Krein Space Numerical Range of Block Matrices - a Unified Approach to the Hyperbolic Case

N. Bebiano[1], R. Lemos[2] and G. Soares[3]

Abstract

In this paper, we investigate the Krein space numerical range of 2-by-2 block matrices, with diagonal blocks as scalar multiples of the identity. For these matrices, we specifically investigate the cases when the respective boundary generating curves consist of hyperbolas. This provides a unified approach to derive established and new results concerning the numerical range hyperbolic shape.

subjclass Primary 47A12; Secondary 15A60
Keywords: Numerical range, Krein space, Indefinite Kippenhahn curve, Hyperbolical Range Theorem

1 Introduction

As usual, let M_n stand for the associative algebra of $n \times n$ complex matrices and let I_n denote the identity matrix. Consider the complex vector space $\mathbf{C}^n$ equipped with the Krein space structure endowed by the *indefinite inner product* $[x, y]_J = y^* J\, x$, for $x, y \in \mathbf{C}^n$, with $J = I_r \oplus -I_{n-r}$, $0 \leq r \leq n$ (for references on Krein spaces see e.g. [1, 17]). In general, we may consider a non-singular Hermitian matrix $H \in M_n$ instead of J. The *Krein space numerical range*, also called the *indefinite numerical range*, of $A \in M_n$ is defined as

$$W^H(A) = -W_-^H(A) \cup W_+^H(A)$$

with

$$W_\pm^H(A) = \{[Ax, x]_H : \; x \in \mathbb{C}^n, \; [x, x]_H = \pm 1\}.$$

It is clear that $W_+^{-H}(-A) = W_-^H(A)$.

If $H = I_n$, then $W^H(A) = W_+^H(A)$ reduces to the well-known classical *numerical range*, introduced a century ago by Toeplitz [28] and Hausdorff [19], and extensively investigated thereafter (see e.g. [21, Chapter 1] and references therein). The set $W^H(A)$ has also been researched and applied by some authors (see e.g. [2–13, 18, 22–24, 27]).

It is well-known that $W(A)$ contains the spectrum of A. In the present context for

$$\sigma_\pm^H(A) = \{\lambda \in \mathbb{C} : \; \exists\, x \in \mathbb{C}^n, \; [x, x]_H = \pm 1, \; A\,x = \lambda\, x\},$$

the following inclusions hold: $\sigma_+^H(A) \subset W_+^J(A)$ and $\sigma_-^H(A) \subset -W_-^H(A)$.

[1]CMUC and Mathematics Department, University of Coimbra, 3001-501 Coimbra (bebiano@mat.uc.pt)
[2]CIDMA, Department of Mathematics, University of Aveiro, Aveiro, Portugal (rute@ua.pt)
[3]CMAT-UTAD, Department of Mathematics, University of Trás-os-Montes e Alto Douro, Vila Real, Portugal (gsoares@utad.pt)

Both sets $W_+^H(A), W_-^H(A)$ are convex, nevertheless they may not be closed or bounded [22]. On the other hand, $W^H(A)$ is *pseudo-convex*, that is, for any pair of distinct points $w_1, w_2 \in W^H(A)$, either $W^H(A)$ contains the closed line segment $\{tw_1 + (1-t)w_2 : \; 0 \leq t \leq 1\}$ or $W^H(A)$ contains the half-lines $\{tw_1 + (1-t)w_2 : \; t \leq 0 \text{ or } t \geq 1\}$.

A matrix $A \in M_n$ is *H-Hermitian* if it equals its *H-adjoint*, which is given by $A^{[*]} = H^{-1}A^*H$, and $W^H(A) \subseteq \mathbb{R}$ if and only of A is H-Hermitian [22].

If r is the number of positive eigenvalues of the matrix H, counting multiplicities, then by Sylvester law of inertia there exists $S \in M_n$ non-singular, such that $S^*HS = J$, where $J = I_r \oplus -I_{n-r}$ is the inertia matrix of H, and $W^H(A) = W^J(S^{-1}AS)$. We also remark that A is H-Hermitian if and only if $S^{-1}AS$ is J-Hermitian. Thus, without loss of generality, we may restrict our attention to the study of $W^J(A)$ and, for simplicity, the notation $A^\# = JA^*J$ for the J-adjoint of A is used.

Any matrix $A \in M_n$ admits the representation $A = \Re^J(A) + i\,\Im^J(A)$, where

$$\Re^J(A) = \frac{A + A^\#}{2} \qquad \text{and} \qquad \Im^J(A) = \frac{A - A^\#}{2i}$$

are J-Hermitian matrices. For each angle $\theta \in \mathbb{R}$, we define the matrix

$$H_\theta(A) \;=\; \Re^H(A)\cos\theta + \Im^H(A)\sin\theta, \tag{1}$$

which is J-Hermitian, so it has real or complex eigenvalues, these occurring in complex conjugate pairs [17]. If $H_\theta(A)$ has non-real eigenvalues, then $W^J(H_\theta(A))$ is the whole real line [5, Proposition 2.1].

To avoid trivial cases of degeneracy of $W^J(A)$, we will focus on the class $\mathcal{J}$ of matrices $A \in M_n$ such that $H_\theta(A)$ for θ in some real interval, has real eigenvalues, satisfying

$$\sigma_+^J\big(H_\theta(A)\big) = \{\lambda_1(\theta), \ldots, \lambda_r(\theta)\}, \quad \lambda_1(\theta) \geq \cdots \geq \lambda_r(\theta), \tag{2}$$

$$\sigma_-^J\big(H_\theta(A)\big) = \{\lambda_{r+1}(\theta), \ldots, \lambda_n(\theta)\}, \quad \lambda_{r+1}(\theta) \geq \cdots \geq \lambda_n(\theta) \tag{3}$$

and the elements in $\sigma_+^J(H_\theta(A))$ and $\sigma_-^J(H_\theta(A))$ do *not interlace*, that is, either $\lambda_r(\theta) > \lambda_{r+1}(\theta)$ or $\lambda_n(\theta) > \lambda_1(\theta)$. If the eigenvalues of $H_\theta(A)$ interlace, then $W^J(H_\theta(A))$ is also the whole real line.

The characteristic polynomial of $H_\theta(A)$, that is,

$$p_A(z,\theta) \;=\; \det\big(\Re^J(A)\cos\theta \;+\; \Im^J(A)\sin\theta \;-\; z\,I_n\big),$$

is called the *indefinite generating polynomial* of $W^J(A)$ and its eigenvalues play a crucial role in our study (for details, see Section 2). For $A \in M_n$, through the equation $\det(u\,\Re^J(A) + v\,\Im^J(A) + w\,I_n) = 0$, it is associated an algebraic curve of class n, in homogeneous line coordinates. Its real part, denoted by $C^J(A)$, is the *indefinite boundary generating curve*, or *indefinite Kippenhahn curve*, of $W^J(A)$ and its pseudo-convex hull generates the set $W^J(A)$ as follows. For any two points $w_1 = [Ax, x]_J$ and $w_2 = [Ay, y]_J$ in $C^J(A)$, we take the line segment joining them, if $[x,x]_J\,[y,y]_J > 0$, or the union of two half-rays given by $\{tw_1 + (1-t)w_2 : t \leq 0 \text{ or } t \geq 1\}$, if $[x,x]_J\,[y,y]_J < 0$ (see e.g. [3] for more information).

In general, not too much is known on the boundary of $W^J(A)$, except for the case $n = 2$ and for certain structured matrices, some of them with low order, when $W^J(A)$ is the pseudo-convex hull of a finite number of hyperbolas [4–7, 10–12]. In this paper, we are interested in investigating to which extent this shape persists.

For $\alpha, \beta \in \mathbb{C}$, we will focus on matrices of the form

$$A = \begin{bmatrix} \alpha I_r & C \\ D & \beta I_{n-r} \end{bmatrix}, \qquad \alpha \neq \beta, \tag{4}$$

with at least a non-zero off-diagonal entry. If $C = D = O$, then $W^J(A)$ is the union of the half-lines $\{t\,\alpha + (1-t)\beta :\ t \leq 0 \text{ or } t \geq 1\}$, with $\alpha \in W^J_+(A)$ and $\beta \in -W^J_-(A)$.

The remaining of this note is organized as follows. Section 2 contains preliminary results used in subsequent discussions. In Sections 3, 4 and 5, counterparts of classical numerical range results (e.g. [14–16, 25, 26]) are presented in the context of indefinite inner product spaces, specifically in the cases when the indefinite boundary generating curve consists of hyperbolas. Previous known results concerning the shape of $W^J(A)$, when $C^J(A)$ contains a family of hyperbolas, are in particular reobtained from the main unified result in Section 3. Most results are illustrated via specific numerical examples. Section 4 is dedicated to the case when arcs of hyperbolas and flat portions appear at the boundary of $W^J(A)$. Section 5 presents families of matrices with hyperbolic $W^J(A)$, derived from the main result, except for the last theorem. Section 6 is applied to the Krein space numerical range of certain tridiagonal matrices. In Section 7, some final comments are stated.

2 Pre-requisites

The *Hyperbolical Range Theorem* [3, 6] describes the indefinite numerical range in the 2-by-2 case. It states that if $A \in M_2$ has eigenvalues λ_1, λ_2 and $J = \operatorname{diag}(1, -1)$, then $W^J(A)$ is bounded by the non-degenerate hyperbola, with foci at λ_1, λ_2, transverse and non-transverse axes of lengths

$$\left(\operatorname{Tr}(A^\# A) - 2\Re(\lambda_1\bar{\lambda}_2)\right)^{\frac{1}{2}} \qquad \text{and} \qquad \left(|\lambda_1|^2 + |\lambda_2|^2 - \operatorname{Tr}(A^\# A)\right)^{\frac{1}{2}},$$

respectively, if and only if

$$2\Re(\lambda_1\bar{\lambda}_2) < \operatorname{Tr}(A^\# A) < |\lambda_1|^2 + |\lambda_2|^2. \tag{5}$$

Moreover, $W^J(A)$ is the disjoint union of two closed half-lines, with endpoints at λ_1, λ_2, on the line containing these points, if and only if

$$2\Re(\lambda_1\bar{\lambda}_2) < \operatorname{Tr}(A^\# A) = |\lambda_1|^2 + |\lambda_2|^2.$$

The reduction to a singleton occurs if and only if A is a scalar matrix. Further, $W^J(A)$ may degenerate into a line, eventually except a point, or the whole complex plane, eventually except a line (see [2] for a detailed description).

The following elementary properties will be used throughout. For $A \in M_n$,

P1. $W^J(\alpha A + \beta I_n) = \alpha W^J(A) + \beta$ for any $\alpha, \beta \in \mathbb{C}$ (*translational property*);

P2. $W^J(A)$ is *J-unitarily invariant*: $W^J(U^\# AU) = W^J(A)$ for any J-unitary $U \in M_n$, that is, $UU^\# = I_n$.

In the sequel, let $J = I_r \oplus -I_{n-r}$, $0 < r < n$. In order to state a criterion of non-degenerate hyperbolicity of $W^J(A)$ obtained in [6], and which plays a key role in the present research, let Ω be the interval, with greatest possible diameter, of angles θ such that $H_\theta(A) \in \mathcal{J}$. For simplicity, recalling (2) and (3), we adopt the following unified notation:

(I) $\lambda_L(\theta) = \lambda_{r+1}(\theta)$ and $\lambda_R(\theta) = \lambda_r(\theta)$ if $\lambda_r(\theta) > \lambda_{r+1}(\theta)$ holds for $\theta \in \Omega$;

(II) $\lambda_L(\theta) = \lambda_1(\theta)$ and $\lambda_R(\theta) = \lambda_n(\theta)$ if $\lambda_n(\theta) > \lambda_1(\theta)$ holds for $\theta \in \Omega$.

If $W^J(A)$ is a hyperbolic disc with horizontal transverse axis, then $x = \lambda_L(\theta)$ and $x = \lambda_R(\theta)$ are support lines, respectively, of $-W^J_-(A)$ and $W^J_+(A)$, in case (I), or of $W^J_+(A)$ and $-W^J_-(A)$, in case (II). The envelopes of the two following family of lines:

$$\mathrm{e}^{-i\theta}(\lambda_R(\theta) + i\mathbb{R}), \qquad \mathrm{e}^{-i\theta}(\lambda_L(\theta) + i\mathbb{R}),$$

for θ ranging over Ω, provide the characterization of the boundaries of $W^J_+(A)$ and $-W^J_-(A)$, and ultimately of the set $W^J(A)$.

Theorem 2.1 (Criterion of hyperbolicity) *Let $\widetilde{a}, \widetilde{b} > 0$ and $A \in M_n$. The set $W^J(A)$ is bounded by the non-degenerate hyperbola centered at the origin, with horizontal transverse and vertical non-transverse semi-axes of length $\widetilde{a}$ and $\widetilde{b}$, respectively, if and only if*

$$\lambda_R(\theta) = \left(\widetilde{a}^2 - \widetilde{c}^2 \sin^2\theta\right)^{\frac{1}{2}} \qquad \textit{and} \qquad \lambda_L(\theta) = -\left(\widetilde{a}^2 - \widetilde{c}^2 \sin^2\theta\right)^{\frac{1}{2}}$$

where $\widetilde{c}^2 = \widetilde{a}^2 + \widetilde{b}^2$, for $\theta \in (-\theta_0, \theta_0)$ with $\theta_0 = \arctan(\widetilde{a}/\widetilde{b})$.

Without loss of generality, for the study of $W^J(A)$ for matrices A of type (4) we can assume throughout $n \leq 2r$. Indeed, if $n > 2r$, we can instead consider $W^{J'}(A')$ for the matrices

$$A' = \begin{bmatrix} \beta I_{n-r} & D \\ C & \alpha I_r \end{bmatrix} \qquad \text{and} \qquad J' = I_{n-r} \oplus -I_r; \tag{6}$$

becasuse by an adequate permutation matrix P, we have $A' = P^T A\, P$ and $J' = P^T(-J)\, P$, which guarantees the equalities:

$$W^J(A) \,=\, W^{-J}(A) \,=\, W^{J'}(A'),$$

although interchanging the roles of the convex components, according to

$$W_+^J(A) = -W_-^{J'}(A') \qquad \text{and} \qquad -W_-^J(A) = W_+^{J'}(A').$$

For matrices of the form (4), the next result gives pertinent information on the eigenvalues of the J-Hermitian matrix $H_\theta(A)$, for each $\theta \in \mathbb{R}$, in terms of the eigenvalues of the positive semidefinite matrix:

$$M(\theta) \,=\, C^*C + D\,D^* - 2\Re(\mathrm{e}^{-2i\theta}DC), \qquad \text{if} \quad n \le 2r. \tag{7}$$

If $n > 2r$, we just need to interchange α, r, C and $\beta, n-r, D$, respectively, having in mind the considerations above. For simplicity of notation, let

$$B = A - \frac{1}{2}(\alpha+\beta)I_n \qquad \text{and} \qquad \omega = \frac{1}{2}(\alpha-\beta). \tag{8}$$

Proposition 2.1 *Let $A \in M_n$ be a block matrix of type (4), $n \le 2r$ and $\theta \in \mathbb{R}$. Let $\mu_1(\theta), \ldots, \mu_{n-r}(\theta)$ be the eigenvalues of the matrix $M(\theta)$ in (7).*

(a) *If $n < 2r$, the eigenvalues of $H_\theta(A)$ are $\Re(\mathrm{e}^{-i\theta}\alpha)$ in $\sigma_+^J(H_\theta(A))$ and*

$$\lambda_{j,\pm}(H_\theta(A)) \,=\, \frac{1}{2}\Re(\mathrm{e}^{-i\theta}(\alpha+\beta)) \pm \sqrt{\big(\Re(\mathrm{e}^{-i\theta}\omega)\big)^2 - \frac{1}{4}\mu_j(\theta)} \tag{9}$$

for $j = 1, \ldots, n-r$. If $n = 2r$, the eigenvalues of $H_\theta(A)$ are those in (9) for $j = 1, \ldots, \frac{n}{2}$.

(b) *If $\mu_j(\theta) < 4\big(\Re(\mathrm{e}^{-i\theta}\omega)\big)^2$, then one of the eigenvalues in (9) belongs to $\sigma_+^J(H_\theta(A))$ and the other to $\sigma_-^J(H_\theta(A))$.*

Proof. (a) Since A is the block matrix (4), considering (8), we have

$$B = \begin{bmatrix} \omega I_r & C \\ D & -\omega I_{n-r} \end{bmatrix}$$

and

$$H_\theta(A) = \frac{1}{2}\Re(\mathrm{e}^{-i\theta}(\alpha+\beta))I_n + H_\theta(B). \tag{10}$$

Let $N_\theta = \frac{1}{2}\left(\mathrm{e}^{-i\theta}C - \mathrm{e}^{i\theta}D^*\right)$ and consider $k = \operatorname{rank} N_\theta$. By the singular value decomposition, there exist unitary matrices $U \in M_r$, $V \in M_{n-r}$, such that

$$U^*\,N_\theta\,V \,=\, \begin{bmatrix} D_\theta \\ O \end{bmatrix},$$

where D_θ is diagonal with the non-zero singular values $s_1(\theta), \ldots, s_k(\theta)$ of N_θ on the first main diagonal entries (and the null block O is absent if $n = 2r$). Since

$$H_\theta(B) \,=\, \begin{bmatrix} \omega_\theta I_r & N_\theta \\ -N_\theta^* & -\omega_\theta I_{n-r} \end{bmatrix}, \qquad \omega_\theta = \Re(\mathrm{e}^{-i\theta}\omega),$$

considering $Z = U \oplus V$, we can see that the matrix

$$Z^* H_\theta(B)\, Z \;=\; \begin{bmatrix} \omega_\theta I_r & U^* N_\theta\, V \\ -\, V^* N_\theta^*\, U & -\omega_\theta I_{n-r} \end{bmatrix}. \tag{11}$$

is permutationally similar to a direct sum of k blocks of order 2 of type

$$\begin{bmatrix} \omega_\theta & s_i(\theta) \\ -s_i(\theta) & -\omega_\theta \end{bmatrix}, \qquad i = 1, \dots, k,$$

and blocks of order 1, namely $r-k$ equal to ω_θ and $n-r-k$ equal to $-\omega_\theta$. Hence, the eigenvalues of $H_\theta(B)$ are given by

$$\lambda_{j,\pm}(\theta) = \pm\sqrt{\omega_\theta^2 - s_j^2(\theta)}, \qquad j = 1, \dots, k, \tag{12}$$

together with ω_θ and $-\omega_\theta$ of multiplicities $r-k$ and $n-r-k$, respectively. We remark that $M(\theta) = 4\, N_\theta^* N_\theta$. Thus, the eigenvalues of $M(\theta)$ are $\mu_j(\theta) = 4 s_j^2(\theta)$, $j = 1, \dots, n-r$. Recalling (10), the eigenvalues of $H_\theta(A)$ are readily obtained.

(b) Under the hypothesis, we have $s_j^2(\theta) < \omega_\theta^2$, so (12) are real and distinct. Suppose that $s_j(\theta) \neq 0$. The vector $u_{j,\pm}(\theta)$ whose jth entry is $\omega_\theta + \lambda_{j,\pm}(\theta)$, the $(r+j)$th entry is $-s_j(\theta)$ and all the others are zero is an eigenvector of (11) associated to the corresponding eigenvalue (12). By some computations, we get

$$\begin{aligned} [u_{j,\pm}(\theta), u_{j,\pm}(\theta)]_J &= \big(\omega_\theta \pm \lambda_{j,+}(\theta)\big)^2 - s_j^2(\theta) \\ &= 2\lambda_{j,+}(\theta)\,\big(\lambda_{j,+}(\theta) \pm \omega_\theta\big). \end{aligned}$$

From $\lambda_{j,+}^2(\theta) - \omega_\theta^2 = -s_j^2(\theta)$, we find that

$$[u_{j,+}(\theta), u_{j,+}(\theta)]_J\, [u_{j,-}(\theta), u_{j,-}(\theta)]_J \;=\; -4\,\lambda_{j,+}^2(\theta)\, s_j^2(\theta) \;<\; 0.$$

Since Z is a J-unitary matrix, we have

$$[Zu_{j,\pm}(\theta), Zu_{j,\pm}(\theta)]_J = [u_{j,\pm}(\theta), u_{j,\pm}(\theta)]_J \;<\; 0.$$

Consequently, $Zu_{j,\pm}(\theta)$ is an eigenvector of $H_\theta(A)$ associated to the eigenvalue $\lambda_{j,\pm}(H_\theta(A))$, such that one belongs to $\sigma_+^J(H_\theta(A))$, the other to $\sigma_-^J(H_\theta(A))$. When $s_j(\theta) = 0$, then (9) reduce trivially to $\Re(\mathrm{e}^{-i\theta}\alpha)$ in $\sigma_+^J(H_\theta(A))$ and to $\Re(\mathrm{e}^{-i\theta}\beta)$ in $\sigma_-^J(H_\theta(A))$. ■

We finish this section with some useful considerations. If $\alpha = \beta$ in Proposition 2.1, then the eigenvalues (9) of $H_\theta(A)$ associated to a non-zero eigenvalue $\mu_j(\theta)$ of $M(\theta)$ are non-real and $H_\theta(A)$ is not in class $\mathcal{J}$. This is the reason for assuming $\alpha \neq \beta$ in (4), to avoid trivial cases of degeneracy of $W^J(A)$.

The following observation is in order too. If $\Re(\mathrm{e}^{-i\theta}\alpha)$, or $\Re(\mathrm{e}^{-i\theta}\beta)$, is an eigenvalue of $H_\theta(A)$, then the point α, or β, respectively, is a component of $C^J(A)$. The remaining curves in $C^J(A)$ form a family which is central symmetric with respect to $\frac{1}{2}(\alpha+\beta)$ as implied by (9). We are interested in imposing conditions on the blocks C and D that ensure that these curves, whenever distinct from singletons, are non-degenerate hyperbolas. Then the boundary of $W^J(A)$ will be formed by hyperbolic arcs and possibly flat portions, produced from the pseudo-convex hull of the different components of $C^J(A)$. For that purpose, we will analyze cases of matrices $M(\theta)$, defined in (7), whose eigenvalues can be explicitly computed.

3 Main Result

Now, we assume that the non-diagonal blocks of the matrix A in (4) are such that DC is normal and commutes with $C^*C + DD^*$, so that they can be diagonalized by the same unitary similarity transformation, inspired by the work [16]. In this event, let us label the eigenvalues of DC and $C^*C + DD^*$, respectively, by z_j and h_j according to the order in which they appear in the respective main diagonals. That is, letting U be the unitary matrix that simultaneously diagonalizes the above matrices, the j-th diagonal entries of $U^*(C^*C + DD^*)\,U$ and U^*DCU are h_j and z_j, respectively. In the sequel, for simplicity of notation, we use

$$\Delta_j = (\alpha - \beta)^2 + 4z_j,$$

and $\hat{\mathcal{H}}_j$ will denote the hyperbola with foci at

$$f_{j,\pm} = \frac{1}{2}(\alpha + \beta) \pm \frac{1}{2}\sqrt{\Delta_j}$$

and transverse axis, non-transverse axis of length, respectively, equal to

$$N_j = \left(\frac{1}{2}|\Delta_j| + \frac{1}{2}\,|\alpha - \beta|^2 - h_j\right)^{\frac{1}{2}}, \quad M_j = \left(\frac{1}{2}|\Delta_j| - \frac{1}{2}\,|\alpha - \beta|^2 + h_j\right)^{\frac{1}{2}}. \quad (13)$$

As usual, $\mathrm{Arg}(z)$ denotes the principal argument of the complex number z.

Theorem 3.1 *Let A be of type (4), $n \leq 2r$, such that DC is normal and commutes with C^*C+DD^*, with eigenvalues z_j and h_j, $j = 1, \ldots, n-r$, respectively, labelled as mentioned above. The following conditions are equivalent:*

- (A) *$C^J(A) = \{\hat{\mathcal{H}}_1, \ldots, \hat{\mathcal{H}}_{n-r}, \alpha\}$, whenever $n < 2r$, or $C^J(A) = \{\hat{\mathcal{H}}_1, \ldots, \hat{\mathcal{H}}_{\frac{n}{2}}\}$, whenever $n = 2r$;*
- (B) *For $j = 1, \ldots, n-r$, we have*

$$|\alpha - \beta|^2 - |\Delta_j| \;\leq\; 2h_j \;<\; |\alpha - \beta|^2 + |\Delta_j|. \quad (14)$$

In this case, $W^J(A)$ is the pseudo-convex hull of $\hat{\mathcal{H}}_1 \cup \cdots \cup \hat{\mathcal{H}}_{n-r}$. Moreover, each hyperbola $\hat{\mathcal{H}}_j$ is nondegenerate if and only if the corresponding LHS inequality in (14) is strict.

Proof. Under the hypothesis, the matrix $M(\theta)$ in (7) is unitary diagonalizable for all values of θ by the same unitary similarity as DC and $C^*C + DD^*$, implying that the eigenvalues of $M(\theta)$ are

$$\mu_j(\theta) \;=\; h_j - 2\,\Re(z_j)\cos(2\theta) - 2\,\Im(z_j)\sin(2\theta), \qquad j = 1, \ldots, n-r. \tag{15}$$

By some computations,

$$\begin{aligned}(\Re(\mathrm{e}^{-i\theta}\omega))^2 \;&=\; \big(\Re(\omega)\cos\theta + \Im(\omega)\sin\theta\big)^2 \\ &=\; (\Re(\omega))^2\cos^2\theta + (\Im(\omega))^2\sin^2\theta + \Re(\omega)\Im(\omega)\sin(2\theta)\end{aligned}$$

and

$$(\Re(\omega))^2 = \frac{\Re(\omega^2)}{2} + \frac{|\omega|^2}{2}, \qquad (\Im(\omega))^2 = -\frac{\Re(\omega^2)}{2} + \frac{|\omega|^2}{2}, \qquad \Re(\omega)\Im(\omega) = \frac{1}{2}\Im(\omega^2).$$

Hence,

$$(\Re(\mathrm{e}^{-i\theta}\omega))^2 \;=\; \frac{|\omega|^2}{2} + \frac{\Re(\omega^2)}{2}\cos(2\theta) + \frac{\Im(\omega^2)}{2}\sin(2\theta). \tag{16}$$

Let $\phi_j = \frac{1}{2}\mathrm{Arg}(\Delta_j)$. We may conclude, by (15) and (16), that

$$\begin{aligned}(\Re(\mathrm{e}^{-i\theta}\omega))^2 - \frac{\mu_j(\theta)}{4} \;&=\; \frac{|\omega|^2}{2} - \frac{h_j}{4} + \frac{\Re(\omega^2+z_j)}{2}\cos(2\theta) + \frac{\Im(\omega^2+z_j)}{2}\sin(2\theta) \\ &=\; \frac{1}{8}\left(4|\omega|^2 - 2h_j + |\Delta_j|\cos(2\theta - 2\phi_j)\right) \\ &=\; \frac{1}{8}\left(4|\omega|^2 - 2h_j + |\Delta_j| - 2|\Delta_j|\sin^2(\theta-\phi_j)\right). \end{aligned} \tag{17}$$

Firstly, assuming (B), suppose that (14) holds with strict LHS inequality for some $j \in \{1, \ldots, n-r\}$. Then M_j, N_j in (13) are positive and we find that (17) is positive for $\theta \in \Omega_j = (\phi_j - \eta_j, \phi_j + \eta_j)$, considering $\eta_j = \arctan\left(N_j/M_j\right)$. Consequently, $H_\theta(B)$ has real eigenvalues of the form

$$\lambda_{j,\pm}\big(H_\theta(B)\big) \;=\; \pm\left(\frac{1}{4}N_j^2 - \frac{1}{4}(M_j^2+N_j^2)\sin^2(\theta-\phi_j)\right)^{\frac{1}{2}}, \qquad \theta \in \Omega_j,$$

according to Proposition 2.1 and the previous computations, such that one of these eigenvalues belongs to $\sigma_+^J(H_\theta(B))$ and the other to $\sigma_-^J(H_\theta(B))$. The equation of the tangent lines of a component of $C^J(\mathrm{e}^{-i\phi_j}B)$, perpendicular to the direction θ, are of the form

$$x\cos\theta + y\sin\theta \;=\; \lambda_{j,\pm}\big(H_{\theta+\phi_j}(B)\big),$$

because $H_\theta(\mathrm{e}^{-i\phi_j}B) = H_{\theta+\phi_j}(B)$. The envelope of this family of tangent lines, with θ ranging over Ω_j, gives this component in $C^J(\mathrm{e}^{-i\phi_j}B)$, which can be computed from its parametric equations

$$\left\{\begin{array}{lcl} x &=& \lambda_{j,\pm}\big(H_{\theta+\phi_j}(B)\big)\,\cos\theta \;-\; \lambda'_{j,\pm}\big(H_{\theta+\phi_j}(B)\big)\,\sin\theta \\ y &=& \lambda_{j,\pm}\big(H_{\theta+\phi_j}(B)\big)\,\sin\theta \;+\; \lambda'_{j,\pm}\big(H_{\theta+\phi_j}(B)\big)\,\cos\theta \end{array}\right. .$$

We easily conclude that this envelope is a non-degenerate hyperbola centered at the origin (see the proof of Theorem 3.1 given in [6, Theorem ?] for an analogous reasoning); each branch of the hyperbola has the parametric equations associated to either $\lambda_{j,+}\big(H_{\theta+\phi_j}(B)\big)$ or $\lambda_{j,-}\big(H_{\theta+\phi_j}(B)\big)$. The transverse and non-transverse axes have length N_j and M_j. The corresponding rotated hyperbola in $C^J(B) = \mathrm{e}^{i\phi_j} C^J(\mathrm{e}^{-i\phi_j} B)$ has transverse axis parallel to e^{ϕ_j} and the foci are $\pm\frac{1}{2}\sqrt{\Delta_j}$, this implying, after applying a translation, that $\hat{\mathcal{H}}_j$ is a non-degenerate hyperbolic component in $C^J(A)$, because $A = B + \frac{1}{2}(\alpha+\beta)I_n$.

Moreover, if (14) holds for $j = 1, \ldots, n-r$, then some components of $C^J(A)$ may (eventually) reduce to points, this occuring if either (i) the LHS inequality in (14) holds as equality for some $\iota \in \{1, \ldots, n-r\}$ or if (ii) $M(\theta)$ is singular. If (i) holds, we have $M_\iota = 0$ and $N_\iota > 0$, then $\hat{\mathcal{H}}_\iota$ reduces to the points $f_{\iota,\pm}$, one in $W_+^J(A)$, the other in $-W_-^J(A)$. In case (ii), we find α in $W_+^J(A)$ and β in $-W_-^J(A)$ as points of $C^J(A)$. Otherwise, if $M(\theta)$ is non-singular and (14) holds with strict LHS inequality for all $j = 1, \ldots, n-r$, then $C^J(A)$ contains at most $n-r$ nondegenerate concentric hyperbolas, $\hat{\mathcal{H}}_1, \ldots, \hat{\mathcal{H}}_{n-r}$, some of these possibly coincident (whenever DC and $C^*C + DD^*$ have multiple eigenvalues).

Thus, we proved that if (B) is assumed, then $C^J(A) = \{\hat{\mathcal{H}}_1, \ldots, \hat{\mathcal{H}}_{n-r}\}$ and the point α is additionally in $C^J(A)$, if $n < 2r$, that is, we find (A). The reverse implication, (A) implies (B), is clearly satisfied too.

In any of these cases, $W^J(A)$ is the pseudo-convex hull of $\hat{\mathcal{H}}_1 \cup \cdots \cup \hat{\mathcal{H}}_{n-r}$, as the point α is already included in this pseudo-convex hull. ■

The main result in Theorem 3.1 can be easily adapted to the case $n > 2r$, just interchanging the roles of C, D, together with those of α, β and $r, n-r$.

The conditions imposed in Theorem 3.1, concerning the non-scalar blocks C, D of the block matrix A of the form (4), are essential to obtain hyperbolic components in $C^J(A)$, as the next example shows.

Example 3.1 *Let $J = I_2 \oplus -I_2$ and A be the block matrix of form (4), with*

$$\alpha = 1, \qquad \beta = -7, \qquad C = \begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 2 \\ 3 & 0 \end{bmatrix}.$$

Then

$$C^*C + DD^* = \begin{bmatrix} 26 & 18 \\ 18 & 18 \end{bmatrix} \qquad \text{and} \qquad DC = \begin{bmatrix} 12 & 6 \\ 6 & 0 \end{bmatrix}.$$

The matrix DC is normal, but it does not commute with $C^*C + DD^*$, so the hypothesis of Theorem 3.1 are not fullfilled. In this case, $C^J(A)$ consists of a hyperbolic-like shaped curve (which is not a hyperbola) and two deltoids in its interior (see Figure 1).

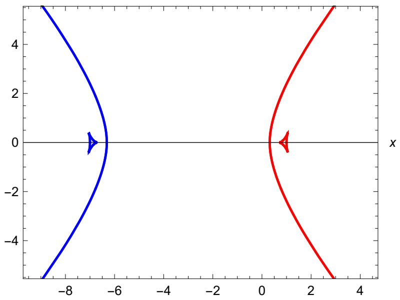

Figure 1: Boundary generating curve of $W^J(A)$ in Example 3.1

4 Arcs of hyperbolas and flat portions on the boundary of $W^J(A)$

Under the hypothesis of Theorem 3.1, if $C^J(A)$ has different hiperbolic components, with points on the boundary of $W^J(A)$, then boundary flat portions will appear too. These boundary line segments are naturally central-symmetric with respect to the point $\frac{1}{2}(\alpha+\beta)$. In other cases, the shape of $W^J(A)$ may be hyperbolic, as discussed in detail in Section 5. Example 4.1 illustrates the existence of boundary flat portions that intersect at a corner of $W^J(A)$, that is, a boundary point of $W^J(A)$ on more than a support line. As it is well known, any corner of $W^J(A)$ is an eigenvalue of A [23].

In all the figures used throughout to illustrate, the branches of the hyperbolas, as well as the points, in $W_+^J(A)$ are represented in blue and those in $-W_-^J(A)$ are colored in red.

Example 4.1 *Let $J = I_3 \oplus -I_3$ and A be a block matrix of the form* (4), *with*

$$\alpha = 10, \qquad \beta = 6, \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then C, D and

$$DC = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = CD$$

are real symmmetric and DC commutes with $C^2 + D^2 = 2D + 4I_2$, that is, the hypothesis of Theorem 3.1 are satisfied. Since $h_1 = 8$, $h_2 = h_3 = 2$, $z_1 = 4$, $z_2 = 1$, $z_3 = -1$, then (14) *holds for $j = 1, 2, 3$, with equality at the LHS of* (14) *for $j = 2$. Therefore, $C^J(A)$ consists of the points $8 \pm \sqrt{3}$ and two nested hyperbolas, with foci $8 \pm \sqrt{8}$ and $8 \pm \sqrt{5}$, respectively, depicted in Figure 2. The set $W^J(A)$ is the pseudo-convex hull of the outer equilateral hyperbola, with foci at $8 \pm \sqrt{8}$ and axes of length 4, and the points $8 + \sqrt{3} \in W_+^J(A)$, $8 - \sqrt{3} \in -W_-^J(A)$, which are corners of $W^J(A)$ (see Figure 3).*

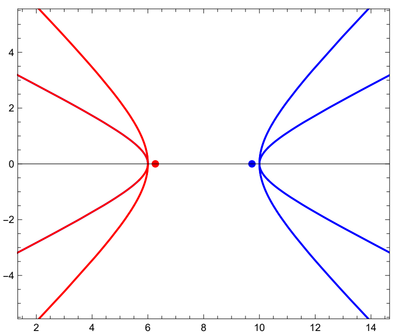

Figure 2: $C^J(A)$ in Example 4.1

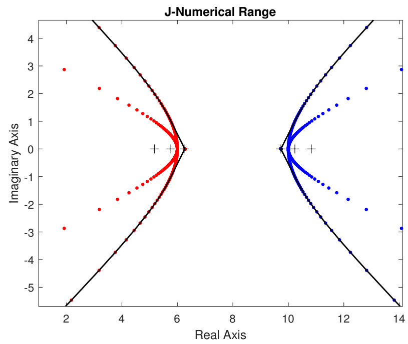

Figure 3: $W^J(A)$ in Example 4.1

Next, Example 4.2 illustrates Theorem 3.1 in the case when non-nested hyperbolas are present in $C^J(A)$.

Example 4.2 *Let $J = I_3 \oplus -I_3$ and A be of the block form* (4) *with*

$$\alpha = 3, \quad \beta = -3, \quad C = \begin{bmatrix} 2+2i & 1-i & 0 \\ -i & -1+i & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} i & 0 & 0 \\ i & 3+i & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

Then

$$C^*C + DD^* = \begin{bmatrix} 10 & -5i & 0 \\ 5i & 15 & 0 \\ 0 & 0 & \frac{257}{16} \end{bmatrix}, \qquad DC = \begin{bmatrix} -2+2i & 1+i & 0 \\ -1-i & -3+3i & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

such that DC is normal and comutes with $C^*C + DD^*$. After some compuations, we can confirm that (14) *holds, with strict LHS inequality, for $j = 1, 2, 3$. By Theorem 3.1, $C^J(A)$ consists of three hyperbolas, all centered at the origin, as displayed in Figure 4. The boundary of $W^J(A)$ contains hyperbolic arcs, from the two non-nested hyperbolas and consequently each convex set $W_+^J(A)$ and $-W_-^J(A)$ will have a line segment on its boundary.*

Theorem 3.1 holds, in particular, for matrices of the form (4) when CD and DC are both normal. The following result was obtained in [3, Theorem 3.1] and is now proved as an easy consequence of Theorem 3.1. In this case, non-nested hyperbolas may be present in $C^J(A)$ too.

Corollary 4.1 *Let A be of type* (4)*, such that CD and DC are both normal, $p = \min\{r, n-r\}$. Let $\sigma_1, \ldots, \sigma_{n-r}$ and $\delta_1, \ldots, \delta_r$ be the singular values of C and D, respectively. If*

$$2\Re(\overline{f_{j+}} f_{j-}) < |\alpha|^2 + |\beta|^2 - \sigma_j^2 - \delta_j^2 < |f_{j+}|^2 + |f_{j-}|^2, \quad j = 1, \ldots, p, \qquad (18)$$

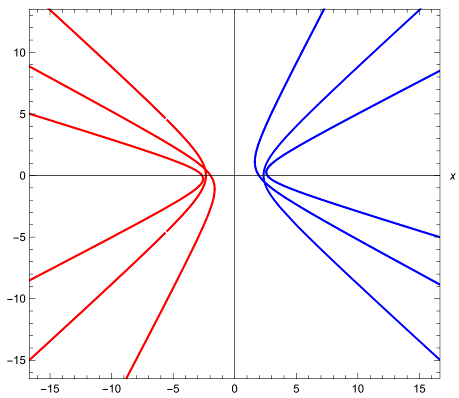

Figure 4: Boundary generating curve of $W^J(A)$ in Example 4.2

then $C^J(A)$ *has at most* p *hyperbolic components (some possibly coincident), with foci at* $f_{j\pm}$ *and non-transverse axis of length*

$$\sqrt{|f_{j+}|^2 + |f_{j-}|^2 - |\alpha|^2 - |\beta|^2 + \sigma_j^2 + \delta_j^2}, \qquad j = 1, \dots, p,$$

and possibly a point, α *if* $n < 2r$ *and* β *if* $n > 2r$. *The set* $W^J(A)$ *is the pseudo-convex hull of these* p *hyperbolas.*

Proof. Without loss of generality, let $n \leq 2r$. The hypothesis on CD and DC being both normal is equivalent to the existence of unitary matrices U, V such that $U^*C^*V = \Sigma$, $U^*D\,V = \Gamma$ are both diagonal [20, p. 426]. This implies that

$$U^*(C^*C + DD^*)U = \Sigma\Sigma^* + \Gamma\Gamma^* \qquad \text{and} \qquad U^*DC\,U = \Gamma\Sigma^*$$

are diagonal matrices too, that is, $C^*C + DD^*$ and DC are simultaneously unitarily diagonalizable, with eigenvalues $h_j = \sigma_j^2 + \delta_j^2$ and $z_j = \sigma_j\delta_j e^{i\chi_j}$, for some $\chi_j \in \mathbb{R}$, for $j = 1, \dots, n-r$, respectively. Then we are under the hypothesis of Theorem 3.1. Moreover,

$$2\Re(\overline{f_{j+}}f_{j-}) = \frac{1}{2}|\alpha+\beta|^2 - \frac{1}{2}|\Delta_j|, \qquad |f_{j+}|^2 + |f_{j-}|^2 = \frac{1}{2}|\alpha+\beta|^2 + \frac{1}{2}|\Delta_j|.$$

Thus, inequalities (18) are equivalent to

$$\big|\,|\alpha-\beta|^2 - 2\sigma_j^2 - 2\delta_j^2\,\big| < |\Delta_j| \qquad j = 1, \dots, n-r.$$

By Theorem 3.1, the result follows. ■

As illustrated by the matrices in Example 4.2 the main result in Theorem 3.1 holds even if CD is not normal, that is, it extends Corollary 4.1. In fact, for C, D as in Example 4.2, we have that

$$CD = \begin{bmatrix} -1+3i & 4-2i & 0 \\ -i & -4+2i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is not normal and Corollary 4.1 does not apply.

In particular, Corollary 4.1 materializes for $A \in M_n$ of the form (4), $n = 2r$, when $C = U - I_r$, $D = U^* + I_r$, $U \in M_r$ unitary. Then $CD = DC = 2i\,\Im(U)$ is skew-Hermitian and $C^*C + DD^* = 4I_r$, corresponding to $h_j = 4$ and z_j zero or pure imaginary with $|z_j| \leq 2$, $j = 1, \ldots, r$, in Theorem 3.1.

Example 4.3 *Let $J = I_3 \oplus -I_3$ and A be of the form (4) with $\alpha = 10$, $\beta = 7$, $C = U - I_4$, $D = U^* + I_4$, considering the unitary matrix*

$$U = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} & 0 \\ \frac{1}{2} - \frac{i}{2} & \frac{1}{2} + \frac{i}{2} & 0 \\ 0 & 0 & -i \end{bmatrix}.$$

In this case, DC has eigenvalues $2i, 0, -2i$ and $C^J(A)$ consists of three non-nested co-centered hyperbolas, depicted in Figure 5. The pseudo-convex hull of these hyperbolas yields a vertical flat portion on the boundary of $W^J(A)$.

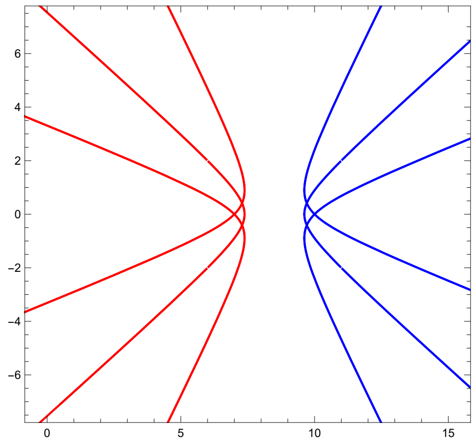

Figure 5: $C^J(A)$ in Example 4.3

In general, it is known that $W^J(A)$ contains a flat portion on the boundary, if $H_\theta(A)$ has a multiple eigenvalue λ, for some $\theta \in \mathbb{R}$, that is a maximum or a minimum of $\sigma_+^J(H_\theta(A))$ or $\sigma_-^J(H_\theta(A))$, $\lambda = \lambda_R(\theta)$ or $\lambda = \lambda_L(\theta)$, and the set of numbers $[H_{\theta+\frac{\pi}{2}}(A)x, x]_J$, when x ranges over the eigenspace $\mathcal{E}_\lambda$ of $H_\theta(A)$ associated to λ, is not a singleton. In this event, if x_1, x_2 are eigenvectors in $\mathcal{E}_\lambda$, satisfying $[x_1, x_1]_J [x_2, x_2]_J > 0$ and

$$[H_{\theta+\frac{\pi}{2}}(A)\, x_1, x_1]_J / [x_1, x_1]_J \neq [H_{\theta+\frac{\pi}{2}}(A)\, x_2, x_2]_J / [x_2, x_2]_J,$$

then the boundary of $W^J(A)$ contains the line segment joining $[Ax_1, x_1]_J$ and $[Ax_2, x_2]_J$.

5 Classes of matrices with hyperbolic $W^J(A)$

In this section, selected classes of block matrices A of the form (4) with hyperbolic shaped $W^J(A)$ are presented. The next corollary applies to Hermitian matrices A of these block form.

Corollary 5.1 *Let $A \in M_n$ be non-diagonal of the form (4) with $\alpha, \beta \in \mathbb{R}$ and $D = C^*$. Let $\sigma_1 > \cdots > \sigma_s$ be the non-zero singular values of C. Then $C^J(A)$ consists of s nested hyperbolas $\tilde{H}_1, \ldots, \tilde{H}_s$ and, additionally, the points α, β, if C is not full rank, the point α if $n < 2r$, and β if $n > 2r$. Each hyperbola $\tilde{H}_j$ is centered at $\frac{1}{2}(\alpha + \beta)$, has horizontal transverse semi-axis of lenght $|\omega|$ and vertical non-transverse semi-axes of length σ_j. The set $W^J(A)$ is the non-degenerate hyperbolic disc bounded by $\tilde{H}_1$.*

Proof. Firstly, let $n \leq 2r$. Clearly, A is under the conditions of Theorem 3.1, $h_j = 2\sigma_j^2$, $z_j = \sigma_j^2$, $j = 1, \ldots, s$, and (14) holds for $j = 1, \ldots, s$, with strict LHS inequality. Hence, if $n = 2r$, then $C^J(A)$ consists of s non-degenerate hyperbolas corresponding to the non-zero singular values of C and the points α, β, if C is not full rank. If $n < 2r$, then $C^J(A)$ consists of these hyperbolas and the point α. By Theorem 3.1, the hyperbolas have foci at

$$\frac{1}{2}(\alpha + \beta) \pm \sqrt{\omega^2 + \sigma_j^2}, \qquad j = 1, \ldots, s,$$

all have the same transverse semi-axis of lenght $|\omega|$ and non-tranverse semi-axis of lenght σ_j, $j = 1, \ldots, s$, respectively. Thus, they are nested and $W_J(A)$ is the non-degenerate hyperbolic disc bounded by the outer hyperbola.

If $n > 2r$, replacing C by D and α by β leads to the same conclusion, because C, C^* have the same non-zero singular values. ■

Theorem 3.1 holds, in particular, when DC is a scalar multiple of the identity, in which case all z_j are the same. The introduced notation for the hyperbola $\hat{\mathcal{H}}_j$ and its foci $f_{j,\pm}$ from Section 3 is reused below.

Corollary 5.2 *Let A be of type (4), $DC = z_1 I_{n-r}$, $n \leq 2r$. The set $W^J(A)$ is a non-degenerate hyperbolic disc with foci at $f_{1\pm}$ and non-transverse axis of length M_1 if and only if*

$$\left| |\alpha - \beta|^2 - 2\|C^*C + DD^*\| \right| < \left| (\alpha - \beta)^2 + 4z_1 \right|. \tag{19}$$

In this case, all the hyperbolic components $\hat{\mathcal{H}}_1, \ldots, \hat{\mathcal{H}}_{n-r}$ in $C_J(A)$ are nested.

Proof. Under the hypothesis, the hyperbolas $\hat{\mathcal{H}}_1, \ldots, \hat{\mathcal{H}}_{n-r}$ share the foci $f_{1\pm}$, and the axes lenghts N_j, M_j in (13) of $\hat{\mathcal{H}}_j$ vary just according to the eigenvalue h_j of $C^*C + DD^*$. Therefore, all the hyperbolas are nested. If $h_1 = \|C^*C + DD^*\|$, then N_1 is the minimum transverse axis and, consequently, M_1 is the maximum non-transverse axis. The result readily follows from Theorem 3.1. ■

Example 5.1 *Let $J = I_3 \oplus -I_2$ and A be of the form* (4) *with*

$$\alpha = 6 + i, \qquad \beta = 13, \qquad C = \begin{bmatrix} -2 & -1 \\ -4 & -2 \\ -3 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & -4 \end{bmatrix}.$$

Then $DC = -6I_2$,

$$C^*C + DD^* = \begin{bmatrix} 33 & 2 \\ 2 & 30 \end{bmatrix}, \qquad \|C^*C + DD^*\| = 34.$$

By simple computations, (19) *holds and, by Corollary 5.2, the set $W_J(A)$ is the hyperbolic disc, with foci at*

$$\frac{1}{2}(19 + i) \pm \frac{1}{2}\sqrt{24 - 14\,i}$$

and non-transverse axis of length $\left(9 + \sqrt{193}\right)^{\frac{1}{2}}$. By Theorem 3.1, $C^J(A)$ consists of two co-centered hyperbolas and a point, $(6, 1)$*, displayed in Figure 7.*

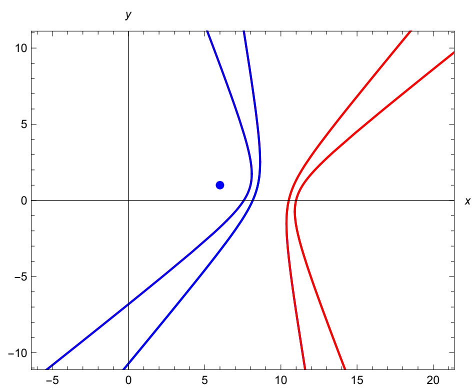

Figure 6: Boundary generating curve of $W^J(A)$ in Example 5.1

A *quadratic matrix* is a matrix with minimal polynomial of degree 2 and it can be reduced, under a unitary similarity transformation, to a matrix A of type (4) with $D = O$. In this case, we easily find the next result (cf. [3, 13]).

Corollary 5.3 *Let A be of type* (4) *with $C \neq O$ and $D = O$. Then $W^J(A)$ is a non-degenerate hyperbolic disc, with foci at α and β, and non-transverse axis of lenght $\|C\|$ if and only if $\|C\| < |\alpha - \beta|$.*

Proof. Since $D = O$, the result follows from Corollary 5.2 with $z_1 = 0$. In this case, $f_{1\pm}$ reduce to α, β. Further, $\|C^*C\| = \|C\|^2 \neq 0$ and the non-degenerate hiperbolicity condition (19) of $W^J(A)$ becomes $\left||\alpha - \beta|^2 - 2\|C\|^2\right| < |\alpha - \beta|^2$, which can equivalently be simplfied to $\|C\| < |\alpha - \beta|$. ■

If $r = n-1$, we have $J = I_{n-1} \oplus -I_1$ and the matrix in (4) reduces to an *arrowhead matrix* of type

$$\left[\begin{array}{ccc|c} \alpha & & & c_1 \\ & \ddots & & \vdots \\ & & \alpha & c_r \\ \hline d_1 & \cdots & d_r & \beta \end{array}\right], \tag{20}$$

with zeros at the omitted entries of the first diagonal block. The next result holds, in parallel to the classical numerical range elliptical case [15, 26].

Corollary 5.4 *For an arrowhead matrix $A \in M_n$ of type (20), $r = n-1$, $\mathbf{c} = (c_1, \ldots, c_r)$, $\mathbf{d} = (d_1, \ldots, d_r)$, the set $W^J(A)$ is a non-degenerate hyperbolic disc, with foci at*

$$\frac{1}{2}(\alpha+\beta) \pm \frac{1}{2}\sqrt{(\alpha-\beta)^2 + 4\,\mathbf{c}^T\mathbf{d}},$$

non-transverse axis of length

$$\left(\frac{1}{2}\left|(\alpha-\beta)^2 + 4\,\mathbf{c}^T\mathbf{d}\right| - \frac{1}{2}|\alpha-\beta|^2 + \|\mathbf{c}\|^2 + \|\mathbf{d}\|^2\right)^{\frac{1}{2}},$$

if and only if

$$\left||\alpha-\beta|^2 - 2\|\mathbf{c}\|^2 - 2\|\mathbf{d}\|^2\right| \;<\; \left|(\alpha-\beta)^2 + 4\,\mathbf{c}^T\mathbf{d}\right|.$$

Proof. The non-diagonal blocks of the arrowhead matrix A are such that

$$DC \;=\; c_1d_1 + \cdots + c_rd_r = \mathbf{c}^T\mathbf{d},$$

$$C^*C + DD^* = |d_1|^2 + \cdots + |d_r|^2 + |c_1|^2 + \cdots + |c_{r1}|^2 = \|\mathbf{c}\|^2 + \|\mathbf{d}\|^2$$

and the result is a direct consequence of Corollary 5.2. ■

Now, we consider the case when the spectrum of $M(\theta)$ is independent of θ and study the condition for non-degenerate hyperbolicity of $W^J(A)$.

Theorem 5.1 *Let $A \in M_n$ be a block matrix of the form (4) with the spectrum of $M(\theta)$ in (7) independent of θ. The set $W^J(A)$ is a hyperbolic disc, with foci at α and β, non-transverse axis of length $\|C - D^*\|$, if and only if*

$$\|C - D^*\| < |\alpha - \beta|.$$

In this case, all the hyperbolic components of $C^J(A)$ are nested, with the same foci, non-transverse axes of lenght $\sqrt{\mu_1}, \ldots, \sqrt{\mu_s}$, where $\mu_1, \ldots, \mu_s$ are the distinct non-zero eigenvalues of $M(0)$.

Proof. By hypothesis, the spectrum of $M(\theta)$ is independent of θ, thus it coincides with the spectrum of

$$M(0) \,=\, C^*C + DD^* - 2\Re(DC) = (C - D^*)^*(C - D^*).$$

Suppose that the maximum eigenvalue of $M(0)$ is μ_1, that is, $\mu_1 = \|C - D^*\|^2$. If $\|C - D^*\| < |\alpha - \beta|$ is assumed and $\theta \in \Omega_j' = (\tau - \theta_j, \tau + \theta_j)$ with $\tau = \mathrm{Arg}(\alpha - \beta)$,

$$\theta_j = \arctan\left(\frac{|\alpha - \beta|^2}{\mu_j} - 1\right)^{\frac{1}{2}},$$

then

$$\mu_j \,<\, \left(\Re(\mathrm{e}^{-i\theta}(\alpha - \beta))\right)^2 \,=\, |\alpha - \beta|^2 \cos^2(\theta - \tau), \qquad j = 1, \ldots, s,$$

and it follows from Proposition 2.1 that $H_\theta(B)$ is in class $\mathcal{J}$, with s pairs of real eigenvalues of the form

$$\lambda_{j,\pm}\big(H_\theta(B)\big) \,=\pm\frac{1}{2}\left(|\alpha - \beta|^2 - \mu_j - |\alpha - \beta|^2 \sin^2(\theta - \tau)\right)^{\frac{1}{2}}, \quad j = 1, \ldots, s.$$

We conclude that $C^J(B)$ contains s hyperbolic components, all with the same foci $\pm\frac{1}{2}(\alpha - \beta)$ and non-transverse axes of lenght $\sqrt{\mu_j}$, $j = 1, \ldots, s$. This corresponds to s hyperbolic nested components in $C^J(A)$, all of them with foci at α and β, with the previous non-transverse axes lenghts.

If $M(0)$ is not full rank, or $n \neq 2r$, then the remaining eigenvalues of $H_\theta(B)$ would additionally give α and β, or just one of these, as points in $C^J(A)$. Clearly, $W^J(A)$ is bounded by the outer hyperbola, obtained for $j = 1$, with non-transverse axis of length $\|C - D^*\|$.

The reverse implication is trivially satisfied. ■

If for a subspace S invariant under $C^*C + DD^*$, the column space of DC is contained in S and S is contained in the null space of DC, then the eigenvalues of (7) are independent of θ, as observed in [16] and then Theorem 5.1 holds.

Example 5.2 *Let $J = I_2 \oplus -I_2$ and A be the block matrix of the form* (4) *with*

$$\alpha = 8 + i, \qquad \beta = 4 - i, \qquad C = \begin{bmatrix} 1 - i & 1 - i \\ -1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 + i & 2 \\ 0 & 0 \end{bmatrix}.$$

Then

$$DC = \begin{bmatrix} 0 & 6 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad C^*C + DD^* = \begin{bmatrix} 9 & 0 \\ 0 & 6 \end{bmatrix}.$$

The matrix in (7) *is*

$$M(\theta) = \begin{bmatrix} 9 & -6\,\mathrm{e}^{-2i\theta} \\ -6\,\mathrm{e}^{2i\theta} & 6 \end{bmatrix},$$

which has eigenvalues $\frac{15}{2} \pm \frac{3}{2}\sqrt{17}$ *independent of* θ*. By Theorem 5.1, the set* $W^J(A)$ *is a non-degenerate hyperbolic disc with foci at* $(8,1)$ *and* $(4,-1)$ *and non-transverse axis of length*

$$\sqrt{\frac{15}{2} + \frac{3}{2}\sqrt{17}}.$$

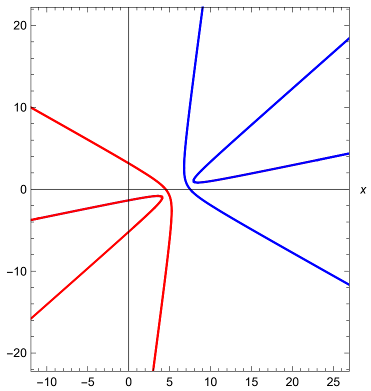

Figure 7: Boundary generating curve of $W^J(A)$ in Example 5.2

6 Results for certain tridiagonal matrices

In this section, we consider $\tilde{J} = \operatorname{diag}(1,-1,1,-1,\dots) \in M_n$ and focus on the class of tridiagonal matrices with biperiodic main diagonal $(\alpha,\beta,\alpha,\beta,\dots) \in \mathbb{C}^n$, denoted by $T_n(\alpha,\beta;\mathbf{c},\mathbf{d})$, where $\mathbf{c} = (c_1,\dots,c_{n-1})$, $\mathbf{d} = (d_1,\dots,d_{n-1}) \in \mathbb{C}^{n-1}$ are, respectively, the first lower and first upper subdiagonals of the matrix.

By [10, Lemma 2.1], the $\tilde{J}$-numerical range of these tridiagonal matrices is invariant under switching of any two corresponding off-diagonal entries c_j and d_j. Consider $K \subset \{1,\dots,n-1\}$ and $\tilde{T}$ the matrix obtained from $T_n(\alpha,\beta;\mathbf{c},\mathbf{d})$ by interchanging c_j and d_j for $j \in K$. Let $r = \left\lceil \frac{n}{2} \right\rceil$ and P be the permutation matrix associated with the permutation $\pi \in S_n$ defined by

$$\pi(i) = 2i-1, \quad 1 \le i \le r, \qquad \text{and} \qquad \pi(r+i) = 2i, \quad 1 \le i \le n-r. \tag{21}$$

Clearly, $\tilde{J} = P^T J P$ and $\tilde{T} = P^T A P$ for $J = I_r \oplus -I_{n-r}$ and A a block matrix of type (4), where $n = 2r$ or $n = 2r-1$. Therefore, $W^{\tilde{J}}(\tilde{T}) = W^J(A)$ and the results of the previous sections may be applied to conveniently derive corresponding characterizations of the indefinite Kippenhahn curve and $\tilde{J}$-numerical range of tridiagonal matrices of type $\tilde{T}$.

The next corollary exemplifies this situation. It was obtained in [6, Theorem 3.1] when $\mathbf{a} = (a,-a,a,-a,\dots)$, $a \in \mathbb{R}$ (see also [10]).

For simplicity, let $\omega = \frac{1}{2}(\alpha - \beta)$ and consider the bidiagonal matrix $X(\mathbf{c})$ defined, in terms of the entries of the vector $\mathbf{c} = (c_1, \ldots, c_{n-1})$, by either

$$\begin{bmatrix} c_1 & & & & \\ c_2 & c_3 & & & \\ & c_4 & c_5 & & \\ & & \ddots & \ddots & \\ & & & c_{n-2} & c_{n-1} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c_1 & & & \\ c_2 & c_3 & & \\ & \ddots & \ddots & \\ & & c_{n-3} & c_{n-2} \\ & & & c_{n-1} \end{bmatrix}, \tag{22}$$

according to n even or n odd, with zeros at the remaining (ommitted) entries.

Corollary 6.1 *Let $T = T_n(\alpha, \beta; \mathbf{c}, \mathbf{d})$ with $\mathbf{d} = \kappa\,\overline{\mathbf{c}}$, $\kappa \in \mathbb{C}$. Let $s_1 \geq \ldots \geq s_{\lfloor \frac{n}{2} \rfloor}$ be the singular values and k be the rank of $X(\mathbf{c})$. Let $\mathcal{H}_j$ be the hyperbola with foci at*

$$\frac{1}{2}(\alpha + \beta) \pm \sqrt{\omega^2 + \kappa\, s_j^2}$$

and non-transverse axis of length

$$\sqrt{2\left|\omega^2 + \kappa\, s_j^2\right| - 2\,|\omega|^2 + (|\kappa|^2 + 1)s_j^2}.$$

Then $C^J(T)$ consits of the nested hyperbolas $\mathcal{H}_1, \ldots, \mathcal{H}_k$ and possibly the points α, β, if $k < \lfloor \frac{n}{2} \rfloor$, or the point α, if n is odd and $k = \lfloor \frac{n}{2} \rfloor$, if and only if

$$\left| |\,\omega|^2 - \frac{|\kappa|^2 + 1}{2} s_1^2 \right| < \left|\omega^2 + \kappa\, s_1^2\right|. \tag{23}$$

In this case, $W^{\tilde{J}}(T)$ is bounded by the non-degenerate hyperbola $\mathcal{H}_1$.

Proof. Let $\tilde{T} = T_n(\alpha, \beta; \widetilde{\mathbf{c}}, \widetilde{\mathbf{d}})$, with $\widetilde{\mathbf{c}} = (c_1, \kappa\overline{c_2}, c_3, \ldots)$, $\widetilde{\mathbf{d}} = (\kappa\overline{c_1}, c_2, \kappa\overline{c_3}, \ldots)$. Considering $J = I_r \oplus -I_{n-r}$, $r = \left\lceil \frac{n}{2} \right\rceil$, and A the block matrix in (4) with $C = X(\mathbf{c})$ and $D = \kappa\, C^*$, having in mind the previous considerations, we may conclude that

$$W^{\tilde{J}}(T) \;=\; W^{\tilde{J}}(\tilde{T}) \;=\; W^J(A).$$

It is obvious that $DC = \kappa\, C^*C$ and $C^*C + DD^* = (1 + |\kappa|^2)\, C^*C$ are under the hypothesis of Theorem 3.1 and have, respectively, the eigenvalues

$$z_j = \kappa s_j^2 \qquad \text{and} \qquad h_j = (1 + |\kappa|^2)\, s_j^2, \qquad j = 1, \ldots, k,$$

all the remaining being zero if $k < \lfloor \frac{n}{2} \rfloor$. The result easily follows from Theorem 3.1, observing that $z_1 \geq \cdots \geq z_k$ and $h_1 \geq \cdots \geq h_k$ allow us to conclude that the hyperbolas $\mathcal{H}_1, \ldots, \mathcal{H}_k$ are nested and non-degenerated if and only if (23) holds; the outer one $\mathcal{H}_1$ forms the hyperbolic boundary of $W^{\tilde{J}}(T)$. ■

7 Final comments

We have investigated ...

References

[1] T. Ya. Azizov and I. S. Iokhvidov, *Linear Operators in Spaces with an Indefinite Metric*, Nauka, Moscow, 1986, English Translation: Wiley, New York, 1989.

[2] N. Bebiano, R. Lemos, J. da Providência and G. Soares, Remarks on generalized numerical ranges of operators on indefinite inner product spaces *Online Proceedings of SIAM Applied Linear Algebra Conference*, College of William and Mary, Williamsburg (2003) 15–19.

[3] N. Bebiano, R. Lemos, J. da Providência and G. Soares, On generalized numerical ranges of operators on a indefinite inner product space, *Linear Multilin. Algebra* **52** (2004), 203–233.

[4] N. Bebiano, R. Lemos and J. da Providência, Numerical ranges of unbounded operators arising in quantum physics, *Linear Algebra Appl.* **381** (2004), 259–279.

[5] N. Bebiano, R. Lemos, J. da Providência and G. Soares, On the geometry of numerical ranges in spaces with an indefinite inner product, *Linear Algebra Appl.* **399** (2005), 17–34.

[6] N. Bebiano, R. Lemos and G. Soares, On the hyperbolicity of the Krein space numerical range, *Linear Multilin. Algebra* **52** (2023), 1–21.

[7] N. Bebiano, R. Lemos and G. Soares, Matrices with hyperbolical Krein space numerical range. *Adv. Oper. Theory* **10**, 13 (2025).

[8] N. Bebiano, H. Nakazato, R. Lemos, J. da Providência and G. Soares, Inequalities for J-Hermitian matrices, *Linear Algebra Appl.* **407** (2005), 67–84.

[9] N. Bebiano, J. da Providência, A. Nata and J. P. da Providência, An inverse indefinite numerical range problem, *Linear Algebra Appl.* **470**, (2015), 200–215.

[10] N. Bebiano, J. da Providência, A. Nata and G. Soares, Krein spaces numerical ranges and their computer generation, *Electr. J. Linear Algebra* **17** (2008), 192–208.

[11] N. Bebiano, J. da Providência and R. Teixeira, Flat portions on the boundary of the indefinite numerical range of 3×3 matrices, *Linear Algebra Appl.* **428** (2008), 2863–2879.

[12] N. Bebiano, J. da Providência and R. Teixeira, Indefinite numerical range of 3×3 matrices. *Czech Math. J.* **59** (2009), 221–239.

[13] N. Bebiano and J. da Providência, Numerical ranges of quadratic operators, *Electr. J. Linear Algebra* **26** (2013), 141–157.

[14] C. C. Chang, H.-L. Gau, Y.-S. Wang, S.-C. Wu, and Y.-T. Yeh, Matrices with defect index one, Operators and Matrices **7** (2013), no. 4, 865–885.

[15] M.-T. Chien and K.-C. Hung, Elliptic numerical ranges of bordered matrices, *Taiwanese Journal of Mathematics*, **16** (3) (2012).

[16] T. Geryba and I. M. Spitkovsky, On the numerical range of some block matrices with scalar diagonal blocks, *Linear Multilin. Algebra* **69**: 5 (2021), 772–785.

[17] I. Gohberg, P. Lancaster and L. Rodman, *Matrices and Indefinite Scalar Products*, OT 8, Birkäuser Verlag, Basel, 1983.

[18] A. Guterman, R. Lemos and G. Soares, More on geometry of Krein space C-numerical range, *Appl. Math. Comput.* **352** (2019), 258–269.

[19] F. Hausdorff, Der Wertvorrat einer Bilinearform. *Math. Z.* **3** (1919), 314–316.

[20] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[21] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[22] C.-K. Li, N.-K. Tsing and F. Uhlig, Numerical ranges of an operator on an indefinite inner product space, *Electr. J. Linear Algebra* **1** (1996), 1–17.

[23] C.-K. Li and L. Rodman, Remarks of numerical ranges of operators in spaces with an indefinite inner product, *Proc. Amer. Math. Soc.* **126** (1998), 973–982.

[24] C.-K. Li and L. Rodman, Shapes and computer generation of numerical ranges of Krein space operators, *Electr. J. Linear Algebra*, **3** (1998), 31–47.

[25] C.-K. Li, Y.-T. Poon and M. Tominaga, Spectra, norms and numerical ranges of generalized quadratic operators, *Linear Multilin. Algebra* **59**: 10 (2011), 1077–1104.

[26] H. Linden, Containment regions for zeros of polynomials from numerical ranges of companion matrices, *Linear Algebra Appl.*, **350**, 1-3, (2002), 125–145.

[27] F. Philipp and C. Trunk, The numerical range of non-negative operators in Krein spaces, *Linear Algebra Appl.* **438** (2013), 2542–2556.

[28] O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejér. *Math. Z.* **2** (1918), 187–197.