THE THIRD COHOMOLOGY GROUP OF A MONOID AND ADMISSIBLE ABSTRACT KERNELS II

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Abstract. We show that, with any admissible abstract kernel of monoids, which is a monoid homomorphism $\Phi\colon M \to \frac{End(A)}{Inn(A)}$, where both M and A are monoids and Φ satisfies a suitable condition, it is possible to associate an obstruction, which is an element of the third cohomology group of M with coefficients in the abelian group $U(Z(A))$ of invertible elements of the center $Z(A)$ of A, whose M-module structure is determined by Φ. We show that the obstruction is zero precisely when the admissible abstract kernel is induced by a regular Schreier extension of monoids. We equip the set of equivalence classes of admissible abstract kernels inducing the same action of M on $U(Z(A))$ with a commutative monoid structure. We prove that the quotient of this monoid with respect to the submonoid of equivalence classes of admissible abstract kernels induced by a regular Schreier extension is isomorphic to the third cohomology group $H^3(M, U(Z(A)))$.

1. Introduction

In [7] Eilenberg and Mac Lane gave an interpretation of the third cohomology group $H^3(\Pi, C)$ of a group Π with coefficients in a Π-module C by means of abstract kernels. An abstract kernel [19, 20] is a group homomorphism $\Phi\colon \Pi \to \frac{Aut(G)}{Inn(G)}$, where Π and G are groups, and $Aut(G)$ and $Inn(G)$ are the groups of automorphisms and inner automorphisms of G, respectively. Eilenberg and Mac Lane associated with every abstract kernel Φ a 3-cocycle of the cohomology of Π with coefficients in the Π-module given by the action of Π on the center $Z(G)$ of G induced by Φ. This 3-cocycle is called the *obstruction* of Φ. The reason of the name relies on the following fact: some abstract kernels are induced by group extensions of the form

$$0 \longrightarrow G \rightarrowtail B \twoheadrightarrow \Pi \longrightarrow 1;$$

it was proved in [7] that the abstract kernels induced by a group extension are precisely those whose obstruction $Obs(\Phi)$ is the zero element of $H^3(\Pi, Z(G))$. Moreover, Eilenberg and Mac Lane considered an equivalence relation on the set of abstract kernels inducing the same module structure, and proved that the set of such equivalence classes, modulo the classes of abstract kernels that are induced by group extensions, can be equipped with a binary operation that turns it isomorphic to the third cohomology group $H^3(\Pi, Z(G))$. They also got a cohomological description of group extensions with non-necessarily abelian kernel: if an abstract kernel $\Phi\colon \Pi \to \frac{Aut(G)}{Inn(G)}$ is induced by some group extension, then the set of isomorphism classes of extensions inducing Φ is in bijection with the second cohomology group $H^2(\Pi, Z(G))$. Similar results have later been obtained for other algebraic structures [9, 10, 11, 14], for categorical groups [8, 4], and in a general categorical

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context [3, 1, 6, 5].

In [12] we extended part of the results of [7] to the case of monoids. An abstract kernel of monoids is a monoid homomorphism $\Phi\colon M \to \frac{End(A)}{Inn(A)}$, where M and A are monoids, $End(A)$ is the monoid of endomorphisms of A, and $Inn(A)$ is the subgroup of inner automorphisms induced by invertible elements of A. We showed that regular Schreier extensions of monoids induce abstract kernels, similarly to what happens for groups. Schreier extensions have been introduced in [18] and studied in [15, 16, 17] in connection with the cohomology of monoids with coefficients in semimodules. We showed that, with any abstract kernel of the form $\Phi\colon M \to \frac{SEnd(A)}{Inn(A)}$, where $SEnd(A)$ is the monoid of surjective endomorphisms of A, it is possible to associate an obstruction, which is a 3-cocycle of the cohomology of M with coefficients in the M-module given by the action of M on $U(Z(A))$ (the subgroup of invertible elements of the center of A) induced by Φ. This obstruction is the zero element of the third cohomology group $H^3(M, U(Z(A)))$ if and only if the abstract kernel is induced by a regular Schreier extension. Furthermore, if an abstract kernel is induced by a regular Schreier extension, then the set of isomorphism classes of Schreier extensions inducing it is in bijection with the second cohomology group $H^2(M, U(Z(A)))$.

Then, in [13], we considered abstract kernels of the form $\Phi\colon M \to \frac{End(G)}{Inn(G)}$, where M is a monoid and G is a group, with the additional condition (introduced in [22, 23]) of *admissibility.* This condition consists in requiring that, for every $x \in M$ and every $\varphi(x) \in \Phi(x)$, the centralizer of $\varphi(x)(G)$ in G coincides with the center $Z(G)$. We showed that any admissible abstract kernel induces an action of M on $Z(G)$. Moreover, generalizing what was done in [7], we defined a commutative monoid structure on the set of suitable equivalence classes of admissible abstract kernels inducing the same action; we showed that the subset of equivalence classes of admissible abstract kernels that are induced by a Schreier extension of the form

$$0 \longrightarrow G \rightarrowtail B \twoheadrightarrow M \longrightarrow 1$$

(those extensions have been called *special Schreier extensions* in [2]) is a submonoid, and that the factor monoid is an abelian group, isomorphic to the third cohomology group $H^3(M, Z(G))$ of the monoid M with coefficient in the given M-module $Z(G)$.

In this paper we extend the results of [13] to the case of abstract kernels of the form $\Phi\colon M \to \frac{End(A)}{Inn(A)}$, where both M and A are monoids. We consider a variation of the admissibility condition, asking that, for every $x \in M$ and every $\varphi(x) \in \Phi(x)$, the subgroup of invertible elements of the centralizer of $\varphi(x)(A)$ in A coincides with the subgroup of invertible elements of the center $Z(A)$. We show that, also in this more general setting, admissible abstract kernels induce actions of M on $U(Z(A))$. Using admissibility instead of the surjectivity of the endomorphisms, we adapt the construction, made in [12], of the 3-cocycle associated with an admissible abstract kernel of the form $\Phi\colon M \to \frac{End(A)}{Inn(A)}$. We define an equivalence relation on the set of admissible abstract kernels inducing the same action; we endow the set of such equivalence classes with a commutative monoid structure; we show that the equivalence classes of the admissible abstract kernels induced by a regular Schreier extension (in the sense of [12]) form a submonoid; we prove that the factor monoid is isomorphic to the third cohomology group $H^3(M, U(Z(A)))$, where the M-module structure on $U(Z(A))$ is the one induced by the admissible abstract kernels. This gives a new interpretation, now purely in terms of monoids, of the third cohomology group of a monoid with coefficients in a module.

2. Admissible abstract kernels

We start by establishing some notation. Given a monoid A, written additively, we will denote by $End(A)$ the monoid of endomorphisms of A and by $Inn(A)$ the subgroup of $End(A)$ whose elements are the inner automorphisms of A determined by the invertible elements of A. The subgroup of invertible elements of A will be denoted by $U(A)$. If B is a submonoid of A, we will write $C_A(B)$ for the centralizer of B in A. The centralizer $C_A(A)$ of A in itself, that is, the center of A will be denoted by $Z(A)$.

Let us recall the following notion. Given a monoid M (with the operation written multiplicatively) and a subgroup H of M, we say that H is *right normal* if $mH \subseteq Hm$ for every $m \in M$. It is *left normal* if $Hm \subseteq mH$ for every $m \in M$, and it is normal if it is both left and right normal. If H is a right normal subgroup of M, the relation defined by

$$m_1 \sim m_2 \quad \iff \quad m_1 = hm_2 \text{ for some } h \in H$$

is a congruence on M. We denote by $\frac{M}{H}$ the factor monoid with respect to this congruence. For every monoid A, one has that $Inn(A)$ is a right normal subgroup of $End(A)$. Hence we can consider the factor monoid $\frac{End(A)}{Inn(A)}$.

The following, rather technical, results will be fundamental in the next sections.

Proposition 2.1. *Given a monoid A, with the monoid operation written additively, and $\alpha \in End(A)$, if $U(C_A(\alpha(A))) = U(Z(A))$, then for all $g \in U(A)$ we have*

$$U(C_A(\mu_g\alpha(A))) = U(Z(A)),$$

where μ_g is the inner automorphism induced by g.

Proof. If $r \in U(C_A(\mu_g\alpha(A)))$, then for all $a' \in A$ we have

$$r + \mu_g\alpha(a') = \mu_g\alpha(a') + r,$$

which means that

$$r + g + \alpha(a') - g = g + \alpha(a') - g + r,$$

which is the same as

$$-g + r + g + \alpha(a') = \alpha(a') - g + r + g.$$

Then $-g + r + g \in C_A(\alpha(A))$, and moreover $-g + r + g \in U(A)$, so $-g + r + g \in U(C_A(\alpha(A))) = U(Z(A))$. This implies that $-g + r + g = c \in U(Z(A))$, and hence

$$r = g + c - g = c + g - g = c \in U(Z(A)).$$

The other inclusion is obvious. □

Proposition 2.2. *If A is a monoid and $\alpha \in End(A)$, then $U(C_A(\alpha(A))) = U(Z(A))$ if and only if for every $g \in U(A)$ the implication*

$$\mu_g\alpha = \alpha \qquad \Rightarrow \qquad \mu_g = id_A$$

holds.

Proof. Suppose that $U(C_A(\alpha(A))) = U(Z(A))$. If $\mu_g\alpha = \alpha$, then for every $a \in A$ one has $g + \alpha(a) - g = \alpha(a)$. In other terms, $g + \alpha(a) = \alpha(a) + g$, so that $g \in C_A(\alpha(A))$. Since, moreover, $g \in U(A)$, we get that $g \in U(C_A(\alpha(A))) = U(Z(A))$; hence $\mu_g = id_A$.

Conversely, if $g \in U(C_A(\alpha(A)))$, then for every $a \in A$ one has $g + \alpha(a) = \alpha(a) + g$ or, in other terms, $g + \alpha(a) - g = \alpha(a)$. This means that $\mu_g \alpha = \alpha$. By assumption this implies $\mu_g = id_A$, and so $g \in Z(A)$. Since g is invertible, we get $g \in U(Z(A))$. □

Corollary 2.3. *If A is a monoid and $\alpha \in End(A)$, then $U(C_A(\alpha(A))) = U(Z(A))$ if and only if for every $g_1, g_2 \in U(A)$ the implication*

$$\mu_{g_1}\alpha = \mu_{g_2}\alpha \qquad \Rightarrow \qquad \mu_{g_1} = \mu_{g_2}$$

holds.

Proof. Suppose that $U(C_A(\alpha(A))) = U(Z(A))$. If $\mu_{g_1}\alpha = \mu_{g_2}\alpha$, then

$$\alpha = \mu_{g_2}^{-1}\mu_{g_1}\alpha = \mu_{-g_2}\mu_{g_1}\alpha = \mu_{-g_2+g_1}\alpha.$$

The previous proposition implies that

$$id_A = \mu_{-g_2+g_1} = \mu_{-g_2}\mu_{g_1},$$

so that $\mu_{g_1} = \mu_{g_2}$.

Conversely, if $\mu_g \alpha = \alpha$, then $\mu_g \alpha = \mu_0 \alpha$. By assumption, we get $\mu_g = \mu_0$, which means that $\mu_g = id_A$. Using the previous proposition we conclude that $U(C_A(\alpha(A))) = U(Z(A))$. □

Definition 2.4. *Given monoids M and A, with the operation in M written multiplicatively and the one in A additively, an abstract kernel is a monoid homomorphism $\Phi\colon M \to \frac{End(A)}{Inn(A)}$.*

We will denote an abstract kernel as (M, A, Φ). In particular, we will consider abstract kernels satisfying the following condition:

Definition 2.5. *An abstract kernel $\Phi\colon M \to \frac{End(A)}{Inn(A)}$ is called* admissible *if, for all $x \in M$ and all $\varphi(x) \in \Phi(x)$, one has that $U(C_A(\varphi(x)(A))) = U(Z(A))$.*

Proposition 2.6. *An abstract kernel $\Phi\colon M \to \frac{End(A)}{Inn(A)}$ is* admissible *if and only if for all $x \in M$ there exists $\varphi(x) \in \Phi(x)$ such that $U(C_A(\varphi(x)(A))) = U(Z(A))$.*

Proof. Thanks to Proposition 2.1, it suffices to observe that, if $\varphi(x), \psi(x) \in \Phi(x)$, then $\varphi(x) = \mu_g \psi(x)$ for some $g \in U(A)$. □

Example 2.7.
(1) *If A is a commutative monoid, then $Inn(A) = \{id_A\}$, and so every monoid homomorphism $\Phi\colon M \to End(A)$ is an admissible abstract kernel.*
(2) *Similarly, if A is a monoid such that $U(A) = \{0\}$, then every monoid homomorphism $\Phi\colon M \to End(A)$ is an admissible abstract kernel.*
(3) *If $\Phi\colon M \to \frac{End(A)}{Inn(A)}$ factors through $\frac{SEnd(A)}{Inn(A)}$, where $SEnd(A)$ is the monoid of surjective endomorphisms of A, then Φ is admissible.*
(4) *Let $F(x, y)$ be the partially free monoid on generators x and y, where x is invertible and y is not. In other terms, $F(x, y)$ is the free product*

$$F(x, y) = C_\infty(x) * M_\infty(y),$$

where $C_\infty(x)$ is the infinite cyclic group generated by x and $M_\infty(y)$ is the infinite cyclic monoid generated by y. Let $\alpha\colon F(x, y) \to F(x, y)$ be defined by $\alpha(x) = 1$, $\alpha(y) = y$, and let $\Phi\colon \mathbb{N} \to \frac{End(F(x,y))}{Inn(F(x,y))}$ be defined by $\Phi(n) = cl(\alpha^n)$, where $\mathbb{N}$ is the monoid of natural numbers with the usual sum. Since $\alpha^n = \alpha$ for $n \neq 0$, we get that $C_{F(x,y)}(\alpha(F(x, y))) = M_\infty(y)$, and hence

$$U(C_{F(x,y)}(\alpha(F(x, y)))) = \{0\} = U(Z(F(x, y))).$$

(5) *Other examples, in the case when A is a group, can be found in* [13] *(Examples* 2.5 *and* 2.6*).*

As observed in [13, Remark 2.7], if $C_A(\alpha(A)) = C_A(\beta(A)) = Z(A)$ for $\alpha, \beta \in End(A)$, it is not true in general that $C_A(\alpha\beta(A)) = Z(A)$ or $C_A(\beta\alpha(A)) = Z(A)$, not even when A is a group.

Proposition 2.8. *Let M and A be monoids. Every admissible abstract kernel $\Phi\colon M \to \frac{End(A)}{Inn(A)}$ induces an M-module structure on the abelian group $U(Z(A))$.*

Proof. We define an action of M on $U(Z(A))$ by putting

$$x \cdot c = \varphi(x)(c) \quad \text{for } x \in M,\ c \in U(Z(A)) \text{ and } \varphi(x) \in \Phi(x).$$

Let us first check that $\varphi(x)(c) \in U(Z(A))$: for every $a \in A$ we have

$$\varphi(x)(a) + \varphi(x)(c) = \varphi(x)(a + c) = \varphi(x)(c + a) = \varphi(x)(c) + \varphi(x)(a),$$

hence $\varphi(x)(c) \in C_A(\varphi(x)(A))$; moreover, $\varphi(x)(c)$ is invertible, so

$$\varphi(x)(c) \in U(C_A(\varphi(x)(A))).$$

By admissibility we conclude that $\varphi(x)(c) \in U(Z(A))$.

Let us then check that the action is well defined, namely that for all possible choices of $\varphi(x), \psi(x) \in \Phi(x)$ one has $\varphi(x)(c) = \psi(x)(c)$ for all $c \in U(Z(A))$. We know that $\psi(x) = \mu_g \varphi(x)$ for some $g \in U(A)$, and so

$$\psi(x)(c) = g + \varphi(x)(c) - g = \varphi(x)(c) + g - g = \varphi(x)(c),$$

where the second equality is due to the fact that $\varphi(x)(c) \in U(Z(A))$.

Finally we check that we actually defined an action of M on $U(Z(A))$: for all $x, y \in M$ and $c \in U(Z(A))$ we have

$$x \cdot (y \cdot c) = x \cdot (\varphi(y)(c)) = \varphi(x)(\varphi(y)(c)) = (\varphi(x)\varphi(y))(c) =$$
$$= \mu_{f(x,y)}\varphi(xy)(c) = f(x, y) + \varphi(xy)(c) - f(x, y) = \varphi(xy)(c) = (xy) \cdot c,$$

where we are using the fact that $\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy)$ for some $f(x, y) \in U(A)$ because Φ is a monoid homomorphism, and the second equality from the end follows from the fact that $\varphi(xy)(c) \in Z(A)$. The equality $1_M \cdot c = c$ for all c is obvious. If $x \in M$ and $c_1, c_2 \in U(Z(A))$, then

$$x \cdot (c_1 + c_2) = \varphi(x)(c_1 + c_2) = \varphi(x)(c_1) + \varphi(x)(c_2) = x \cdot c_1 + x \cdot c_2.$$

This concludes the proof. □

3. The monoid of admissible abstract kernels

From now on, let M be a fixed monoid, C a fixed abelian group, and $\Phi_0\colon M \to End(C)$ a fixed monoid homomorphism, which is necessarily an admissible abstract kernel thanks to Example 2.7 (1). We will consider abstract kernels of the form $\Phi\colon M \to \frac{End(A)}{Inn(A)}$ such that $U(Z(A)) = C$ and the action of M on C induced by Φ is Φ_0, writing briefly $U(Z(A)) \overset{M}{=} C$.

Given two admissible abstract kernels (M, A_1, Φ_1) and (M, A_2, Φ_2), with $\Phi_i\colon M \to \frac{End(A_i)}{Inn(A_i)}$, $i = 1, 2$, such that $U(Z(A_1)) \overset{M}{=} C \overset{M}{=} U(Z(A_2))$, we want to define a product (M, A, Φ) of (M, A_1, Φ_1) and (M, A_2, Φ_2). In order to do that, we first observe that the set $S = \{(c, -c) \mid c \in C\}$ is a normal subgroup of the monoid $A_1 \times A_2$; this allows us to consider the factor monoid $A = \frac{A_1 \times A_2}{S}$. Moreover, we have an injective homomorphism $j\colon C \to A$ defined by $j(c) = cl(c, 0) = cl(0, c)$.

Lemma 3.1. *Given an element $cl(u_1, u_2) \in A$, if $cl(u_1, u_2) \in U(A)$ then $u_1 \in U(A_1)$ and $u_2 \in U(A_2)$.*

Proof. If $cl(u_1, u_2) \in U(A)$, there exists $cl(v_1, v_2) \in A$ such that

$$cl(u_1, u_2) + cl(v_1, v_2) = cl(0, 0) = cl(v_1, v_2) + cl(u_1, u_2),$$

which means that

$$cl(u_1 + v_1, u_2 + v_2) = cl(0, 0) = cl(v_1 + u_1, v_2 + u_2).$$

Then

$$(u_1 + v_1, u_2 + v_2) = (c, -c), \quad (v_1 + u_1, v_2 + u_2) = (c', -c')$$

for some $c, c' \in C$. Hence

$$u_1 + v_1 = c, \quad u_2 + v_2 = -c, \quad v_1 + u_1 = c', \quad v_2 + u_2 = -c'$$

and, using that $c, c' \in Z(A_i), i = 1, 2$, we get that

$$u_1 + v_1 - c = 0 = v_1 - c' + u_1, \quad u_2 + v_2 + c = 0 = v_2 + c' + u_2,$$

which means that $u_1 \in U(A_1)$ and $u_2 \in U(A_2)$. □

Lemma 3.2. *If $cl(u_1, u_2) \in U(Z(A))$ then $u_1 \in U(Z(A_1))$ and $u_2 \in U(Z(A_2))$.*

Proof. If $cl(u_1, u_2) \in U(Z(A))$, then for any $a_1 \in A_1$ we have

$$cl(u_1, u_2) + cl(a_1, 0) = cl(a_1, 0) + cl(u_1, u_2),$$

or, in other terms,

$$cl(u_1 + a_1, u_2) = cl(a_1 + u_1, u_2).$$

Then there exists $c \in C$ such that

$$(u_1 + a_1, u_2) = (c, -c) + (a_1 + u_1, u_2).$$

Hence

$$u_1 + a_1 = c + a_1 + u_1, \quad u_2 = -c + u_2.$$

But, thanks to the previous lemma, $u_2 \in U(A_2)$, so $c = 0$ and $u_1 + a_1 = a_1 + u_1$, whence $u_1 \in Z(A_1)$. By the same lemma, $u_1 \in U(A_1)$. Hence $u_1 \in U(Z(A_1))$. The proof for u_2 is completely analogous. □

Corollary 3.3. $j(C) = U(Z(A))$.

Proof. Given $cl(u_1, u_2) \in U(Z(A))$, from Lemma 3.2 we get that $u_i \in U(Z(A_i))$, $i = 1, 2$. But $U(Z(A_1)) = U(Z(A_2)) = C$, so $u_1, u_2 \in C$ and hence $u_1 + u_2 \in C$. Moreover,

$$(u_1 + u_2, 0) = (u_2, -u_2) + (u_1, u_2)$$

and hence

$$j(u_1 + u_2) = cl(u_1 + u_2, 0) = cl(u_1, u_2),$$

therefore $U(Z(A)) \subseteq j(C)$. The other inclusion is obvious. □

Thanks to this fact, we can identify $U(Z(A))$ with C.

Now, starting from the admissible abstract kernels (M, A_1, Φ_1) and (M, A_2, Φ_2), for every $x \in M$, $\varphi_1(x) \in \Phi_1(x)$, $\varphi_2(x) \in \Phi_2(x)$, we define a map

$$\varphi_1(x) \times \varphi_2(x) \colon A_1 \times A_2 \to A_1 \times A_2$$

by putting

$$(\varphi_1(x) \times \varphi_2(x))(a_1, a_2) = (\varphi_1(x)(a_1), \varphi_2(x)(a_2)).$$

Then we have

$$(\varphi_1(x) \times \varphi_2(x))(c, -c) = (\varphi_1(x)(c), \varphi_2(x)(-c)) = (x \cdot c, -x \cdot c) \in S,$$

where $x \cdot c = \Phi_0(x)(c)$, so that $(\varphi_1(x) \times \varphi_2(x))(S) \subseteq S$. Thus we get a well-defined homomorphism $\varphi(x) \colon A \to A$ given by

$$\varphi(x)(cl(a_1, a_2)) = cl(\varphi_1(x)(a_1), \varphi_2(x)(a_2)).$$

Choosing different representatives $\psi_1(x) \in \Phi_1(x)$, $\psi_2(x) \in \Phi_2(x)$, we would have $\varphi_i(x) = \mu_{g_i}\psi_i(x)$ for some $g_i \in U(A_i)$, $i = 1, 2$. Considering the corresponding homomorphism $\psi(x)\colon A \to A$ defined by

$$\psi(x)(cl(a_1, a_2)) = cl(\psi_1(x)(a_1), \psi_2(x)(a_2)),$$

we have that $\varphi(x) = \mu_{cl(g_1,g_2)}\psi(x)$, indeed:

$$\begin{gathered}\mu_{cl(g_1,g_2)}\psi(x)(cl(a_1, a_2)) = \mu_{cl(g_1,g_2)}(cl(\psi_1(x)(a_1), \psi_2(x)(a_2))) = \\ = cl(g_1, g_2) + cl(\psi_1(x)(a_1), \psi_2(x)(a_2)) - cl(g_1, g_2) = \\ = cl(g_1 + \psi_1(x)(a_1) - g_1, g_2 + \psi_2(x)(a_2) - g_2) = cl(\mu_{g_1}\psi_1(x)(a_1), \mu_{g_2}\psi_2(x)(a_2)) = \\ = cl(\varphi_1(x)(a_1), \varphi_2(x)(a_2)) = \varphi(x)(cl(a_1, a_2)).\end{gathered}$$

Thus we get a well-defined map $\Phi\colon M \to \frac{End(A)}{Inn(A)}$ given by $\Phi(x) = cl(\varphi(x))$.

Proposition 3.4. *The map Φ is a monoid homomorphism.*

Proof. Given $x, y \in M$, $\varphi_i(x) \in \Phi_i(x)$, $\varphi_i(y) \in \Phi_i(y)$, $i = 1, 2$, we have

$$\varphi_i(x)\varphi_i(y) = \mu_{g_i}\varphi_i(xy)$$

for some $g_i \in U(A_i)$. Hence

$$\begin{gathered}\varphi(x)\varphi(y)(cl(a_1, a_2)) = \varphi(x)(cl(\varphi_1(y)(a_1), \varphi_2(y)(a_2))) = \\ = cl(\varphi_1(x)\varphi_1(y)(a_1), \varphi_2(x)\varphi_2(y)(a_2)) = cl(\mu_{g_1}\varphi_1(xy)(a_1), \mu_{g_2}\varphi_2(xy)(a_2)) = \\ = cl(g_1 + \varphi_1(xy)(a_1) - g_1, g_2 + \varphi_2(xy)(a_2) - g_2) = \\ = cl(g_1, g_2) + cl(\varphi_1(xy)(a_1), \varphi_2(xy)(a_2)) - cl(g_1, g_2) = \\ = cl(g_1, g_2) + \varphi(xy)(cl(a_1, a_2)) - cl(g_1, g_2) = \mu_{cl(g_1,g_2)}\varphi(xy)(cl(a_1, a_2)).\end{gathered}$$

Thus $\Phi(xy) = cl(\varphi(xy)) = cl(\varphi(x)\varphi(y)) = cl(\varphi(x))cl(\varphi(y)) = \Phi(x)\Phi(y)$. Moreover, $\Phi(1) = cl(\varphi(1)) = cl(id_A) = id_{\frac{End(A)}{Inn(A)}}$. □

Proposition 3.5. *The abstract kernel $\Phi\colon M \to \frac{End(A)}{Inn(A)}$ is admissible, and $U(Z(A)) \overset{M}{=} C$.*

Proof. For every $x \in M$, $\varphi(x) \in End(A)$ and $cl(r_1, r_2) \in U(C_A(\varphi(x)(A)))$, we know that $r_1 \in U(A_1)$ and $r_2 \in U(A_2)$ thanks to Lemma 3.1. Moreover, for every $a_1 \in A_1$ we have

$$cl(r_1, r_2) + \varphi(x)(cl(a_1, 0)) = \varphi(x)(cl(a_1, 0)) + cl(r_1, r_2).$$

This equality can be rewritten as

$$cl(r_1, r_2) + cl(\varphi_1(x)(a_1), 0) = cl(\varphi_1(x)(a_1), 0) + cl(r_1, r_2).$$

Hence

$$cl(r_1 + \varphi_1(x)(a_1), r_2) = cl(\varphi_1(x)(a_1) + r_1, r_2),$$

and so there exists $c \in C$ such that

$$(r_1 + \varphi_1(x)(a_1), r_2) = (c, -c) + (\varphi_1(x)(a_1) + r_1, r_2).$$

From the equalities

$$r_1 + \varphi_1(x)(a_1) = c + \varphi_1(x)(a_1) + r_1, \quad r_2 = -c + r_2,$$

using the invertibility of r_2, we get $c = 0$, and therefore $r_1 + \varphi_1(x)(a_1) = \varphi_1(x)(a_1) + r_1$. Then $r_1 \in U(C_{A_1}(\varphi_1(x)(A_1))) = U(Z(A_1)) = C$. In a similar way we can show that also $r_2 \in C$. Hence $cl(r_1, r_2) \in Z(A)$ because, for all $a_i \in A_i$, $i = 1, 2$, we have

$$cl(a_1, a_2) + cl(r_1, r_2) = cl(a_1 + r_1, a_2 + r_2) = cl(r_1 + a_1, r_2 + a_2) = cl(r_1, r_2) + cl(a_1, a_2).$$

Since, moreover, $cl(r_1, r_2) \in U(A)$, we obtain that $cl(r_1, r_2) \in U(Z(A))$. Identifying $U(Z(A))$ with C, we get $cl(r_1, r_2) \in C$. Thus $U(C_A(\varphi(x)(A))) = U(Z(A)) = C$ and Φ is admissible.

Finally, the induced action on C is

$$x \cdot j(c) = x \cdot cl(c,0) = \varphi(x)(cl(c,0)) = cl(\varphi_1(x)(c),0) = cl(x \cdot c, 0) = j(x \cdot c).$$

□

The previous observations allow us to define a binary operation on the class of admissible abstract kernels inducing the fixed action Φ_0 of M on C. We will denote such operation as

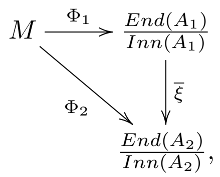
$$(M, A_1, \Phi_1) \otimes (M, A_2, \Phi_2) = (M, A, \Phi).$$

We want to turn such a binary operation into a monoid operation. In order to do that, we need to identify our abstract kernels up to an equivalence relation, which is an adaptation to the monoid case of the one introduced in [13, Definition 3.1]:

Definition 3.6. *Given two admissible abstract kernels $\Phi_1 \colon M \to \frac{End(A_1)}{Inn(A_1)}$ and $\Phi_2 \colon M \to \frac{End(A_2)}{Inn(A_2)}$ inducing the fixed M-action $\Phi_0 \colon M \to End(C)$ on $C = U(Z(A_1)) = U(Z(A_2))$, we say that Φ_1 is C-*equivalent *to Φ_2, and write $\Phi_1 \overset{C}{\cong} \Phi_2$, if there exists a monoid isomorphism $\xi \colon A_1 \to A_2$ satisfying the following two conditions:*

(i) *for all $c \in C$, $\xi(c) = c$;*
(ii) *for all $x \in M$ and all $\varphi_1(x) \in \Phi_1(x)$, $\xi\varphi_1(x)\xi^{-1} \in \Phi_2(x)$.*

Condition (ii) can be expressed by the commutativity of the following triangle:



where $\overline{\xi}(cl(\alpha)) = cl(\xi\alpha\xi^{-1})$.

It is clear that $\overset{C}{\cong}$ is an equivalence relation. The proof that the binary operation $\otimes$ defined above is compatible with the equivalence relation $\overset{C}{\cong}$ is the same as the one of [13, Proposition 3.2]. Let $\overline{\mathcal{M}}(M, C)$ be the set of C-equivalence classes $[M, A, \Phi]$ of admissible abstract kernels inducing the fixed M-action $\Phi_0 \colon M \to End(C)$ on the group $C = U(Z(A))$. Our aim is to show that this set, equipped with the operation $\otimes$, is a commutative monoid.

Proposition 3.7. *The fixed M-action on C, $\Phi_0 \colon M \to End(C)$, is the neutral element of $\otimes$.*

Proof. We have to show that, for any admissible abstract kernel (M, A, Φ), the product $(M, A, \Phi) \otimes (M, C, \Phi_0)$ is C-equivalent to (M, A, Φ) (the equivalence of the other composite can be verified in a similar way). Let us denote by $\Psi \colon M \to \frac{End(((A \times C)/S)}{Inn((A \times C)/S)}$ the product of Φ and Φ_0. The map $\xi \colon A \to \frac{A \times C}{S}$ defined by $\xi(a) = cl(a, 0)$ is clearly a monoid homomorphism and it fixes C. Its inverse is the map $\xi' \colon \frac{A \times C}{S} \to A$ defined by $\xi'(cl(a,c)) = a + c$. It is well defined since $cl(a,c) = cl(a',c')$ if and only if $(a,c) = (c_1, -c_1) + (a', c')$ for some $c_1 \in C$. Then $a = c_1 + a'$, $c = -c_1 + c'$ and so

$$a + c = c_1 + a' - c_1 + c' = c_1 - c_1 + a' + c' = a' + c'.$$

Clearly $\xi'\xi = id_A$. Let us consider the other composite:

$$\xi\xi'(cl(a,c)) = \xi(a + c) = cl(a + c, 0) = cl(a, c),$$

where the last equality holds because $(a+c,0)=(c,-c)+(a,c)$. The remaining part of the proof is analogous to the one of [13, Proposition 3.3], so we omit it. $\square$

In order to prove the associativity of $\otimes$, we need the following lemmas:

Lemma 3.8. *Given three admissible abstract kernels $\Phi_i\colon M\to\frac{End(A_i)}{Inn(A_i)}$, $i=1,2,3$, inducing the fixed M-action $\Phi_0\colon M\to End(C)$ on $C=U(Z(A_i))$, consider the product $\Phi\colon M\to\frac{End(A)}{Inn(A)}$ of Φ_1 and Φ_2 and the product $\Phi^\sharp\colon M\to\frac{End(A^\sharp)}{Inn(A^\sharp)}$ of Φ and Φ_3, so that*

$$(M,A^\sharp,\Phi^\sharp)=((M,A_1,\Phi_1)\otimes(M,A_2,\Phi_2))\otimes(M,A_3,\Phi_3).$$

Then, in $A^\sharp=\frac{((A_1\times A_2)/S)\times A_3}{S}$, we have that $cl(cl(a_1,a_2),a_3)=cl(cl(a_1',a_2'),a_3')$ if and only if

$$a_1=c_1+a_1',\ \ a_2=c_2+a_2',\ \ a_3=-(c_1+c_2)+a_3' \text{ for some } \ c_1,c_2\in C.$$

Proof. If $cl(cl(a_1,a_2),a_3)=cl(cl(a_1',a_2'),a_3')$, then for some $c\in C$ we have

$$(cl(a_1,a_2),a_3)=(c,-c)+(cl(a_1',a_2'),a_3').$$

This means that

$$cl(a_1,a_2)=c+cl(a_1',a_2'),\quad a_3=-c+a_3'$$

or, in other terms,

$$cl(a_1,a_2)=cl(c,0)+cl(a_1',a_2'),\quad a_3=-c+a_3'.$$

The first equality can be reformulated as

$$(a_1,a_2)=(-c',c')+(c+a_1',a_2')\quad\text{for some } c'\in C.$$

Then

$$a_1=-c'+c+a_1',\quad a_2=c'+a_2',\quad a_3=-c+a_3'.$$

Hence, putting $c_1=-c'+c$ and $c_2=c'$, we get the desired result.

Conversely, if

$$a_1=c_1+a_1',\ \ a_2=c_2+a_2',\ \ a_3=-(c_1+c_2)+a_3' \text{ for some } \ c_1,c_2\in C,$$

then

$$(cl(a_1,a_2),a_3)=(cl(c_1+a_1',c_2+a_2'),-(c_1+c_2)+a_3')=$$
$$=(cl(c_1,c_2)+cl(a_1',a_2'),-(c_1+c_2)+a_3')=(cl(c_1,c_2),-(c_1+c_2))+(cl(a_1',a_2'),a_3')=$$
$$=(cl(c_1+c_2,0),-(c_1+c_2))+(cl(a_1',a_2'),a_3')=(c_1+c_2,-(c_1+c_2))+(cl(a_1',a_2'),a_3'),$$

and so $cl(cl(a_1,a_2),a_3)=cl(cl(a_1',a_2'),a_3')$. $\square$

Similarly we get

Lemma 3.9. *Given three admissible abstract kernels $\Phi_i\colon M\to\frac{End(A_i)}{Inn(A_i)}$, $i=1,2,3$, inducing the fixed M-action $\Phi_0\colon M\to End(C)$ on $C=U(Z(A_i))$, consider the product $\Psi\colon M\to\frac{End(A)}{Inn(A)}$ of Φ_2 and Φ_3 and the product $\Psi^\flat\colon M\to\frac{End(A^\flat)}{Inn(A^\flat)}$ of Φ_1 and Ψ, so that*

$$(M,A^\flat,\Psi^\flat)=(M,A_1,\Phi_1)\otimes((M,A_2,\Phi_2)\otimes(M,A_3,\Phi_3)).$$

Then, in $A^\flat=\frac{A_1\times((A_2\times A_3)/S)}{S}$, we have that $cl(a_1,cl(a_2,a_3))=cl(a_1',cl(a_2',a_3'))$ if and only if

$$a_2=c_2'+a_2',\ \ a_3=c_3'+a_3',\ \ a_1=-(c_2'+c_3')+a_1' \text{ for some } \ c_2',c_3'\in C.$$

The key step to prove associativity of $\otimes$ is the following:

Proposition 3.10. *Given three admissible abstract kernels $\Phi_i \colon M \to \frac{End(A_i)}{Inn(A_i)}$, $i = 1, 2, 3$, inducing the fixed M-action $\Phi_0 \colon M \to End(C)$ on $C = U(Z(A_i))$, the map*

$$\xi \colon A^\sharp = \frac{((A_1 \times A_2)/S) \times A_3}{S} \to A^\flat = \frac{A_1 \times ((A_2 \times A_3)/S)}{S},$$

defined by $\xi(cl(cl(a_1, a_2), a_3)) = cl(a_1, cl(a_2, a_3))$, is a monoid isomorphism fixing C and, for all $x \in M$ and all $\varphi^\sharp(x) \in \Phi^\sharp(x)$, one has $\xi\varphi^\sharp\xi^{-1} \in \Psi^\flat(x)$.

Proof. Let us check that ξ is well defined: if $cl(cl(a_1, a_2), a_3) = cl(cl(a'_1, a'_2), a'_3)$, then by Lemma 3.8 there exist $c_1, c_2 \in C$ such that

$$a_1 = c_1 + a'_1, \;\; a_2 = c_2 + a'_2, \;\; a_3 = -(c_1 + c_2) + a'_3.$$

Putting $c'_2 = c_2$ and $c'_3 = -(c_1 + c_2)$, we get that

$$a_2 = c'_2 + a'_2, \quad a_3 = c'_3 + a'_3, \quad a_1 = c_1 + a'_1 = -c_2 - c'_3 + a'_1 = -(c_2 + c'_3) + a'_1,$$

and then, using Lemma 3.9, we conclude that $cl(a_1, cl(a_2, a_3)) = cl(a'_1, cl(a'_2, a'_3))$.

It is obvious that ξ is a monoid isomorphism. Moreover, it is an isomorphism with inverse ξ' defined by

$$\xi'(cl(a_1, cl(a_2, a_3))) = cl(cl(a_1, a_2), a_3).$$

Thanks again to Lemmas 3.8 and 3.9, the proof that ξ' is a well-defined map is similar to the one for ξ. Clearly $\xi'\xi = id_{A^\sharp}$ and $\xi\xi' = id_{A^\flat}$. Moreover,

$$\xi(c) = \xi(cl(c, 0)) = \xi(cl(cl(c, 0), 0)) = cl(c, cl(0, 0)) = cl(c, 0) = c$$

for all $c \in C$. The remaining part of the proof is analogous to the one of [13, Proposition 3.7], so we omit it. □

The previous proposition immediately implies the associativity of $\otimes$. The fact that $\otimes$ is commutative can be proved in the same way as in [13, Proposition 3.8]. Therefore we have:

Proposition 3.11. *The set $\overline{\mathcal{M}}(M, C)$, equipped with the operation $\otimes$, is a commutative monoid.*

4. Schreier extensions and extendable abstract kernels

We need now to consider a particular kind of admissible abstract kernels, that are induced by suitable monoid extensions. In order to do that, let us first recall from [18] the following definition:

Definition 4.1. *Let*

$$E \colon \; 0 \longrightarrow A \overset{\kappa}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} M \longrightarrow 1 \tag{1}$$

be a sequence of monoids and monoid homomorphisms such that σ is a surjection, κ is an injection and $\kappa(A) = \{b \in B \,|\, \sigma(b) = 1\}$ (i.e. κ is the kernel of σ). The sequence E is a (right) Schreier extension *of M by A (some authors would instead say "A by M") if, for every $x \in M$, there exists an element $u_x \in \sigma^{-1}(x)$ such that for every $b \in \sigma^{-1}(x)$ there exists a unique $a \in A$ such that*

$$b = \kappa(a) + u_x.$$

The elements u_x, for $x \in M$, will be called the representatives *of E. We will always choose $u_1 = 0$ (we use the multiplicative notation for M and the additive one for the other monoids involved).*

We observed in [12] that some Schreier extensions (called *regular* there) induce abstract kernels of the form $\Phi\colon M \to \frac{End(A)}{Inn(A)}$. Let us briefly recall how. Given a Schreier extension (1), with representatives u_x, $x \in M$, we have that, for all $a \in A$, there exists a unique element $a' \in A$ such that $u_x + a = a' + u_x$. This defines a map $\varphi(x)\colon A \to A$ sending a to a'.

Proposition 4.2 ([12], Proposition 3.6)**.** *The following statements hold:*

(a) *for every $x \in M$ we have that $\varphi(x) \in End(A)$;*
(b) *if v_x is another representative, and $\psi(x)\colon A \to A$ is the induced endomorphism of A, then $\psi(x) = \mu_g \varphi(x)$ with $g \in U(A)$.*

Hence, every Schreier extension (1) induces a well-defined map

$$\Phi\colon M \to \frac{End(A)}{Inn(A)}, \tag{2}$$

given by $\Phi(x) = cl(\varphi(x))$. In order to guarantee that Φ is a monoid homomorphism, an additional assumption on (1) is needed:

Definition 4.3 ([12], Definition 3.7)**.** *A Schreier extension E as in* (1) *is a* regular *Schreier extension if, whenever u_x and u_y are representatives for E, then so is $u_x + u_y$.*

Proposition 4.4 ([12], Proposition 3.8)**.** *If E is a regular Schreier extension, then the map* (2) *is a monoid homomorphism.*

The following definition is an extension to the monoid case of Definition 4.6 in [13]:

Definition 4.5. *An admissible abstract kernel $\Phi\colon M \to \frac{End(A)}{Inn(A)}$ is* extendable *if it is induced by a regular Schreier extension* (1)*.*

In [13] we observed, for abstract kernels of the form $\Phi\colon M \to \frac{End(G)}{Inn(G)}$, where G was a group, that if two admissible abstract kernels (M, G, Φ) and (M, G', Φ') inducing the same M-action on $C = Z(G) = Z(G')$ are C-equivalent, then the first one is extendable if and only if the second one is. The proof there extends in a straightforward way to the general monoid case. We can then consider the set $\overline{\mathcal{L}}(M, C)$ of C-equivalence classes of extendable admissible abstract kernels inducing the same M-action on C. The main goal of this section is to show that $\overline{\mathcal{L}}(M, C)$ is a submonoid of $\overline{\mathcal{M}}(M, C)$. It is immediate to observe that the neutral element of $\overline{\mathcal{M}}(M, C)$, namely the homomorphism $\Phi_0\colon M \to End(C)$ representing the action of M on C, is extendable, because it is induced by the semidirect product of M and C with respect to the given action (for a detailed account on monoid actions and semidirect products of monoids we refer to [2]). Concerning the product $\otimes$, we have:

Proposition 4.6. *Let (M, A_1, Φ_1) and (M, A_2, Φ_2) be admissible abstract kernels with $U(Z(A_i)) \overset{M}{=} C$, $i = 1, 2$. If they are both extendable, then their product*

$$(M, A, \Phi) = (M, A_1, \Phi_1) \otimes (M, A_2, \Phi_2)$$

is extendable, too.

Proof. Given $\Phi_1\colon M \to \frac{End(A_1)}{Inn(A_1)}$ and $\Phi_2\colon M \to \frac{End(A_2)}{Inn(A_2)}$, we denoted their product as $\Phi\colon M \to \frac{End(A)}{Inn(A)}$, where $A = \frac{A_1 \times A_2}{S}$ and $S = \{(c, -c) \mid c \in C\}$, with $C = U(Z(A_1)) = U(Z(A_2))$. Let

$$E_1 : 0 \longrightarrow A_1 \overset{\kappa_1}{\rightarrowtail} B_1 \overset{\sigma_1}{\twoheadrightarrow} M \longrightarrow 1$$

and

$$E_2 :\ 0 \longrightarrow A_2 \overset{\kappa_2}{\rightarrowtail} B_2 \overset{\sigma_2}{\twoheadrightarrow} M \longrightarrow 1$$

be regular Schreier extensions inducing Φ_1 and Φ_2, respectively. We consider the pullback

$$\begin{array}{ccc} R & \xrightarrow{\pi_2} & B_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\ B_1 & \xrightarrow[\sigma_1]{} & M. \end{array}$$

Explicitly, $R = \{ (b_1, b_2) \in B_1 \times B_2 \,|\, \sigma_1(b_1) = \sigma_2(b_2) \}$. Clearly S is a submonoid of R. Let us check that S is a right normal subgroup of R: for any $(b_1, b_2) \in R$ and any $(c, -c) \in S$, let us put $x := \sigma_1(b_1) = \sigma_2(b_2)$. Let then u_x and v_x be representatives of x in E_1 and E_2, respectively, and let $\varphi_1(x)$ and $\varphi_2(x)$ be the corresponding endomorphisms of A_1 and A_2, respectively. Then we have

$$(b_1, b_2) + (c, -c) = (b_1 + c, b_2 - c) = (a_1 + u_x + c, a_2 + v_x + (-c))$$

for unique $a_i \in A_i$. The last expression is equal to

$$(a_1 + \varphi_1(x)(c) + u_x, a_2 + \varphi_2(x)(-c) + v_x) = (\varphi_1(x)(c) + a_1 + u_x, -\varphi_2(x)(c) + a_2 + v_x) =$$

$$= (x \cdot c + b_1, -x \cdot c + b_2) = (x \cdot c, -x \cdot c) + (b_1, b_2),$$

where the first equality is due to the fact that $\varphi_i(x)(c) \in U(Z(A_i))$, $i = 1, 2$. Thus $(b_1, b_2) + S \subseteq S + (b_1, b_2)$, proving that S is right normal in R.

Therefore we can consider the factor monoid $B = \frac{R}{S}$ and build a sequence

$$0 \longrightarrow A \overset{\kappa}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} M \longrightarrow 1$$

where $\kappa(cl(a_1, a_2)) = cl(a_1, a_2)$ and $\sigma(cl(b_1, b_2)) = \sigma_1(b_1) = \sigma_2(b_2)$. The map κ is obviously well-defined; let us check that σ is, too: if $cl(b_1, b_2) = cl(b_1', b_2')$, then there exists $c \in C$ such that

$$(b_1', b_2') = (c, -c) + (b_1, b_2).$$

Hence $b_1' = c + b_1$, $b_2' = -c + b_2$ and so $\sigma_i(b_i') = \sigma_i(b_i)$, $i = 1, 2$. The fact that κ and σ are monoid homomorphisms and that κ is injective are obvious. Moreover, σ is surjective: for every $x \in M$, let $b_1 \in B_1$, $b_2 \in B_2$ be such that $\sigma_1(b_1) = \sigma_2(b_2) = x$; then $\sigma(cl(b_1, b_2)) = x$. The kernel of σ is $\kappa(A)$: indeed, if $\sigma(cl(b_1, b_2)) = 1$, then $\sigma_1(b_1) = \sigma_2(b_2) = 1$; hence $b_i = \kappa_i(a_i)$ for some $a_i \in A_i$ and so $\kappa(cl(a_1, a_2)) = cl(a_1, a_2) = cl(b_1, b_2)$ (treating κ_1 and κ_2 as inclusions). The other inclusion is obvious.

Now we prove that E is a regular Schreier extension. For every $x \in M$ and every representative u_x of E_1 and v_x of E_2, the element $w_x = cl(u_x, v_x)$ is a representative of E. In fact, if $\sigma(x_1, x_2) = x$, then $\sigma_1(b_1) = \sigma_2(b_2) = x$, and $b_1 = a_1 + u_x$, $b_2 = a_2 + v_x$ for unique $a_i \in A_i$. Then

$$cl(b_1, b_2) = cl(a_1 + u_x, a_2 + v_x) = cl(a_1, a_2) + cl(u_x, v_x) = cl(a_1, a_2) + w_x.$$

Concerning uniqueness: if $cl(a_1, a_2) + cl(u_x, v_x) = cl(a_1', a_2') + cl(u_x, v_x)$, then $cl(a_1 + u_x, a_2 + v_x) = cl(a_1' + u_x, a_2' + v_x)$. This means that there exists $c \in C$ such that

$$(a_1 + u_x, a_2 + v_x) = (c, -c) + (a_1' + u_x, a_2' + v_x),$$

namely

$$a_1 + u_x = c + a_1' + u_x, \quad a_2 + v_x = -c + a_2' + v_x.$$

Since E_1 and E_2 are Schreier extensions, we get

$$a_1 = c + a_1', \quad a_2 = -c + a_2',$$

which means that $cl(a_1, a_2) = cl(a'_1, a'_2)$. The fact that E is regular is an immediate consequence of the fact that E_1 and E_2 are.

It remains to show that E induces the abstract kernel Φ. We have

$$cl(u_x, v_x) + cl(a_1, a_2) = cl(u_x + a_1, v_x + a_2) = cl(\varphi_1(x)(a_1) + u_x, \varphi_2(x)(a_2) + v_x) =$$
$$= cl(\varphi_1(x)(a_1), \varphi_2(x)(a_2)) + cl(u_x, v_x),$$

so the abstract kernel Φ_E induced by E is given by $\Phi_E(x) = cl(\psi(x))$, with $\psi(x)(cl(a_1, a_2)) = cl(\varphi_1(x)(a_1), \varphi_2(x)(a_2))$. But this is precisely the abstract kernel Φ. □

So we completed the proof that $\overline{\mathcal{L}}(M, C)$ is a submonoid of $\overline{\mathcal{M}}(M, C)$. We can then consider the factor monoid $\overline{\mathcal{A}}(M, C) = \frac{\overline{\mathcal{M}}(M,C)}{\overline{\mathcal{L}}(M,C)}$. In fact, a well-known result says that, if A is a commutative monoid and $B \subseteq A$ is a submonoid, the relation $\sim$ on A defined by

$$a_1 \sim a_2 \quad \Longleftrightarrow \quad \exists\, b_1, b_2 \in B \text{ such that } a_1 + b_1 = a_2 + b_2$$

is a congruence on A. So, we can consider the factor monoid $\frac{A}{\sim}$ and denote it by $\frac{A}{B}$. The goal of the next section is to show that the factor monoid $\overline{\mathcal{A}}(M, C)$ is an abelian group which is isomorphic to the third cohomology group $H^3(M, C)$ of the monoid M with coefficients in the M-module C.

5. The connection between admissible abstract kernels and cohomology

In [12] we showed that, to every abstract kernel of the form $\Phi\colon M \to \frac{SEnd(A)}{Inn(A)}$, where A and M are monoids and $SEnd(A)$ is the monoid of surjective endomorphisms of A, it is possible to associate a 3-cocycle of the cohomology of M with coefficients in the induced M-module $C = U(Z(A))$. Thanks to Corollary 2.3, we can replace the condition on surjectivity of the endomorphisms with the assumption that the abstract kernel Φ is admissible. Let us describe this explicitly (the construction is a slight adaptation of the one we considered in [12]).

Let $\Phi\colon M \to \frac{End(A)}{Inn(A)}$ be an admissible abstract kernel such that $U(Z(A)) \overset{M}{=} C$. Choose a representative $\varphi(x) \in \Phi(x)$ for any $x \in M$, with $\varphi(1) = id_A$. We have that

$$\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy)$$

for some $f(x, y) \in U(A)$, with $f(x, 1) = f(1, y) = 0$. Moreover, given $x, y, z \in M$, associating the ternary product $\varphi(x)\varphi(y)\varphi(z)$ in the two possible ways we have, on one hand

$$\varphi(x)\varphi(y)\varphi(z) = \varphi(x)\mu_{f(y,z)}\varphi(yz) = \mu_{\varphi(x)(f(y,z))}\varphi(x)\varphi(yz) =$$
$$= \mu_{\varphi(x)(f(y,z))}\mu_{f(x,yz)}\varphi(xyz) = \mu_{\varphi(x)(f(y,z))+f(x,yz)}\varphi(xyz),$$

and, on the other hand

$$\varphi(x)\varphi(y)\varphi(z) = \mu_{f(x,y)}\varphi(xy)\varphi(z) = \mu_{f(x,y)}\mu_{f(xy,z)}\varphi(xyz) = \mu_{f(x,y)+f(xy,z)}\varphi(xyz).$$

Comparing the two expressions, we get

$$\mu_{\varphi(x)(f(y,z))+f(x,yz)}\varphi(xyz) = \mu_{f(x,y)+f(xy,z)}\varphi(xyz).$$

Now, since Φ is an admissible asbtract kernel, we can apply Corollary 2.3, and we obtain the equality

$$\mu_{\varphi(x)(f(y,z))+f(x,yz)} = \mu_{f(x,y)+f(xy,z)}.$$

Hence

$$\mu_{\varphi(x)(f(y,z))+f(x,yz)-(f(x,y)+f(xy,z))} = id_A,$$

and so

$$\varphi(x)(f(y,z)) + f(x,yz) - (f(x,y) + f(xy,z)) \in U(Z(A)).$$

This gives a unique element $k(x,y,z) \in U(Z(A))$ such that

$$\varphi(x)(f(y,z)) + f(x,yz) = k(x,y,z) + f(x,y) + f(xy,z).$$

Clearly, $k(x,y,1) = k(x,1,z) = k(1,y,z) = 0$. We call this element k the *obstruction* of the admissible abstract kernel Φ. The obstruction of an admissible abstract kernel Φ is a 3-cocycle of the cohomology of M with coefficients in the M-module $C = U(Z(A))$: the proof of this fact is exactly the same as the one of Proposition 5.4 in [12].

The cohomology class $Obs(\Phi)$ of the obstruction of an admissible abstract kernel Φ does not depend on the choice of the element $f(x,y) \in U(A)$ such that $\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy)$. In fact:

Proposition 5.1. *Consider an admissible abstract kernel $\Phi\colon M \to \frac{End(A)}{Inn(A)}$, with chosen representatives $\varphi(x) \in \Phi(x)$ for any $x \in M$, with $\varphi(1) = id_A$. If, for any $x, y \in M$, we have*

$$\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy) = \mu_{f'(x,y)}\varphi(xy)$$

with $f(x,1) = 0 = f(1,y)$ and $f'(x,1) = 0 = f'(1,y)$, then the 3-cocycles k and k' constructed using f and f' are cohomologous.

Proof. Since Φ is admissible, we can apply Corollary 2.3 to the equality

$$\mu_{f(x,y)}\varphi(xy) = \mu_{f'(x,y)}\varphi(xy),$$

getting $\mu_{f(x,y)} = \mu_{f'(x,y)}$. The rest of the proof works like the one of [12, Proposition 5.5]. □

Concerning the other possible choices in our construction, we have the following two propositions, that we state without proofs, since they are analogous to the ones of Propositions 5.6 and 5.7 in [12]:

Proposition 5.2. *Consider an admissible abstract kernel $\Phi\colon M \to \frac{End(A)}{Inn(A)}$, with chosen representatives $\varphi(x) \in \Phi(x)$ for any $x \in M$, with $\varphi(1) = id_A$. Let $f : M \times M \to U(A)$ be a map with $\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy)$ and $f(x,1) = 0 = f(1,y)$ for any $x, y \in M$, and let $k\colon M \times M \times M \to U(Z(A))$ be the 3-cocycle induced by f. If k'' is a 3-cocycle which is cohomologous to k, then there exists a map $f''\colon M \times M \to U(A)$, with $f''(x,1) = 0 = f''(1,y)$, such that*

$$\varphi(x)\varphi(y) = \mu_{f''(x,y)}\varphi(xy)$$

and the 3-cocycle induced by f'' is precisely k''.

Proposition 5.3. *Consider an admissible abstract kernel $\Phi\colon M \to \frac{End(A)}{Inn(A)}$, with chosen representatives $\varphi(x) \in \Phi(x)$ for any $x \in M$, with $\varphi(1) = id_A$. Let $f\colon M \times M \to U(A)$ be a map with $\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy)$ and $f(x,1) = 0 = f(1,y)$ for any $x, y \in M$, and let $k\colon M \times M \times M \to U(Z(A))$ be the 3-cocycle induced by f. If one chooses other representatives $\varphi'(x) \in \Phi(x)$, again with $\varphi'(1) = id_A$, then there exists a map $f'\colon M \times M \to U(A)$, with $f'(x,1) = 0 = f'(1,y)$, such that*

$$\varphi'(x)\varphi'(y) = \mu_{f'(x,y)}\varphi'(xy)$$

and its induced 3-cocycle is precisely k.

Finally, we need to observe that C-equivalent admissible abstract kernels determine cohomologous 3-cocycles. In [13] we showed that, given two admissible abstract kernels $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ and $\Phi'\colon M \to \frac{End(G')}{Inn(G')}$, where G and G' are groups and $(M, G, \Phi) \overset{C}{\cong} (M, G', \Phi')$, the obstructions of Φ and Φ' are cohomologous. The same argument applies also to the case of admissible abstract kernels of the form $\Phi\colon M \to \frac{End(A)}{Inn(A)}$, where now A is a monoid. Thus we get a well-defined map

$$\overline{\zeta}\colon \overline{\mathcal{M}}(M, C) \to H^3(M, C),$$

given by

$$\overline{\zeta}([M, A, \Phi]) = Obs(\Phi) = cl(k).$$

Proposition 5.4. *The map $\overline{\zeta}\colon \overline{\mathcal{M}}(M, C) \to H^3(M, C)$ is a surjective monoid homomorphism.*

Proof. The proof that $\overline{\zeta}$ is a monoid homomorphism is the same as the one of [13, Proposition 6.1]. Moreover, if we denote by $\mathcal{M}(M, C)$ the monoid of C-equivalence classes of admissible asbtract kernels of the form $\Phi\colon M \to \frac{End(G)}{Inn(G)}$, where G is a group and $Z(G) \overset{M}{\cong} C$, Proposition 6.2 in [13] tells us that the map $\zeta\colon \mathcal{M}(M, C) \to H^3(M, C)$, defined exactly as $\overline{\zeta}$, is a surjective monoid homomorphism. Since clearly $\mathcal{M}(M, C) \subseteq \overline{\mathcal{M}}(M, C)$ and $\overline{\zeta}$ extends ζ, we conclude that $\overline{\zeta}$ is surjective, too. $\square$

We now want to describe the kernel of $\overline{\zeta}$.

Proposition 5.5. *Let $\Phi\colon M \to \frac{End(A)}{Inn(A)}$ be an admissible abstract kernel. Then $\overline{\zeta}([M, A, \Phi]) = 0$ if and only if Φ is extendable.*

Proof. If Φ is extendable, then there exists a regular Schreier extension

$$E:\ 0 \longrightarrow A \overset{\kappa}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} M \longrightarrow 1$$

which induces Φ. For $x \in M$, let us choose representatives $u_x \in \sigma^{-1}(x)$, with $u_1 = 0$, and consider $\varphi(x) \in End(A)$ defined by

$$u_x + a = \varphi(x)(a) + u_x$$

for any $a \in A$, so that $\Phi(x) = cl(\varphi(x))$, with $\Phi(1) = cl(id_A)$. Given $x, y \in M$, we have $\varphi(x), \varphi(y), \varphi(xy) \in End(A)$ given by

$$u_x + a = \varphi(x)(a) + u_x, \quad u_y + a = \varphi(y)(a) + u_y, \quad u_{xy} + a = \varphi(xy)(a) + u_{xy}.$$

Moreover, $u_x + u_y = f(x, y) + u_{xy}$ for some $f(x, y) \in U(A)$. So, on one hand

$$u_x + u_y + a = f(x, y) + u_{xy} + a = f(x, y) + \varphi(xy)(a) + u_{xy}$$

and, on the other hand,

$$u_x + u_y + a = u_x + \varphi(y)(a) + u_y = \varphi(x)(\varphi(y)(a)) + u_x + u_y = \varphi(x)(\varphi(y)(a)) + f(x, y) + u_{xy}.$$

From the uniqueness in the Schreier condition for E we obtain

$$f(x, y) + \varphi(xy)(a) = \varphi(x)(\varphi(y)(a)) + f(x, y).$$

Since $f(x, y)$ is invertible, the last equality can be rewritten as

$$\varphi(x)(\varphi(y)(a)) = f(x, y) + \varphi(xy)(a) - f(x, y),$$

so that

$$\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy) \quad \text{for all } x, y \in M.$$

Then $\overline{\zeta}([M, A, \Phi]) = cl(k)$, where k is the 3-cocycle determined by the equality

$$(3)\quad \varphi(x)(f(y, z)) + f(x, yz) = k(x, y, z) + f(x, y) + f(xy, z) \quad \text{for all } x, y, z \in M.$$

Moreover, associating the sum $u_x + u_y + u_z$ in the two possible ways, we get on one hand

$$u_x + u_y + u_z = f(x, y) + u_{xy} + u_z = f(x, y) + f(xy, z) + u_{xyz}$$

and, on the other hand,

$$u_x+u_y+u_z = u_x+f(y, z)+u_{yz} = \varphi(x)(f(y, z))+u_x+u_{yz} = \varphi(x)(f(y, z))+f(x, yz)+u_{xyz}.$$

Hence

$$f(x, y) + f(xy, z) = \varphi(x)(f(y, z)) + f(x, yz)$$

and comparing this last equality with (3) we get $k = 0$, and so $\overline{\zeta}([M, A, \Phi]) = 0$.

Conversely, suppose that $\overline{\zeta}([M, A, \Phi]) = 0$. By Proposition 5.2, for any $x \in M$ there exist $\varphi(x) \in \Phi(x)$, with $\varphi(1) = id_A$, and $f(x, y) \in U(A)$ such that $f(x, 1) = 0 = f(1, y)$,

$$\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy) \quad \text{for all } x, y \in M,$$

and

$$f(x, y) + f(xy, z) = \varphi(x)(f(y, z)) + f(x, yz) \quad \text{for all } x, y, z \in M.$$

Let us denote by $[A, \varphi, f, M]$ the set $A \times M$ equipped with the binary operation defined by

$$(a_1, x) + (a_2, y) = (a_1 + \varphi(x)(a_2) + f(x, y), xy).$$

We have that $[A, \varphi, f, M]$ is a monoid. Indeed

$$((a_1, x) + (a_2, y)) + (a_3, z) = (a_1 + \varphi(x)(a_2) + f(x, y), xy) + (a_3, z) =$$
$$= (a_1 + \varphi(x)(a_2) + f(x, y) + \varphi(xy)(a_3) + f(xy, z), xyz),$$

while

$$(a_1, x) + ((a_2, y) + (a_3, z)) = (a_1, x) + (a_2 + \varphi(y)(a_3) + f(y, z), yz) =$$
$$= (a_1 + \varphi(x)(a_2) + \varphi(x)\varphi(y)(a_3) + \varphi(x)(f(y, z)) + f(x, yz), xyz) =$$
$$= (a_1 + \varphi(x)(a_2) + f(x, y) + \varphi(xy)(a_3) - f(x, y) + \varphi(x)(f(y, z)) + f(x, yz), xyz) =$$
$$= (a_1 + \varphi(x)(a_2) + f(x, y) + \varphi(xy)(a_3) + f(xy, z), xyz),$$

so the operation is associative. Moreover, $(0, 1)$ is the neutral element, indeed:

$$(a, x) + (0, 1) = (a + \varphi(x)(0) + f(x, 1), x) = (a, x)$$

and

$$(0, 1) + (a, x) = (0 + \varphi(1)(a) + f(1, x), x) = (a, x).$$

We can therefore build the sequence

$$E:\ 0 \longrightarrow A \rightarrowtail^{i} [A, \varphi, f, M] \overset{p}{\twoheadrightarrow} M \longrightarrow 1,$$

where $i(a) = (a, 1)$ and $p(a, x) = x$. The map i is a (clearly injective) monoid homomorphism:

$$i(a_1)+i(a_2) = (a_1, 1)+(a_2, 1) = (a_1+\varphi(1)(a_2)+f(1, 1), 1) = (a_1+a_2, 1) = i(a_1+a_2),$$

and $i(0) = (0, 1)$. It is immediate to see that p is a surjective monoid homomorphism, and $i(A)$ is its kernel. Let us check that E is a Schreier extension: we choose as representatives the elements of the form $u_x = (0, x)$, for $x \in M$. Then, for every $(a, x) \in A \times M$, we have:

$$(a, x) = (a, 1) + (0, x) = i(a) + u_x.$$

Concerning uniqueness: if $i(a_1) + u_x = i(a_2) + u_x$, then

$$(a_1, x) = i(a_1) + u_x = i(a_2) + u_x = (a_2, x),$$

and so $a_1 = a_2$. In order to show that E is a regular Schreier extension, let us first prove that (a, x) is a representative for E if and only if $a \in U(A)$. If $a \in U(A)$, then for every $a_1 \in A$ we get

$$(a_1, x) = (a_1 - a, 1) + (a, x) = i(a_1 - a) + (a, x)$$

and, if $(a_1, 1) + (a, x) = (a_2, 1) + (a, x)$, then $(a_1 + a, x) = (a_2 + a, x)$. Hence $a_1 + a = a_2 + a$, and invertibility of a gives $a_1 = a_2$. Conversely, if (a, x) is a representative for E, then

$$(0, x) = i(a_1) + (a, x) = (a_1, 1) + (a, x) = (a_1 + a, x)$$

for some $a_1 \in A$; then $a_1 + a = 0$. Moreover

$$(a, x) = (a, 1) + (0, x) = (a, 1) + (a_1, 1) + (a, x) = (a + a_1, 1) + (a, x),$$

and since (a, x) is a representative, this gives $a + a_1 = 0$, so a is invertible. This allows to show that E is regular: given representatives $u_x = (a_1, x)$ and $u_y = (a_2, y)$, we have that $a_1, a_2 \in U(A)$. Therefore

$$u_x + u_y = (a_1, x) + (a_2, y) = (a_1 + \varphi(x)(a_2) + f(x, y), xy)$$

and $a_1+\varphi(x)(a_2)+f(x, y) \in U(A)$ because it is the sum of three invertible elements of A. Hence $u_x + u_y$ is a representative.

It remains to prove that E induces the abstract kernel Φ. To do this, we observe that

$$(0, x)+i(a) = (0, x)+(a, 1) = (\varphi(x)(a), x) = (\varphi(x)(a), 1)+(0, x) = i(\varphi(x)(a))+(0, x),$$

so the induced abstract kernel Φ_E is given by $\Phi_E(x) = cl(\varphi(x)) = \Phi(x)$. □

Now we can show that the factor monoid $\overline{\mathcal{A}}(M, C)$ is an abelian group which is isomorphic to the third cohomology group $H^3(M, C)$ of the monoid M with coefficients in the M-module C. Let us start by recalling the following:

Definition 5.6 ([21], Definition 4.2)**.** *A homomorphism of commutative monoids* $f : T \to T'$ *is called* kernel regular *(or, briefly, k-regular) if the equality* $f(t_1) = f(t_2)$ *implies the existence of* $k_1, k_2 \in Ker(f)$ *such that* $t_1 + k_1 = t_2 + k_2$.

Proposition 5.7. *A surjective homomorphism* $f : T \to G$*, where* G *is an abelian group, is necessarily k-regular.*

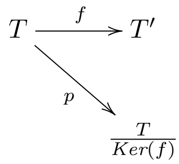
Proof. If $f(t_1) = f(t_2) = g \in G$, the surjectivity of f implies the existence of $t \in T$ such that $f(t) = -g$. Thus $f(t_1 + t) = 0 = f(t_2 + t)$, i.e. $t_1 + t, t_2 + t \in \mathrm{Ker}(f)$, and $t_1 + t_2 + t = t_2 + t_1 + t$. □

Recall that, if A is a commutative monoid and $B \subseteq A$ is a submonoid, then the relation $\sim$ on A defined by

$$a_1 \sim a_2 \iff \exists\, b_1, b_2 \in B \text{ such that } a_1 + b_1 = a_2 + b_2$$

is a congruence on A, and we denote by $\frac{A}{B}$ the corresponding factor monoid $\frac{A}{\sim}$.

Proposition 5.8. *Given a diagram*



where f *is a k-regular homomorphism of commutative monoids and* p *is the canonical projection, there exists a unique monoid homomorphism* $f' : \frac{T}{Ker(f)} \to T'$ *such that* $f'p = f$*. Moreover,* f' *is always injective; so, if* f *is surjective, then* f' *is an isomorphism.*

Proof. Define f' by putting $f'(cl(t)) = f(t)$. It is well-defined, because if $cl(t_1) = cl(t_2)$, then $t_1 + k_1 = t_2 + k_2$ for some $k_1, k_2 \in$ Ker(f). Then

$$f(t_1) = f(t_1) + f(k_1) = f(t_2) + f(k_2) = f(t_2).$$

Moreover, f' is injective: if $f'(cl(t_1)) = f'(cl(t_2))$, then $f(t_1) = f(t_2)$. Since f is k-regular, this implies that there exist $k_1, k_2 \in$ Ker(f) such that $t_1 + k_1 = t_2 + k_2$, and so $cl(t_1) = cl(t_2)$. Clearly, if f is surjective then so is f', and hence f' is in fact an isomorphism. □

The previous proposition says, in particular, that a surjective k-regular homomorphism is the cokernel of its kernel.

Corollary 5.9. *Let $f\colon T \to G$ be a surjective homomorphism of commutative monoids, with G a group. Then $\frac{T}{Ker(f)}$ is a group and the map $f'\colon \frac{T}{Ker(f)} \to G$ of the previous proposition is a group isomorphism.*

Summing up, we get:

Theorem 5.10. *The map $\overline{\zeta}'\colon \overline{\mathcal{A}}(M,C) = \frac{\overline{\mathcal{M}}(M,C)}{\overline{\mathcal{L}}(M,C)} \to H^3(M,C)$ defined by*

$$\overline{\zeta}'(cl([M,A,\Phi])) = \overline{\zeta}([M,A,\Phi]) = Obs(\Phi)$$

is an isomorphism of abelian groups.

Proof. The monoid homomorphism $\overline{\zeta}\colon \overline{\mathcal{M}}(M,C) \to H^3(M,C)$ is surjective (Proposition 5.4) and its kernel is $\overline{\mathcal{L}}(M,C)$ (Proposition 5.5). Then the result follows from the previous corollary. □

In [13] we observed that the factor monoid $\mathcal{A}(M,C) = \frac{\mathcal{M}(M,C)}{\mathcal{L}(M,C)}$, where $\mathcal{M}(M,C)$ is the monoid of C-equivalence classes of admissible abstract kernels of the form $\Phi\colon M \to \frac{End(G)}{Inn(G)}$, with G a group, and $\mathcal{L}(M,C)$ is the submonoid of C-equivalence classes of extendable admissible abstract kernels, is itself isomorphic to $H^3(M,C)$ [13, Theorem 6.4]. So we have an isomorphism between the groups $\mathcal{A}(M,C)$ and $\overline{\mathcal{A}}(M,C)$, which is given by sending $cl([M,G,\Phi]) \in \mathcal{A}(M,C)$ to $\overline{cl}([M,G,\Phi]) \in \overline{\mathcal{A}}(M,C)$.

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