\ T_gG \ni Z &\mapsto (T_eL_g)^{-1}(Z).\end{aligned}$$

In these two cases we will skip ω in $\pounds^{\omega}_{\widetilde X}$ and write $\pounds_{\widetilde X} h$ instead of $\pounds^{\mathrm{vpr}_V}_{\widetilde X} h$ or $\pounds^{\omega_{MC}}_{\widetilde X} h$. Similarly, we write $\mathcal{L}^{\nabla}_X h$ for $\mathcal{L}^{\mathrm{vpr}_V,\nabla}_X h$ and for $\mathcal{L}^{\omega_{MC},\nabla}_X h$.

The Darboux-Lie derivatives $\pounds^{\omega}_{\widetilde X}$ can be expressed in terms of flows.

Proposition 4.4. *Let $h\colon F(M) \to P \times_G N$ be a fiber bundle map over M. For all $x \in M$, $p \in P_x$ and $y \in F(M)_x$*

$$\pounds^{\omega}_{\widetilde X} h(y) = \left[p, \omega\left(\left.\frac{d}{dt}\right|_0 (\Phi^t_{\widetilde X} p)\backslash h(\Phi^t_{\mathcal{F}(X)}(y))\right)\right] \in P \times_G V. \tag{11}$$

Proof. From (6), we get

$$\begin{aligned}\tilde{\pounds}_{\widetilde X} h(y) &= \left.\frac{d}{dt}\right|_0 \Phi^{-t}_{\widetilde X_N} \circ h \circ \Phi^t_{\mathcal{F}(X)}(y) \\ &= \left.\frac{d}{dt}\right|_0 (\Phi^{-t}_{\widetilde X} \times_G \mathrm{id})[\Phi^t_{\widetilde X} p, (\Phi^t_{\widetilde X} p)\backslash h(\Phi^t_{\mathcal{F}(X)}(y))] \\ &= \left.\frac{d}{dt}\right|_0 [p, (\Phi^t_{\widetilde X} p)\backslash h(\Phi^t_{\mathcal{F}(X)}(y))].\end{aligned}$$

Now the result follows from $\pounds^{\omega}_{\widetilde X} h = (\mathrm{id} \times_G \omega) \circ \nu \circ \tilde{\pounds}_{\widetilde X} h$. □

5. Examples and relation to other definitions

Many known derivatives can be expressed in terms of Darboux-Lie derivative.

First we consider the case of Lie derivatives as defined in Example 3.4. Let F be a natural bundle and E a natural vector bundle. Then E is isomorphic to, and thus can be replaced by, the functor $\mathrm{inv}J^k_0(\mathbb{R}^n, -) \times_{G^k_n} V$ for a suitable G^k_n-module V. For $X \in \mathfrak{X}(M)$, denote by X^c the canonical lift of X to $\mathrm{inv}J^k_0(\mathbb{R}^n, M)$. Then

X^c is G_n^k-invariant, and X_V^c coincides with the canonical lift $\mathcal{E}(X)$ of X to $E(M)$. Hence for $h\colon F(M) \to E(M) = \mathrm{inv}J_0^k(\mathbb{R}^n, M)\times_{G_n^k} V$

$$\mathcal{L}_X h = \mathcal{L}_{(\mathcal{F}(X),\mathcal{E}(X))}^{\mathrm{vpr}_{E(M)}} h = \pounds_{X^c}^{\mathrm{vpr}_V} h. \tag{12}$$

5.1. **Covariant derivative.** Let M be a manifold and E an m-dimensional vector bundle on M. For an m-dimensional vector space W, we denote by $L(W, E)$ the bundle of W-frames in E, thus $L(W, E)$ is the set of maps $f\colon W \to E$ that induce an isomorphism between W and E_x for some $x \in M$ depending on f. The set $L(W, E)$ is a principal $\mathrm{GL}(W)$-bundle over M and E is canonically isomorphic to the associated bundle $L(W, E) \times_{\mathrm{GL}(W)} W$ via the evaluation map

$$\begin{aligned} \mathrm{ev}_E\colon L(W, E) \times_{\mathrm{GL}(W)} W &\to E \\ [f, w] &\mapsto f(w). \end{aligned}$$

Let ∇ be a linear connection on E. For a vector field X on M, denote its horizontal lift to E by X^∇. We get an induced connection on the principal $\mathrm{GL}(W)$-bundle $L(W, E)$, for which the horizontal lift X^H of X has the flow

$$\begin{aligned} \Phi_{X^H}^t\colon L(W, E) &\to L(W, E) \\ f &\mapsto (w \mapsto \Phi_{X^\nabla}^t(f(w))). \end{aligned}$$

We can express the covariant derivatives from Example 3.5 in terms of covariant-Darboux-Lie derivative.

Proposition 5.1. *Let ∇ be a linear connection on E. If $h\colon M \to E$ is a section of a vector bundle E, then for every $X \in \mathfrak{X}(M)$*

$$\nabla_X h = \mathrm{ev}_E \circ \mathcal{L}_X^\nabla(\mathrm{ev}_E^{-1} \circ h). \tag{13}$$

Proof. As we already explained in Example 3.5, we have

$$\nabla_X h = \mathcal{L}_{(X,X^\nabla)}^{\mathrm{vpr}_E} h = \mathrm{vpr}_E \circ \tilde{\mathcal{L}}_{(X,X^\nabla)} h = \mathrm{pr}_2 \circ \mathrm{vl}_E^{-1} \circ \tilde{\mathcal{L}}_{(X,X^\nabla)} h.$$

Hence, for every $x \in M$, we get by (6)

$$\begin{aligned} \nabla_X h(x) &= \mathrm{vpr}_E \left(\left.\frac{d}{dt}\right|_{t=0} \Phi_{X^\nabla}^{-t} \circ h \circ \Phi_X^t(x)) \right) \\ &= \mathrm{vpr}_E \left(\left.\frac{d}{dt}\right|_{t=0} \Phi_{X^\nabla}^{-t} \circ h \circ \Phi_X^t(x)) \right). \end{aligned} \tag{14}$$

On the other hand

$$\mathcal{L}_X^\nabla(\mathrm{ev}_E^{-1} \circ h) = (\mathrm{id} \times \mathrm{vpr}_W) \circ \nu \circ \tilde{\mathcal{L}}_{(X,X_W^H)}(\mathrm{ev}_E^{-1} \circ h), \tag{15}$$

where $\nu\colon \mathcal{V}(L(W, E) \times_G W) \xrightarrow{\cong} L(W, E) \times_G TW$ is given by (10). By (6), we get

$$\tilde{\mathcal{L}}_{(X,X_W^H)}(\mathrm{ev}_E^{-1} \circ h)(x) = \left.\frac{d}{dt}\right|_{t=0} \left(\Phi_{X_W^H}^{-t} \circ \mathrm{ev}_E^{-1} \circ h \circ \Phi_X^t\right)(x). \tag{16}$$

Write $c(t)$ for $\Phi^t_X(x)$. Observe that $h(\Phi^t_X(x)) \in E_{c(t)}$ and $\Phi^t_{X^H} f \colon W \xrightarrow{\cong} E_{c(t)}$ is a frame at $c(t)$. Thence

$$\mathrm{ev}_E^{-1}\left(h\left(\Phi^t_X(x)\right)\right) = \left[\Phi^t_{X^H} f, (\Phi^t_{X^H} f)^{-1} \circ h \circ \Phi^t_X(x)\right].$$

Plug in this into (16) and using the definition of X^H_W, we get

$$\tilde{\mathcal{L}}_{(X,X^H_W)}(\mathrm{ev}_E^{-1} \circ h)(x) = \left.\frac{d}{dt}\right|_{t=0} \left[f, (\Phi^t_{X^H} f)^{-1} \circ h \circ \Phi^t_X(x)\right].$$

Applying (10) and (15), we get

$$\mathcal{L}^\nabla_X(\mathrm{ev}_E^{-1} \circ h)(x) = \left[f, \mathrm{vpr}_W\left(\left.\frac{d}{dt}\right|_{t=0} (\Phi^t_{X^H} f)^{-1} \circ h \circ \Phi^t_X(x)\right)\right].$$

Therefore

$$\mathrm{ev}_E\left(\mathcal{L}^\nabla_X(\mathrm{ev}_E^{-1} \circ h)(x)\right) = f\left(\mathrm{vpr}_W\left(\left.\frac{d}{dt}\right|_{t=0} (\Phi^t_{X^H} f)^{-1} \circ h \circ \Phi^t_X(x)\right)\right). \tag{17}$$

Write $\tilde{c}(t)$ for $h(\Phi^t_X(x))$. We have $\tilde{c}(t) \in E_{c(t)}$. Since $\Phi^t_{X^H} f(w) = \Phi^t_{X^\nabla}(f(w))$ for all $w \in W$, we get

$$(\Phi^t_{X^H} f)^{-1}(\tilde{c}(t)) = f^{-1}\left(\Phi^{-t}_{X^\nabla}(\tilde{c}(t))\right).$$

Therefore

$$\mathrm{ev}_E\left(\mathcal{L}^\nabla_X(\mathrm{ev}_E^{-1} \circ h)(x)\right) = f\left(\mathrm{vpr}_W\left(\left.\frac{d}{dt}\right|_{t=0} f^{-1}\left(\Phi^{-t}_{X^\nabla} \circ h \circ \Phi^t_X(x)\right)\right)\right). \tag{18}$$

It is left to check that the right hand sides of (14) and (18) coincide, i.e. that

$$f\left(\mathrm{vpr}_W\left(\left.\frac{d}{dt}\right|_{t=0} f^{-1}\left(\Phi^{-t}_{X^\nabla} \circ h \circ \Phi^t_X(x)\right)\right)\right) = \mathrm{vpr}_E\left(\left.\frac{d}{dt}\right|_{t=0} \Phi^{-t}_{X^\nabla} \circ h \circ \Phi^t_X(x)\right).$$

Let $\gamma(t)$ be the curve $f^{-1}(\Phi^{-t}_{X^\nabla} \circ h \circ \Phi^t_X(x))$ in W. Then the last equation can be written as

$$f\left(\mathrm{vpr}_W(\dot{\gamma}(0))\right) = \mathrm{vpr}_E\left(\left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma)(t)\right)$$

or, equivalently, as

$$f\left(\mathrm{vpr}_W(\dot{\gamma}(0))\right) = \mathrm{vpr}_E\left(Tf\left(\dot{\gamma}(0)\right)\right).$$

Hence, we only have to ensure that $f \circ \mathrm{vpr}_W = \mathrm{vpr}_E \circ Tf$. Every element of TW can be written as $\left.\frac{d}{dt}\right|_{t=0}(w_0 + w_1 t)$ for suitable w_0, $w_1 \in W$. We get

$$\begin{aligned} f \circ \mathrm{vpr}_W\left(\left.\frac{d}{dt}\right|_{t=0} (w_0 + w_1 t)\right) &= f \circ \mathrm{pr}_2 \circ \mathrm{vl}_W^{-1}\left(\left.\frac{d}{dt}\right|_{t=0} (w_0 + w_1 t)\right) \\ &= f(\mathrm{pr}_2(w_0, w_1)) = f(w_1) \end{aligned}$$

and, using that f is linear,

$$\begin{aligned}\mathrm{vpr}_E\left(Tf\left(\left.\frac{d}{dt}\right|_{t=0}(w_0+w_1t)\right)\right) &= \mathrm{pr}_2\circ \mathrm{vl}_E^{-1}\left(\left.\frac{d}{dt}\right|_{t=0}(f(w_0)+f(w_1)t\right)\\ &= \mathrm{pr}_2\left(f(w_0),f(w_1)\right)=f(w_1).\end{aligned}$$

Hence the result. □

6. Darboux-Lie derivatives with respect to vertical right-invariant vector fields

Write $\mathfrak{g}$ for the Lie algebra of the group G. For each section a of $P\times_G\mathfrak{g}$, we denote by X^a the vertical vector field on P whose flow is given by

$$\Phi^t_{X^a}p = p\exp((p\backslash a(x))t), \quad \text{for } x\in M, p\in P_x.$$

It is a G-invariant vector field on P. Conversely, one can show that if $\widetilde{X}$ is a G-invariant vertical vector field on P then it coincides with X^a, where

$$\begin{aligned} a\colon M &\to P\times_G\mathfrak{g}\\ x &\mapsto \left[p, \left.\frac{d}{dt}\right|_0 p\backslash\Phi^t_{\widetilde{X}}p\right],\end{aligned}$$

for an arbitrary $p\in P_x$.

Let V be a G-module. The induced action of $\dot\gamma(0)\in\mathfrak{g}$ on $v\in V$ is defined by

$$\dot\gamma(0)v = \mathrm{vpr}_V\left(\left.\frac{d}{dt}\right|_0\gamma(t)v\right). \tag{19}$$

For $a\in\Gamma(M,P\times_G\mathfrak{g})$ and $h\colon F(M)\to P\times_G V$, we define $a\cdot h\colon F(M)\to P\times_G V$ by

$$a\cdot h(y) = [p,(p\backslash a(x))(p\backslash h(y))],$$

where $y\in F(M)$, x is the base point of the fiber of y and p is an arbitrary point in the fiber P_x.

Proposition 6.1. *Let V be a G-module. For every $a\in\Gamma(M,P\times_G\mathfrak{g})$ and every map $h\colon F(M)\to P\times_G V$ of fiber bundles over M, we have $\pounds_{X^a}h = -a\cdot h$.*

Proof. Let $x\in M$, $y\in F(M)_x$ and $p\in P_x$. From Proposition 4.4, it follows that

$$p\backslash\pounds_{X^a}h(y) = \mathrm{vpr}_V\left(\left.\frac{d}{dt}\right|_0(\Phi^t_{X^a}p)\backslash h(y)\right).$$

Applying the definition of $\Phi^t_{X^a}$, we get

$$p\backslash\pounds_{X^a}h(y) = \mathrm{vpr}_V\left(\left.\frac{d}{dt}\right|_0(p\exp((p\backslash a(x))t))\backslash h(y)\right).$$

As for every $g \in G$ and $z \in (P \times_G V)_x$, we have $(pg)\backslash z = g^{-1}(p\backslash z)$, the above formula can be rewritten in the form

$$p\backslash \pounds_{X^a} h(y) = \mathrm{vpr}_V \left(\left.\frac{d}{dt}\right|_0 \exp(-(p\backslash a(x))t)(p\backslash h(y)) \right).$$

By (19), we get $p\backslash \pounds_{X^a} h(y) = -(p\backslash a(x))(p\backslash h(y))$. Hence $\pounds_{X^a} h = -a \cdot h$. $\square$

When the target bundle is $P \times_{G\ c} G$ instead of $P \times_G V$ the resulting formula is more interesting. To state the result, we need additional notation. For $h \colon F(M) \to P \times_{G\ c} G$, we deviate from the standard conventions and denote by h^{-1} the map from $F(M)$ to $P \times_{G\ c} G$ given by $h^{-1}(y) = [p, (p\backslash h(y))^{-1}]$, where $p \in P$ lies over the same point in M as y.

Further, for $a \in \Gamma(M, P \times_G \mathfrak{g})$, we define $\mathrm{Ad}_h(a) \colon F(M) \to P \times_G \mathfrak{g}$ by

$$\mathrm{Ad}_h(a)(y) = [p, \mathrm{Ad}_{p\backslash h(y)}(p\backslash a(x))],$$

for all $x \in M$, $y \in F(M)_x$ and any $p \in P_x$.

Proposition 6.2. *For every $a \in \Gamma(M, P \times_G \mathfrak{g})$ and $h \colon F(M) \to P \times_{G\,c} G$, we have $\pounds_{X^a} h = a \circ p_M - \mathrm{Ad}_{h^{-1}}(a)$, where $p_M \colon F(M) \to M$ is the bundle projection.*

Proof. Write $*_c$ for the conjugation action of G on itself, i.e. $h *_c g = hgh^{-1}$ for all $h, g \in G$. Let $x \in M$, $p \in P_x$ and $y \in F(M)_x$. By the same chain of arguments as in the proof of Proposition 6.1, we get

$$p\backslash \pounds_{X^a} h(y) = \omega_{MC} \left(\left.\frac{d}{dt}\right|_0 \exp(-(p\backslash a(x))t) *_c (p\backslash h(y)) \right).$$

For any $b \in \mathfrak{g}$ and $g \in G$, we have $\exp(-bt) *_c g = \exp(-bt) g \exp(bt)$. Next

$$\begin{aligned} \omega_{MC} \left(\left.\frac{d}{dt}\right|_0 \exp(-bt) g \exp(bt) \right) &= \left.\frac{d}{dt}\right|_0 g^{-1} \exp(-bt) g \exp(bt) \\ &= -\mathrm{Ad}_{g^{-1}}(b) + b. \end{aligned}$$

Therefore with $b = p\backslash a(x)$ and $g = p\backslash h(y)$, we get

$$p\backslash \pounds_{X^a} h(y) = -\mathrm{Ad}_{(p\backslash h(y))^{-1}}(p\backslash a(x)) + (p\backslash a(x)).$$

This proves the proposition. $\square$

More generally, for a G-equivariant V-valued 1-form ω on N, we define $*_\omega \colon \mathfrak{g} \times N \to V$ by

$$b *_\omega z = \omega \left(\left.\frac{d}{dt}\right|_0 \exp(bt) z \right). \tag{20}$$

Then adapting the first part of the proof of Proposition 6.1, we get

$$p\backslash \pounds^{\omega}_{X^a} h(y) = -(p\backslash a(x)) *_\omega (p\backslash h(y)), \tag{21}$$

for all $y \in F(M)$. This equation can be rewritten as

$$\mathcal{L}^{\omega}_{X^a} h = -a \hat{*}_{\omega} h, \tag{22}$$

for a suitably defined $\hat{*}_{\omega}$.

In the case P is equipped with two G-principal connections, we can compare the corresponding covariant Darboux-Lie derivatives $\mathcal{L}^{\nabla}_{X}$ and $\mathcal{L}^{\widetilde{\nabla}}_{X}$. Write X^H and $X^{\widetilde{H}}$ for the horizontal lifts of the vector field $X \in \mathfrak{X}(M)$ to P with respect to these connections. Then $X^H - X^{\widetilde{H}}$ is a vertical vector field on P and there is a unique section $a \in \Gamma(M, P \times_G \mathfrak{g})$ such that $X^H - X^{\widetilde{H}} = X^a$. If V is a G-module and $h\colon F(M) \to P \times_G V$ is a vector bundle map, Proposition 6.1 implies that

$$\mathcal{L}^{\nabla}_{X} h - \mathcal{L}^{\widetilde{\nabla}}_{X} h = -a \cdot h. \tag{23}$$

If $h\colon F(M) \to P \times_G {}_cG$ is a fiber bundle map, Proposition 6.2 implies that

$$\mathcal{L}^{\nabla}_{X} h - \mathcal{L}^{\widetilde{\nabla}}_{X} h = a \circ p_M - \mathrm{Ad}_{h^{-1}}(a), \tag{24}$$

where $p_M\colon F(M) \to M$ is the projection of the fiber bundle $F(M)$.

7. Leibniz rule

In future articles, we will need the expressions for $\mathcal{L}_{\widetilde{X}}(s \cdot \beta)$, where $s \in \Gamma(M, P \times_G {}_cG)$ and $\beta \in \Omega^1(M, P \times_G V)$, and for $\mathcal{L}_{\widetilde{X}}(\alpha \wedge \beta)$, where β is as before and $\alpha \in \Omega^1(M, P \times_G \mathfrak{g})$. This section aims to explore the Leibniz rule for the Darboux-Lie derivative with respect to some binary operations.

Notice that the map $s \cdot \beta$ can be described as the composition

$$TM \xrightarrow{\cong} M \times_M TM \xrightarrow{s \times_M \beta} P \times_G ({}_cG \times V) \to P \times_G V$$

and $\alpha \wedge \beta$ as the composition

$$\Lambda^2 TM \to TM \otimes TM \xrightarrow{\alpha \otimes \beta} P \times_G (\mathfrak{g} \otimes V) \to P \times_G V.$$

Thus in both cases we have a composition of

- a natural map $\eta_M\colon F(M) \to F_1(M) * F_2(M)$, where $*$ is either $\times$ or $\otimes$;
- a suitably defined product $h_1 * h_2\colon F_1(M) * F_2(M) \to P \times_G (N_1 * N_2)$ of $h_1\colon F_1(M) \to P \times_G N_1$ and $h_2\colon F_2(M) \to P \times_G N_2$;
- the map from $P \times_G (N_1 * N_2) \to P \times_G N_3$ induced by a map $N_1 * N_2 \to N_3$.

Below we develop a machinery to deal with Darboux-Lie derivatives of compositions of the above type.

Applying Proposition 3.6 and (5), we conclude that for every natural map $\eta_M\colon F'(M) \to F(M)$ and for $h\colon F(M) \to P \times_G N$

$$\tilde{\mathcal{L}}_{\widetilde{X}}(h \circ \eta_M) = Th \circ \tilde{\mathcal{L}}_{(\mathcal{F}'(X), \mathcal{F}(X))} \eta_M + (\tilde{\mathcal{L}}_{\widetilde{X}} h) \circ \eta_M = (\tilde{\mathcal{L}}_{\widetilde{X}} h) \circ \eta_M.$$

Hence

$$\mathcal{L}^{\omega}_{\widetilde{X}}(h \circ \eta_M) = (\mathcal{L}^{\omega}_{\widetilde{X}} h) \circ \eta_M. \tag{25}$$

7.1. **Darboux-Lie derivative of a product.** Suppose F_1 and F_2 are natural bundles. Define the natural bundle $F_1 \times F_2$ by $(F_1 \times F_2)(M) = F_1(M) \times_M F_2(M)$.

Given $h_1 \colon F_1(M) \to P \times_G N_1$ and $h_2 \colon F_2(M) \to P \times_G N_2$, define

$$\begin{aligned} h_1 \times_M h_2 \colon (F_1 \times F_2)(M) &\to P \times_G (N_1 \times N_2) \\ (y_1, y_2) &\mapsto [p, (p\backslash h_1(y_1), p\backslash h_2(y_2))], \end{aligned}$$

where $y_1 \in F_1(M)_x$, $y_2 \in F_2(M)_x$, $p \in P_x$ for some $x \in M$. Let $\omega_1 \colon TN_1 \to V_1$ and $\omega_2 \colon TN_2 \to V_2$ be G-equivariant 1-forms with values in G-modules V_1 and V_2. Define $\omega_1 \times \omega_2 \colon T(N_1 \times N_2) \to V_1 \oplus V_2$ by

$$\omega_1 \times \omega_2 \left(\frac{d}{dt}\bigg|_0 (\gamma_1(t), \gamma_2(t)) \right) = (\omega_1(\dot\gamma_1(0)), \omega_2(\dot\gamma_2(0))) .$$

By Proposition 4.4, we get for $\widetilde{X} \in \mathfrak{X}(P)^G$ over $X \in \mathfrak{X}(M)$

$$\begin{aligned} &p\backslash \pounds^{\omega_1 \times \omega_2}_{\widetilde{X}} (h_1 \times_M h_2)(y_1, y_2) = \\ &= \omega_1 \times \omega_2 \left(\frac{d}{dt}\bigg|_0 \Phi^t_{\widetilde{X}} p\backslash (h_1 \times_M h_2)(\Phi^t_{\mathcal{F}_1(X)} y_1, \Phi^t_{\mathcal{F}_2(X)} y_2) \right) \\ &= \omega_1 \times \omega_2 \left(\frac{d}{dt}\bigg|_0 \left(\Phi^t_{\widetilde{X}} p\backslash h_1 \circ \Phi^t_{\mathcal{F}_1(X)}(y_1) \,,\, \Phi^t_{\widetilde{X}} p\backslash h_2 \circ \Phi^t_{\mathcal{F}_2(X)}(y_2) \right) \right) \\ &= \left(\omega_1 \left(\frac{d}{dt}\bigg|_0 \Phi^t_{\widetilde{X}} p\backslash h_1 \circ \Phi^t_{\mathcal{F}_1(X)}(y_1) \right) , \omega_2 \left(\frac{d}{dt}\bigg|_0 \Phi^t_{\widetilde{X}} p\backslash h_2 \circ \Phi^t_{\mathcal{F}_2(X)}(y_2) \right) \right) \\ &= \left(p\backslash \pounds^{\omega_1}_{\widetilde{X}} h_1(y_1), p\backslash \pounds^{\omega_2}_{\widetilde{X}} h_2(y_2) \right) . \end{aligned}$$

Hence

$$\pounds^{\omega_1 \times \omega_2}_{\widetilde{X}} (h_1 \times_M h_2) = \pounds^{\omega_1}_{\widetilde{X}} h_1 \times_M \pounds^{\omega_2}_{\widetilde{X}} h_2. \tag{26}$$

7.2. **Darboux-Lie derivative of a tensor product.** For natural vector bundles E_1, E_2, we define the natural vector bundle $E_1 \otimes E_2$ by $(E_1 \otimes E_2)(M) = E_1(M) \otimes E_2(M)$. Given two vector bundle maps $h_1 \colon E_1(M) \to P \times_G V_1$ and $h_2 \colon E_2(M) \to P \times_G V_2$, we define

$$\begin{aligned} h_1 \otimes h_2 \colon (E_1 \otimes E_2)(M) &\to P \times_G (V_1 \otimes V_2) \\ y_1 \otimes y_2 &\mapsto [p, (p\backslash h_1(y_1)) \otimes (p\backslash h_2(y_2))], \end{aligned}$$

where $y_1 \in E_1(M)_x$, $y_2 \in E_2(M)_x$, $p \in P_x$ for some $x \in M$. We have for all $x \in M$, $y_1 \in E_1(M)_x$, $y_2 \in E_2(M)_x$

$$\Phi^t_{(\mathcal{E}_1 \otimes \mathcal{E}_2)(X)}(y_1 \otimes y_2) = \Phi^t_{\mathcal{E}_1(X)}(y_1) \otimes \Phi^t_{\mathcal{E}_2(X)}(y_2).$$

By Proposition 4.4, we get for $\widetilde{X} \in \mathfrak{X}(P)^G$ over $X \in \mathfrak{X}(M)$

$$
\begin{aligned}
&p\backslash \pounds_{\widetilde{X}}(h_1 \otimes h_2)(y_1 \otimes y_2) = \\
&\quad = \mathrm{vpr}_{V_1 \otimes V_2}\left(\left.\frac{d}{dt}\right|_0 \Phi^t_{\widetilde{X}} p\backslash (h_1 \otimes h_2)(\Phi^t_{\mathcal{F}_1(X)} y_1 \otimes \Phi^t_{\mathcal{F}_2(X)} y_2)\right) \\
&\quad = \mathrm{vpr}_{V_1 \otimes V_2}\left(\left.\frac{d}{dt}\right|_0 \Phi^t_{\widetilde{X}} p\backslash h_1 \circ \Phi^t_{\mathcal{F}_1(X)}(y_1) \otimes \Phi^t_{\widetilde{X}} p\backslash h_2 \circ \Phi^t_{\mathcal{F}_2(X)}(y_2) \right).
\end{aligned}
$$

Given curves $\gamma_1\colon I \to V_1$ and $\gamma_2\colon I \to V_2$, we have

$$
\begin{aligned}
\mathrm{vpr}_{V_1 \otimes V_2}\left(\left.\frac{d}{dt}\right|_0 \gamma_1(t) \otimes \gamma_2(t) \right) &= \mathrm{vpr}_{V_1}(\dot{\gamma}_1(0)) \otimes \gamma_2(0) \\
&\quad + \gamma_1(0) \otimes \mathrm{vpr}_{V_2}(\dot{\gamma}_2(0)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
p\backslash \pounds_{\widetilde{X}}(h_1 \otimes h_2)(y_1 \otimes y_2) &= (p\backslash \pounds_{\widetilde{X}} h_1(y_1)) \otimes (p\backslash h_2(y_2)) \\
&\quad + (p\backslash h_1(y_1)) \otimes (p\backslash \pounds_{\widetilde{X}} h_2(y_2)).
\end{aligned} \tag{27}
$$

Thence

$$
\pounds_{\widetilde{X}}(h_1 \otimes h_2) = \pounds_{\widetilde{X}} h_1 \otimes h_2 + h_1 \otimes \pounds_{\widetilde{X}} h_2. \tag{28}
$$

7.3. **Darboux-Lie derivative of a composition.** In this subsection we deal with the Darboux-Lie derivative of $(\mathrm{id} \times_G f) \circ h$, where $h\colon F(M) \to P \times_G N$ and f is a G-equivariant map from N to N'. Suppose $\omega'\colon TN' \to V'$ is a G-equivariant 1-form with values in a G-module V'. By Proposition 4.4, for $h\colon F(M) \to P \times_G N$ and every $x \in M$, $y \in F(M)_x$ and $p \in P_x$

$$
\begin{aligned}
p\backslash \pounds^{\omega'}_{\widetilde{X}}((\mathrm{id} \times_G f) \circ h)(y) &= \omega' \left(\left.\frac{d}{dt}\right|_0 \Phi^t_{\widetilde{X}} p\backslash ((\mathrm{id} \times_G f) \circ h \circ \Phi^t_{\mathcal{F}(X)}(y)) \right) \\
&= f^*\omega' \left(\left.\frac{d}{dt}\right|_0 \Phi^t_{\widetilde{X}} p\backslash (h \circ \Phi^t_{\mathcal{F}(X)}(y)) \right) \\
&= p\backslash \pounds^{f^*\omega'}_{\widetilde{X}} h(y).
\end{aligned}
$$

Hence

$$
\pounds^{\omega'}_{\widetilde{X}}((\mathrm{id} \times_G f) \circ h) = \pounds^{f^*\omega'}_{\widetilde{X}} h. \tag{29}
$$

The ability to write the above formula is the reason we don't require ω to be a parallelization form in the definition of $\pounds^{\omega}_{\widetilde{X}}$, since even if ω' is a parallelization form its pull-back $f^*\omega'$, in general, is not.

Formula (29) becomes more useful if there is a relation between $f^*\omega'$ and a G-equivariant 1-form $\omega\colon TN \to V$, where V is a G-module. The vector space $\mathrm{Hom}_{\mathbb{R}}(V, V')$ has the G-module structure defined by $(g\alpha)(v) = g\alpha(g^{-1}v)$ for all

$g \in G$, $\alpha \in \mathrm{Hom}_{\mathbb{R}}(V, V')$ and $v \in V$. Now, assume that there is a G-equivariant map $\varphi\colon N \to \mathrm{Hom}_{\mathbb{R}}(V, V')$ such that

$$(f^*\omega')_z = \varphi(z) \circ \omega_z \tag{30}$$

for each $z \in N$. Let $x \in M$, $y \in F(M)_x$ and $p \in P_x$. Denote the curve $\Phi^t_{\widetilde{X}} p\backslash(h \circ \Phi^t_{\mathcal{F}(X)}(y)$ by γ. By Proposition 4.4, we get

$$p\backslash \pounds^{f^*\omega'}_{\widetilde{X}} h(y) = f^*\omega'(\dot\gamma(0)) = \varphi(p\backslash h(y))(\omega(\dot\gamma(0))) = \varphi(p\backslash h(y))(p\backslash \pounds^{\omega}_{\widetilde{X}} h(y)). \tag{31}$$

Hence

$$\pounds^{\omega'}_{\widetilde{X}}((\mathrm{id} \times_G f) \circ h)(y) = \pounds^{f^*\omega'}_{\widetilde{X}} h(y) = ((\mathrm{id} \times_G \varphi)\, h(y))\, (\pounds^{\omega}_{X} h(y)), \tag{32}$$

where we define an action of $\Gamma(M, P \times_G \mathrm{Hom}_{\mathbb{R}}(V, V'))$ on $\Gamma(M, P \times_G V)$ by

$$(\psi(s))(x) = [p, (p\backslash\psi(x))(p\backslash s(x))],$$

for all $\psi \in \Gamma(M, P \times_G \mathrm{Hom}_{\mathbb{R}}(V, V'))$, $s \in \Gamma(M, P \times_G V)$, $x \in M$ and $p \in P_x$.

Next we give several examples of N, N', ω, ω' and f for which there exists a map φ satisfying (30).

Example 7.1. Take $N = V$, $N' = V'$, $f\colon V \to V'$ a linear G-equivariant map, $\omega = \mathrm{vpr}_V$ and $\omega' = \mathrm{vpr}_{V'}$. In this case $f^*\omega' = f \circ \omega$. Hence $\varphi\colon V \to \mathrm{Hom}_{\mathbb{R}}(V, V')$ is the constant map that sends each $v \in V$ into f and (32) becomes

$$\pounds_{\widetilde{X}}((\mathrm{id} \times_G f) \circ h)(y) = (\mathrm{id} \times_G f)(\pounds_{\widetilde{X}} h(y)). \tag{33}$$

Example 7.2. Take $N = G \times V$, $N' = V$ and $f\colon G \times V \to V$ to be the action of G on V. We equip N with the form $\omega := \omega_{MC} \times \mathrm{vpr}_V$ and N' with the form $\omega' := \mathrm{vpr}_V$. Fix $g \in G$ and $v \in V$. Every element of $T_{(g,v)}(G \times V)$ is of the form $\frac{d}{dt}\big|_0 (g\exp(at), v + ut)$ for suitable $a \in \mathfrak{g}$ and $u \in V$. We get

$$\begin{aligned} f^*\omega'\left(\frac{d}{dt}\bigg|_0 (ge^{at}, v + ut)\right) &= \mathrm{vpr}_V\left(\frac{d}{dt}\bigg|_0 \left(ge^{at}v + ge^{at}ut\right)\right) \\ &= g(av) + gu = g(av + u). \end{aligned} \tag{34}$$

Notice that

$$a = \frac{d}{dt}\bigg|_0 \exp(at) = (T_e L_g)^{-1} \frac{d}{dt}\bigg|_0 g\exp(at) = \omega_{MC}\left(\frac{d}{dt}\bigg|_0 g\exp(at)\right)$$

and

$$u = \mathrm{vpr}_V\left(\frac{d}{dt}\bigg|_0 (v + ut)\right).$$

Hence

$$(\omega_{MC} \times \mathrm{vpr}_V)\left(\frac{d}{dt}\bigg|_0 (ge^{at}, v + ut)\right) = (a, u)\,. \tag{35}$$

Now (34) and (35) imply that for all $(g, v) \in G \times V$

$$(f^*\omega')_{(g,v)} = \varphi(g, v) \circ ((\omega_{MC})_g \times (\mathrm{vpr}_V)_v),$$

where

$$\begin{aligned}\varphi\colon G\times V &\to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}\oplus V, V)\\ (g,v)&\mapsto ((a,u)\mapsto g(av+u)).\end{aligned}\tag{36}$$

Example 7.3. Let H be a Lie group equipped with a left G-action by Lie automorphisms. Take $N = H\times H$, $N' = H$ and $f\colon H\times H\to H$ the multiplication map. Write ω_H for the Maurer-Cartan form on H. It is G-equivariant, since G acts on H by automorphisms. The Maurer-Cartan form on the direct product $H\times H$ coincides with $\omega_H\times\omega_H$.

For h_1, $h_2\in H$ and a_1, $a_2\in\mathfrak{h} = T_eH$, we have

$$\begin{aligned} f^*\omega_H\left(\left.\frac{d}{dt}\right|_0 (h_1e^{a_1t}, h_2e^{a_2t})\right) &= \omega_H\left(\left.\frac{d}{dt}\right|_0 h_1h_2h_2^{-1}e^{a_1t}h_2e^{a_2t}\right)\\ &= \omega_H\left(\left.\frac{d}{dt}\right|_0 h_1h_2\exp(\mathrm{Ad}_{h_2^{-1}}(a_1t))e^{a_2t}\right)\\ &= \left.\frac{d}{dt}\right|_0 \exp(\mathrm{Ad}_{h_2^{-1}}(a_1)t)\exp(a_2t)\\ &= \mathrm{Ad}_{h_2^{-1}}(a_1)+a_2.\end{aligned}$$

Hence $(f^*\omega_H)_{h_1,h_2} = \varphi(h_1,h_2)\circ(\omega_H\times\omega_H)_{h_1,h_2}$, where

$$\begin{aligned}\varphi\colon H\times H &\to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}\oplus\mathfrak{h},\mathfrak{h})\\ (h_1,h_2)&\mapsto ((a_1,a_2)\mapsto \mathrm{Ad}_{h_2^{-1}}(a_1)+a_2).\end{aligned}\tag{37}$$

7.4. **Synthesis.** The aim of this subsection is to obtain rather general Lie-type formulas by combining the results already obtained in this section. We will treat the special cases in examples.

Let F_1, F_2 be natural bundles and $h_1\colon F_1(M)\to P\times_G N_1$, $h_2\colon F_2(M)\to P\times_G N_2$ morphisms of fibred manifolds over M. Suppose $\omega_1\colon TN_1\to V_1$, $\omega_2\colon TN_2\to V_2$, $\omega\colon TN\to V$ are G-equivariant forms, $f\colon N_1\times N_2\to N$ is a G-equivariant map, and $\varphi\colon N_1\times N_2\to\mathrm{Hom}_{\mathbb{R}}(V_1\oplus V_2, V)$ is a smooth map such that

$$(f^*\omega)_z = \varphi(z)\circ(\omega_1\times\omega_2)_z.$$

Let $\eta\colon F\to F_1\times F_2$ be a natural bundle map. Define

$$h_1\times_{f,\eta} h_2\colon F(M)\to P\times_G N$$

to be the composition $(\mathrm{id}\times_G f)\circ(h_1\times_M h_2)\circ\eta_M$.

Fix $y\in F(M)$. Then $\eta_M(y) = (y_1,y_2)$ for suitable $y_1\in F_1(M)$, $y_2\in F_2(M)$. Using (25) and (32), we get

$$\begin{aligned}\pounds^{\omega}_{\widetilde{X}}(h_1\times_{f,\eta}h_2)(y) &= \pounds^{\omega}_{\widetilde{X}}((\mathrm{id}\times_G f)\circ(h_1\times_M h_2)\circ\eta_M)(y)\\ &= \pounds^{\omega}_{\widetilde{X}}((\mathrm{id}\times_G f)\circ(h_1\times_M h_2))(y_1,y_2)\\ &= (\mathrm{id}\times_G\varphi)((h_1\times_M h_2)(y_1,y_2))(\pounds^{\omega_1\times\omega_2}_{\widetilde{X}}(h_1\times_M h_2)(y_1,y_2)).\end{aligned}\tag{38}$$

By definition of $h_1 \times_M h_2$, we have

$$(h_1 \times_M h_2)(y_1, y_2) = [p, (p\backslash h_1(y_1), p\backslash h_2(y_2))].$$

Hence

$$(\mathrm{id} \times_G \varphi)((h_1 \times_M h_2)(y_1, y_2)) = [p, \varphi(p\backslash h_1(y_1), p\backslash h_2(y_2))].$$

By (26), we have

$$\pounds^{\omega_1 \times \omega_2}_{\widetilde{X}}(h_1 \times_M h_2)(y_1, y_2) = [p, (p\backslash \pounds^{\omega_1}_{\widetilde{X}} h_1(y_1), p\backslash \pounds^{\omega_2}_{\widetilde{X}} h_2(y_2))].$$

Thence (38) becomes

$$\begin{aligned} \pounds^{\omega}_{\widetilde{X}}(h_1 \times_{f,\eta} h_2)(y) = [p, \varphi(p\backslash h_1(y_1), p\backslash h_2(y_2)) \\ (p\backslash \pounds^{\omega_1}_{\widetilde{X}} h_1(y_1), p\backslash \pounds^{\omega_2}_{\widetilde{X}} h_2(y_2))]. \end{aligned} \tag{39}$$

Example 7.4. Suppose $s \in \Gamma(M, P \times_G {}_cG)$ and $\beta\colon F(M) \to P \times_G V$, where V is a G-module. We define $\eta_M\colon F(M) \to M \times_M F(M)$ to be $Z \mapsto (x, Z)$ for every $x \in M$ and $Z \in F(M)_x$. Write f for the action map $G \times V \to V$. Then $s \cdot \beta$ defined at the beginning of the section coincides with $s \times_{f,\eta} \beta$. Taking φ defined by (36) and applying (39), we get for $x \in M$, $p \in P_x$ and $Z \in T_xM$

$$\begin{aligned} p\backslash \pounds_{\widetilde{X}}(s \cdot \beta)(Z) &= \varphi(p\backslash s(x), p\backslash \beta(Z))(p\backslash \pounds_{\widetilde{X}} s(x), p\backslash \pounds_{\widetilde{X}} \beta(Z)) \\ &= (p\backslash s(x)) \left((p\backslash \pounds_{\widetilde{X}} s(x))(p\backslash \beta(Z)) + p\backslash \pounds_{\widetilde{X}} \beta(Z) \right). \end{aligned}$$

Hence

$$\pounds_{\widetilde{X}}(s \cdot \beta) = s \cdot (\pounds_{\widetilde{X}} s \cdot \beta + \pounds_{\widetilde{X}} \beta). \tag{40}$$

If P is equipped with a G-principal connection and $X \in \mathfrak{X}(M)$, then we get the following property for the covariant Darboux-Lie derivative

$$\mathcal{L}^{\nabla}_X(s \cdot \beta) = s \cdot (\mathcal{L}^{\nabla}_X s \cdot \beta + \mathcal{L}^{\nabla}_X \beta). \tag{41}$$

Example 7.5. Now let s_1, $s_2 \in \Gamma(M, P \times_G {}_cG)$. Notice that the multiplication map $\mu_G\colon {}_cG \times {}_cG \to {}_cG$ is G-equivariant. Define $\eta_M\colon M \to M \times_M M$ by $\eta_M(x) = (x, x)$. We write $s_1 \cdot s_2$ for $s_1 \times_{\mu_G,\eta} s_2$. Taking φ defined by (37) and applying (39), we get for all $x \in M$ and $p \in P_x$,

$$\begin{aligned} p\backslash \pounds_{\widetilde{X}}(s_1 \cdot s_2)(x) &= \varphi(p\backslash s_1(x), p\backslash s_2(x))(p\backslash \pounds_{\widetilde{X}} s_1(x), p\backslash \pounds_{\widetilde{X}} s_2(x)) \\ &= \mathrm{Ad}_{(p\backslash s_2(x))^{-1}} \left(p\backslash \pounds_{\widetilde{X}} s_1(x) \right) + (p\backslash \pounds_{\widetilde{X}} s_2(x)). \end{aligned}$$

Thus with appropriate definitions for s_2^{-1} and $\mathrm{Ad}_{s_2^{-1}}$, we get

$$\pounds_{\widetilde{X}}(s_1 \cdot s_2) = \mathrm{Ad}_{s_2^{-1}} \left(\pounds_{\widetilde{X}} s_1 \right) + \pounds_{\widetilde{X}} s_2. \tag{42}$$

In the case P is equipped with a G-principal connection, this implies

$$\mathcal{L}^{\nabla}_X(s_1 \cdot s_2) = \mathrm{Ad}_{s_2^{-1}}(\mathcal{L}^{\nabla}_X s_1) + \mathcal{L}^{\nabla}_X s_2. \tag{43}$$

Now we turn our attention to the Leibniz rule involving tensor product. Let E_1, E_2 be natural vector bundles and $h_1\colon E_1(M) \to P \times_G V_1$, $h_2\colon E_2(M) \to P \times_G V_2$ morphisms of vector bundles, $f\colon V_1 \otimes V_2 \to V$ a homomorphism of G-modules, and $\eta\colon E \to E_1 \otimes E_2$ a natural transformation of natural vector bundles. Write $h_1 \otimes_{f,\eta} h_2$ for the composition $(\mathrm{id} \times_G f) \circ (h_1 \otimes h_2) \circ \eta_M$.

Fix $x \in M$, $y \in E(M)_x$, and $p \in P_x$. Then $\eta_M(y)$ can be written as $\sum_{i\in I} y_{1,i} \otimes y_{2,i}$ for suitable elements $y_{1,i} \in E_1(M)_x$ and $y_{2,i} \in E_2(M)_x$. Using (25) and (33), we get

$$\begin{aligned}\pounds_{\widetilde{X}}(h_1 \otimes_{f,\eta} h_2)(y) &= \pounds_{\widetilde{X}}((\mathrm{id} \times_G f) \circ (h_1 \otimes h_2) \circ \eta_M)(y)\\ &= \pounds_{\widetilde{X}}((\mathrm{id} \times_G f) \circ (h_1 \otimes h_2))\,(\eta_M(y))\\ &= (\mathrm{id} \times_G f)\left(\pounds_{\widetilde{X}}(h_1 \otimes h_2)\,(\eta_M(y))\right).\end{aligned}$$

Applying (27), we get

$$\begin{aligned}p\backslash \pounds_{\widetilde{X}}(h_1 \otimes h_2)\,(\eta_M(y)) = \sum_{i\in I}&(p\backslash \pounds_{\widetilde{X}} h_1(y_{1,i}) \otimes p\backslash h_2(y_{2,i})\\ &+ p\backslash h_1(y_{1,i}) \otimes p\backslash \pounds_{\widetilde{X}} h_2(y_{2,i})).\end{aligned}$$

Thence

$$\begin{aligned}\pounds_{\widetilde{X}}(h_1 \otimes_{f,\eta} h_2)(y) = \sum_{i\in I}&[p, f(p\backslash \pounds_{\widetilde{X}} h_1(y_{1,i}) \otimes p\backslash h_2(y_{2,i}))]\\ &+ [p, f(p\backslash h_1(y_{1,i}) \otimes p\backslash \pounds_{\widetilde{X}} h_2(y_{2,i}))].\end{aligned} \tag{44}$$

Example 7.6. Let $\alpha \in \Omega^1(M, P \times_G \mathfrak{g})$ and $\beta \in \Omega^1(M, P \times_G V)$, where V is a G-module. Define $\alpha \wedge \beta \in \Omega^2(M, P \times_G V)$ by

$$(\alpha \wedge \beta)(Z_1, Z_2) = \alpha(Z_1) \cdot \beta(Z_2) - \alpha(Z_2) \cdot \beta(Z_1).$$

Alternatively, it can be written as the composition $(\mathrm{id} \times_G f) \circ (\alpha \otimes \beta) \circ \eta_M$, where

$$\begin{aligned}\eta_M\colon \Lambda^2 TM &\to TM \otimes TM\\ Z_1 \wedge Z_2 &\mapsto Z_1 \otimes Z_2 - Z_2 \otimes Z_1\end{aligned}$$

and $f\colon \mathfrak{g} \otimes V \to V$ is the action of $\mathfrak{g}$ on V. Notice that f is G-equivariant. Thence $\alpha \wedge \beta = \alpha \otimes_{f,\eta} \beta$. Applying (44), we get

$$\begin{aligned}p\backslash \pounds_{\widetilde{X}}(\alpha \wedge \beta)(Z_1, Z_2) &= (p\backslash \pounds_{\widetilde{X}} \alpha(Z_1))(p\backslash \beta(Z_2)) + (p\backslash \alpha(Z_1))(p\backslash \pounds_{\widetilde{X}} \beta(Z_2))\\ &\quad - (p\backslash \pounds_{\widetilde{X}} \alpha(Z_2))(p\backslash \beta(Z_1)) - (p\backslash \alpha(Z_2))(p\backslash \pounds_{\widetilde{X}} \beta(Z_1)).\end{aligned}$$

Hence

$$\pounds_{\widetilde{X}}(\alpha \wedge \beta) = \pounds_{\widetilde{X}} \alpha \wedge \beta + \alpha \wedge \pounds_{\widetilde{X}} \beta. \tag{45}$$

Remark 7.7. More generally we can take $\alpha \in \Omega^k(M, P \times_G \mathfrak{g})$ and $\beta \in \Omega^\ell(M, P \times_G V)$. Then $\alpha \wedge \beta$ can be defined as $\alpha \otimes_{f,\eta} \beta$, where f is the same as above and

$$\begin{aligned} \eta_M \colon \Lambda^{k+\ell} TM &\to \Lambda^k TM \otimes \Lambda^\ell TM \\ Z_1 \wedge \cdots \wedge Z_{k+\ell} &\mapsto \sum_{\sigma \in \mathrm{Sh}_{k,\ell}} (-1)^\sigma Z_{\sigma(1)} \wedge \cdots \wedge Z_{\sigma(k)} \otimes Z_{\sigma(k+1)} \wedge \cdots \wedge Z_{\sigma(k+\ell)}, \end{aligned}$$

where $\mathrm{Sh}_{k,l}$ is the set of all (k,ℓ)-shuffles. As expected, for every $\widetilde{X} \in \mathfrak{X}(P)^G$ one gets

$$\pounds_{\widetilde{X}}(\alpha \wedge \beta) = \pounds_{\widetilde{X}}\alpha \wedge \beta + \alpha \wedge \pounds_{\widetilde{X}}\beta. \tag{46}$$

In the case P is equipped with a G-principal connection and $X \in \mathfrak{X}(M)$, this implies

$$\mathcal{L}^\nabla_X(\alpha \wedge \beta) = \mathcal{L}^\nabla_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}^\nabla_X \beta. \tag{47}$$

8. Cartan magic formula

Suppose ∇ is a covariant derivative on a vector bundle E Then the exterior covariant derivative d^∇ of $\beta \in \Omega^k(M, E)$ is given by

$$\begin{aligned} d^\nabla \beta(X_0, \dots, X_k) := &\sum_{j=0}^k (-1)^j \nabla_{X_j}(\beta(X_0, \dots, \widehat{X}_j, \dots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \beta([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k). \end{aligned}$$

In the special case when ∇ is a covariant derivative on $P \times_G V$ induced by a G-principal connection on P, we get

$$\nabla_{X_j}(\beta(X_0, \dots, \widehat{X}_j, \dots, X_k)) = \mathcal{L}^\nabla_{X_j}(\beta(X_0, \dots, \widehat{X}_j, \dots, X_k)).$$

Indeed, let $s \in \Gamma(M, P \times_G V)$ and X^H the horizontal lift of X to P. Then X^H_V it the ∇-horizontal lift of X to $P \times_G V$ and by the definition of covariant Darboux-Lie derivative

$$\mathcal{L}^\nabla_X(s)(x) = \mathcal{L}_{(X, X^H_V)}(s)(x) = \mathrm{vpr}_V \left(\left.\frac{d}{dt}\right|_{t=0} \Phi^{-t}_{X^H_V} \circ s \circ \Phi^t_X(x) \right) = (\nabla_X s)(x).$$

Hence

$$\begin{aligned} d^\nabla \beta(X_0, \dots, X_k) = &\sum_{j=0}^k (-1)^j \mathcal{L}^\nabla_{X_j}(\beta(X_0, \dots, \widehat{X}_j, \dots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \beta([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k). \end{aligned} \tag{48}$$

Let $Y_1, \dots, Y_k$ and Z be vector fields on M. The section $\beta(Y_1, \dots, Y_k)$ of $P \times_G V$ is the composition of $\wedge_{j=1}^k Y_j \colon M \to \wedge^k TM$ and $\beta \colon \wedge^k TM \to P \times_G V$. Write $\widetilde{Z}$ for the canonical lift of Z to $\wedge^k TM$. Applying Proposition 3.6, we get

$$\tilde{\mathfrak{L}}_{Z^H}(\beta \circ \wedge_{j=1}^k Y_j) = T\beta \circ \tilde{\mathcal{L}}_{(Z,\widetilde{Z})}(\wedge_{j=1}^k Y_j) + (\tilde{\mathcal{L}}_{(\widetilde{Z}, Z_V^H)}\beta) \circ \wedge_{j=1}^k Y_j.$$

Applying $\mathrm{id} \times_G \mathrm{vpr}_V$ to the both sides, we obtain

$$\mathcal{L}_Z^\nabla(\beta \circ \wedge_{j=1}^k Y_j) = \beta \circ \mathcal{L}_{(Z,\widetilde{Z})}(\wedge_{j=1}^k Y_j) + (\mathcal{L}_{(\widetilde{Z}, Z_V^H)}\beta) \circ \wedge_{j=1}^k Y_j.$$

Now $\mathcal{L}_{(Z,\widetilde{Z})}$ is the usual Lie derivative $\mathcal{L}_Z$ of tensor fields and $\mathcal{L}_{(\widetilde{Z}, Z_V^H)}$ is the covariant Darboux-Lie derivative $\mathcal{L}_Z^\nabla$. Hence

$$\begin{aligned} \mathcal{L}_Z^\nabla(\beta \circ \wedge_{j=1}^k Y_j) &= \beta \circ \mathcal{L}_Z(\wedge_{j=1}^k Y_j) + (\mathcal{L}_Z^\nabla\beta) \circ \wedge_{j=1}^k Y_j \\ &= \sum_{j=1}^k \beta(Y_1, \dots, [Z, Y_j], \dots, Y_k) + \mathcal{L}_Z^\nabla\beta(Y_1, \dots, Y_k). \end{aligned} \tag{49}$$

Taking $X_0 = Z$ and $X_i = Y_i$ for $1 \le i \le k$ in (48), we get

$$\begin{aligned} i_Z d^\nabla\beta(Y_1, \dots, Y_k) &= \mathcal{L}_Z^\nabla(\beta(Y_1, \dots, Y_k)) - \sum_{j=1}^k (-1)^{j-1} \mathcal{L}_{Y_j}^\nabla(\beta(Z, Y_1, \dots, \widehat{Y}_j, \dots, Y_k)) \\ &\quad + \sum_{j=1}^k (-1)^j \beta([Z, Y_j], Y_1, \dots, \widehat{Y}_j, \dots, k) \\ &\quad - \sum_{i<j} (-1)^{i+j} \beta(Z, [Y_i, Y_j], Y_1, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_k) \\ &= (\mathcal{L}_Z^\nabla\beta)(Y_1, \dots, Y_k) - d^\nabla(i_Z\beta)(Y_1, \dots, Y_k). \end{aligned}$$

Hence we obtain the Cartan magic formula for covariant Darboux-Lie derivative

$$\mathcal{L}_Z^\nabla\beta = i_Z(d^\nabla\beta) + d^\nabla(i_Z\beta). \tag{50}$$

9. Future work on G-structures

The principal motivation for introducing the covariant Darboux–Lie derivative was its anticipated application to the theory of G-structures. Recall that a *G-structure* on an n-dimensional manifold M is a reduction of the structure group of TM to $G < \mathrm{GL}_n(\mathbb{R})$. This can be described in several equivalent ways:

- by choosing an open covering of M such that the transition maps of TM with respect to this covering take values in G;
- by specifying a principal G-subbundle of the frame bundle $L(\mathbb{R}^n, TM)$;
- by giving an isomorphism $TM \cong P \times_G V$, for a suitable principal G-bundle P and G-module V.

The last formulation is particularly suited for analytic treatments, in contrast with the more geometric viewpoint of describing the structure as a subbundle[2]. It should be noted that two distinct isomorphisms β, $\beta'\colon TM \to P \times_G V$ may determine the same G-structure. In fact, we will formally verify that β and β' define the same G-structure if and only if there is $s \in \Gamma(M, P \times_G {}_cG)$ such that $\beta' = s \cdot \beta$. Thus a G-structure on M can be described as a gauge equivalence class $[\beta]$ of soldering forms $\beta\colon TM \to P \times_G V$. We conclude this section by announcing two results that will be proved in forthcoming work. Suppose P is equipped with a fixed G-principal connection.

Proposition 9.1. *A vector field $X \in \mathfrak{X}(M)$ is an infinitesimal automorphism of a G-structure $[\beta]$ on M if and only if $\mathcal{L}^\nabla_X \beta = a \cdot \beta$ for some $a \in \Gamma(M, P \times_G \mathfrak{g})$.*

Proposition 9.2. *A G-structure $[\beta]$ is torsion-free if and only if $d^\nabla \beta = \alpha \wedge \beta$ for some $\alpha \in \Omega^1(M, P \times_G \mathfrak{g})$.*

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[2]By "analytic" we mean a formulation adapted to symbolic manipulation (compare *analytic geometry*). This use of the word "analytic" should not be confused with "analytic" in the sense of real or complex analysis.