MATRIX POLYNOMIALS WHOSE EIGENVALUES ARE THE DIFFERENCES OF THE EIGENVALUES OF A GIVEN MATRIX POLYNOMIAL

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Abstract. Matrix roots of matrix polynomials and block-eigenvalues of matrices partitioned into blocks are considered in this paper. We address the questions of root-finding, root-localization and root-separation for polynomials over matrices by using matricial norms of matrices and matrix norms of matrices. Block-versions of Cayley-Hamilton Theorem and Leverrier-Faddeev Algorithm and, as well, the Kronecker Sum of matrices partitioned into blocks assist in building a matrix polynomial ($\lambda-$matrix) whose eigenvalues are the differences of the eigenvalues of a given matrix polynomial ($\lambda-$matrix). Some numerical examples are presented in the context of second degree polynomials over commutative matrices.

1. Introduction

In the present paper, we deal with polynomials having matrix coefficients and with matrices partitioned into blocks.

We are concerned with problems arising in two types of matrix polynomials

$$M(X) = X^n + M_1 X^{n-1} + \cdots + M_{n-1} X + M_n$$

and the corresponding $\lambda-$matrix

$$M(\lambda) = I\lambda^n + M_1 \lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n.$$

In much of the text, the coefficients of the polynomials and blocks of matrices pairwisely commute.

This type of matrices partitioned into commuting blocks has been considered earlier [7, 9, 48].

With respect to matrix polynomials, two approaches are taken into consideration: to look for matrix roots (or solvents) or for eigenvalues (or latent roots).

With respect to matrices partitioned into blocks, two approaches are taken into consideration: to look for block-eigenvalues or for (scalar) eigenvalues.

For the sake of simplicity, in the rest of this paper, we just use the concepts of matrix root and eigenvalue.

Matrix roots of matrix polynomials and block-eigenvalues of partitioned matrices are intertwined [3].

The eigenvalues of matrix roots are eigenvalues of matrix polynomials ($\lambda-$matrices) [3, p. 3 and p. 15].

2020 *Mathematics Subject Classification.* 15A18, 15A27, 65F15, 65Y20.

Key words and phrases. Matrix Polynomials ($\lambda-$matrices), Block-Eigenvalues, Block-Kronecker Sum, Block-Leverrier-Faddeev Algorithm, Block-Cayley-Hamilton Theorem.

M. A. Facas Vicente acknowledges financial support by the Centre for Mathematics of the University of Coimbra (CMUC, https://doi.org/10.54499/UID/00324/2025) under the Portuguese Foundation for Science and Technology (FCT), Grants UID/00324/2025 and UID/PRR/00324/2025 and is partially supported by Instituto de Engenharia de Sistemas e Computadores de Coimbra - INESC Coimbra (https://doi.org/10.54499/UID/00308/2025) under the Portuguese Foundation for Science and Technology (FCT), Grant UID/00308/2025.

The eigenvalues of block-eigenvalues of matrices partitioned into blocks are (scalar) eigenvalues of such matrices [3, Theorem 8.1, p. 73]. A comprehensive treatment of matrix roots has been considered [3, 4, 5].

Regarding the computation of matrix roots, in [3, p. 45] is presented an algorithm that uses a generalization of a much referred to algorithm of J. Sebastião e Silva [33].

The concept of matrix root of a matrix polynomial turned to be useful when dealing with the computation of the zeros of quaternionic polynomials, by adapting the referred to J. Sebastião e Silva's algorithm [34].

Theory and computation of block-eigenvalues (and block-eigenvectors) has seen a certain development based on [3] and [21].

Block-eigenvalues and matrix roots are useful because they have all the spectral informations of the block-matrix [28, 29].

Computation of matrix roots, block-eigenvalues and of eigenvalues is sometimes cumbersome, even due to theoretical factors: there exist matrix polynomials with no matrix roots [3].

In practical problems, it may suffice to have a localization of the eigenvalues of the matrix polynomials or the eigenvalues of the associated block-companion matrices. A steady development in the field of localization re-started with the paper [11]. A recent approach, in the context of matrix polynomials with commuting coefficients, is [1]. Both [11] and [1] overlooked results obtained almost thirty years earlier in [38, 39, 40, 41, 42, 43].

Recent work [18, p. 78] in Biomedicine explored the use block-decomposition of partitioned matrices. The papers [29] and [45] are mentioned in this work, where it is stated that block-diagonalization and block-eigenvalue decomposition has not been used to a great extent in biomedical signals analysis.

The study of polynomial matrices keeps being actively pursued (as in [17] and [49]).

Most of the results connecting matrix roots of matrix polynomials and block-eigenvalues of block-companion matrices are established in [3]. The connection between eigenvalues of matrix polynomials and eigenvalues of the associated block-companion matrix is found in [2]. Further work on matrix roots includes [4, 5, 28, 29, 41].

Block-eigenvalues of general matrices (not exclusively in block-companion form) partitioned into blocks were dealt with in [27].

When specifying for matrices partitioned into pairwisely-commuting blocks some steps forward were taken [20, 44, 45].

In some sections of the present paper, commutativity of blocks of the partitioned matrices is an essential condition. In the context of commutativity, we are able to exhibit the characteristic (matrix) polynomial of a general square matrix partitioned into commuting blocks. The obtained matrix polynomial $M(X)$ helps in finding results on the associated matrix polynomial ($\lambda-$matrix) $M(\lambda)$.

The block-Cayley-Hamilton Theorem is applied twice to the characteristic (matrix) polynomial: both for the given matrix A and for its block-companion matrix.

When dealing with matrix polynomials ($\lambda-$matrices), we have to tackle the very same questions that occur when dealing with scalar (real or complex) polynomials. That is to say we need to be aware of problems on root-finding, root-localization and root-separation.

The paper is organized as follows. Section 1: introduction. Section 2: fundamental concepts and extensions for matrix polynomials of classical results on (scalar) polynomials. Section 3: on eigenvalues finding, eigenvalues localization and eigenvalues separation. Section 4: the main results in the commutative context. Section 5: illustrative numerical examples. Section 6: concluding remarks.

2. Fundamental Auxiliary Concepts and Results

In this section, we treat concepts and results that extend, to the context of polynomials with matrix coefficients and of matrices partitioned into blocks, classic results for (scalar) polynomials. Fundamental references are [3, 4].

We address the existence of matrix roots and of block-eigenvalues: all block-matrices have block-eigenvalues; there exist matrix polynomials with no matrix roots (see [3, Theorem 2.6, p. 23] and [47]). For example, $X^2 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has no matrix roots [47]. Hence, the Fundamental Theorem of Algebra is not valid for matrix polynomials.

The block-companion matrix C defined by

$$C = \begin{bmatrix} 0 & I_s & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_s \\ -M_n & -M_{n-1} & -M_{n-2} & \cdots & -M_2 & -M_1 \end{bmatrix} \in M_{ns,ns}\left(\mathbb{R}\right)$$

is associated with the matrix polynomial

$$M\left(X\right) = X^n + M_1 X^{n-1} + \cdots + M_{n-1}X + M_n$$

as well as with the matrix polynomial ($\lambda-$matrix)

$$M\left(\lambda\right) = I\lambda^n + M_1\lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n.$$

2.1. **Matrix Roots Revisited: context and applications.**

Let us consider the matrix polynomial

$$M\left(X\right) = X^n + M_1 X^{n-1} + \cdots + M_{n-1}X + M_n.$$

We seek **matrix roots** for the matrix polynomial $M\left(X\right)$, that is to say, matrices S such that

$$M\left(S\right) = S^n + M_1 S^{n-1} + \cdots + M_{n-1}S + M_n = 0.$$

The concept of complete set [8] of matrix roots plays an important rôle.

Definition 2.1. The matrices $S_1, S_2, \ldots, S_n$ form a complete set of matrix roots of the matrix polynomial $M\left(X\right)$ if the block-Vandermonde matrix

$$V(S_1,\ldots,S_n) = \begin{bmatrix} I & \ldots & I \\ S_1 & \ldots & S_n \\ \vdots & & \vdots \\ S_1^{n-1} & \ldots & S_n^{n-1} \end{bmatrix}$$

is nonsingular.

A complete set of matrix roots is an important tool in block-diagonalization.

Theorem 2.2. *Let $S_1, \ldots, S_n$ be a complete set of matrix roots of the matrix polynomial*

$$M\left(X\right) = X^n + M_1 X^{n-1} + \cdots + M_{n-1}X + M_n.$$

Then

$$C = V(S_1,\ldots,S_n)\,\mathrm{diag}(S_1,\ldots,S_n)V(S_1,\ldots,S_n)^{-1},$$

where C is the block-companion matrix associated to $M(X)$.

Proof. See [16, p. 524]. □

Based upon the preceding block-diagonalization, we are going to use matrix roots in the solution of differential matrix equations and difference matrix equations.

Theorem 2.3. *Consider the differential matrix equation*

$$M\left(\frac{d}{dt}\right)x = x^{(n)}(t) + M_1 x^{(n-1)}(t) + \cdots + M_n x(t) = 0.$$

Let $S_1, \ldots, S_n$ be a complete set of matrix roots of the matrix polynomial

$$M(X) = X^n + M_1 X^{n-1} + \cdots + M_{n-1}X + M_n,$$

with $M_j \in M_{s,s}(\mathbb{R})$, $j = 1, 2, \ldots, n$. Then every solution of

$$M\left(\frac{d}{dt}\right)x = 0$$

is given by

$$x(t) = e^{S_1 t}c_1 + e^{S_2 t}c_2 + \cdots + e^{S_n t}c_n,$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}^s$.

Proof. See [16, p. 525] and [27, p. 46]. □

For difference matrix equations, we have a similar result in terms of power of matrices.

Theorem 2.4. *Consider the difference matrix equation*

$$m_j = x_{j+n} + M_1 x_{j+n-1} + \cdots + M_n x_j = 0, \quad j = 0, 1, \ldots, n.$$

Let $S_1, \ldots, S_n$ be a complete set of matrix roots of the matrix polynomial

$$M(X) = X^n + M_1 X^{n-1} + \cdots + M_{n-1}X + M_n,$$

with $M_j \in M_{s,s}(\mathbb{R})$, $j = 1, 2, \ldots, n$. Then the solution of the difference matrix equation is given by

$$x_j = S_1^j c_1 + S_2^j c_2 + \cdots + S_n^j c_n, \quad j = 0, 1, 2, \ldots, n,$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}^s$.

Proof. See [30, p. 517]. □

2.2. **Block-eigenvalues Revisited: context and applications.** We look for the **eigenvalues** of the matrix polynomial ($\lambda-$matrix)

$$M(\lambda) = I\lambda^n + M_1\lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n,\ M_j \in M_{s,s}(\mathbb{R}),\ j = 1, 2, \ldots, n,$$

that is to say, the roots of the generalized characteristic polynomial

$$\det(M(\lambda)) = \det\left(I\lambda^n + M_1\lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n\right).$$

We need the concept of block-eigenvalue (see [3, p. 72] and [45]).

Definition 2.5. Let A be a block-matrix of order ns. If

$$AX_1 = X_1\Lambda_1,$$

where Λ_1 is a block (a matrix of order s) and the block-vector X_1 (a matrix of type $ns \times s$) is full rank, then Λ_1 is called a **block-eigenvalue** of A and X_1 is the corresponding **block-eigenvector**.

The connection between matrix polynomials and block-companion matrices is established in three aspects: eigenvalues, matrix roots and block-eigenvalues.

Now we present a classical result.

Theorem 2.6. *Let $M(\lambda)$ be the matrix polynomial ($\lambda-$matrix) defined by*

$$M(\lambda) = I\lambda^n + M_1\lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n,$$

with $M_j \in M_{s,s}(\mathbb{R})$, $j = 1, 2, \ldots, n$. Let C be the block-companion matrix

$$C = \begin{bmatrix} 0_s & I_s & 0_s & \cdots & 0_s \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0_s \\ 0_s & \cdots & \cdots & 0_s & I_s \\ -M_n & -M_{n-1} & \cdots & -M_2 & -M_1 \end{bmatrix}.$$

Then the eigenvalues of C are the eigenvalues of $M(\lambda)$.

Proof. See [2, Theorem 1.7, p. 6]. □

With respect to matrix roots, things are more intricate.

A matrix root of a matrix polynomial is a block-eigenvalue of its block-companion matrix.

Theorem 2.7. *If the matrix S_1, of order s, is a matrix root of the matrix polynomial*

$$M(X) = X^n + M_1X^{n-1} + \cdots + M_{n-1}X + M_n,$$

then

$$CX_1 = X_1S_1,$$

where C is the block-companion matrix of $M(X)$ and

$$X_1 = \begin{bmatrix} I_s \\ S_1 \\ \vdots \\ S_1^{n-1} \end{bmatrix}.$$

Proof. See [3, p. 79]. □

Throughout block-eigenvalues, we can construct matrix roots.

Theorem 2.8. *Let C be the block-companion matrix associated with the matrix polynomial*

$$M(X) = X^n + M_1X^{n-1} + \cdots + M_{n-1}X + M_n$$

and let Λ_1 be a block-eigenvalue of C associated with the full rank block-eigenvector X_1, i.e.,

$$CX_1 = X_1\Lambda_1.$$

Under these conditions, if the first block $(X_1)_1$, of X_1, is nonsingular, then

$$S_1 = (X_1)_1\Lambda_1(X_1)_1^{-1}$$

is a matrix root of $M(X)$.

Proof. See [3, p. 85]. □

Definition 2.9. Let A be a block-matrix of order ns and let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ be a set of block-eigenvalues of A. This set is said to be a **complete set of block-eigenvalues** when all the eigenvalues, and respective partial multiplicities, of these block-eigenvalues are the eigenvalues, with the same partial multiplicities, of the matrix A.

A complete set of block-eigenvalues plays an important rôle in block-diagonalization of block-partitioned matrices [27].

Theorem 2.10. *Let Λ_1, Λ_2, ..., Λ_n be a complete set of block-eigenvalues of a partitioned matrix A and let $X_1, \ldots, X_n$ be the associated block-eigenvectors. If $Y = (X_1, X_2, \ldots, X_n)$ is nonsingular, then*

$$Y^{-1}AY = \operatorname{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n).$$

Proof. See [3, Theorem 8.4, p. 77] and [27, Theorem 2]. □

Assuming commutativity of the blocks and of the block-eigenvalues, we have a similar result with the help of a block-Vandermonde matrix [45, Proposition 2.2].

Based again upon a convenient block-diagonalization, we are going to use block-eigenvalues in the solution of differential matrix equations and difference matrix equations.

Theorem 2.11. *Consider the differential matrix equation*

$$M\left(\frac{d}{dt}\right)x = x^{(n)}(t) + M_1 x^{(n-1)}(t) + \cdots + M_n x(t) = 0$$

and let

$$M(X) = X^n + M_1 X^{n-1} + \cdots + M_{n-1}X + M_n$$

be its associated matrix polynomial. Let $\Lambda_1, \ldots, \Lambda_n$ be a complete set of block-eigenvalues of the block-companion matrix C of $M(X)$. Let $X_1, \ldots, X_n$ be the corresponding block-eigenvectors. Then the solution of

$$M\left(\frac{d}{dt}\right)x = 0$$

is given by

$$x(t) = (X_1)_1 e^{\Lambda_1 t} c_1 + \cdots + (X_n)_1 e^{\Lambda_n t} c_n,$$

where $(X_i)_1$ is the top submatrix of s rows of X_i, for $i = 1, 2, \ldots, n$ and $c_1, \ldots, c_n \in \mathbb{R}^s$.

Proof. See [27, p. 48]. □

Theorem 2.12. *Consider the difference matrix equation*

$$m_j = x_{j+n} + M_1 x_{j+n-1} + \cdots + M_n x_j = 0, \quad j = 0, 1, \ldots, n.$$

Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ be a complete set of block-eigenvalues of the block-companion matrix C of the matrix polynomial

$$M(X) = X^n + M_1 X^{n-1} + \cdots + M_{n-1}X + M_n,$$

with $M_j \in M_{s,s}(\mathbb{R})$, $j = 1, 2, \ldots, n$. Then the solution of the difference matrix equation is given by

$$x_j = (X_1)_1 \Lambda_1^j c_1 + (X_2)_1 \Lambda_2^j c_2 + \cdots + (X_n)_1 \Lambda_n^j c_n, \quad j = 0, 1, 2, \ldots, n,$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}^s$ are arbitrary and $(X_i)_1, i = 1, 2, \ldots, n$, is the $s \times s$ submatrix in the top of the block-eigenvector X_i.

Proof. See [30, p. 519]. □

3. On Eigenvalues Finding, Eigenvalues Localization and Eigenvalues Separation

In order to find upper bounds for the eigenvalues of partitioned matrices, use is made of matricial norms. A matricial norm $\mu(A)$ of a square matrix A is a non-negative matrix, which obeys the usual properties of a norm.

A matricial norm $\mu(A)$ of a matrix A is obtained in the following way: we partition the given square matrix A into blocks, where the diagonal blocks are square (not necessarily of the same order); then we take a subordinate matrix norm of each block [6].

There is a relation between the spectral radius $\rho(A)$ and the spectral radius $\rho(\mu(A))$ of the matricial norm $\mu(A)$ and, as well, the matrix norm $\|A\|_j$, $j = 1, 2, \infty$. Since

$$\rho(A) \leq \|A\|_j$$

and

$$\rho(A) \leq \rho(\mu(A)),$$

we get

$$\rho(A) \leq \|\mu(A)\|_j,$$

for $j = 1, 2, \infty$.

3.1. Eigenvalues Finding.

Eigenvalues Finding. The eigenvalues of matrix roots are eigenvalues of matrix polynomials ($\lambda-$ matrices) [3]. The eigenvalues of block-eigenvalues of matrices partitioned into blocks are (scalar) eigenvalues of such matrices [3, Theorem 8.1, p. 73].

The above assertions are the motivation for finding matrix roots and block-eigenvalues.

On matrix roots, earlier references are [3, 4, 5]. Further work has been developed in [29, 28, 30, 22]. On block-eigenvalues, a seminal paper is [3]. Further work has been developed in [44, 29, 27, 20, 30, 21, 22].

Concerning root-finding we can use several approaches.

3.1.1. The zeros of $\det(M(\lambda))$ are, by definition, the eigenvalues of the matrix polynomial ($\lambda-$matrix) $M(\lambda)$. In order to find $\det(M(\lambda))$, we proceed as follows. Considering a matrix polynomial $M(\lambda)$ as a square matrix whose elements are polynomials and noticing that scalar polynomials are associated to adequate lower triangular Toeplitz matrices, we may accomplish the computation of $\det(M(\lambda))$ by setting the problem into the context of matrices partitioned into commuting blocks [46, pp. 130-132]. Once getting $\det(M(\lambda))$ – a scalar polynomial – we dispose of a vast arsenal for computing the zeros of scalar polynomials [23, 24].

3.1.2. Considering the use of a complete set of matrix roots $S_1, S_2, \ldots, S_n$ of a matrix polynomial $M(X) = X^n + M_1X^{n-1} + \cdots + M_{n-1}X + M_n$ and reminding that every eigenvalue of each S_j, $j = 1, \ldots, n$, is an eigenvalue of $M(\lambda)$ [3, p. 3 and p. 15].

3.1.3. Considering the use of a complete set of block-eigenvalues $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ and reminding that every eigenvalue of Λ_j, $j = 1, \ldots, n$, is a eigenvalue of $M(\lambda)$ [3, Theorem 8.1, p. 73]. There exists always a complete set of block-eigenvalues [3, Theorem 8.2, p. 74].

3.2. **Eigenvalues Localization.** Concerning root-localization, we may refer to a great deal of results on upper bounds for the eigenvalues of a matrix polynomial ($\lambda-$matrix). These upper bounds are obtained throughout the use of matricial norms of a matrix and matrix norms of a matrix [11, 38].

A lot of upper bounds for the eigenvalues of a matrix polynomial can be obtained either by generalizing results in [6] in the context of complex polynomials or by adapting for general matrices results in [36] obtained for octonionic polynomials.

Next, we have just a sample of the referred to results.

Proposition 3.1. *Let $M(\lambda)$ be the matrix polynomial ($\lambda-$matrix)*

$$M(\lambda) = I\lambda^n + M_1\lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n.$$

Let λ be any eigenvalue of $M(\lambda)$.

Then

$$|\lambda| \leq \max\{2, M + \|M_1\|_1\}$$

and

$$|\lambda| \leq \max\{1 + M, 1 + \|M_1\|_1\},$$

where $M = \max\{\|M_n\|_1, \|M_{n-1}\|_1, \ldots, \|M_2\|_1\}$.

Proof. Take the $1-$norm of each of the four blocks of the block-companion matrix C partitioned as follows

$$C = \left[\begin{array}{cccc|c} 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & I \\ \hline -M_n & -M_{n-1} & \cdots & -M_2 & -M_1 \end{array}\right].$$

We obtain the matricial norm

$$\mu(C) = \left[\begin{array}{cc} 1 & 1 \\ \|-M_n \;\; -M_{n-1} \;\; \cdots \;\; -M_2\|_1 & \|-M_1\|_1 \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ M & \|M_1\|_1 \end{array}\right],$$

where

$$M = \|-M_n \;\; -M_{n-1} \;\; \cdots \;\; -M_2\|_1 = \max\{\|M_n\|_1, \|M_{n-1}\|_1, \ldots, \|M_2\|_1\}.$$

Using matricial norms and matrix norms, we have successively

$$\rho(C) \leq \rho(\mu(C)) \leq \|\mu(C)\|_{1,\infty}.$$

Hence

$$\rho(C) \leq \|\mu(C)\|_\infty := \max\{2, M + \|M_1\|_1\}$$

and

$$\rho(C) \leq \|\mu(C)\|_1 := \max\{1 + M, 1 + \|M_1\|_1\}.$$

□

Next, we present a very simple result.

Proposition 3.2. *Let $M(\lambda)$ be the matrix polynomial ($\lambda-$matrix)*

$$M(\lambda) = I\lambda^n + M_1\lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n.$$

Let λ be any eigenvalue of $M(\lambda)$.

Then

$$|\lambda| \leq \max\left\{\|M_n\|_j, 1 + \|M_k\|_j\right\},$$

$k = 1, 2, 3, \ldots, n-1$*;* $j = 1, 2, \infty$.

Proof. Build the trivial matricial norm $\mu(C)$ by taking the subordinate norm $\|\cdot\|_j$, $j = 1, 2, \infty$, of each block. We obtain a non-negative companion matrix $\mu(C)$. Taking the 1−norm of each one of $n \times n$ blocks, we obtain $|\lambda| \le \max\left\{\|M_n\|_j, 1 + \|M_k\|_j\right\}$. □

3.3. Eigenvalues Separation.

Concerning root-separation, mention is to be made of results by [41] with the help of matricial norms.

Proposition 3.3. *Let $M(\lambda)$ be the matrix polynomial*

$$M(\lambda) = I\lambda^n + M_1\lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n.$$

Let the spread (gap) of $M(\lambda)$ be defined by

$$spr\,(M(\lambda)) = \max|\mu_j - \mu_k|,$$

where μ_j and μ_k are eigenvalues of $M(\lambda)$.
Then

$$spr\,(M(\lambda)) \le \rho\left(\begin{bmatrix} 1+\alpha & 1 \\ \beta & \gamma \end{bmatrix}\right) \le \left\|\begin{matrix} 1+\alpha & 1 \\ \beta & \gamma \end{matrix}\right\|_\infty \le \max\{2+\alpha, \beta+\gamma\},$$

where

$$\alpha = \max_{i=1,2,\ldots,n-1}\{1 + \|M_i\|, \|M_n\|\},$$

$$\beta = \max_{i=2,\ldots,n}\{\|M_i\|\}$$

and

$$\gamma = \max_{i=2,\ldots,n-1}\{1 + \|M_1\| + \|M_i\|, \|M_1\| + \|M_n\|\},$$

where $\|\cdot\|$ stands for the norms $1, 2, \infty$ and with $\rho(A)$ standing for the spectral radius of a square matrix A.

In the next result, we are using the Perron root, the dominant eigenvalue.

Proposition 3.4. *Let $M(\lambda)$ be the matrix polynomial*

$$M(\lambda) = I\lambda^n + M_1\lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n.$$

Let G be the positive matrix

$$G = \begin{bmatrix} 1+\alpha & 1 \\ \beta & \gamma \end{bmatrix},$$

where

$$\alpha = \max_{i=1,2,\ldots,n-1}\{1 + \|M_i\|, \|M_n\|\},$$

$$\beta = \max_{i=2,\ldots,n}\{\|M_i\|\}$$

and

$$\gamma = \max_{i=2,\ldots,n-1}\{1 + \|M_1\| + \|M_i\|, \|M_1\| + \|M_n\|\},$$

where $\|\cdot\|$ stands for the norms $1, 2, \infty$.
Let $r^\#$ be the Perron root of G given by

$$r^\# = \frac{1}{2}\left(\operatorname{tr}(G) + \sqrt{(\operatorname{tr}(G))^2 - 4\det(G)}\right).$$

Then

$$|\mu_j - \mu_k| \le r^\#,$$

where μ_j and μ_k are eigenvalues of $M(\lambda)$.

4. The Commutative Setting: Main Results

In this section – in the context of commutativity – we construct a matrix polynomial ($\lambda-$matrix) $P(\lambda)$, whose eigenvalues are the differences between any two eigenvalues of the matrix polynomial ($\lambda-$matrix) $M(\lambda)$. We use a block-Kronecker sum of two matrices.

Let $P_s(\mathbb{R})$ denote a set of $s-$order square real matrices that commute into pairs and let $M_{p,q}(P_s(\mathbb{R}))$ stand for the set of real matrices partitioned into $p \times q$ blocks each belonging to $P_s(\mathbb{R})$, and, if $p = q = n$, we simply write $M_n(P_s(\mathbb{R}))$.

The **block-determinant** of a matrix $A \in M_n(P_s(\mathbb{R}))$ is the matrix (block) obtained by developing the determinant of A, by considering the blocks as elements; it is denoted as $\det_b(A)$ [45]. The **block-trace** of a matrix $A \in M_n(P_s(\mathbb{R}))$ is the matrix (block) that is the sum of the blocks A_{ii}, $i = 1, \ldots, n$, and is denoted as $tr_b(A)$ [46]. Moreover, $A \otimes B$ is the Kronecker product of the matrices A and B.

In the commutative context, we are able to extend the concept of characteristic polynomial of a matrix [44].

Definition 4.1. The matrix polynomial $\det_b(I_n \otimes X - A) = X^n + M_1X^{n-1} + \cdots + M_{n-1}X + M_n$ is called the **characteristic matrix polynomial** of the matrix $A \in M_n(P_s(\mathbb{R}))$, where the indeterminate X belongs to $P_s(\mathbb{R})$.

Each matrix partitioned into commuting blocks satisfies its characteristic matrix polynomial. The next result is the block-Cayley-Hamilton Theorem [44].

Theorem 4.2. *Let*

$$M(X) = X^n + M_1X^{n-1} + \cdots + M_{n-1}X + M_n$$

be the characteristic matrix polynomial of a matrix A partitioned into commuting blocks. Consider the relation

$$I \otimes M(X) = (I \otimes X)^n + (I \otimes M_1)(I \otimes X)^{n-1} + \cdots + (I \otimes M_{n-1})(I \otimes X) + (I \otimes M_n).$$

Then

$$A^n + (I \otimes M_1)A^{n-1} + \cdots + (I \otimes M_{n-1})A + (I \otimes M_n) = 0.$$

Remark 4.3. Theorem 4.2 plays a rôle in the control of the computational reliability of the block-Leverrier-Faddeev Algorithm.

The following result [46] is an algorithm that provides the matrix coefficients of the characteristic matrix polynomial $M(X) = X^n + M_1X^{n-1} + \cdots + M_{n-1}X + M_n$ and the correspondent coefficients in the block-Cayley-Hamilton Theorem.

Theorem 4.4. *Let $A \in M_n(P_s(\mathbb{R}))$ be a block-matrix. Then*

$$M(X) = X^n + (-E_1)X^{n-1} + \cdots + (-E_{n-1})X + (-E_n), E_i \in P_s(\mathbb{R}),$$

is the characteristic matrix polynomial of A, where the matrix coefficients E_i are constructed by the following algorithm (block-Leverrier-Faddeev Algorithm)

$$\begin{array}{lll}
D_1 = A \rightarrow & E_1 = tr_b(D_1) \rightarrow & B_1 = D_1 - (I_n \otimes E_1) \rightarrow \\
D_2 = AB_1 \rightarrow & E_2 = \frac{1}{2}tr_b(D_2) \rightarrow & B_2 = D_2 - (I_n \otimes E_2) \rightarrow \\
\cdots & \cdots & \cdots \\
D_{n-1} = AB_{n-2} \rightarrow & E_{n-1} = \frac{1}{n-1}tr_b(D_{n-1}) \rightarrow & B_{n-1} = D_{n-1} - (I_n \otimes E_{n-1}) \rightarrow \\
D_n = AB_{n-1} \rightarrow & E_n = \frac{1}{n}tr_b(D_n) \rightarrow & B_n = D_n - (I_n \otimes E_n) = 0_{ns}.
\end{array}$$

Remark 4.5. The matrix polynomial

$$M(X) = X^n + M_1X^{n-1} + \cdots + M_{n-1}X + M_n$$

and the matrix polynomial ($\lambda-$matrix)

$$M(\lambda) = I\lambda^n + M_1\lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n$$

have the same matrix coefficients. Hence, the block-Leverrier-Faddeev Algorithm that was derived for the matrix polynomial $M(X)$ is applied to the matrix polynomial ($\lambda-$matrix) $M(\lambda)$.

The next proposition is the main result of this paper.

Proposition 4.6. *Let the matrix polynomial ($\lambda-$matrix) $M(\lambda)$ be given by*

$$M(\lambda) = I_s\lambda^n + M_1\lambda^{n-1} + \cdots + M_{n-1}\lambda + M_n,$$

with M_j $(j = 1, 2, \ldots, n)$ being real $s \times s$ matrices, $M_j \in M_{s,s}(\mathbb{R})$, and I_s denotes the $s \times s$ unit matrix.

Consider the matrix polynomial ($\lambda-$matrix) $M_1(\lambda)$ given by

$$M_1(\lambda) = I_s\lambda^n - M_1\lambda^{n-1} + \cdots + (-1)^{n-1} M_{n-1}\lambda + (-1)^n M_n.$$

Let C be the block-companion matrix of the matrix polynomial $M(\lambda)$.
Let C_1 be the block-companion matrix of the matrix polynomial $M_1(\lambda)$.
Consider the Kronecker sum

$$C \oplus C_1 = C \otimes I_{ns} + I_{ns} \otimes C_1 \in M_{n^2s^2,n^2s^2}(\mathbb{R}).$$

Let the matrix polynomial $P(\lambda)$ be the characteristic matrix polynomial ($\lambda-$matrix) of the Kronecker sum $C \oplus C_1$.

Then, the eigenvalues of the matrix polynomial ($\lambda-$matrix) $P(\lambda)$ are the differences, taken two by two, between the eigenvalues of the given matrix polynomial $M(\lambda)$.

Proof. We consider the Kronecker sum $C \oplus C_1 \in M_{n^2s^2,n^2s^2}(\mathbb{R})$, given by

$$C\oplus C_1=\begin{bmatrix} \Delta C_1 & \Delta I_{ns} & \Delta 0 & \cdots & \Delta 0 & \Delta 0 \\ \Delta 0 & \Delta C_1 & \Delta I_{ns} & \cdots & \Delta 0 & \Delta 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta 0 & \Delta 0 & \Delta 0 & \cdots & \Delta C_1 & \Delta I_{ns} \\ -M_n \otimes I_{ns} & -M_{n-1} \otimes I_{ns} & -M_{n-2} \otimes I_{ns} & \cdots & -M_2 \otimes I_{ns} & -M_1 \otimes I_{ns} + \Delta C_1 \end{bmatrix},$$

where

$$\Delta A = diag\,\underbrace{(A, A, \ldots, A)}_{s \text{ times})},$$

with

$$C = \begin{bmatrix} 0 & I_s & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_s \\ -M_n & -M_{n-1} & -M_{n-2} & \cdots & -M_2 & -M_1 \end{bmatrix} \in M_{ns,ns}(\mathbb{R})$$

and

$$C_1 = \begin{bmatrix} 0 & I_s & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_s \\ (-1)^{n+1} M_n & (-1)^n M_{n-1} & (-1)^{n-1} M_{n-2} & \cdots & -M_2 & M_1 \end{bmatrix} \in M_{ns,ns}(\mathbb{R}).$$

We notice the following. The eigenvalues of C are the eigenvalues of $M(\lambda)$. The eigenvalues of C_1 are the eigenvalues of $M_1(\lambda)$ and are the negative of the eigenvalues of C [26]. The n^2s^2 differences between the eigenvalues of C are the eigenvalues of the matrix $C \oplus C_1$ [26, 32]. The eigenvalues of the characteristic matrix polynomial ($\lambda-$matrix) $P(\lambda)$ are the eigenvalues of $C \oplus C_1$. $\square$

The matrix polynomial ($\lambda-$matrix) $P(\lambda)$, which is the characteristic matrix polynomial ($\lambda-$matrix) of the Kronecker sum $C \oplus C_1$, is obtained by a block-Leverrier-Faddeev Algorithm ($k < n^2 s$):

$$
\begin{aligned}
&A_1 = C \oplus C_1 = A,\ P_1 = \text{ block-trace}\,(A_1),\ B_1 = A_1 - I_{n^2 s} \otimes P_1 \\
&A_2 = AB_1,\ P_2 = \tfrac{1}{2} \text{ block-trace}\,(A_2),\ B_2 = A_2 - I_{n^2 s} \otimes P_2 \\
&\vdots \\
&A_k = AB_{k-1},\ P_k = \tfrac{1}{k} \text{ block-trace}\,(A_k),\ B_k = A_k - I_{n^2 s} \otimes P_k
\end{aligned}
$$

Algorithm 1: Block-Leverrier-Faddeev Algorithm

Result: Coefficients of the Characteristic Matrix Polynomial of the Kronecker sum $A = C \oplus C_1$

Given a matrix $A \in M_{n^2 s^2, n^2 s^2}(\mathbb{R})$;

$A_1 = A$;

$P_1 = $ block-trace$\,(A_1)$;

$B_1 = A_1 - I_{n^2 s} \otimes P_1$;

for $j = 2 : n$ **do**

$A_j = A_1 * B_{j-1}$;

$P_j = \dfrac{\text{block-trace}\,(A_j)}{j}$;

$B_j = A_j - I_{n^2 s} \otimes P_j$

end

5. Computational Experiments

In this section, we present some computational experiments, on finding the matrix polynomial ($\lambda-$matrix) $P(\lambda)$ whose eigenvalues are the differences between the eigenvalues of a given matrix polynomial ($\lambda-$matrix) $M(\lambda)$. We present cases where the computing of the matrix coefficients of $P(\lambda)$ is stable (subsections 5.1, 5.3.1, 5.3.2, 5.3.3 and 5.3.4). In subsection 5.2 we present a case where finding the matrix coefficients of the looked for matrix polynomial $P(\lambda)$ leads to a not stable procedure.

We do expect some computational perturbation due to two facts: the statement in [13, p. 30, Lemma 1.5.6; p. 88; p. 258] about computational costs and stability in the Leverrier-Faddeev Algorithm and in the Cayley-Hamilton Theorem; the remark and subsequent algorithm by Golub and Van Loan [10, pp. 568-569, Algorithm 11.2.1] on the computational sensitivity on the computing of powers of matrices. The above references [10, 13] justify, by giving the appropriate technical background, the predictable computational drawbacks during our extensive computational experiments involving mainly matrices in companion form.

Even in structured cases, our intensive computational experiments show that we have to pay special attention to the numerical stability of the calculations, mainly when using the Block-Cayley-Hamilton Theorem in order to control the Kronecker sum $C \oplus C_1$ and the computed matrix coefficients of the matrix polynomial $P(\lambda)$.

Even though we did not manage questions of computational costs and stability, our experience leads us to be convinced that working in the block context allows some benefits, either on computational costs and either on computational stability. This goes in line with the assertion in [31, Remark 5.1], where the authors stressed that the block-Leverrier-Faddeev Algorithm is s times faster than the usual Leverrier-Faddeev Algorithm, where s stands for the dimension of the blocks in the block-partitioned matrix.

Along the steps of the block-Leverrier-Faddeev Algorithm, we use matrix norms $\|A\|_{1,\infty} := \max(|a_{ij}|)$, in order to graphically visualize the behaviour of the algorithm.

Our computational wisdom suggests we control the intermediate and the final computed results, namely the Kronecker sum $C \oplus C_1$ and the block-companion matrix F of the sought-after matrix polynomial $P(\lambda)$. In both cases, we (Theorem 4.2) make use of a block-Cayley-Hamilton Theorem.

5.1. **A Good Example.** In the following example, for a given matrix polynomial $M(\lambda)$, we construct (and exhibit) the Kronecker sum $C \oplus C_1$, whose characteristic matrix polynomial is the looked for matrix polynomial $P(\lambda)$.

Let us consider the matrix polynomial

$$M(\lambda) = I_2\lambda^2 + M_1\lambda + M_2 = I_2\lambda^2 + \begin{bmatrix} -14 & 4 \\ -2 & -20 \end{bmatrix}\lambda + \begin{bmatrix} 46 & -34 \\ 17 & 97 \end{bmatrix}.$$

The matrix $C \oplus C_1$, whose eigenvalues are the differences between all the eigenvalues of the matrix polynomial $M(\lambda)$, is given by

$$C \oplus C_1 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-46 & 34 & -14 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-17 & -97 & -2 & -20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -46 & 34 & -14 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -17 & -97 & -2 & -20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-46 & 0 & 0 & 0 & 34 & 0 & 0 & 0 & 14 & 0 & 1 & 0 & -4 & 0 & 0 & 0 \\
0 & -46 & 0 & 0 & 0 & 34 & 0 & 0 & 0 & 14 & 0 & 1 & 0 & -4 & 0 & 0 \\
0 & 0 & -46 & 0 & 0 & 0 & 34 & 0 & -46 & 34 & 0 & 4 & 0 & 0 & -4 & 0 \\
0 & 0 & 0 & -46 & 0 & 0 & 0 & 34 & -17 & -97 & -2 & -6 & 0 & 0 & 0 & -4 \\
-17 & 0 & 0 & 0 & -97 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 20 & 0 & 1 & 0 \\
0 & -17 & 0 & 0 & 0 & -97 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 20 & 0 & 1 \\
0 & 0 & -17 & 0 & 0 & 0 & -97 & 0 & 0 & 0 & 2 & 0 & -46 & 34 & 6 & 4 \\
0 & 0 & 0 & -17 & 0 & 0 & 0 & -97 & 0 & 0 & 0 & 2 & -17 & -97 & -2 & 0
\end{bmatrix}.$$

The characteristic matrix polynomial $P(\lambda)$ of $C \oplus C_1$, obtained through the block-Leverrier-Faddeev Algorithm, is as follows

$$\begin{aligned} P(\lambda) = I_2\lambda^8 &+ \begin{bmatrix} 12 & 16 \\ -8 & -12 \end{bmatrix}\lambda^7 + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\lambda^6 + \begin{bmatrix} -60 & -80 \\ 40 & 60 \end{bmatrix}\lambda^5 \\ &+ \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}\lambda^4 + \begin{bmatrix} 48 & 64 \\ -32 & -48 \end{bmatrix}\lambda^3 + \begin{bmatrix} -12 & 0 \\ 0 & -12 \end{bmatrix}\lambda^2 \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\lambda + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

By using Theorem 4.2, we must obtain

$$(C \oplus C_1)^8 + (I_8 \otimes P_1)(C \oplus C_1)^7 + \cdots + (I_8 \otimes P_7)(C \oplus C_1) + (I_8 \otimes P_8) = 0.$$

Analogously, Theorem 4.2 leads us to verify

$$F^8 + (I_8 \otimes P_1) F^7 + \cdots + (I_8 \otimes P_7) F + (I_8 \otimes P_8) = 0.$$

We are using some graphic illustration that indicates when the block-Leverrier-Faddeev Algorithm must stop (figure 1 shows the maximum absolute value observed on matrix $B_k = A_k - I_{n^2 s} \otimes P_k$ and the same representation in a logarithmic scale).

Figure 1 shows that the algorithm should stop on the 7th step, where the matrix B_7 is the null matrix, this meaning that it should not go beyond the 8th step.

5.2. **A Not So Good Example.** Next, we show an example where one can see the computational perturbation. Consider the matrix polynomial

$$\begin{aligned} M(\lambda) &= I_3\lambda^3 + M_1\lambda^2 + M_2\lambda + M_3 \\ &= I_3\lambda^3 + \begin{bmatrix} -20 & 10 \\ -5 & -35 \end{bmatrix}\lambda^2 + \begin{bmatrix} 120 & -220 \\ 110 & 450 \end{bmatrix}\lambda + \begin{bmatrix} -100 & 1700 \\ -850 & -2650 \end{bmatrix}. \end{aligned}$$

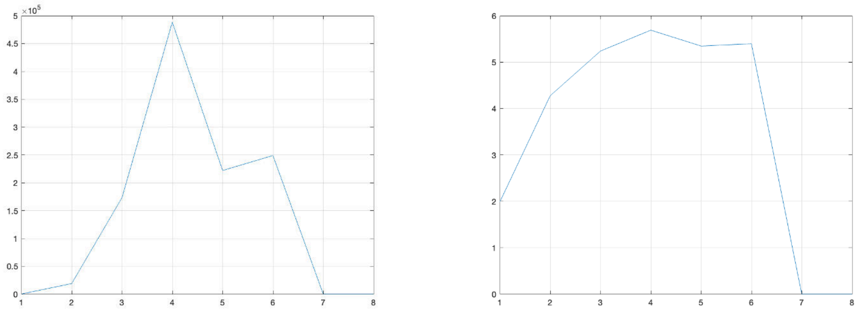

FIGURE 1. Maximum absolute value (left) and its logarithm (right) observed on matrix $B_k = A_k - I_{n^2s} \otimes P_k$ (8 steps).

The characteristic matrix polynomial of $C \oplus C_1$ is

$$P(\lambda) = I_3\lambda^{18} + P_1\lambda^{17} + \cdots + P_{17}\lambda + P_{18},\ P_i \in P_2(\mathbb{R}),$$

with P_i computed by using the block-Leverrier-Faddeev Algorithm. We note that the matrix coefficients P_i are, for all the steps of the block-Leverrier-Faddeev Algorithm, non-null matrices. In the example presented in subsection 5.1, there are some (the tail ones) null matrix coefficients, as theoretically expected. In the present case, at the end of the $n^2s = 18$th step, the matrix coefficient P_{18} is not null:

$$P_{18} = 10^6 \begin{bmatrix} 0.9215 & -1.1724 \\ -0.6231 & 2.9690 \end{bmatrix}.$$

In figure 2, we see that a perturbation occurs at step 16, thus hindering the convergence of the algorithm.

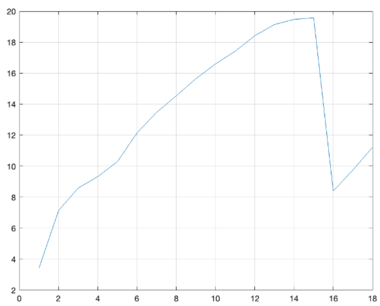

FIGURE 2. Logarithm of the maximum absolute value observed on matrix $B_k = A_k - I_{n^2s} \otimes P_k$ (18 steps).

It is not worth performing further steps, as we see in figure 3.

5.3. **Examples with structured matrices of 4-dimensional commutative algebras.** In the following illustrative examples, we zoom in on very structured matrices belonging to commutative algebras on $\mathbb{R}^{4\times 4}$. They are the Tessarines, the Cotessarines, the Tangerines and the Cotangerines [12], respectively

$$\begin{bmatrix} a & -b & c & -d \\ b & a & d & c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} a & b & -c & -d \\ b & a & -d & -c \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} a & -b & -c & d \\ b & a & -d & -c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix},$$

where a, b, c and d are real numbers.

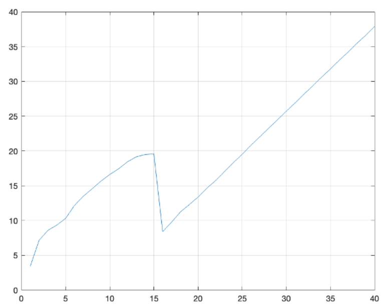

FIGURE 3. Logarithm of the maximum absolute value observed on matrix $B_k = A_k - I_{n^2 s} \otimes P_k$ (40 steps).

5.3.1. *Tessarines.* Let us consider the matrix polynomial

$$M(\lambda) = I_4\lambda^2 + \begin{bmatrix} 1 & -5 & 3 & -4 \\ 5 & 1 & 4 & 3 \\ 3 & -4 & 1 & -5 \\ 4 & 3 & 5 & 1 \end{bmatrix}\lambda + \begin{bmatrix} 2 & 2 & 3 & 3 \\ -2 & 2 & -3 & 3 \\ 3 & 3 & 2 & 2 \\ -3 & 3 & -2 & 2 \end{bmatrix}.$$

The characteristic matrix polynomial $P(\lambda)$ of $C \oplus C_1$, whose eigenvalues are the differences between the eigenvalues of $M(\lambda)$, obtained through the block-Leverrier-Faddeev Algorithm, is as follows

$$P(\lambda) = I_4\lambda^{16} + \sum_{i=1}^{16} P_i\lambda^{16-i},$$

where the matrix coefficients P_i are obtained in a stable way, as shown in figure 4.

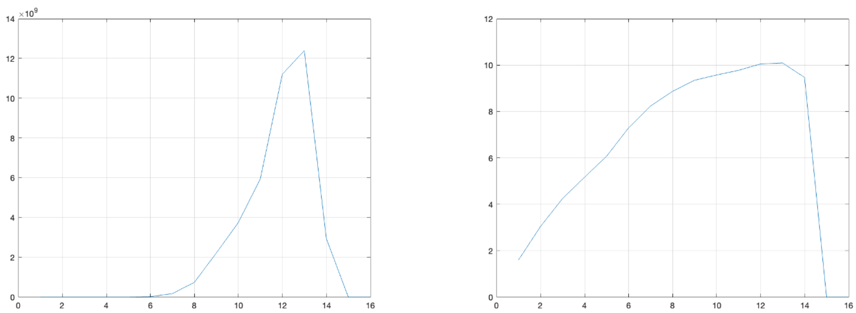

FIGURE 4. Maximum absolute value (left) and its logarithm (right) observed on matrix $B_k = A_k - I_{n^2 s} \otimes P_k$ (16 steps).

5.3.2. *Cotessarines.* For an example with cotessarine coefficients, we use the matrix polynomial

$$M(\lambda) = I_4\lambda^2 + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\lambda + \begin{bmatrix} 2 & -2 & 3 & -3 \\ -2 & 2 & -3 & 3 \\ 3 & -3 & 2 & -2 \\ -3 & 3 & -2 & 2 \end{bmatrix}.$$

The graphics representing the convergence of the block-Leverrier-Faddeev Algorithm are presented in figure 5.

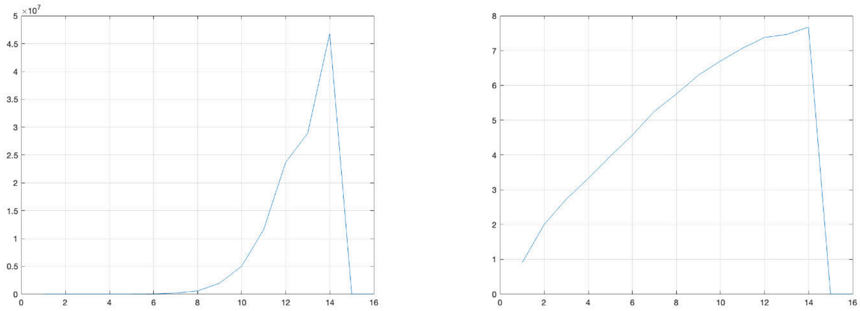

FIGURE 5. Maximum absolute value (left) and its logarithm (right) observed on matrix $B_k = A_k - I_{n^2 s} \otimes P_k$ (16 steps).

5.3.3. *Tangerines.* The use of tangerines for coefficients is here illustrated by an example where

$$M(\lambda) = I_4\lambda^2 + \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \lambda + \begin{bmatrix} 2 & -2 & -3 & 3 \\ -2 & 2 & 3 & -3 \\ 3 & -3 & 2 & -2 \\ -3 & 3 & -2 & 2 \end{bmatrix}.$$

The convergence occurs, as we can see in figure 6.

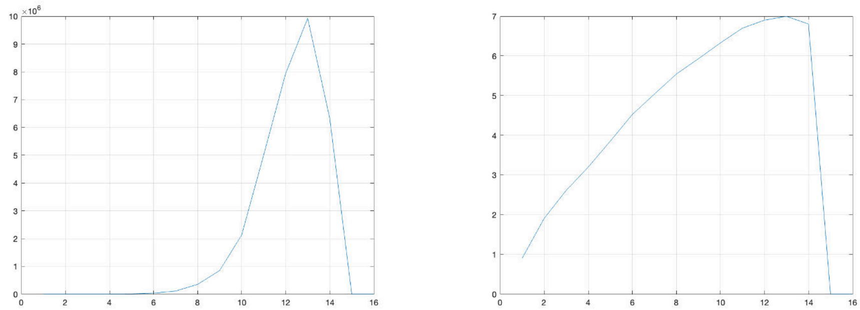

FIGURE 6. Maximum absolute value (left) and its logarithm (right) observed on matrix $B_k = A_k - I_{n^2 s} \otimes P_k$ (16 steps).

5.3.4. *Cotangerines.* In this last case,

$$M(\lambda) = I_4\lambda^2 + \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \lambda + \begin{bmatrix} 2 & 2 & -3 & -3 \\ -2 & 2 & 3 & -3 \\ 3 & 3 & 2 & 2 \\ -3 & 3 & -2 & 2 \end{bmatrix}$$

and the algorithm's evolution is presented in figure 7.

6. Concluding Remarks

A block-generalized Cayley-Hamilton theorem plays a rôle in the present paper. This theorem has been generalized for wider contexts [14, 15, 50].

We dealt with matrix roots of matrix polynomials and block-eigenvalues of block-matrices.

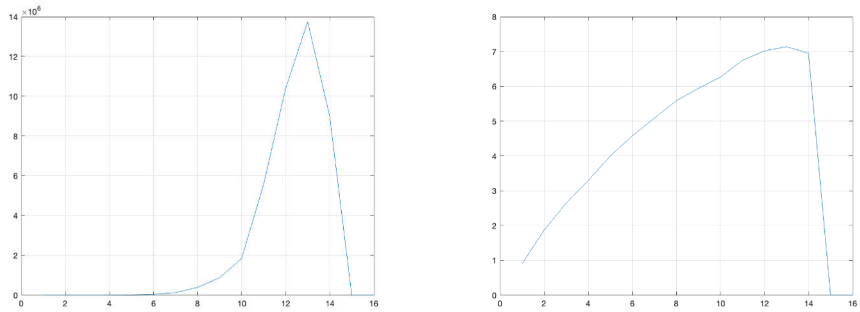

Figure 7. Maximum absolute value (left) and its logarithm (right) observed on matrix $B_k = A_k - I_{n^2 s} \otimes P_k$ (16 steps).

We did not consider the use of generalized Viète formulae — which are considered in [27, 29]. For considerations on Viète formulae, we mention [8].

For solving our problem, we could assume the existence of a complete set of block-eigenvalues [or a complete set of matrix roots], concerning the given matrix polynomial and, then, apply Viète formulae for all the differences, taken two-by-two, of the matrix roots and of the block-eigenvalues.

We stressed the Kronecker sum approach.

There is no balance between the two approaches, for two reasons:

(i) The consideration of a complete set of commuting matrix roots sets two orders of questions – existence and computation;

(ii) The use of the Kronecker sum allows for a more simple starting point, as there is no need for knowing neither any matrix roots nor eigenvalues of the given matrix polynomial.

Eigenvalues of matrix polynomials ($\lambda-$matrices) keep being studied, both in the context of matrix differential equations [22, 28] and in the context of matrix difference equations [30].

When revisiting the expressions of the solutions of differential matrix equations in terms of matrix roots and block-eigenvalues, we did not focus our attention to computational aspects. Computing matrix exponentials is quite a demanding task [25].

Expectedly bounds for the zeros of octonionic (and quaternionic) polynomials did appear [36]. Some extensions for polynomials whose entries are quaternion balls are dealt with [35], where the existence and computation of zeros is shown.

Some considerations about numerical stability and algorithm convergence are in order, alongside what was developed in [37].

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