THE SYMPLECTIC LEFT COMPANION OF A LITTLEWOOD-RICHARDSON-SUNDARAM TABLEAU AND THE KWON PROPERTY

OLGA AZENHAS

ABSTRACT. As a consequence of the Littlewood-Richardson (LR) commuters coincidence and the Kumar-Torres branching model via Kushwaha-Raghavan-Viswanath flagged hives, we have solved the Lecouvey-Lenart conjecture on the bijections between the Kwon and Sundaram branching models for the pair $(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$ consisting of the general linear group $GL_{2n}(\mathbb{C})$ and the symplectic group $Sp_{2n}(\mathbb{C})$. In particular, thanks to the Henriques-Kamnitzer gl_n-crystal commuter, we have recognized that the left companion of an LR-Sundaram tableau is characterized by the Kwon symplectic condition. We now show that the construction of the left companion tableau of a LR-Sundaram tableau exhibits in fact the Kwon symplectic property.

CONTENTS

1. INTRODUCTION

Consider G a group and $\hat{G}$ a complete set of representatives of the equivalence classes of certain irreducible G-modules [Wat25]. Given H a subgroup of G, a natural and interesting problem is to determine how and if a given irreducible G-module $V \in \hat{G}$ decomposes into irreducible H-submodules [Lit44, Sun86, HTW05, Kwo18a, Wat25, KT25]:

$$V \simeq \bigoplus_{W \in \hat{H}} W^{c_W^V}.$$

The multiplicities numbers c_W^V are called *branching coefficients* of the pair (G, H). An explicit description of the branching coefficients is called a *branching rule* for the pair (G, H). For example, the Littlewood-Richardson rule [LR34] gives a branching rule for the pair $(GL_m(\mathbb{C}) \times GL_m(\mathbb{C}), GL_m(\mathbb{C}))$. In this note one considers the pair $(G, H) = (GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$. The polynomial irreducible representations of $GL_{2n}(\mathbb{C})$ respectively $Sp_{2n}(\mathbb{C})$ are parameterized by partitions λ of length $\leq 2n$ respectively partitions μ of $\leq n$, and we denote the corresponding branching coefficient by c_μ^λ.

2000 *Mathematics Subject Classification.* 05E05, 05E10, 05E14, 17B37, 68Q17.
Key words and phrases. Kwon and Sundaram branching model, Littlewood-Richardson-Sundaram tableaux, left companion.

Littlewood [Lit44] has given a branching rule only in the case where both partitions have length $\leq n$. Sundaram has given a complete branching rule [Sun86, Sun90] by counting certain Littlewood-Richardson (LR) tableaux, called Littlewood-Richardson-Sundaram tableaux in [LL20, KT25], and symplectic LR tableaux in [Wat25]. Schumann and Torres [ST] proved a conjectural branching rule by Naito and Sagaki [NS05] in terms of Littelmann paths.

Kwon [Kwo18a] has provided branching rules for various pairs in particular the pair $(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$. To express the branching coefficients c_μ^λ he enumerates certain sets in its combinatorial spinor model [Kwo18b] for crystals of classical type. As expected the problem of comparing the Sundaram and the Kwon branching rules addresses. This has been considered by Lecouvey and Lenart [LL20] by conjecturing an explicit bijection between the two models via the combinatorial R-matrix realized by the Henriques-Kamnitzer LR commuter [HK06a, HK06b] which has several realizations [KTW04, Buc00, TY08, AKT16, Aze25, TKA18, ACM25, Aze25] depending also on the LR model. Recently Kumar and Torres [KT25] use the LR commuter by Kushwaha–Raghavan–Viswanath for flagged hives [KRV21, KRKV24] to establish a bijection between the Sundram and Kwon branching models. As conjectured by Pack and Vallejo [PV10] and proved in [PV10, DK08, AKT16, ACM25, Aze25] all these realizations coincide and henceforth the Lecouvey-Lenart conjecture [LL20] is solved.

As one shows in [Aze25] the objects counted by the Kwon branching model, after a rephrasing of that model by Lecouvey-Lenart, are precisely the left companions of LR-Sundaram tableaux. While Kumar-Torres use the right companion tableau of an LR-Sundaram tableau in their bijection via flagged hives, here, we use the left companion to directly show that the Sundaram flag condition on an LR tableau is mirrored on the left companion as a symplectic Kwon property . In other words, the construction of the left companion tableau of an LR-Sundaram tableau gives a symplectic semistandard tableau, equivalently, a semistandard tableau satisfying the Kwon property. To summarize we exhibit the branching models of Kwon and Sundaram via the left companion of a Littlewood-Richardson-Sundaram (LRS) tableau.

More precisely, for partitions $\mu \subset \lambda$ with μ of length at most n and λ of length at most $2n$, the Sundaram branching rule says: the branching coefficient c_μ^λ equals the cardinality of the set

$$LRS(\lambda, \mu) := \bigcup_\nu LRS(\lambda/\mu, \nu)$$

where the union is taken over all even partitions ν, and $LRS(\lambda/\mu, \nu)$ denotes the set consisting of the Littlewood-Richardson-Sundaram tableaux of shape λ/μ and weight the even partition ν.

According to a reformulation of the Kwon's branching rule by Lecouvey–Lenart [LL20, Section 8] (see also [KT25]), the branching coefficient c_μ^λ equals the cardinality of the set

$$LRK(\lambda, \mu) := \bigcup_\nu LRK_{\nu,\mu}^\lambda$$

where the union is taken over all even partitions ν, and $LRK_{\nu,\mu}^\lambda$ denotes the set of right companions T (of shape μ) of the LR tableaux in $LR(\lambda/\nu, \mu)$ (note ν and μ are swapped) whose evacuation (or Schützenberger involution) $S(T)$ satisfy the Kwon property (also called symplectic property). Those tableaux $S(T)$ are precisely the left companions of the set Littlewood-Richardson-Sundaram tableaux $LRS(\lambda/\mu, \nu)$.

Our main result is stated as follows and illustrated in Examples 4.4. It describes how the Sundaram property violation mirrors on its left companion.

Main Theorem1 Let $T \in LR(\lambda/\mu, \nu)$ with ν even, on the alphabet $[2n]$, and $\ell(\mu) \leq n$. T does not satisfy the Sundaram property if and only if $G_\mu(T)$, the left companion of T, is not symplectic.

Moreover, in this case, there exists a unique $t \geq 0$ such that the following are equivalent

(1) $n + t + 1$ is the minimal row of T where the Sundaram property violation occurs.

(2)

$$T(n+t, 1) = 2t,\ T(n+t+1, 1) = 2t+1,\ T(n+t+2, 1) = 2(t+1),$$
$$T(n+1, 1) \geq 2,\ T(n+2, 1) \geq 4, \ldots, T(n+t+1-2, 1) \geq 2(t+1-2).$$

(3) the maximal row of $G_\mu(T)$ where a symplectic violation occurs is either in the cell $(\ell(\mu), 1)$, or $(\ell(\mu)-1, 1), \ldots,$ or $(\ell(\mu)-t, 1)$ of the first column of $G_\mu(T)$.

Watanabe [Wat25] recently has also established a new branching rule for the pair $(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$. It is an interesting question to ask how the Watanabe branching rule bijects to Sundaram or Kwon branching rules.

This paper is organized in four sections. Section 2 introduces the relevant notation. Section 3 introduces semistandard symplectic tableaux (King tableaux rephrased in the alphabet $[2n]$) [Wat25] also called tableaux satisfying the Kwon property [KT25]. The left companion of an LR tableau and Littlewood-Richardson-Sundaram tableaux are defined in Section 4. The main result and examples also appear in Section 4.

Acknowledgements

The author acknowledges financial support by the Centre for Mathematics of the University of Coimbra (CMUC, https://doi.org/10.54499/UID/00324/2025) under the Portuguese Foundation for Science and Technology (FCT), Grants UID/00324/2025 and UID/PRR/00324/2025.

2. Preliminaries

2.1. **Partitions and semistandard tableaux.** A partition is a weakly decreasing sequence of nonnegative integers $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots$ such that $\lambda_k = 0$ for some $k \geq 1$. The length of λ is the maximal i such that $\lambda_i > 0$ also called the number of parts or length of λ. We write the partition λ as a vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ for $k \geq \ell(\lambda)$. A partition λ is identified with its Young diagram $Y(\lambda)$ which is a left and top justified collection of boxes (or cells) with λ_k many boxes in the kth row for all $k \in \mathbb{Z}_{>}0$. In particular, The empty Young diagram and the partition (0) are identified. The boxes or cells of the Young diagram of λ are identified by its coordinates (i, j) in the matrix style, that is, $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$.

Let λ, μ partitions with $\mu \subset \lambda$, that is, $\mu_i \leq \lambda_i$ for all $i \in \mathbb{Z}_{>}0$, or the Young diagram of μ is a subset of the Young diagram of λ. A semistandard tableau T of (skew) shape λ/μ is a map (or a filling of $Y(\lambda)$)

$$T : Y(\lambda) \to \mathbb{Z}_{\geq}0,\ T(i,j) \mapsto T(i,j),$$

assigning a nonnegative integer to each box of λ such that it is weakly increasing as we go from left to right along a row and strictly increasing as we go from top to bottom along a column, and equal to 0 in the boxes corresponding to μ,

$$\begin{aligned} &T(i,j) \leq T(i,j+1),\ T(i,j) < T(i+1,j),\ \text{for all } (i,j) \in Y(\lambda) \setminus Y(\mu),\\ &\text{and } T(i,j) = 0,\ \text{for } (i,j) \in Y(\mu), \end{aligned}$$

where we set $T(a,b) := \infty$ if $(a,b) \notin Y(\lambda)$. Usually $T(i,j)$ is just referred as the entry in the box (i,j) and we omit the zeroes in the boxes of μ. A positive integer $k \geq \ell(\lambda$ will be fixed and $[0,k] := \{0, 1, \ldots, k\}$ will be used as a co-domain for map T. We call $[k] := \{1, \ldots, k\}$ the alphabet of the semistandard tableau T. In this case, we will denote the set of semistandard tableaux of shape λ/μ by $SST_k(\lambda/\mu)$. When $\mu = (0)$, we just write $SST_k(\lambda)$. The *weight* or *content* of T is the nonnegative vector $(m_1, \ldots, m_k)$, where $m_i := \#\{(a,b) \in Y(\lambda) : T(a,b) = i\}$ for $i \in [k]$, that is, m_i is the number of occurrences of i in the tableau T.

A semistandard Young tableau $G \in SST_k(\lambda)$ is also realized by the sequence of nested partitions

$$\lambda = \lambda^{(k)} \supseteq \cdots \supseteq \lambda^{(1)} \supseteq \lambda^{(0)} = 0$$

where $\lambda^{(m)}$ defines the filling of the boxes of T on the alphabet $[m]$, for $1 \leq m \leq k$. Equivalently, $\lambda^{(m)} = G^{-1}([m])$ the pre-image of $[m]$, for $1 \leq m \leq k$. The semistandard condition translates to the condition that $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip, that is in any column of the Young diagram of $\lambda^{(i)}$, there is at most one box of $\lambda^{(i)}$ that is not a box of $\lambda^{(i-1)}$. In other words, $0 \leq \ell(\lambda^{(i)}) - \ell(\lambda^{(i-1)}) \leq 1$. Henceforth, the first column of G records in strictly decreasing order, from bottom to top, the i's in $[k]$ such that

$\ell(\lambda^{(i)}) - 1 = \ell(\lambda^{(i-1)})$. In other words,

$$G(s,1) = b > G(s-1,1) = a, \text{ for some } s \in [2, \ell(\lambda)] \tag{1}$$

$$\Leftrightarrow$$

$$\ell(\lambda^{(b)}) - 1 = \ell(\lambda^{(a)}), \text{ and } \ell(\lambda^{(x)}) = \ell(\lambda^{(a)}), \text{ for all } a \leq x < b. \tag{2}$$

Example 1. Let $G \in SST_8(\mu)$ with $\mu = (4,3,2,1)$

$$G = \begin{array}{|c|c|c|c|} \hline 3 & 4 & 4 & 6 \\ \hline 4 & 5 & 7 & \\ \cline{1-3} 6 & 8 & & \\ \cline{1-2} 8 & & & \\ \cline{1-1} \end{array} \quad \text{and its defining nested sequence of partitions} \quad \begin{array}{c} 0 \\ 0 \quad 0 \\ \mathbf{1} \quad 0 \quad 0 \\ \mathbf{3} \quad \mathbf{1} \quad 0 \quad 0 \\ 3 \quad 2 \quad 0 \quad 0 \quad 0 \\ \mathbf{4} \quad \mathbf{2} \quad \mathbf{1} \quad 0 \quad 0 \quad 0 \\ 4 \quad 3 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \\ \mathbf{4} \quad \mathbf{3} \quad \mathbf{2} \quad \mathbf{1} \quad 0 \quad 0 \quad 0 \quad 0 \end{array}$$

$$\mu = \mu^{(8)} = (4,3,2,1,0^4) \supseteq \mu^{(7)} = (4,3,1,0^4) \supseteq \mu^{(6)} = (4,2,1,0^3) \supseteq \mu^{(5)} = (3,2,0^3) \supseteq \mu^{(4)} = (3,1,0^2)$$
$$\supseteq \mu^{(3)} = (1,0^2) \supseteq \mu^{(2)} = (0^2) = \mu^{(1)} = (0).$$

The first column of G records the supra indices of the subsequence $\mu^{(8)} \supseteq \mu^{(6)} \supseteq \mu^{(4)} \supseteq \mu^{(3)}$

$$\ell(\mu^{(8)}) = 4 \quad > \ell(\mu^{(7)}) = \ell(\mu^{(6)}) = 3 > \ell(\mu^{(5)}) = \ell(\mu^{(4)}) = 2 \quad > \ell(\mu^{(3)}) = 1 > \ell(\mu^{(2)}) = \ell(\mu^{(1)}) = 0$$

The sequence of numbers $8 > 6 > 4 > 3$ defines the entries of the first column of G.

The reverse row word of a semistandard tableau T, denoted $w(T) = w_1 \cdots w_l$, l the number of non zero entries of T, is obtained by reading the entries of its rows (excluding the entry 0) right to left starting from the top row and proceeding downward. The weight of $w(T)$ is the weight of T. A Yamanouchi word is a word $w = w_1 \cdots w_l$ such that, for each $1 \leq k \leq l$, the the weight of the subword $w_1 \cdots w_k$ is a partition.

3. Symplectic tableaux

From now on, we fix $n \in \mathbb{N}$. Let γ be a partition with length $\ell(\gamma) \leq 2n$. The partition γ is said to be *even* if $\gamma_{2i-1} = \gamma_{2i}$ for all $i \geq 1$. In other words, all columns of γ have even length and necessarily the length of γ is even. Let $SST_{2n}(\gamma)$ be the set of all semistandard tableaux of shape γ with entries in $[2n] := \{1, \ldots, 2n\}$.

Definition 1. [Kin76] A semistandard tableau $G \in SST_{2n}(\gamma)$ is said to be symplectic if

$$G(k,1) \geq 2k-1, \text{ for all } k \in [\ell(\gamma)].$$

Let $SpT_{2n}(\gamma)$ denote the set of all symplectic tableaux of shape γ on the alphabet $[2n]$.

The following proposition due to Watanabe [Wat25] describes the minimal row of $G \in SpT_{2n}(\gamma)$ where a symplectic violation occurs. Our main result Theorem 1 describes the maximal row of G, as a companion of an LR Sundaram tableau, where a symplectic violation occurs. See Examples 4.4.

Proposition 1. [Wat25] Let $G \in SST_{2n}(\gamma)$.

(1) If $G \in SpT_{2n}(\gamma)$ then $\ell(\gamma) \leq n$.
(2) If G is not symplectic, then there exists a unique $i \in [2, 2n]$ such that

$$G(i,1) < 2i-1 \text{ and } G(k,1) \geq 2k-1 \text{ for all } k \in [1, i-1]. \tag{3}$$

Moreover, we have

$$G(i-1,1) = 2i-3 = 2(i-1)-1 \text{ and } G(i,1) = 2i-2 = 2(i-1)\ . \tag{4}$$

4. Symplectic Littlewood-Richardson tableaux and the left companion

Let λ be a partition with $\ell(\lambda) \leq 2n$. Let $\mu, \nu \subseteq \lambda$ with $\ell(\mu) \leq n$ and ν an even partition.

4.1. **LR tableaux of even weight.** A tableau $T \in SST_{2n}(\lambda/\mu)$ of weight ν is said to be Littlewood-Richardson (LR) tableau if its reverse row word is a Yamanouchi word of weight ν. Let $LR(\lambda/\mu, \nu)$ be the set of all LR tableaux of shape λ/μ and weight ν on the alphabet $[2n]$.

Remark 1. Note $2n \geq \ell(\lambda) \Leftrightarrow \ell(\lambda) - n \leq n$.

Lemma 1. Let $T \in LR(\lambda/\mu, \nu)$. Let $i \geq 0$.

(1) If $T(k,j) = 2i+1$ then $T(k',j') = 2(i+1)$ for some $k' > k$ and j'.
(2) If $T(k,1) = 2i+1$ then $T(k+1,1) = 2(i+1)$.
(3) For any $1 \leq k, j$ $T(k,j) \leq k$.

Proof. By assumption ν is an even partition and the reverse row word of T is a Yamanouchi word. □

4.2. **The left companion of an LR tableau.** In [KT25] the right companion of a Littlewood-Richardson-Sundaram tableau has been characterized by a flag condition. Next section does it for the left companion by the symplectic property. We recall first the definition of left companion of an LR tableau.

Definition 2. [AKT16] Let $T \in LR(\lambda/\mu, \nu)$. The left companion of T, $G_\mu(T) \in SST_{2n}(\mu)$ of shape μ in the alphabet $[2n]$ and content $rev(\lambda - \nu)$ the reverse of $\lambda - \nu$, is obtained from T by recording the sequence of partitions $\mu^{(2n-r+1)}$ giving the shapes occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \dots, 2n$ of T, for $r = 1, 2, \dots, 2n$. We then get the nested sequence of partitions $\mu = \mu^{(2n)} \supseteq \mu^{(2n-1)} \supseteq \cdots \supseteq \mu^{(1)}$ defining $G_\mu(T)$.

Since the shape of $G_\mu(T)$ is μ and the first column has length $\ell(\mu)$, the following is an immediate consequence of the previous definition and a rephrasing of (1), (2). It is an equivalent description of the left companion $G_\mu(T)$ of T.

Lemma 2. Let $G = G_\mu(T)$ be defined by the nested sequence of partitions

$$\mu = \mu^{(2n)} \supseteq \mu^{(2n-1)} \supseteq \cdots \supseteq \mu^{(2)} \supseteq \mu^{(1)}. \tag{5}$$

The first column of G of length $\ell(\mu)$ read bottom to top is equal to $r_{\ell(\mu)} > \cdots > r_i > \cdots > r_1$, that is,

$$G(\ell(\mu),1) = r_{\ell(\mu)} > G(\ell(\mu)-1,1) = r_{\ell(\mu)-1} > \cdots > G(i,1) = r_i > \cdots > G(1,1) = r_1$$

such that
(i)

$$2n \geq r_{\ell(\mu)} > \cdots > r_i > \cdots > r_1 \geq 1, \tag{6}$$

and
(ii)

$$\mu^{(r_{\ell(\mu)})} \supset \mu^{(r_{\ell(\mu)-1})} \supset \cdots \supset \mu^{(r_i)} \supset \cdots \supset \mu^{(r_1)} \tag{7}$$

is the maximal subsequence of (5) of length $\ell(\mu)$ such that

$$\ell(\mu) = \ell(\mu^{(r_{\ell(\mu)})}) > \ell(\mu^{(r_{\ell(\mu)-1})}) > \cdots > \ell(\mu^{(r_{i+1})}) > \ell(\mu^{(r_i)}) > \cdots > \ell(\mu^{(r_1)}) = 1$$
$$\ell(\mu^{(r_i)}) - \ell(\mu^{(r_{i-1})}) = 1, \quad 1 \leq i \leq \ell(\mu),$$

where we set $\mu^{(r_0)} = 0$, $r_{\ell\mu)+1} = 2n+1$, and, for $i = 0, 1, \dots, \ell(\mu)$,

$$\ell(\mu^{(s)}) = \ell(\mu^{(r_i)}), \text{ for any } r_{i+1} > s \geq r_i. \tag{8}$$

Example 2. Let $n=4$, $\lambda=(6,5,5,4,3,1,0^2)$, $\nu=(4,4,3,3,0^4)$, $\mu=(4,3,2,1,0^4)$ even and $T\in LR(\lambda/\mu,\nu)$ as below. We illustrate T with its left companion $G_\mu(T)=G$ in Example 1,

$$T=\begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 & 1\\ \hline \ & \ & \ & 1 & 2 \\ \cline{1-5} \ & \ & 1 & 2 & 3\\ \cline{1-5} \ & 2 & 3 & 4 \\ \cline{1-4} 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}\quad , \quad G_\mu(T)=\begin{array}{|c|c|c|c|}\hline 3 & 4 & 4 & 6\\ \hline 4 & 5 & 7 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}\quad .$$

$G_\mu(T)$ has weight $rev(\lambda-\nu)=(0^2,1,3,1,2,1,2)$ and as the left companion of T is defined by the nested sequence of partitions constructed as in Definition 2,

$$\begin{aligned}&\mu=\mu^{(8)}=(4,3,2,1,0^4)\supseteq\mu^{(8-1)}=(4,3,1,0^4)\supseteq\mu^{(8-2)}=(4,2,1,0^3)\supseteq\mu^{(8-3)}=(3,2,0^3)\\ &\supseteq\mu^{(8-4)}=(3,1,0^2)\supseteq\mu^{(8-5)}=(1,0^2)\supseteq\mu^{(8-6)}=(0^2)=\mu^{(8-7)}=(0).\end{aligned}\tag{9}$$

The subsequence $\mu^{(8)}\supseteq\mu^{(6)}\supseteq\mu^{(4)}\supseteq\mu^{(3)}$ is obtained according to Lemma 2

$$\begin{array}{cccc}\ell(\mu^{(8)})=4 & >\ell(\mu^{(7)})=\ell(\mu^{(6)})=3>\ell(\mu^{(5)})=\ell(\mu^{(4)})=2 & >\ell(\mu^{(3)})=1>\ell(\mu^{(2)})=\ell(\mu^{(1)})=0\\ 8 & 6 \qquad\qquad\qquad 4 & 3\end{array}\tag{10}$$

The sequence of numbers $8>6>4>3$ (10) are the entries of the first column of $G_\mu(T)$. Iterating this procedure, by subtracting one unity to each positive entry of the nested sequence (9), we get

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 0 & & & & & & & \\
 & & & & & & 0 & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & 0 & & & & & \\
 & & & & 2 & & 0 & & 0 & & 0 & & & & \\
 & & & 2 & & 1 & & 0 & & 0 & & 0 & & & \\
 & & 3 & & 1 & & 0 & & 0 & & 0 & & 0 & & \\
 & 3 & & 2 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
3 & & 2 & & 1 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

By the same reasoning as above, the sequence of numbers $8>5>4$ are the entries of the second column of $G_\mu(T)$. Again subtracting 1 to each positive entry we get by the same reasoning the sequence of numbers $7>4$ that are the entries of the third column of $G_\mu(T)$. Again subtracting 1 to each positive entry, and by the same reasoning we get 4 which gives the fourth column of $G_\mu(T)$.

The next proposition collects a few properties of $G_\mu(T)$ in relation with T.

Proposition 2. Let $T\in LR(\lambda/\mu,\nu)$ and the left companion of $G_\mu(T)$ defined by (5). The following holds

(1) $T(\ell(\mu)+s,1)=\ell(\mu)+s$, for some $s\geq 1$, or not defined if and only if
$$\ell(\mu^{(2n-(\ell(\mu)+t)+1)})=0 \text{ for all } t\geq s.\tag{11}$$

(2) Either $T(k,1)$ is not defined for $k\geq 1$, in which case $\ell(\lambda)=\ell(\mu)$, or $T(k,1)=k$ for $\ell(\mu)<k\leq\ell(\lambda)\leq 2n$ if and only if
$$\ell(\mu^{(2n)})>\ell(\mu^{(2n-1)})>\cdots>\ell(\mu^{(2n-\ell(\mu)+1)})>\ell(\mu^{(2n-\ell(\mu))})=\cdots=\ell(\mu^{(1)})=0.\tag{12}$$

(3) Let $T(\ell(\mu)+1,1)=s\in[\ell(\mu)]$. Either $T(\ell(\mu)+1,1)=s$ even, or s odd and for some even $2\leq t\leq\ell(\mu)-s+2$, $T(\ell(\mu)+i,1)=s+i-1$, $1\leq i\leq t$. Moreover, this is equivalent to

$$\ell(\mu^{(2n)}) > \ell(\mu^{(2n-1)}) > \cdots > \ell(\mu^{(2n-s+1)}) = \ell(\mu^{(2n-s)}),\ \text{if } s \text{ even}, \tag{13}$$

and, for some even $2 \leq t \leq \ell(\mu) + 1$,

$$\ell(\mu^{(2n)}) > \ell(\mu^{(2n-1)}) > \cdots > \ell(\mu^{(2n-s+1)}) = \ell(\mu^{(2n-s)}) = \cdots = \ell(\mu^{(2n-(s+t-1))}),\ \text{if } s \text{ odd}. \tag{14}$$

Proof. Recall since T is LR, above or in a row k of T there are no larger entries than k, that is, $T(k,j)$ is not defined or $T(k,j) \leq k$, for any $1 \leq k, j$. In particular, $T(k,1) \leq k$ or not defined for $k > \ell(\mu)$.

(1) The partition $\mu^{(2n-(\ell(\mu)+t)+1)}$ gives the shape occupied by the entries $< \ell(\mu)+t$, including the empty entries identified with 0, in rows $\ell(\mu)+t, \ell(\mu)+t+1, \ldots, 2n$ of T. Since T is LR and $T(\ell(\mu)+s,1) = \ell(\mu)+s < T(\ell(\mu)+t,1)$, it forces $T(\ell(\mu)+t,1) = \ell(\mu)+t \leq T(\ell(\mu)+t,j)$, for $t \geq s$ and $j \geq 1$. Therefore, there are no entries $< \ell(\mu)+t$ in rows $\ell(\mu)+t, \ell(\mu)+t+1, \ldots, 2n$ of T, and $\ell(\mu^{(2n-(\ell(\mu)+t)+1)}) = 0$ for all $t \geq s$.

(2) It is a consequence of (1). For $r = \ell(\mu)+1, \ldots, 2n$, the partition $\mu^{(2n-r+1)}$ giving the shape occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \ldots, 2n$ of T, is empty.

For $1 \leq r \leq \ell(\mu)$, the partition $\mu^{(2n-r+1)}$ giving the shape occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \ldots, 2n$ of T, has length $\ell(\mu) - r + 1$.

We then get the nested sequence of partitions defining $G_\mu(T)$ to be

$$\mu = \mu^{(2n)} \supsetneqq \mu^{(2n-1)} \supsetneqq \cdots \supsetneqq \mu^{(2n-\ell(\mu)+1)} \supsetneqq \mu^{(2n-\ell(\mu))} = \cdots = \mu^{(1)} = \emptyset,$$

and the result follows by definition of $G_\mu(T)$.

(3) Note $T(\ell(\mu)+i,1) = s+i-1 \leq \ell(\mu)+i-1 < \ell(\mu)+i$, for $i \geq 1$.

Let $T(\ell(\mu)+1,1) = s$ even $\in [\ell(\mu)]$. Then, for $1 \leq r \leq s \leq \ell(\mu)$, the partition $\mu^{(2n-r+1)}$ gives the shape occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \ldots, \ell(\mu)$ of T, and $\ell(\mu^{(2n-r+1)}) = \ell(\mu) - r + 1$. In particular, $\mu^{(2n-s+1)}$ gives the shape occupied by the entires $< s$ in rows $s, \ldots, \ell(\mu)$, and $\mu^{(2n-s)}$ gives the shape occupied by the entires $\leq s$ in rows $s+1, \ldots, \ell(\mu), \ell(\mu)+1$. This means, $\ell(\mu^{(2n-s+1)}) = \ell(\mu) - s + 1$ and

$$\ell(\mu^{(2n-s)}) = \ell(\mu^{(2n-s+1)}) - 1 + 1 = \ell(\mu) - s + 1 = \ell(\mu^{(2n-s+1)}).$$

Let $T(\ell(\mu)+1,1) = s$ odd $\in [\ell(\mu)]$. Since the partition ν is even, then one also has $T(\ell(\mu)+2,1) = s+1$. Assume, for some even $2 \leq t \leq \ell(\mu)+1$, $T(\ell(\mu)+i,1) = s+i-1$, $1 \leq i \leq t$.

For $s = 1$, one has for some even $2 \leq t \leq \ell(\mu)+1$, $T(\ell(\mu)+i,1) = i$, $1 \leq i \leq t$. We show that this is equivalent to

$$\ell(\mu) = \ell(\mu^{(2n)}) = \ell(\mu^{(2n-1)}) = \cdots = \ell(\mu^{(2n-t)} \tag{15}$$

The partition $\mu^{(2n-r+1)}$ gives the shape occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \ldots, 2n$ of T, for $r = 1, 2, \ldots, 2n$

For $i = 1, 2, \ldots, t \leq \ell(\mu)+1$ with t even, since $T(\ell(\mu)+i,1) = i$, $1 \leq i \leq t$, the partition $\mu^{(2n-i+1)}$ gives the shape occupied by the entries $< i$, including the empty entries of the shape μ identified with 0, in rows $i, i+1, \ldots, \ell(\mu), \ell(\mu)+1, \ldots, \ell(\mu)+i-1$ of T. Therefore, $\ell(\mu^{(2n-i+1)}) = \ell(\mu)$,for $i = 1, \ldots t$.

Let $3 \leq s \leq \ell(\mu)$ with s odd. Since for some even $2 \leq t \leq \ell(\mu)-s+2$, $T(\ell(\mu)+i,1) = s+i-1$, $1 \leq i \leq t$, then indeed

$$\ell(\mu^{(2n)}) > \ell(\mu^{(2n-1)}) > \cdots > \ell(\mu^{(2n-s+1)}) = \ell(\mu) - s + 1 = \ell(\mu^{(2n-s)}) = \cdots = \ell(\mu^{(2n-(s+t-1))})$$

□

4.3. **The left companion of a symplectic LR tableau.** We now recall the definition of LR-Sundaram tableau also called symplectic LR tableau in [Wat25].

Definition 3. [Sun86, Sun90] Let $\mu, \nu \subseteq \lambda$ such that $\ell(\mu) \leq n$ and ν an even partition. A Littlewood–Richardson tableau T of shape λ/μ and weight ν on the alphabet $[2n]$ satisfies the *Sundaram property* if for each $i = 0, \ldots, \ell(\nu)/2$, the odd entry $2i+1$ appears in row $n+i$ or above in the Young diagram of λ. In other words, if $T(k,j) = 2i+1$ for some cell (k,j) of T and $i \in \mathbb{Z}_{\geq} 0$, then we have $k \leq n+i$.

The set of $T \in LR(\lambda/\mu, \nu)$ satisfying the Sundaram property is denoted by $LRS(\lambda/\mu, \nu)$ and called the set of *LR-Sundaram tableaux* or *symplectic LR tableaux* in [Wat25].

Remark 2. Let $T \in LR(\lambda/\mu, \nu)$.

(1) A Sundaram property violation never occurs in the first n rows of T.
(2) If $T \in LRS(\lambda/\mu, \nu)$, the possible odd numbers in row $n+t$ of T are larger or equal than $2t+1$, for $t \geq 0$. In other words, for $t \geq 1$, the possible odd numbers in row $n+t$ of T are $2i+1$ with $t \leq i$: 1 or larger in row n; 3 or larger in row $n+1$, 5 or larger in row $n+2$, etc.
(3) For $n \geq \ell(\lambda)$ any $T \in LR(\lambda/\mu, \nu)$ is LR-Sundaram.

Lemma 3. Let $T \in LR(\lambda/\mu, \nu)$ with ν even, on the alphabet $[2n]$, and $n \geq \ell(\mu)$.

(1) If T satisfies the Sundaram property, it holds

$$T(n+t, 1) = 2i+1, \text{ for some } i \geq 0 \text{ and } t \geq 1 \;\Rightarrow i \geq t \geq 1. \tag{16}$$

(2) If T satisfies (16), $T(n+t, 1) \geq 2t$, for all $t \geq 1$.

Proof. 1) It is a consequence of the definition of LRS tableau with $j = 1$. If $T \in LRS(\lambda/\mu, \nu)$ then, in particular, for $j = 1$, (16) holds.

(2) From (16), indeed $T(n+1, 1) \geq 2$. By induction on $t \geq 1$, assume $T(n+t, 1) \geq 2t$. Then either

$$T(n+t+1, 1) = odd \geq 2(t+1) + 1 > 2(t+1) \text{ or },$$
$$T(n+t+1, 1) = even > T(n+t, 1) \geq 2t \Rightarrow even \geq 2(t+1).$$

□

The proposition below asserts that to verify the Sundaram property in an LR tableau it is enough to check the odd entries in the rows below row n in the first column of T.

Proposition 3. Let $T \in LR(\lambda/\mu, \nu)$ with ν even, on the alphabet $[2n]$, and $n \geq \ell(\mu)$. Then, $T \in LRS(\lambda/\mu, \nu)$ if and only if T satisfies (16)

$$T(n+t, 1) = 2i+1, \text{ for some } i \geq 0 \text{ and } t \geq 1 \;\Rightarrow i \geq t \geq 1.$$

In other words $T \in LRS(\lambda/\mu, \nu)$ if and only for all $t \geq 1$, either $T(n+t, 1) = even \geq 2t$ or $T(n+t, 1) = odd \geq 2t+1$.

Proof. The "Only if part" was proved in Lemma 3.

"If part". Assume that condition (16) holds for $T \in LR(\lambda/\mu, \nu)$. We want to show that for $t \geq 1$, the possible odd numbers in row $n+t$ of T are $2i+1$ with $t \leq i$. Indeed, from (16), $T(k, 1) \geq 2$ for $k \geq n+1$.

If $T(n+t, 1) = 2i+1$ and $T(n+t, j) = 2i'+1$ for some $j > 1, t \geq 1$ then from the semistandard property of T, $i' \geq i \geq t$ and $i' \geq t$.

If $T(n+t, 1)$ is even and $T(n+t, j) = 2i+1$ is odd for some $j > 1$ and $i \geq 1$, then $T(n+t, 1) < 2i+1$ and $T(n+t, 1) = even \leq 2i$. From Lemma 3, (2), one has $2t \leq T(n+t, 1) = even \leq 2i$ which implies $t \leq i$. □

Corollary 1. Let $T \in LR(\lambda/\mu, \nu)$ with ν even, on the alphabet $[2n]$, and $n \geq \ell(\mu)$. T does not satisfy Sundaram property if and only if $T(n+t, 1) = 2i+1$, for some $t > i \geq 0$.

A more detailed description of the Sundaram property violation on a Littlewood-Richardson tableau of even weight is the following.

Corollary 2. Let $T \in LR(\lambda/\mu, \nu)$ with ν even, on the alphabet $[2n]$, and $n \geq \ell(\mu)$. Then

(1) $T(n+t, 1) = even < 2t$, for some $t \geq 1$ only if $T(n+s, 1) = odd < 2s+1$, for some $1 \leq s < t$.
(2) $T(n+s, 1) = odd < 2s+1$ for some $s \geq 1$, only if $T(n+t, 1) = even < 2t$, for some $1 \leq s < t$.

Proof. (1) If $T(n+t, 1) = 2a$ for some $1 \leq a < t$, and $t \geq 2$, then there exists $1 \leq s < t$ such that $T(n+s, 1) = odd < 2s+1$. One has only $a-1$ positive even numbers to distribute on $t-1 > a-1$ cells $(n+1, 1), \ldots, (n+t-1, 1)$. Hence, there exists at least one cell $(n+s, 1)$ such that $T(n+s, 1) = odd$ with $1 \leq s \leq t-1$. Let $(n+s, 1)$ be the first cell above the cell $(n+t, 1)$ (seen from the bottom) where this occurs. If $odd \geq 2s+1$, since ν is even, it follows $T(n+s+1, 1) = 2(s+1) < \cdots < T(n+t, 1)$ and $T(n+t, 1) \geq 2t > 2a$ which is absurd.

(2) If $s \geq 1$ is such that $(n+s,1)$ is the first cell, seen from the top, where $T(n+s,1) = odd < 2s+1$ then from the previous implication, all previous cells in the first column of T satisfy $T(n+1,1) \geq 2, \dots, T(n+s-1,1) \geq 2(s-1)$. Then standard-ness forces $T(n+s,1) = 2(s-1)+1$ and ν even forces $T(n+s+1,1) = 2s < 2(s+1)$. □

Theorem 1. (Main result) Let $T \in LR(\lambda/\mu,\nu)$ with ν even, on the alphabet $[2n]$, and $\ell(\mu) \leq n$. T does not satisfy the Sundaram property if and only if $G_\mu(T)$ is not symplectic.

Moreover, in this case, there exists a unique $t \geq 0$ such that

(1) $n+t+1$ is the minimal row of T where the Sundaram property violation occurs.

(2)
$$T(n+t,1) = 2t,\ T(n+t+1,1) = 2t+1,\ T(n+t+2,1) = 2(t+1), \tag{17}$$
$$T(n+1,1) \geq 2,\ T(n+2,1) \geq 4, \dots, T(n+t+1-2,1) \geq 2(t+1-2). \tag{18}$$

(3) the maximal row of $G_\mu(T)$ where a symplectic violation occurs is among the bottom most $t+1$ cells, $(\ell(\mu),1)$, $(\ell(\mu)-1,1), \dots,$ $(\ell(\mu)-t,1)$ of the first column of $G_\mu(T)$.

Proof. Let $G = G_\mu(T)$ and recall $\ell(\lambda) - n \leq n$. From Corollary 1, let $t \geq 1$ be minimal such that $T(n+t,1) = 2i+1$ for some $0 \leq i < t$.

For readability we start to spelling out the cases $t = 1, 2$.

If $\mathbf{t=1}$, $T(n+1,1) = 1$ and $n = \ell(\mu)$. Since ν is even, $T(n+2,1) = 2$. Hence
$$2 \leq \ell(\lambda) - n \leq n \Rightarrow n = \ell(\mu) \geq 2.$$
We show that $T(n+1,1) = 1$ means a symplectic violation in the cell $(\ell(\mu),1)$ of $G_\mu(T)$.

One has, $\ell(\mu) \geq 2$, and $T(\ell(\mu)+1,1) = 1$, $T(\ell(\mu)+2,1) = 2$ is equivalent to
$$\ell(\mu) = \ell(\mu^{2n}) = \ell(\mu^{2n-1}) = \ell(\mu^{2n-2}) \geq \ell(\mu^{2n-3}) \tag{19}$$
$$\Leftrightarrow \tag{20}$$
$$G(\ell(\mu),1) \leq 2\ell(\mu) - 2 = 2(\ell(\mu)-1) \Leftrightarrow G(\ell(\mu),1) \ngeq 2\ell(\mu) - 1. \tag{21}$$
Hence, $G_\mu(T)$ is not symplectic with bottom most symplectic violation in the cell $(\ell(\mu),1)$.

If $\mathbf{t=2}$, by definition of t, $T(n+2,1) = 3$ and $T(n+1,1) = 2$. On the other hand, ν is even, so $T(n+3,1) = 4$ and $3 \leq \ell(\lambda) - n \leq n$ and thus $n \geq 3$, and
$$T(n+1,1) = 2,\ T(n+2,1) = 3, T(n+3,1) = 4.$$
We show that $T(n+2,1) = 3$ means a symplectic violation in a cell of the first column of $G_\mu(T)$.

Case $n = \ell(\mu) \geq 3$: $T(\ell(\mu)+1,1) = 2$, $T(\ell(\mu)+2,1) = 3$, $T(\ell(\mu)+3,1) = 4$.

This translates to
$$\ell(\mu) = \ell(\mu^{2n}) > \ell(\mu^{2n-1}) = \ell(\mu^{2n-2}) = \ell(\mu^{2n-3}) = \ell(\mu^{2n-4}) \geq \cdots$$
$$\Leftrightarrow$$
$$G(\ell(\mu),1) = 2n \geq 2n-1,\ \ G(\ell(\mu)-1,1) \leq 2n-4 = 2(\ell(\mu)-1) - 2 \ngeq 2(\ell(\mu)-1) - 1$$
So G is not symplectic with bottom most symplectic violation in the cell $(\ell(\mu)-1,1)$.

Case $n = \ell(\mu)+1 \geq 3$ in which case and $\ell(\mu) \geq 2$, $T(\ell(\mu)+1,1) = 1$, $T((\ell(\mu)+1)+1,1) = 2$, $T((\ell(\mu)+1)+2,1) = 3$, $T((\ell(\mu)+1)+3,1) = 4$.

This translates to
$$\ell(\mu) = \ell(\mu^{2n}) = \ell(\mu^{2n-1}) = \ell(\mu^{2n-2} = \ell(\mu^{2n-3}) = \ell(\mu^{2n-4} \geq \cdots$$
$$\Leftrightarrow$$
$$G(\ell(\mu),1) \leq 2n-4 = 2(\ell(\mu)+1) - 4 = 2\ell(\mu) - 2 \ngeq 2\ell(\mu) - 1.$$
Hence, $G_\mu(T)$ is not symplectic with bottom most symplectic violation in the cell $(\ell(\mu),1)$.

Le $\mathbf{t+1} \geq 1$ with $t \geq 0$, be minimal such that $T(n+t+1,1) = 2i+1$ for some $0 \leq i \leq t$. From Corollary 2, this means that $T(n+k,1)$ does not violate the Sundaram property for $1 \leq k \leq t$ and $T(n+k,1) \geq 2k$, for $1 \leq k \leq t$. Therefore, since $T(n+t,1) \geq 2t$, $T(n+t+1,1) \leq 2t+1$ and

$$2 \leq T(n+1,1) < \cdots < T(n+t,1) < T(n+t+1,1) \leq 2t+1,$$

it follows that

$$T(n+t,1) = 2t < T(n+t+1,1) = 2t+1$$

On the other hand, ν is even, so $T(n+t+2,1) = 2(t+1)$. Thus $\mathbf{n+t+1} \geq n+1$, with $t \geq 0$, is the minimal row of T where the Sundaram property violation occurs if and only if

$$T(n+1,1) < \cdots < T(n+t+1-2,1) < T(n+t,1) = 2t,\ T(n+t+1,1) = 2t+1,\ T(n+t+2,1) = 2(t+1),$$
$$T(n+1,1) \geq 2\times 1,\ T(n+2,1) \geq 2\times 2, \ldots, T(n+t+1-2,1) \geq 2(t+1-2).$$

Moreover $t+2 \leq \ell(\lambda) - n \leq n$ and thus $n \geq t+2$.

Case $n = \ell(\mu) \geq t+1+1$:

Let $T(n+1,1) = 2, T(n+2,1) = 4, \ldots, T(n+t-1),1) = 2(t-1)$, and $T(n+t,1) = 2t$, $T(n+t+1,1) = 2t+1$, $T(n+t+2,1) = 2(t+1)$. Note $2, 4, \ldots, 2(t-1)$ are the first positive $t-1$ even numbers.

This translates to

$$\begin{aligned}
&\ell(\mu) = \ell(\mu^{2n}) > \qquad (22)\\
&\ell(\mu^{2n-1}) = \ell(\mu^{2n-2}) >\\
&\ell(\mu^{2n-3}) = \ell(\mu^{2n-4}) >\\
&\vdots\\
&\ell(\mu^{2n-(2(t+1-2)-1)}) = \ell(\mu^{2n-2(t+1-2)}) >\\
&\ell(\mu^{2n-(2(t+1-2)+1)}) = \ell(\mu^{2n-(2(t+1-2)+2)}) = \ell(\mu^{2n-(2(t+1-2)+3)}) = \ell(\mu^{2n-(2(t+1-2)+4)}) \geq \cdots \qquad (23)\\
&\Leftrightarrow\\
&G(\ell(\mu),1) = 2n \geq 2n-1\ , G(\ell(\mu)-1,1) = 2n-2 \geq 2(n-1)-1, \ldots,\\
&G(\ell(\mu)-(t+1-2),1) = 2\ell(\mu) - 2(t+1-2) = 2(\ell(\mu)-(t+1-2)) \geq 2(\ell(\mu)-(t+1-2))-1,\\
&G(\ell(\mu)-(t+1-1),1) \leq 2n - (2(t+1-2)+4) = 2\ell(\mu) - (2(t+1-2)+4) = 2(\ell(\mu)-t)-2\\
&\not\geq 2(\ell(\mu)-t)-1 \qquad (24)
\end{aligned}$$

Note $\ell(\mu^{2n-2(t+1-2)}) = \ell(\mu)-(t+1-2)-1) = \ell(\mu)-(t+1-1) = \ell(\mu)-t$. Furthermore, t is the number of $>$ in (22) before arriving to the flat sequence in (23) which is also the number of the first t non negative even numbers, $t = \#\{0, 2, 4, \ldots, 2(t+1-2)\}$.

Thus $G_\mu(T)$ is not symplectic with bottom most symplectic violation in the cell $(\ell(\mu)-t, 1)$.

In the remaining cases there exists at least one odd number among

$$T(n+1,1) \geq 2\times 1, T(n+2,1) \geq 2\times 2, \ldots, T(n+t-2),1) \geq 2\times(t-2), T(n+t-1),1) \geq 2\times(t-1).$$

This means, at least one of the even numbers $2, 4, \ldots, 2(t-1)$ is replaced by an odd number, that is, for some $1 \leq i \leq t-1$, $2i$ is replaced by $2i+1$ and $2(i+1)$ is preserved in the list. Indeed such odd numbers do not violate the Sundaram condition. Each time we do this we glue two next flats subsequences in (22) and reduce by one unity the number of $>$. The flat tail (23) can be longer but the flat portion $\ell(\mu^{2n-(2(t+1-2)+1)}) = \ell(\mu^{2n-(2(t+1-2)+2)}) = \ell(\mu^{2n-(2(t+1-2)+3)}) = \ell(\mu^{2n-(2(t+1-2)+4)})$ is preserved. Therefore, in the first column of $G_\mu(T)$, from the bottom, the first symplectic violation occurs among the bottom most $t+1$ cells $(\ell(\mu),1)$, $(\ell(\mu)-1,1), \ldots, (\ell(\mu)-t,1)$.

A situation where the bottom most failing is in cell $(\ell(\mu),1)$ of G, is the following

Case $n = \ell(\mu)+t \geq t+2 \Rightarrow \ell(\mu) \geq 2$ in which case one has

$T(\ell(\mu)+1,1)=1$, $T(\ell(\mu)+2,1)=2$, $T(\ell(\mu)+3,1)=3$, $\dots$, $T(\ell(\mu)+t,1)=t$, $T((\ell(\mu)+t)+1,1)=t+1$, $\dots$, $T((\ell(\mu)+t)+t,1)=2t$, $T((\ell(\mu)+t)+t+1,1)=2t+1$, $T((\ell(\mu)+t)+t+2,1)=2(t+1)$.

This translate to a sequence where all $>$ in (22) disappear and we are reduced to a longest final flat sequence in (23)

$$\ell(\mu^{2n})=\ell(\mu^{2n-1})=\cdots=\ell(\mu^{2n-t+1})=\ell(\mu^{2n-t})=\ell(\mu^{2n-t-1})=\ell(\mu^{2n-t-2})=\cdots\geq .$$

□

4.4. **Examples.** In all examples below T is an LR tableau with even weight.

(1) Let $n=5$ and

$$T=\begin{array}{ccccc} \, & \, & \, & \, & 1\\ \, & \, & \, & 1 & 2\\ \, & \, & 1 & 2 & 3\\ \, & 2 & 4 & & \\ \, & 5 & & & \\ \color{red}{\mathbf{1}} & 6 & & & \\ 2 & & & & \end{array} \quad \text{or } T=\begin{array}{ccccc} \, & \, & \, & \, & 1\\ \, & \, & \, & 1 & 2\\ \, & \, & 1 & 2 & 3\\ \, & 2 & 4 & & \\ \, & 5 & & & \\ \color{red}{\mathbf{1}} & 6 & & & \\ 2 & 7 & & & \\ 8 & & & & \end{array} \quad \text{and } G_\mu(T)=\begin{array}{cccc} 4 & \, & \, & \,\\ 5 & \, & \, & \\ 6 & \, & & \\ 7 & & & \\ \color{red}{\mathbf{8}} & & & \end{array} \quad \text{with the first column exhibited} \tag{25}$$

$$\begin{aligned} \ell(\mu)=\ell(\mu^{(10)})=5=\ell(\mu^{(10-1)})&={\color{red}\ell(\mu^{(\mathbf{10-2})})=\mathbf{5}} \\ >\ell(\mu^{(10-3)})&=4\\ >\ell(\mu^{(10-4)})&=3\\ >\ell(\mu^{(10-5)})&=2\\ >\ell(\mu^{10-6})&=1\\ >\ell(\mu^{10-7})&=0\\ &=\ell(\mu^{10-8})=\ell(\mu^{10-9})=0 \end{aligned} \tag{26}$$

T fails the Sundarm property and $G_\mu(T)$ fails the symplectic property (in red).

(2) Let $n=7$,

$$T=\begin{array}{ccccc} \, & \, & \, & \, & 1\\ \, & \, & \, & 1 & 2\\ \, & \, & 1 & 2 & 3\\ \, & 2 & 3 & 4 & \\ \, & 3 & 4 & 5 & \\ \mathbf{1} & 6 & & & \\ 2 & 7 & & & \\ 4 & 8 & & & \\ \mathbf{5} & 9 & & & \\ 6 & 10 & & & \\ \mathbf{11} & & & & \\ 12 & & & & \end{array} \qquad G_\mu(T)=\begin{array}{cccc} 5 & \, & \, & \,\\ 6 & \, & \, & \\ 7 & \, & & \\ 8 & & & \\ 12 & & & \end{array} \tag{27}$$

$$\begin{aligned}
\ell(\mu) = \ell(\mu^{(14)}) = 5 = \ell(\mu^{(14-1)}) = 5 = \ell(\mu^{(14-2)}) = 5 \\
> \ell(\mu^{14-3}) = 4 \\
= \ell(\mu^{14-4}) = 4 = \ell(\mu^{14-5}) = 4 = \ell(\mu^{14-6}) = 4 \\
> \ell(\mu^{14-7}) = 3 \\
> \ell(\mu^{14-8}) = 2 \\
> \ell(\mu^{14-9}) = 1 \\
> \ell(\mu^{14-10}) = 0 \\
= \ell(\mu^{14-11}) = \ell(\mu^{14-12}) = \ell(\mu^{14-13}) = 0
\end{aligned}$$

For $n = 7$, T satisfies the Sundaram property and G is symplectic.

(3) For $n = 5$

$$T = \begin{array}{|c|c|c|c|c|}
\hline
 & & & & 1 \\ \hline
 & & & 1 & 2 \\ \hline
 & & 1 & 2 & 3 \\ \hline
 & 2 & 3 & 4 \\ \cline{1-4}
 & 3 & 4 & 5 \\ \cline{1-4}
\mathbf{1} & 6 \\ \cline{1-2}
2 & 7 \\ \cline{1-2}
4 & 8 \\ \cline{1-2}
\mathbf{5} & 9 \\ \cline{1-2}
6 & 10 \\ \cline{1-2}
\end{array}
\qquad G_\mu(T) = \begin{array}{|c|c|c|c|}
\hline
1 & & & \\ \hline
2 & & \\ \cline{1-3}
3 & \\ \cline{1-2}
4 \\ \cline{1-1}
8 \\ \cline{1-1}
\end{array} \tag{28}$$

$$\ell(\mu) = \ell(\mu^{(10)}) = 5 = \ell(\mu^{(10-1)}) = \ell(\mu^{(\mathbf{10-2})}) = \mathbf{5} \tag{29}$$

$$> \ell(\mu^{(10-3)}) = 4 \tag{30}$$

$$= \ell(\mu^{(10-4)}) = \ell(\mu^{(10-5)}) = \ell(\mu^{\mathbf{10-6}}) = \mathbf{4} \tag{31}$$

$$> \ell(\mu^{\mathbf{10-7}}) = \mathbf{3} \tag{32}$$

$$> \ell(\mu^{\mathbf{10-8}}) = \mathbf{2} \tag{33}$$

$$> \ell(\mu^{10-9}) = 1 \tag{34}$$

$T(5+1,1) = 1 \not\geq 2\times 1+1 = 3$, $T(5+4,1) = 5 \not\geq 2\times 4+1$ and T fails the Sundaram property, and $G_\mu(T)$ is not symplectic.

For $n = 6$, T is not Sundaram: if $T(6+1,j) = odd$ for some j then $odd \geq 3 > 2 = T(6+1,1)$; $T(6+2,1) = 4even$, and if $T(6+2,j) = odd \geq 5 \geq 2\times 2+1$ for some j; $\mathbf{T(6+3,1) = 5 \not\geq 2\times 3+1}$; $T(6+4,1) = 6$, $T(6+5) = 11 \geq 2\times 5+1$ and $G(T) = \begin{array}{|c|c|c|c|}
\hline
3 & & & \\ \hline
4 & & \\ \cline{1-3}
5 & \\ \cline{1-2}
\mathbf{6} \\ \cline{1-1}
10 \\ \cline{1-1}
\end{array}$ is not symplectic:

$i = 4$, $\quad G(3,1) = 5 = 2\times 3-1$; $\mathbf{G(4,1) = 6 \not\geq 2\times 4-1}$, $\quad G(5,1) = 10 > 2\times 5-1$.

$$
\begin{aligned}
\ell(\mu) = \ell(\mu^{(12)}) = 5 &= \ell(\mu^{(11)}) = \ell(\mu^{(10)}) \\
&> \ell(\mu^{(9)}) = 4 \\
&\qquad = \ell(\mu^{(8)}) = \ell(\mu^{(7)}) = \boldsymbol{\ell(\mu^{6}) = 4} \\
&> \ell(\mu^{5}) = 3 \\
&> \ell(\mu^{4}) = 2 \\
&> \ell(\mu^{3}) = 1 \\
&> \ell(\mu^{2}) = \ell(\mu^{1}) = 0
\end{aligned} \tag{35}
$$

For $n = 7$, T is Sundaram and G is symplectic.

(4) Let $n = 5$ and

$$
T = \begin{array}{|c|c|c|c|c|}
\hline
 & & & & 1 \\ \hline
 & & & 1 & 2 \\ \hline
 & & 1 & 2 & 3 \\ \hline
 & 1 & 3 & 4 \\ \cline{1-4}
 & 2 & 4 & 5 \\ \cline{1-4}
2 & 3 \\ \cline{1-2}
4 & 6 \\ \cline{1-2}
\mathbf{5} & 7 \\ \cline{1-2}
6 & 8 \\ \cline{1-2}
\end{array}
\qquad
G_\mu(T) = \begin{array}{|c|c|c|c|}
\hline
2 & & & \\ \hline
3 & & \\ \cline{1-3}
\mathbf{4} & \\ \cline{1-2}
8 \\ \cline{1-1}
10 \\ \cline{1-1}
\end{array} \tag{36}
$$

$$
\begin{aligned}
\ell(\mu) = \ell(\mu^{(10)}) &= 5 \\
> \ell(\mu^{(10-1)}) &= 4 \\
&= \ell(\mu^{(10-2)}) = 4 \\
> \ell(\mu^{(10-3)}) &= 3 \\
&= \ell(\mu^{(10-4)}) = \ell(\mu^{(10-5)}) = \boldsymbol{\ell(\mu^{10-6}) = 3} \\
> \ell(\mu^{10-7}) = 2 &> \ell(\mu^{10-8}) = 1 > \ell(\mu^{10-9}) = 0
\end{aligned} \tag{37}
$$

References

[ACM25] O. Azenhas, A. Conflitti, and R. Mamede. A uniform action of the dihedral group on Littlewood–Richardson coefficients. *arXiv:2501.01947*, pages 1–59, 2025.

[AKT16] O. Azenhas, R. C. King, and I. Terada. The involutive nature of the commutativity of Littlewood-Richardson coefficients. *arXiv:1603.05037*, pages 1–109, 2016.

[Aze25] O. Azenhas. Skew RSK and the switching on ballot tableau pairs. *arXiv:1808.06095v5*, pages 1–47, 2018, 2025.

[Buc00] A. S. Buch. The saturation conjecture (after A. Knutson and T. Tao). With an appendix by William Fulton. *Enseign. Math.*, 46(1–2)((2)):43—-60, 2000.

[DK08] V. I. Danilov and G. A. Koshevoi. Robinson-Schensted-Knuth correspondence and the bijections of commutativity and associativity. *Izvestiya: Math.*, (72(4)):689–716, 2008.

[HK06a] A. Henriques and J. Kamnitzer. Crystals and coboundary categories. *Duke Math. J.*, 132(2):191–216, 2006.

[HK06b] A. Henriques and J. Kamnitzer. The octahedron recurrence and $\mathfrak{gl}_n$-crystals. *Adv. Math.*, 206(1):211–249, 2006.

[HTW05] R. Howe, E. C. Tan, and J. Willenbring. Stable branching rules for classical symmetric pairs. *Transactions of the American Mathematical Soc.*, 357(4):1601–1626, 2005.

[Kin76] R. C. King. Weight multiplicities for the classical groups. *Group theoretical methods in physics, (Fourth Internat. Colloq., Nijmegen, 1975), Lecture Notes in Phys., Springer, Berlin*, Vol. 50:490–499, 1976.

[KRKV24] Siddheswar Kundu, K. N. Raghavan, V. Sathish Kumar, and Sankaran Viswanath. Saturation for flagged skew Littlewood–Richardson coefficients. *Algebraic Combinatorics*, 7((3)):659—-678, 2024.

[KRV21] Mrigendra Singh Kushwaha, K. N. Raghavan, and Sankaran Viswanath. The saturation problem for refined Littlewood-Richardson coefficients. *Sém. Lothar. Combin., arXiv:2204.03399v2*, 85B(52):1–12, 2021.

[KT25] V. S. Kumar and J. Torres. The branching models of Kwon and Sudaram via flagged hives. *Journal of Algebraic Combinatorics*, 62(5):1–14, 2025.

[KTW04] A. Knutson, T. Tao, and C. Woodward. The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determine facets of the Littlewood–Richardson cone. *J. Amer. Math. Soc.*, 17:19–48, 2004.

[Kwo18a] J.-H. Kwon. Combinatorial extension of stable branching rules for classical groups. *Transactions of the American Mathematical Society*, 370(9):6125–6152, 2018.

[Kwo18b] J.-H. Kwon. Lusztig data of Kashiwara–Nakashima tableaux in types B and C. *Journal of Algebra*, (503):222–264, 2018.

[Lit44] D. E. Littlewood. On invariant theory under restricted groups. *Philos. Trans. Roy. Soc. London Ser. A, Math. Phys. Sci.*, (239–809):387–417, 1944.

[LL20] C. Lecouvey and C. Lenart. Combinatorics of generalized exponents. *International Mathematics Research Notices*, (16):4942–4992, 2020.

[LR34] D. E. Littlewood and A. Richardson. Group characters and algebra. *Phil. Trans. Roy. Soc. London Ser. A*, (233):99–141, 1934.

[NS05] S. Naito and D. Sagaki. An approach to the branching rule from sl2n(c) to sp2n(c) via littelmann's path model. *J. Algebra*, 286(1):187—-212, 2005.

[PV10] I. Pak and E. Vallejo. Reductions of Young tableau bijections. *Siam J. Discrete Math.*, (24):113–145, 2010.

[ST] B. Schumann and J. Torres. A non-levi branching rule in terms of littelmann paths. *Proc. Lond. Math. Soc.*, (5, (3)).

[Sun86] Sheila Sundaram. *On the combinatorics of representations of $Sp(2n, \mathbb{C})$. PhD thesis.* Massachusetts Institute of Technology. 1986.

[Sun90] S. Sundaram. Tableaux in the representation theory of the classical lie groups. *Invariant theory and tableaux (Minneapolis, MN, 1988) IMA Vol. Math. Appl., 19, Springer, New York,*, pages 191–225, 1990.

[TKA18] I. Terada, R. C. King, and O. Azenhas. The symmetry of Littlewood-Richardson coefficients: a new hive model involutory bijection. *SIAM Journal on Discrete Mathematics*, 32(4):2850–2899, 2018.

[TY08] H. Thomas and A. Yong. An S_3–symmetric Littlewood–Richardson rule. *Math. Res. Lett.*, (15):1027–1037, 2008.

[Wat25] H. Watanabe. Symplectic tableaux and quantum symmetric pairs. *J. Comb. Algebra DOI 10.4171/JCA/113*, page 57, 2025.

University of Coimbra, CMUC, Department of Mathematics, Portugal
Email address: oazenhas@mat.uc.pt