

ON THE ELLIPTICITY OF THE HIGHER RANK NUMERICAL RANGE

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ABSTRACT. The higher rank numerical range is a concept that generalizes the classical numerical range, and it has application in quantum error correction. We investigate these sets for 2-by-2 block matrices with associated Kippenhahn curves consisting of ellipses (and eventually points). As a consequence, elliptical higher rank numerical range results are derived in a unified way, using an approach developed by Spitkovsky *et al.*

1. INTRODUCTION

Let $M_{m,n}$ be the set of $m \times n$ complex matrices, with $M_{n,n}$ abbreviated to M_n .

Let $B(\mathcal{H})$ be the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$, identified with M_n if $\mathcal{H}$ is n -dimensional. The classical *numerical range* of $A \in B(\mathcal{H})$ is defined and denoted as

$$W(A) := \{ \langle A, x \rangle x : x \in \mathcal{H}, \langle x, x \rangle = 1 \}.$$

This concept was introduced in the second decade of the 20th century by Toeplitz [31] and Hausdorff [22], and it has been extensively investigated by pure and applied scientists. The numerical range has attractive properties (see e.g. [21]), such as its convexity, asserted by the Toeplitz-Hausdorff theorem [22,31], and the closure of $W(A)$ contains the spectrum of A . Some incidences in Physics were considered in [2,4].

Motivated by problems in quantum error correction, Choi, Kribs and Życzkowski [13,14] introduced for $A \in M_n$ and any integer $k = 1, \dots, n$, the *rank- k* or *higher rank numerical range* of A . For $A \in B(\mathcal{H})$ and $k \leq \dim \mathcal{H}$, this set is defined and denoted as

$$\Lambda_k(A) := \{ \lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank-}k \text{ orthogonal projection } P \in B(\mathcal{H}) \}.$$

An equivalent definition for $A \in M_n$ is

$$(1.1) \quad \Lambda_k(A) = \{ \lambda \in \mathbb{C} : X^*AX = \lambda I_k \text{ for some } X \in M_{n,k} \text{ such that } X^*X = I_k \}.$$

From the previous characterization, it is clear that the rank-one numerical range $\Lambda_1(A)$ coincides with $W(A)$ and that the higher rank numerical ranges of $A \in M_n$ form a decreasing sequence of compact sets:

$$\Lambda_n(A) \subseteq \dots \subseteq \Lambda_1(A) = W(A).$$

In particular, $\Lambda_n(A) \neq \emptyset$ if and only if $A = \lambda I_n$, in which case all the sets $\Lambda_k(A) = \{\lambda\}$. For $k > n/2$, $\Lambda_k(A)$ is either the empty set or a singleton $\{\lambda_0\}$, being in the later case λ_0 an eigenvalue of A with geometric multiplicity at least $2k - n$ [13, Proposition 2.2]. Li, Poon and Sze [27] showed that $\Lambda_k(A)$ is non-empty if $k < n/3 + 1$.

As usual, for $A \in M_n$ we write $A = \Re(A) + i\Im(A)$, where

$$\Re(A) := (A + A^*)/2 \quad \text{and} \quad \Im(A) := (A - A^*)/2i$$

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are the *real* and *imaginary parts* of A , respectively. We denote by $\lambda_1(\theta), \dots, \lambda_n(\theta)$ the eigenvalues, counting the multiplicities, of the Hermitian matrix

$$\Re(e^{-i\theta}A) = \Re(A) \cos \theta + \Im(A) \sin \theta, \quad \theta \in [0, 2\pi).$$

The convexity of higher rank numerical ranges holds for $k = 1$ and $k > n/2$, and the convexity for intermediate values of k was proved by Woerdeman in [33], and independently by Li and Sze in [26]. In this paper, the authors obtained the following description of the higher rank numerical range of $A \in M_n$ as an intersection of half-planes:

$$(1.2) \quad \Lambda_k(A) = \bigcap_{\theta \in [0, 2\pi)} \{z \in \mathbb{C} : \Re(e^{-i\theta}z) \leq \lambda_k(\theta)\},$$

where $\lambda_k(\theta)$ denotes the k -th largest eigenvalue, counting the multiplicities, of the Hermitian matrix $\Re(e^{-i\theta}A)$, for each $\theta \in [0, 2\pi)$, and $k = 1, \dots, n$. For $A \in M_n$ normal with eigenvalues $\lambda_1, \dots, \lambda_n$, then (1.2) yields

$$\Lambda_k(A) = \bigcap_{1 \leq j_1 \leq \dots \leq j_{n-k+1} \leq n} \text{conv} \{\lambda_{j_1}, \dots, \lambda_{j_{n-k+1}}\},$$

where $\text{conv } S$ denotes the convex hull of the set S . For $A \in M_n$ Hermitian with eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$, then

$$\Lambda_k(A) = [\lambda_{n-k+1}, \lambda_k]$$

when $\lambda_{n-k+1} < \lambda_k$, $\Lambda_k(A)$ is a singleton if $\lambda_k = \lambda_{n-k+1}$ and an empty set, otherwise.

Kippenhahn [24] proved that $W(A)$ is the convex hull of a certain algebraic curve $C(A)$, associated with the matrix A . Not only $W(A)$ but also any $\Lambda_k(A)$ can be described in terms of $C(A)$ (see, e.g. [8, 19, 29]). The boundary lines of the half-planes in the right hand side of the representation (1.2), that is,

$$\ell_{\theta,k} = \{z \in \mathbb{C} : \Re(e^{-i\theta}z) = \lambda_k(\theta)\}, \quad \theta \in [0, 2\pi),$$

when taken for all $k = 1, \dots, n$, form a family, the envelope of which may determine the boundary generating curve $C(A)$ of the numerical range of A .

In general, it is difficult to characterize $\Lambda_k(A)$ for any k and, in particular, to describe the boundary of $\Lambda_k(A)$.

We mainly focus on complex matrices of the form

$$(1.3) \quad A = \begin{bmatrix} \alpha I_r & C \\ D & \beta I_{n-r} \end{bmatrix}, \quad 0 < r < n.$$

Clearly, $\Lambda_k(A) \neq \emptyset$ for $k \leq \max\{r, n-r\}$. Our goal is to investigate the higher rank numerical range of matrices of this form (1.3).

The remaining of this note is organized as follows. Section 2 contains useful preliminary results on block matrices of type (1.3). In Section 3, the ellipticity of higher rank numerical ranges of block matrices of type (1.3), such that $\Re(e^{-i\theta}A)$ happens to be unitarily reducible to the direct sum of 2-by-2 (and eventually 1-by-1) matrices, is investigated. In such cases, $C(A)$ splits into at most $n/2$ elliptical components, each solely responsible for the respective higher rank numerical range. Using the obtained results, the ellipticity of the higher rank numerical ranges of several classes of matrices is derived in an unified way.

2. PRELIMINARIES

For $A \in M_n$ and $1 \leq k \leq n$, the following elementary properties of $\Lambda_k(A)$ can be easily checked.

- P1. $\Lambda_k(\alpha A + \beta I_n) = \alpha \Lambda_k(A) + \beta$ for any $\alpha, \beta \in \mathbb{C}$ (*translational property*).
- P2. $\Lambda_k(A)$ is *unitarily invariant*: $\Lambda_k(U^* A U) = \Lambda_k(A)$ for any unitary matrix $U \in M_n$.
- P3. $\Lambda_k(A_1 \oplus A_2) \supseteq \text{conv}(\Lambda_k(A_1) \cup \Lambda_k(A_2))$ for any $A_1, A_2 \in M_n$.
- P4. $\Lambda_{k_1+k_2}(A_1 \oplus A_2) \supseteq \Lambda_{k_1}(A_1) \cap \Lambda_{k_2}(A_2)$ for any $k_1, k_2 \in \{1, \dots, n\}$.

We will concentrate on matrices of the form (1.3). By property P_1 , when investigating $\Lambda_k(A)$ it suffices to consider matrices with $\alpha + \beta = 0$. Without loss of generality, we can moreover assume $n \leq 2r$, because if $n > 2r$ by an adequate unitary similarity of A , which by property P_2 preserves $\Lambda_k(A)$, we may switch the matrix to the following one:

$$(2.1) \quad \begin{bmatrix} \beta I_{n-r} & D \\ C & \alpha I_r \end{bmatrix}.$$

All the matrices $A \in M_2$ have this form, and the *Elliptical Range Theorem* (see e.g. [25]) states that for $A \in M_2$ with eigenvalues λ_1, λ_2 , $W(A)$ is an elliptical disc centered at $\frac{1}{2}\text{Tr } A$, with foci at the eigenvalues, major and minor axes of lengths, respectively

$$(\text{Tr}(A^* A) - 2\Re(\lambda_1 \bar{\lambda}_2))^{\frac{1}{2}} \quad \text{and} \quad (\text{Tr}(A^* A) - |\lambda_1|^2 - |\lambda_2|^2)^{\frac{1}{2}}.$$

In this case, $W(A)$ degenerates into the line segment joining λ_1 and λ_2 if and only if A is normal, and it reduces to a singleton if and only if A is a scalar matrix. The elliptical shape of the 2-by-2 case persists for the numerical range of certain classes of matrices independently of the matrix size (see e.g. [12, 16, 20]), other structured matrix classes [9, 11] have different geometric behaviour when the order increases, and several matrices whose numerical range is the convex hull of ellipses were investigated (see e.g. [3, 7, 23]).

As already observed, the eigenvalues of the Hermitian matrix

$$H_\theta(A) := \Re(e^{-i\theta} A), \quad \theta \in [0, 2\pi),$$

play a central role in our study. For matrices of type (1.3) they have been obtained in [20] in terms of those of the matrix

$$(2.2) \quad M_{C,D}(\theta) := C^* C + D D^* + 2\Re(e^{-2i\theta} DC), \quad \theta \in [0, 2\pi), \quad \text{if } n \leq 2r.$$

If $n > 2r$, switching to the matrix (2.1), they may be obtained from the eigenvalues of $M_{D,C}(\theta)$, recalling that we may switch to the matrix in (2.1). We observe that $M_{C,D}(\theta)$ and $M_{D,C}(\theta)$ share the same non-zero eigenvalues. The proof here presented exhibits the unitary similarity of $H_\theta(B)$, where

$$(2.3) \quad B = A - \frac{1}{2}(\alpha + \beta) I_n,$$

to a direct sum of small sized blocks, which is crucial to derive our results. For simplicity of notation, we write $M(\theta) = M_{C,D}(\theta)$ if $n \leq 2r$ and $M(\theta) = M_{D,C}(\theta)$ if $n > 2r$,

$$(2.4) \quad w = \frac{1}{2}(\alpha - \beta) \quad \text{and} \quad w_\theta = \Re(w e^{-i\theta}).$$

Lemma 2.1. *Let A be of the form (1.3) and $\theta \in [0, 2\pi)$. The following holds.*

- (i) The eigenvalues of $M(\theta)$ are $\mu_j(\theta) = 4s_j^2(\theta)$, $1 \leq j \leq p$, and $\mu_j(\theta) = 0$, $j > p$, with $s_1(\theta), \dots, s_p(\theta)$ the non-zero singular values, counting the multiplicities, of

$$N_\theta = \frac{1}{2} (e^{-i\theta} C + e^{i\theta} D^*).$$

- (ii) For B given by (2.3), the matrix $H_\theta(B)$ is unitarily similar to the direct sum

$$B_1 \oplus \dots \oplus B_p \oplus w_\theta I_{r-p} \oplus (-w_\theta) I_{n-r-p},$$

where the non-scalar blocks have the form

$$B_j = \begin{bmatrix} w_\theta & s_j(\theta) \\ s_j(\theta) & -w_\theta \end{bmatrix}, \quad j = 1, \dots, p.$$

- (iii) If $n < 2r$, ($n > 2r$), the eigenvalues of $H_\theta(A)$ are $\Re(e^{-i\theta}\alpha)$ (resp. $\Re(e^{-i\theta}\beta)$) and

$$(2.5) \quad \frac{1}{2} \Re(e^{-i\theta}(\alpha + \beta)) \pm \frac{1}{2} \sqrt{4w_\theta^2 + \mu_j(\theta)},$$

for $j = 1, \dots, n - r$ (resp. $j = 1, \dots, r$). If $n = 2r$, the eigenvalues of $H_\theta(A)$ are those of the form (2.5) for $j = 1, \dots, n/2$.

Proof. (i) If $n \leq 2r$, it is an obvious consequence of the fact $M(\theta) = 4N_\theta^* N_\theta$. Otherwise, interchanging C and D , the conclusion follows.

- (ii) Letting $A = \frac{1}{2}(\alpha + \beta)I_n + B$, we have

$$B = \begin{bmatrix} wI_r & C \\ D & -wI_{n-r} \end{bmatrix}$$

and we obtain

$$H_\theta(B) = \begin{bmatrix} w_\theta I_r & N_\theta \\ N_\theta^* & -w_\theta I_{n-r} \end{bmatrix}$$

with $N_\theta = \frac{1}{2} (e^{-i\theta} C + e^{i\theta} D^*)$. By the singular value decomposition, there exist unitary matrices $U \in M_r$, $V \in M_{n-r}$, such that

$$U^* N_\theta V = D_\theta \in M_{r, n-r}$$

contains a diagonal principal submatrix with the nonzero singular values $s_1(\theta), \dots, s_p(\theta)$ of N_θ in its main diagonal. Since the matrix $W = U \oplus V$ is unitary and $W^* = U^* \oplus V^*$, we easily get

$$W^* H_\theta(B) W = \begin{bmatrix} w_\theta I_r & U^* N_\theta V \\ V^* N_\theta^* U & -w_\theta I_{n-r} \end{bmatrix} = \begin{bmatrix} w_\theta I_r & D_\theta \\ D_\theta & -w_\theta I_{n-r} \end{bmatrix}.$$

Thus, $W^* H_\theta(B) W$ is permutationally similar to the direct sum of the p blocks $B_j \in M_2$, plus $r - p$ blocks w_θ and $n - r - p$ blocks $-w_\theta$ of size 1.

- (iii) If $n < 2r$ ($n > 2r$), then it follows from (ii) that the eigenvalues of $H_\theta(B)$ are w_θ (resp. $-w_\theta$) and

$$(2.6) \quad \pm \sqrt{w_\theta^2 + s_j^2(\theta)}, \quad j = 1, \dots, n - r,$$

(resp. $j = 1, \dots, r$) and if $n = 2r$, they are those in (2.6). Having in mind (i) and that

$$(2.7) \quad H_\theta(A) = \frac{1}{2} \Re(e^{-i\theta}(\alpha + \beta)) I_n + H_\theta(B),$$

the claimed eigenvalues of $H_\theta(A)$ are readily obtained. $\square$

Some considerations are in order. If $\Re(e^{-i\theta}\alpha)$ or $\Re(e^{-i\theta}\beta)$ are eigenvalues of $H_\theta(A)$, then they correspond to tangent lines of $C(A)$ passing through the point α or β , respectively, this meaning that these points belong to $C(A)$. Moreover, the remaining tangent lines form a family central symmetric relatively to $(\alpha + \beta)/2$ as implied by (2.5).

The following technical lemma is used in the proof of the next theorem.

Lemma 2.2. *Let $A \in M_n$ with two of the eigenvalues of $H_\theta(A)$ of the form*

$$\Re(e^{-i\theta}c) \pm \frac{1}{2}\sqrt{a^2 - (a^2 - b^2)\sin^2(\theta - \phi)}, \quad \theta \in [0, 2\pi),$$

for some $a, b, \phi \in \mathbb{R}$ and $c \in \mathbb{C}$. Then $C(A)$ contains an ellipse centered at c , with the major axis parallel to the vector $e^{i\phi}$ of length $|a|$ and the minor axis of length $|b|$.

If the non-diagonal blocks C, D in (1.3) are such that $Z = DC$ is normal and commutes with $H = C^*C + DD^*$, then these matrices can be simultaneously diagonalized by the same unitary matrix, and we denote their eigenvalues by z_j and h_j , $j = 1, \dots, n - r$, labeled according to the order in which they appear in the respective diagonal matrices unitarily similar to Z and H , respectively.

The next result is a refinement of [20, Theorem 3.1] and is instrumental for our purposes. The proof takes the original ideas there combined with the introduced refinement in Lemma 2.1 ensuring the unitary reducibility of $\Re(e^{-i\theta}A)$ to a direct sum of small sized blocks.

As usual, $\text{Arg } z$ denotes the principal argument of the complex number z .

Theorem 2.3. *Let $A \in M_n$ be of type (1.3), $n \leq 2r$, such that $Z = DC$ is normal and commutes with $H = C^*C + DD^*$, and let their eigenvalues be z_j and h_j , $j = 1, \dots, n - r$, respectively, labeled as mentioned above. Let $\hat{\mathcal{E}}_j$ be the ellipse with foci at $\frac{1}{2}(\alpha + \beta) \pm \frac{1}{2}\sqrt{\Delta_j}$, for $\Delta_j = (\alpha - \beta)^2 + 4z_j$, with major and minor axes of length*

$$(2.8) \quad \sqrt{\frac{1}{2}|\alpha - \beta|^2 + h_j + \frac{1}{2}|\Delta_j|} \quad \text{and} \quad \sqrt{\frac{1}{2}|\alpha - \beta|^2 + h_j - \frac{1}{2}|\Delta_j|},$$

respectively. Then the following statements hold.

(a) *The boundary generating curve of $W(A)$ is given by*

$$C(A) = \begin{cases} \hat{\mathcal{E}}_1 \cup \dots \cup \hat{\mathcal{E}}_{n-r} \cup \{\alpha\}, & \text{if } n < 2r \\ \hat{\mathcal{E}}_1 \cup \dots \cup \hat{\mathcal{E}}_{n/2}, & \text{if } n = 2r \end{cases}$$

and $W(A)$ is the convex hull of $\hat{\mathcal{E}}_1, \dots, \hat{\mathcal{E}}_{n-r}$.

(b) *The higher rank numerical range of A is given by*

$$(2.9) \quad \Lambda_k(A) = \bigcap_{1 \leq j_1 \leq \dots \leq j_{n-r-k+1} \leq n-r} \text{conv} \{\hat{\mathcal{E}}_{j_1}, \dots, \hat{\mathcal{E}}_{j_{n-r-k+1}}\}, \quad 1 \leq k \leq n - r,$$

and

$$(2.10) \quad \text{either } \Lambda_k(A) = \begin{cases} \{\alpha\}, & n - r < k \leq r \\ \emptyset, & r < k \leq n < 2r \end{cases} \quad \text{or} \quad \Lambda_k(A) = \emptyset, \quad \frac{n}{2} < k \leq n = 2r.$$

Proof. (a) As DC is normal and commutes with $C^*C + DD^*$, they can be diagonalized by the same unitary similarity. Under the theorem hypothesis, if $n \leq 2r$, then the matrix $M_{C,D}(\theta)$ is unitary diagonalizable for all values of θ by the same unitary similarity of DC and $C^*C + DD^*$, and so its eigenvalues are

$$(2.11) \quad \mu_j(\theta) = h_j + 2\Re(z_j)\cos(2\theta) + 2\Im(z_j)\sin(2\theta), \quad j = 1, \dots, n - r.$$

For $\theta \in [0, 2\pi)$, recalling (2.4) and using

$$2(\Re(w))^2 = \Re(w^2) + |w|^2, \quad 2(\Im(w))^2 = -\Re(w^2) + |w|^2, \quad 2\Re(w)\Im(w) = \Im(w^2),$$

by simple computations we get

$$\begin{aligned} 4w_\theta^2 + \mu_j(\theta) &= 4(\Re(w)\cos\theta + \Im(w)\sin\theta)^2 + \mu_j(\theta) \\ &= 4(\Re(w))^2\cos^2\theta + 4(\Im(w))^2\sin^2\theta + 4\Re(w)\Im(w)\sin(2\theta) + \mu_j(\theta) \\ &= 2|w|^2 + 2\Re(w^2)\cos(2\theta) + 2\Im(w^2)\sin(2\theta) + \mu_j(\theta) \\ &= 2|w|^2 + h_j + 2\Re(w^2 + z_j)\cos(2\theta) + 2\Im(w^2 + z_j)\sin(2\theta) \\ &= 2|w|^2 + h_j + 2|w^2 + z_j|\cos(2\theta - \text{Arg}(w^2 + z_j)) \\ &= 2|w|^2 + h_j + 2|w^2 + z_j| - 4|w^2 + z_j|\sin^2(\theta - \phi_j), \end{aligned}$$

with $2\phi_j = \text{Arg}(w^2 + z_j)$. Recalling the expressions of the eigenvalues of $H_\theta(A)$ in Lemma 2.1 (iii) and by Lemma 2.2, we easily conclude that for each $j \in \{1, \dots, n-r\}$ to the pair of eigenvalues in (2.1) with $\theta \in [0, 2\pi)$ corresponds the component of $C(A)$, which is the ellipse $\hat{\mathcal{E}}_j$ centered at $\frac{1}{2}(\alpha + \beta)$, with major and minor axes of lengths

$$\sqrt{2|w|^2 + h_j + 2|w^2 + z_j|} \quad \text{and} \quad \sqrt{2|w|^2 + h_j + 2|w^2 + z_j|},$$

respectively, and whose major axis is parallel to $e^{i\phi_j}$, this implying that $\hat{\mathcal{E}}_j$ has the asserted foci. If $n < 2r$, then $\Re(e^{-i\theta}\alpha)$ is an eigenvalue of $H_\theta(A)$, and so the point α is also in $C(A)$. Therefore, the statement on $C(A)$ holds. We easily see that α belongs to the elliptical discs bounded by $\hat{\mathcal{E}}_j$, $j = 1, \dots, n-r$, and so, by Kippenhahn result, we have

$$W(A) = \text{conv } C(A) = \text{conv } \{\hat{\mathcal{E}}_1, \dots, \hat{\mathcal{E}}_{n-r}\}.$$

(b) If $k = 1$, the result is clear. Otherwise, the characterization of the higher rank numerical range in (1.2) is equivalent to

$$\Lambda_k(A) = \bigcap_{\theta \in [0, \pi]} \{z \in W(A) : \lambda_{n-k+1}(\theta) \leq \Re(e^{-i\theta}z) \leq \lambda_k(\theta)\}.$$

By Lemma 2.1 (iii), if $k \leq n-r$, then $\lambda_k(\theta)$ is the k -th largest number in the set

$$\left\{ \frac{1}{2}\Re(e^{-i\theta}(\alpha + \beta)) + \frac{1}{2}\sqrt{4w_\theta^2 + \mu_j(\theta)} : j = 1, \dots, n-r \right\}$$

and $\lambda_{n-k+1}(\theta)$ is the k -th smallest number in the set

$$\left\{ \frac{1}{2}\Re(e^{-i\theta}(\alpha + \beta)) - \frac{1}{2}\sqrt{4w_\theta^2 + \mu_j(\theta)} : j = 1, \dots, n-r \right\}.$$

Moreover, if $n-r < k \leq r$, then $\lambda_k(\theta) = \lambda_{n-k+1}(\theta) = \Re(e^{-i\theta}\alpha)$ and if $r < k \leq n$, then $\lambda_k(\theta) < \lambda_{n-k+1}(\theta)$. Thus, (2.10) holds. From the proof in (a), we conclude that

$$(2.12) \quad \Re(e^{-i\theta}z) = \lambda_k(\theta) \quad \text{and} \quad \Re(e^{-i\theta}z) = \lambda_{n-k+1}(\theta)$$

are the support lines of the elliptical component $\hat{\mathcal{E}}_k$ in $C(A)$ perpendicular to the direction θ , when $k \leq n-r$, and the smallest higher rank numerical range of A with nonempty interior is clearly

$$\Lambda_{n-r}(A) = \bigcap_{j=1}^{n-r} \text{conv } \hat{\mathcal{E}}_j.$$

If $1 < k < n-r$, taking all the possible convex hulls

$$\text{conv } \{\hat{\mathcal{E}}_{j_1}, \dots, \hat{\mathcal{E}}_{j_{n-r-k+1}}\}, \quad 1 \leq j_1 \leq \dots \leq j_{n-r-k+1} \leq n-r,$$

and then intersecting them is equivalent to consider all the elements z in $W(A)$ in the stripes defined by the lines (2.12), for all $\theta \in [0, 2\pi)$. This means that (2.9) holds too. $\square$

We remark that Theorem 2.3 holds, in particular, if the non-diagonal blocks of A in (1.3) are such that CD and DC are both normal matrices.

Moreover, if the ellipses in $C(A)$ form a nested chain, then elliptical higher rank numerical ranges of A are obtained.

Corollary 2.4. *Let A be under the hypothesis of Theorem 2.3 and let $\mathcal{E}_j$ be the closed elliptical disc bounded by the ellipse $\hat{\mathcal{E}}_j$ there described. If $\mathcal{E}_{n-r} \subseteq \cdots \subseteq \mathcal{E}_1$, then*

$$\Lambda_k(A) = \mathcal{E}_k, \quad 1 \leq k \leq n-r.$$

Proof. Let $1 \leq k \leq n-r$. Since by hypothesis the elliptical discs are nested, we have

$$\text{conv} \{ \hat{\mathcal{E}}_{j_1}, \dots, \hat{\mathcal{E}}_{j_{n-r-k+1}} \} = \mathcal{E}_{j_1}, \quad 1 \leq j_1 \leq \cdots \leq j_{n-r-k+1} \leq n-r.$$

From Theorem 2.3 (b), it follows that the intersection in (2.9) reduces to

$$\Lambda_k(A) = \bigcap_{j_1=1}^k \mathcal{E}_{j_1} = \mathcal{E}_k.$$

$\square$

Remark 2.1. Recalling the proof of Theorem 2.3 (a) and the role of the eigenvalues of $H_\theta(A)$ in Lemma 2.1 (iii), by Lemma 2.2, it can easily be seen that Corollary 2.4 holds for $n \leq 2r$, whenever the eigenvalues of $M(\theta)$ in (2.11) satisfy $\mu_1(\theta) \geq \cdots \geq \mu_{n-r}(\theta)$ for all $\theta \in [0, 2\pi)$.

As previously noticed, the statements of Theorem 2.3, Corollary 2.4 and Remark 2.1 may be formulated for $n > 2r$, interchanging C, D , as well as α, β and $r, n-r$.

3. MATRICES WITH ELLIPTICAL HIGHER RANK NUMERICAL RANGE

In this section, classes of block matrices of the form (1.3), yielding elliptical higher rank numerical ranges are presented. If DC is a scalar multiple of the identity, the following result holds.

Corollary 3.1. *Let A be of type (1.3) with $DC = z_1 I_{n-r}$ and $h_1 \geq \cdots \geq h_{n-r}$ be the eigenvalues, counting the multiplicities, of $C^*C + DD^*$. Let $\mathcal{E}_k^1$ be the elliptical disc with foci at $\frac{1}{2}(\alpha + \beta) \pm \frac{1}{2}\sqrt{\Delta}$, for $\Delta = (\alpha - \beta)^2 + 4z_1$, major and minor axes of lengths*

$$\sqrt{\frac{1}{2}|\alpha - \beta|^2 + h_k + \frac{1}{2}|\Delta|} \quad \text{and} \quad \sqrt{\frac{1}{2}|\alpha - \beta|^2 + h_k - \frac{1}{2}|\Delta|}.$$

If $n < 2r$, then

$$\Lambda_k(A) = \begin{cases} \mathcal{E}_k^1, & \text{if } 1 \leq k \leq n-r \\ \{\alpha\}, & \text{if } n-r < k \leq r \\ \emptyset, & \text{if } r < k \leq n \end{cases}.$$

If $n = 2r$, then

$$\Lambda_k(A) = \begin{cases} \mathcal{E}_k^1, & \text{if } 1 \leq k \leq n/2 \\ \emptyset, & \text{if } n/2 < k \leq n \end{cases}.$$

Proof. By hypothesis, z_1 is the unique eigenvalue of DC and Theorem 2.3 (a) characterizes $C(A)$ when $n \leq 2r$. All the elliptical components of $C(A)$, which are the boundaries of $\mathcal{E}_j^1$, $j = 1, \dots, n-r$, have the same foci. Since $h_1 \geq \dots \geq h_{n-r}$, the chain of inclusions

$$\{\alpha\} \subseteq \mathcal{E}_{n-r}^1 \subseteq \dots \subseteq \mathcal{E}_1^1$$

is ensured by the axes length of the ellipses. By Corollary 2.4, we get $\Lambda_k(A) = \mathcal{E}_k^1$, for $1 \leq k \leq n-r$. By Theorem 2.3 (b), the cases of $\Lambda_k(A)$ being a singleton or an emptyset are obtained. $\square$

If $n > 2r$, then the previous result holds, interchanging C, D , as well as α, β and $n-r, r$, in which case we consider $h_1 \geq \dots \geq h_r$ as the eigenvalues of $CC^* + D^*D$.

If $\Delta = 0$ in Corollary 3.1, then $\Lambda_k(A)$ is the circular disc centered at $\frac{1}{2}(\alpha + \beta)$ with radius of length

$$\sqrt{\frac{1}{8}|\alpha - \beta|^2 + \frac{1}{4}h_k}, \quad \text{if } 1 \leq k \leq \min\{r, n-r\}.$$

For an arrowhead matrix of the form

$$(3.1) \quad \left[\begin{array}{ccc|c} \alpha & & & c_1 \\ & \ddots & & \vdots \\ & & \alpha & c_{n-1} \\ \hline d_1 & \dots & d_{n-1} & \beta \end{array} \right] \quad \text{or} \quad \left[\begin{array}{c|ccc} \alpha & c_1 & \dots & c_{n-1} \\ \hline d_1 & \beta & & \\ \vdots & & \ddots & \\ d_{n-1} & & & \beta \end{array} \right]$$

with zeros at the omitted entries, the next case is immediate (cf. [16]).

Corollary 3.2. *Let A be an arrowhead matrix of type (1.3), $n \geq 3$, with $C = \mathbf{c}$ and $D = \mathbf{d}^T$ (or $C = \mathbf{c}^T$ and $D = \mathbf{d}$) for $\mathbf{c}, \mathbf{d} \in \mathbb{C}^{n-1}$. Then*

$$\Lambda_k(A) = \begin{cases} \mathcal{E}, & \text{if } k = 1 \\ \{\alpha\} \text{ (or } \{\beta\}), & \text{if } 2 \leq k \leq n-1 \\ \emptyset, & \text{if } k = n \end{cases},$$

where $\mathcal{E}$ is the elliptical disc with foci at $\frac{1}{2}(\alpha + \beta) \pm \frac{1}{2}\sqrt{\Delta}$, for $\Delta = (\alpha - \beta)^2 + 4\mathbf{c}^T\mathbf{d}$, major and minor axis of length

$$\sqrt{\frac{1}{2}|\alpha - \beta|^2 + \|\mathbf{c}\|^2 + \|\mathbf{d}\|^2 + \frac{1}{2}|\Delta|} \quad \text{and} \quad \sqrt{\frac{1}{2}|\alpha - \beta|^2 + \|\mathbf{c}\|^2 + \|\mathbf{d}\|^2 - \frac{1}{2}|\Delta|}.$$

Proof. This is an immediate consequence of Theorem 3.1 with $z = \mathbf{c}^T\mathbf{d}$ and $r = n-1$ if $C = \mathbf{c}$, $D = \mathbf{d}^T$ (or $r = 1$ if $C = \mathbf{c}^T$, $D = \mathbf{d}$), which trivially applies as $DC = \mathbf{c}^T\mathbf{d} = CD$ and $C^*C + DD^* = \|\mathbf{c}\|^2 + \|\mathbf{d}\|^2$ ($CC^* + D^*D = \|\mathbf{c}\|^2 + \|\mathbf{d}\|^2$, respectively). $\square$

Let $[\alpha, \beta]$ denote the line segment joining the numbers α, β if $\alpha \neq \beta$, reduced to the singleton $\{\alpha\}$ if $\alpha = \beta$.

Corollary 3.3. *Let A be of type (1.3) with $D = \zeta C^*$ for some $\zeta \in \mathbb{C}$. Let $s_1 \geq \dots \geq s_p$ be the non-zero singular values of C , $p = \text{rank}(C)$, $m = \min\{n-r, r\}$ and $M = \max\{n-r, r\}$. Let $\mathcal{E}_k$ be the elliptical disc with foci at $\frac{1}{2}(\alpha + \beta) \pm \frac{1}{2}\sqrt{\Delta_k}$, for $\Delta_k = (\alpha - \beta)^2 + 4\zeta s_k^2$, major and minor axes of length*

$$\sqrt{\frac{1}{2}|\alpha - \beta|^2 + (1 + |\zeta|^2)s_k^2 + \frac{1}{2}|\Delta_k|} \quad \text{and} \quad \sqrt{\frac{1}{2}|\alpha - \beta|^2 + (1 + |\zeta|^2)s_k^2 - \frac{1}{2}|\Delta_k|}.$$

If $n < 2r$ ($n > 2r$), then

$$\Lambda_k(A) = \begin{cases} \mathcal{E}_k & \text{if } 1 \leq k \leq p \\ [\alpha, \beta] & \text{if } p < k \leq m \\ \{\alpha\} \text{ (or } \{\beta\}) & \text{if } m < k \leq M \\ \emptyset & \text{if } M < k \leq n \end{cases}.$$

If $n = 2r$, then

$$\Lambda_k(A) = \begin{cases} \mathcal{E}_k & \text{if } 1 \leq k \leq p \\ [\alpha, \beta] & \text{if } p < k \leq n/2 \\ \emptyset & \text{if } n/2 < k \leq n \end{cases}.$$

Proof. Under the hypothesis, the matrices Z, H defined as in Theorem 2.3 for $n \leq 2r$ are

$$Z = \zeta C^* C \quad \text{and} \quad H = (1 + |\zeta|^2) C^* C.$$

and the matrix in (2.2) becomes a scalar multiple of $C^* C$. Then $C(A)$ is characterized in Theorem 2.3 (a), being $\hat{\mathcal{E}}_1, \dots, \hat{\mathcal{E}}_p$ the non-degenerate ellipses there described, with

$$z_j = \zeta s_j^2, \quad h_j = (1 + |\zeta|^2) s_j^2, \quad j = 1, \dots, p,$$

which are the non-zero eigenvalues of Z, H , respectively, and $\hat{\mathcal{E}}_{p+1} = \dots = \hat{\mathcal{E}}_{n-r} = \{\alpha, \beta\}$, since the remaining eigenvalues of Z, H are $z_j = h_j = 0$, $j = p+1, \dots, n-r$.

Moreover, the non-zero eigenvalues of $M(\theta)$ are of the form

$$\mu_j(\theta) = ((1 + |\zeta|^2 + 2\Re(\zeta) \cos(2\theta) + 2\Im(\zeta) \sin(2\theta)) s_j^2), \quad j = 1, \dots, p,$$

where $s_1 \geq \dots \geq s_p$ are the non-zero singular values of C . Then

$$\mu_1(\theta) \geq \dots \geq \mu_p(\theta) > \mu_{p+1}(\theta) = \dots = \mu_{n-r}(\theta) = 0, \quad \theta \in [0, 2\pi).$$

By Remark 2.3, the non-degenerate elliptical discs $\mathcal{E}_j$ bounded by $\hat{\mathcal{E}}_j$, $j = 1, \dots, p$, are nested and containing the line segment joining α and β :

$$\mathcal{E}_{n-r} = \dots = \mathcal{E}_{p+1} = [\alpha, \beta] \subset \mathcal{E}_p \subseteq \dots \subseteq \mathcal{E}_1.$$

By Corollary 2.4, we conclude that $\Lambda_k(A) = \mathcal{E}_k$ if $1 \leq k \leq p$ and $\Lambda_k(A) = [\alpha, \beta]$ if $p < k \leq n-r$. Otherwise, the result follows by Theorem 2.3 (b).

If $n > 2r$, interchanging α, β , as well as $n-r, r$ and C, C^* in the above proof, the result is obtained, because the singular values of C and C^* coincide. $\square$

The higher rank numerical range of tridiagonal matrices with special structure was characterized in [1], generalizing known results in the literature for the classical numerical range.

Let $T(\mathbf{d}, \mathbf{a}, \mathbf{c})$ denote a tridiagonal matrix, with main diagonal $\mathbf{a}$, the first upper descending diagonal $\mathbf{c}$ and the first lower descending diagonal $\mathbf{d}$. Let $B(\tilde{\mathbf{c}}) = T(\tilde{\mathbf{c}}_{\mathbf{e}}, \tilde{\mathbf{c}}_{\mathbf{o}}, 0)$ be the bidiagonal matrix associated to the vector $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \dots)$, where

$$\tilde{\mathbf{c}}_{\mathbf{o}} = (\tilde{c}_1, \tilde{c}_3, \dots) \quad \text{and} \quad \tilde{\mathbf{c}}_{\mathbf{e}} = (\tilde{c}_2, \tilde{c}_4, \dots).$$

Corollary 3.4 provides a direct proof of the numerical range result in [12, Theorem 3.3] and also the characterization of the higher rank numerical range.

Corollary 3.4. *Let $T = T(\mathbf{d}, \mathbf{a}, \mathbf{c}) \in M_n$ with $\mathbf{a} = (\alpha, \beta, \alpha, \beta, \dots)$, $\mathbf{c} = (c_1, \dots, c_{n-1})$, $\mathbf{d} = (d_1, \dots, d_{n-1})$, such that*

$$c_j = \zeta \overline{d_j}, \quad j \in J_1 \subseteq \{1, \dots, n-1\} \quad \text{and} \quad d_j = \zeta \overline{c_j}, \quad j \in J_2 = \{1, \dots, n-1\} \setminus J_1,$$

for some $\zeta \in \mathbb{C}$. For the vector $\tilde{\mathbf{c}}$ with components

$$\tilde{c}_j = d_j, \quad j \in J_1 \quad \text{and} \quad \tilde{c}_j = c_j, \quad j \in J_2,$$

let the elliptical discs $\mathcal{E}_1, \dots, \mathcal{E}_p$ be described as in Corollary 3.3, with $s_1 \geq \dots \geq s_p$ the non-zero singular values of the bidiagonal matrix $B(\tilde{\mathbf{c}})$ and p its rank. If $n = 2r + 1$, then

$$\Lambda_k(T) = \begin{cases} \mathcal{E}_k, & \text{if } 1 \leq k \leq p \\ [\alpha, \beta], & \text{if } p < k \leq r \\ \{\alpha\}, & \text{if } r < k \leq r + 1 \\ \emptyset, & \text{if } r + 1 < k \leq n \end{cases}.$$

If $n = 2r$, then

$$\Lambda_k(T) = \begin{cases} \mathcal{E}_k, & \text{if } 1 \leq k \leq p \\ [\alpha, \beta], & \text{if } p < k \leq n/2 \\ \emptyset, & \text{if } n/2 < k \leq n \end{cases}.$$

Proof. Without loss of generality, we may suppose that J_1 is the subset of odd numbers in $\{1, \dots, n-1\}$, because interchanging any pair of corresponding off-diagonal entries of the tridiagonal matrix T , resulting into $\tilde{T}$, the higher rank numerical range remains unchanged, observing that $H_\theta(T)$ and $H_\theta(\tilde{T})$ are both tridiagonal matrices with the same spectra, for every $\theta \in [0, 2\pi)$.

Then the result follows easily by Corollary 3.3, because T is permutationally similar, via the permutation matrix whose columns are the vectors of the canonical basis of $\mathbb{R}^n$ reordered into the odd indexed $e_1, e_3, \dots$, followed by the even indexed ones $e_2, e_4, \dots$, to

$$\begin{bmatrix} \alpha I_r & C \\ \zeta C^* & \beta I_{n-r} \end{bmatrix}$$

with $r = \lceil \frac{n}{2} \rceil$ and $C = B(\tilde{\mathbf{c}})$. □

Considering in Corollary 3.4 that $\mathbf{c} = \mathbf{1}$ is the vector with all the entries equal to 1 and $\zeta = -1$, then the result in [15, Theorem 2] for the numerical range of a continuant matrix with biperiodic main diagonal is extended to the rank- k numerical range, since the singular values of the bidiagonal matrix $B(\mathbf{1})$ of order $\lceil \frac{n}{2} \rceil \times \lfloor \frac{n}{2} \rfloor$ are (see e.g. [10]):

$$s_k = 2 \cos \frac{k\pi}{n+1}, \quad k = 1, \dots, \lfloor \frac{n}{2} \rfloor.$$

A *Toeplitz matrix* is the one with constant entries along each descending diagonal. A *tridiagonal 2-Toeplitz matrix* is of the form $T(\mathbf{d}, \mathbf{a}, \mathbf{c})$ with biperiodic descending diagonals $\mathbf{a}, \mathbf{c}, \mathbf{d}$. If the matrix T in Corollary 3.4 is a tridiagonal 2-Toeplitz matrix, such that either $c_1 \bar{c}_2 = \bar{d}_1 d_2$ or $\bar{c}_1 d_2 = \bar{c}_2 d_1$, then we find [1, Theorem 6 and 11], where the singular values s_k of the corresponding bidiagonal matrices $B(\tilde{\mathbf{c}})$ are explicitly given by

$$s_k^2 = |c_1|^2 + |d_2|^2 + 2|c_1 d_2| \cos \frac{2k\pi}{n+1}, \quad k = 1, \dots, \lfloor \frac{n}{2} \rfloor.$$

Let $A(\mathbf{d}, \mathbf{a}, \mathbf{c})$ denote the antitridiagonal matrix, with main antidiagonal $\mathbf{a}$, first lower and upper ascending diagonals $\mathbf{c}$ and $\mathbf{d}$, respectively, and zeros elsewhere. Antitridiagonal matrices $A(\mathbf{d}, \mathbf{a}, \mathbf{c})$ with at most two non-zero antidiagonals, under the conditions of [6, Theorem 2 or 3], have also elliptical higher rank numerical ranges, since they are permutationally similar to tridiagonal matrices with zero main diagonal, satisfying the hypothesis of the last corollary, as a consequence of [5, Theorems 1 and 2].

The (*backward*) *shift* operator on $\mathbb{C}^n$ is represented by the n -square matrix with ones on the subdiagonal (resp. superdiagonal) and zeros elsewhere. Henceforth, as an obvious consequence of Corollary 3.4, we get the following result obtained in [18, 30].

Corollary 3.5. *The rank- k numerical range of the n -dimensional shift operator is the circular disc centered at the origin with radius $\cos \frac{k\pi}{n+1}$, if $1 < k < \lfloor \frac{n+1}{2} \rfloor$, and the emptyset, otherwise.*

Next, we consider the case when the spectrum of $M(\theta)$ is independent of θ .

Denote by $E(\alpha, \beta; s)$ the elliptical disc with foci at α and β , and minor axis of length s if $\alpha \neq \beta$, reduced to a circular disc centered at α of radius $s/2$ if $\alpha = \beta$.

Theorem 3.6. *Let $A \in M_n$ be the block matrix in (1.3), such that the spectrum of $M(\theta)$ is independent of θ with non-zero eigenvalues, counting the multiplicities, $\mu_1 \geq \dots \geq \mu_p$. If $n < 2r$, then*

$$\Lambda_k(A) = \begin{cases} E(\alpha, \beta; \sqrt{\mu_k}), & \text{if } 1 \leq k \leq p \\ [\alpha, \beta], & \text{if } p < k \leq n - r \\ \{\alpha\}, & \text{if } n - r < k \leq r \\ \emptyset, & \text{if } r < k \leq n \end{cases}.$$

If $n = 2r$, then

$$\Lambda_k(A) = \begin{cases} E(\alpha, \beta; \sqrt{\mu_k}), & \text{if } 1 \leq k \leq p \\ [\alpha, \beta], & \text{if } p < k \leq n/2 \\ \emptyset, & \text{if } n/2 < k \leq n \end{cases}.$$

In particular, $W(A)$ is the elliptical disc with foci α, β and minor axis of length $\|C + D^*\|$.

Proof. By hypothesis, the spectrum of $M(\theta)$ is independent of θ , thus it coincides with the spectrum of $M = M(0)$. If $n \leq 2r$, the non-zero eigenvalues of

$$M = C^*C + DD^* + 2\Re(DC) = (C + D^*)^*(C + D^*)$$

are $\mu_1 \geq \dots \geq \mu_p$ and, by Lemma 2.1 (iii), the k -th largest eigenvalue of $H_\theta(A)$, which determines the boundary line $\ell_{\theta,k}$ in the representation (1.2), is

$$\lambda_k(\theta) = \begin{cases} \frac{1}{2}\Re(e^{-i\theta}(\alpha + \beta)) + \frac{1}{2}\sqrt{4w_\theta^2 + \mu_k}, & \text{if } 1 \leq k \leq p \\ \max\{\Re(e^{-i\theta}\alpha), \Re(e^{-i\theta}\beta)\}, & \text{if } p < k \leq n - r \\ \Re(e^{-i\theta}\alpha), & \text{if } n - r < k \leq r \\ \min\{\Re(e^{-i\theta}\alpha), \Re(e^{-i\theta}\beta)\}, & \text{if } r < k \leq n - p \\ \frac{1}{2}\Re(e^{-i\theta}(\alpha + \beta)) - \frac{1}{2}\sqrt{4w_\theta^2 + \mu_{n-k+1}}, & \text{if } n - p < k \leq n \end{cases}.$$

If $n < 2r$, it is clear that $\Lambda_k(A) = \{\alpha\}$, whenever $n - r < k \leq r$. By Lemma 2.1, $\Re(e^{-i\theta}\alpha)$ and $\Re(e^{-i\theta}\beta)$ are eigenvalues of $H_\theta(A)$ of multiplicities $r - p$ and $n - r - p$, respectively; concerning the remaining eigenvalues of $H_\theta(A)$, having in mind that

$$\begin{aligned} 4w_\theta^2 + \mu_j &= 2|w|^2 + 2\Re(w^2)\cos(2\theta) + 2\Im(w^2)\sin(2\theta) + \mu_j \\ &= 2|w|^2 + 2|w|^2\cos(2\theta - 2\tau) + \mu_j \\ &= 4|w|^2 - 4|w|^2\sin^2(\theta - \tau) + \mu_j, \end{aligned}$$

with $\tau = \text{Arg } w$, they are of the form

$$\frac{1}{2}\Re(e^{-i\theta}(\alpha + \beta)) \pm \frac{1}{2}\sqrt{|\alpha - \beta|^2 + \mu_j - |\alpha - \beta|^2\sin^2(\theta - \tau)}, \quad j = 1, \dots, p.$$

By Lemma 2.2, we conclude that $C(A)$ contains (at most) p non-degenerate (distinct) elliptical components, all centered at $\frac{1}{2}(\alpha + \beta)$, with major and minor axes of lengths

$$\sqrt{|\alpha - \beta|^2 + \mu_j} \quad \text{and} \quad \sqrt{\mu_j},$$

respectively, being the major one parallel to the vector $e^{i\tau}$, $j = 1, \dots, p$. Then all the ellipses have the same foci at α, β and so

$$E(\alpha, \beta; \sqrt{\mu_p}) \subseteq \dots \subseteq E(\alpha, \beta; \sqrt{\mu_1}) = W(A)$$

is obtained. In particular, the maximum eigenvalue of M is $\mu_1 = \|C + D^*\|^2$. By the characterization of the higher rank numerical range in (1.2), we easily conclude that

$$\Lambda_k(A) = E(\alpha, \beta; \sqrt{\mu_k}), \quad 1 \leq k \leq p.$$

If M is not full rank and $p < k \leq n - r$, then the k -th and $(n - k + 1)$ -th eigenvalues of $H_\theta(A)$ are the maximum and minimum of $\{\Re(e^{-i\theta}\alpha), \Re(e^{-i\theta}\beta)\}$, respectively, yielding an additional degenerated component in $C(A)$. Hence, $\Lambda_k(A) = [\alpha, \beta]$.

Finally, it is obvious that $\Lambda_k(A) = \emptyset$, whenever $r < k \leq n$. $\square$

The *quadratic matrices* are those with minimal polynomial of degree two, which are unitarily similar to matrices of type (1.3), where C can be chosen to be positive semidefinite, $D = O$ and $\alpha \neq \beta$. Corollary 3.8 stated below includes this specific case, for which it is well known [28, 32] that $W(A)$ is the elliptical disc $E(\alpha, \beta; \|C\|)$. According to the next result, the last theorem characterizes the higher rank numerical range of quadratic matrices.

Theorem 3.7. *Let A be the block matrix in (1.3) with $\alpha \neq \beta$. If the spectrum of $M(\theta)$ is independent of θ , then A is a quadratic matrix.*

Proof. If A is of type (1.3) we conclude that the polynomial $p_{H_\theta(A)}(t) = \det(H_\theta(A) - tI_n)$ is a product of quadratic and eventually linear factors associated to the eigenvalues given in Lemma 2.5. If the spectrum of $M(\theta)$ is independent of θ and $n \leq 2r$, then according to the proof of Theorem 3.6, $C(A)$ contains the boundaries of $E(\alpha, \beta; \sqrt{\mu_j})$, $j = 1, \dots, n - r$, and the additional point $\{\alpha\}$, for $n < 2r$. Since the foci of the previous ellipses are eigenvalues of A , then A has at least these two distinct eigenvalues α and β . Suppose that A has another eigenvalue, say γ . Then either $t - \Re(e^{-i\theta}\gamma)$ is a linear factor of the polynomial $p_{H_\theta(A)}(t)$ or there exists a quadratic factor, whose corresponding ellipse in $C(A)$ would have γ as one of its foci. This implies that either $\gamma = \alpha$ or $\gamma = \beta$. This means that the minimal polynomial of A has precisely degree two, that is, A is a quadratic matrix. $\square$

Let $\nu \subseteq \{1, \dots, r\}$ and $\bar{\nu} = \{1, \dots, r\} \setminus \nu$. We have $DC = O$, when the i -th rows of C are zero, for $i \in \nu$ and the j -th columns of D are zero for $j \in \bar{\nu}$. Thus, the results involving the block matrices in [16, Theorems 5–7] follow as corollaries of Theorem 3.6 or Corollary 3.1, and their elliptical higher rank numerical ranges can be explicitly obtained. These matrices are in fact quadratic matrices, due to Proposition 3.7.

Corollary 3.8. *Let A be of type (1.3) with zeros on the i -th rows (columns) of C , for $i \in \nu$, and the j -th columns (resp. rows) of D , for $j \in \bar{\nu}$, and let $p = \text{rank}(C + D^*)$. If $1 \leq k \leq p$, then $\Lambda_k(A)$ is the elliptical disc $E(\alpha, \beta; s_k)$, with $s_1 \geq \dots \geq s_p$ the non-zero singular values of $C + D^*$. In particular,*

$$W(A) = E(\alpha, \beta; \|C + D^*\|).$$

Otherwise, $\Lambda_k(A)$ is either $[\alpha, \beta]$, a point or the empty set, as in Theorem 3.6.

Proof. The statements follow readily from Theorem 3.6, observing that $DC = O$ implies that either $M(\theta) = C^*C + DD^*$ if $n \leq 2r$ or $M(\theta) = CC^* + D^*D$ if $n > 2r$, so $M(\theta)$ is independent of θ and the square roots of the non-zero eigenvalues of $M(\theta)$ coincide with the non-zero singular values of $C + D^*$. $\square$

The previous corollary is also a consequence of Corollary 3.1 for $z = 0$.

As the next example shows, block matrices of type (2.1) can have elliptical higher rank numerical range even if the blocks C, D are not under the conditions of Theorem 2.3.

Example 3.1. The non empty higher rank numerical ranges of the matrix

$$(3.2) \quad A = \begin{bmatrix} 2 & 0 & 0 & 2+2i & 1-i & 0 \\ 0 & 2 & 0 & -i & -1+i & 0 \\ 0 & 0 & 2 & 0 & 0 & 4 \\ \frac{i}{4} & 0 & 0 & 3 & 0 & 0 \\ \frac{i}{4} & \frac{3}{4} + \frac{i}{4} & 0 & 0 & 3 & 0 \\ 0 & 0 & \frac{1}{16} & 0 & 0 & 3 \end{bmatrix}$$

form a nested chain of elliptical discs. In Figure 3.1, the boundaries of $\Lambda_k(A)$ and their foci are represented in black for $k = 3$, blue for $k = 2$ and red for $k = 1$.

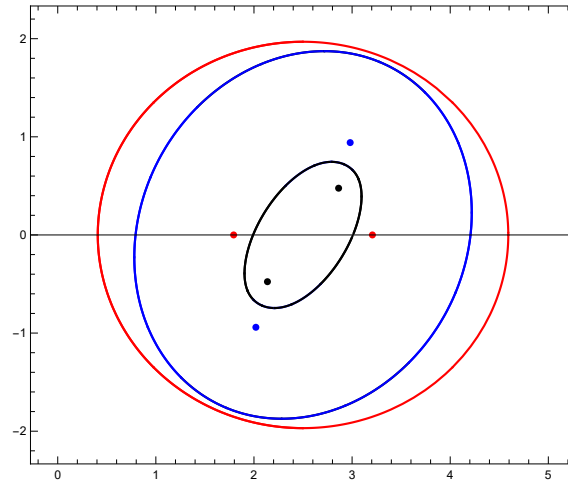


FIGURE 3.1. Boundaries and foci of $\Lambda_3(A) \subset \Lambda_2(A) \subset \Lambda_1(A)$ for A in (3.2)

Thus, the problem of finding necessary and sufficient conditions for occurring elliptical shaped higher rank numerical ranges for block matrices of type (1.3), that might include those of Theorem 2.3, can still be raised. Another question concerns the study of the higher rank numerical ranges when the matrices (1.3) are extended to bi-infinite matrices with two scalar infinite main diagonal blocks.

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