HALF-STEP-INVARIANT SETS, ADMISSIBLE MAPS, AND CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. This paper revisits the notion of classical orthogonal polynomials from a broader functional-analytic point of view. It is intended neither as a survey of known results nor as a review of the literature, but rather as a conceptual reappraisal of the subject from a perspective in which certain persistent distortions become plainly visible. The theory is developed on subsets of the complex plane that are stable under half-step translations, and both classicality and orthogonality are understood in the continuous dual of a suitable locally convex space of polynomials. The repeated reappearance of ostensibly new families of classical orthogonal polynomials arising from exotic maps, algebraically equivalent families artificially separated, unnecessary parameter restrictions inherited from positive-definite models, apparently distinct phenomena associated with root-of-unity values of q in the q-exponential case, naive $q \to -1$ limits in that same setting, finite truncations of otherwise infinite orthogonal polynomial sequences, and geometric recastings of the local half-step relation in terms of plane conic curves all make the need for a broader structural framework increasingly clear. The present paper seeks to articulate such a framework in a way that allows the reader to distinguish the genuinely new from the merely artificial, without resorting to an exhaustive case-by-case examination of prior work.

Contents

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1. Introduction

Hoffman records in *The Man Who Loved Only Numbers* [11, p. 49] an anecdote about Erdős, recounted by Purdy, who had Erdős number 1. Whether or not one wishes to regard the story as literally true in every detail, it deserves to be taken seriously in spirit: during a visit to Texas A&M in the mid-1970s, Erdős is said to have encountered, over coffee in the mathematics lounge, a problem in functional analysis seemingly far removed from his natural territory. Two analysts had already produced a lengthy thirty-page argument, of which they were understandably rather proud. After asking for the meaning of a few symbols, Erdős reportedly returned with a proof in two lines. Apocryphal or not in its exact literary form, the story points to something mathematically substantial: genuine understanding lies not in passive familiarity with a formalism, but in the ability to see through it to the structure that governs it. That distinction between formal appearance and structural reality is precisely what is at issue in the present paper, in the setting of classical orthogonal polynomials, whatever meaning the reader may presently attach to that expression.

In [5], we argued that the inherited conception of classical orthogonal polynomials reflected in the *NIST Handbook of Mathematical Functions* (2010), in the version of the *NIST Digital Library of Mathematical Functions* consulted at the time of writing, and in much of the surrounding literature, has been distorted by an interpretation that is increasingly difficult to sustain, especially since the features by which classicality is usually recognised are algebraic in nature. In that sense, [5] may be read as exhibiting a phenomenon that becomes progressively more dramatic once one moves beyond the ordinary differential setting treated in the introduction of that work, and even beyond the better-known discrete classical situations examined there in detail, into the more general framework described by Nikiforov and Uvarov in the early 1980s [24]. As [5] makes clear, once one leaves behind the tacit identification of orthogonality with the existence of a representing positive measure, the algebraic structure of the problem comes properly into view. From that standpoint, many distinctions that had seemed canonical reveal themselves to be contingent artefacts of a narrow realisation: algebraically equivalent families are no longer artificially separated, parameter restrictions inherited from positive-definite models lose their conceptual privilege, and the continuous and discrete theories can be understood within a common dual-topological framework.

The issue, however, does not lie solely in the way orthogonality is interpreted. A recent example from the literature, discussed in detail in [4] and closely aligned with the concerns of the present work, illustrates this point quite vividly. Let N be a fixed odd integer, and consider the map

$$W : \mathbb{N}_N = \{0, 1, \dots, N\} \longrightarrow \mathbb{R}, \quad W(s) = s + \frac{1}{2}(\gamma - 1)\big(1 - (-1)^s\big),$$

where $0 < \gamma < 2$. In [34], in the course of an analysis of XX spin chains with perfect state transfer, a physical setting whose details are immaterial here, the authors define a certain positive discrete weight ω on the image of W. The associated positive-definite linear functional $\mathbf{w}$ on the space of complex polynomials may then be written in the form

$$\mathbf{w}(p) = \sum_{s \in \mathbb{N}_N} p\big(W(s)\big)\, \omega\big(W(s)\big).$$

The resulting finite orthogonal polynomial sequence was presented as new and introduced under the name para–Krawtchouk polynomials, the terminology being motivated by an apparent connection with the Krawtchouk polynomials. The suggestion, explicitly advanced by the authors of [34], that these polynomials share features commonly regarded, within the Nikiforov–Uvarov setting, as characteristic of the classical orthogonal polynomial sequences might at first suggest the presence of some structural fissure in that framework

[24, 25, 21, 22, 23, 3], since the map W is not among the admissible forms allowed there. Yet this is not in fact the case, nor is the intuition underlying the claim made in [34] simply mistaken.

The apparent tension is therefore the following: how can these two facts, seemingly irreconcilable, both be true at once? To answer this, no extensive familiarity with the subject is required. Following the line of thought suggested by the Erdős anecdote, the natural question is in fact a simple one: what forms may the map have if the sequence is genuinely to belong to the Nikiforov–Uvarov framework? Once one knows the answer, and in particular that affine maps arise among the admissible possibilities, the matter becomes almost immediate, since the image of W admits a reparametrisation by a map of a type already allowed in that framework. Indeed, define

$$X : \mathbb{S}_N \longrightarrow \mathbb{R}, \quad X(s) = 2s,$$

where

$$\mathbb{S}_N = \left\{0,\, \frac{\gamma}{2},\, 1,\, \frac{\gamma}{2}+1,\, \dots,\, \frac{N-1}{2},\, \frac{\gamma}{2}+\frac{N-1}{2}\right\}.$$

Then

$$W(\mathbb{N}_N) = \{0, \gamma, 2, \gamma+2, \dots, N-1, \gamma+N-1\} = X(\mathbb{S}_N).$$

With the map X in place, and orthogonality understood in a substantially broader framework, there is no longer any compelling reason to keep s tied to the integers, nor to preserve the original positivity-driven restriction $0 < \gamma < 2$, apart from the non-degeneracy conditions required by the chosen functional representation. What initially appeared to be a new construction then begins to reveal its genuine structural meaning. Accordingly, $\mathbf{w}$ can be written as

$$\mathbf{w}(p) = \sum_{s\in\mathbb{S}_N} p\big(X(s)\big)\,\omega\big(X(s)\big). \tag{1.1}$$

Therefore, even the slightest suspicion that the orthogonal polynomial sequence with respect to $\mathbf{w}$ satisfies some property rendering it classical is enough to cast serious doubt on its actual novelty, whether or not one is as yet able to recognise it as such. In the positive finite framework of [34], this identification is in fact rather simple, precisely because the relevant facts are well documented [12]: if the connection with the Krawtchouk polynomials is not especially transparent, the link with the Hahn polynomials is not difficult to discern. In a broader setting, we identified in [5] a much wider class, of which this one is an explicit instance, and showed, through Examples 3.6, 4.4, 5.1, and 8.6, step by step, how naturally and completely it fits into the Nikiforov–Uvarov framework, while at the same time illustrating how the corresponding functional representation may be recovered in a particularly simple manner.

It remains to understand what led the authors of [34] to formulate the matter in precisely those terms. The answer, at least in part, lies in the fact that, once one starts from a prespecified set—in [34], $\mathbb{Z}$—that set acquires an artificial autonomy. It thereby comes to appear as the primary object, whereas the parametrising map is treated as secondary, if not altogether incidental. From a structural point of view, however, the opposite is the case. The initial set is more naturally regarded as

$$\mathbb{S} = \mathbb{Z} \cup \left(\frac{\gamma}{2} + \mathbb{Z}\right),$$

that is, in the non-degenerate case, as the union of two distinct translates of the same arithmetic progression. The finite set $\mathbb{S}_N$ is only the portion on which the non-zero weights sit; the ambient half-step-invariant set is $\mathbb{S}$. Thus the finiteness belongs to the weight, not to the underlying domain. At this structural level, the essential feature is a proper description of the underlying set, one that allows a parametrising map compatible with the theory. This distinction is easy to underestimate, especially because, for many readers, the set $\mathbb{S}$ appears to stand in tension with the traditional way of thinking about the matter. Yet the theory is not without internal latitude: it admits a genuine margin of structural freedom, but not to the point of compromising its overall coherence. In that sense, what is at stake in [34] is not a matter of calculation but one of perspective: the formulas are correct, the orthogonality is genuine, but the structural level at which the sequence ought to be interpreted has been misidentified. The reader may verify that the same phenomenon occurs in [13, 14].

This is also the appropriate place to clarify the status of the conics frequently invoked in the literature in this context. Such objects naturally give rise to an algebraic relation involving neighbouring values of a map, and this relation may be encoded by a curve in the plane. In the family presented in [34], for instance, this phenomenon already appears as soon as the support is reparametrised by the affine map X: the neighbouring half-step values become algebraically governed by X, and one is thereby led, in the general formalism, to a conic relation of the kind underlying what may be called the Magnus conic; in the affine case this relation is degenerate [15, 16]. The point is not that such a conic fails to appear; on the contrary, it arises naturally. However, its role must be interpreted correctly. The conic is not the source of the structure, but one of its algebraic consequences. By itself, it neither determines the domain on which the parameter moves, nor reveals the decomposition of that domain into two translates of the same arithmetic progression, nor decides whether two apparently different supports arise from the same underlying mechanism after reparametrisation. In short, the essential point in the situation under discussion is that the support arises from a set that decomposes into two such arithmetic progressions under an affine map. Once this has been recognised, the conic is seen for what it is: one further trace of the same underlying mechanism.

Another source of confusion appears in the q-setting, where the map is of the form

$$X : U \subseteq \mathbb{C} \longrightarrow \mathbb{C}, \quad X(s) = a\, q^{-s} + b\, q^{s} + c,$$

with $a, b, c \in \mathbb{C}$ and $(a, b) \neq (0, 0)$. The difficulty becomes especially acute when q is a root of unity. This situation has sometimes been treated as though it gave rise, from within the theory itself, to a genuinely different kind of object; see, for instance, [27, 28]. Here, however, some care is needed. The values $q = 1$ and $q = -1$ are not merely special cases of the general torsion phenomenon: the corresponding quadratic and alternating regimes are not obtained by direct limiting procedures applied to the q-exponential formulas, but rather require separate analysis. The case $q = 1$ leads to the quadratic regime, where the q-geometric mechanism has collapsed into an additive, second-degree structure. The case $q = -1$, by contrast, leads to an alternating regime, in which the even and odd parts of the parametrisation interact in a genuinely different way. As we shall see, the latter case has given rise to a substantial and partly self-contained literature, which, from our perspective, is often interpreted at a structurally inappropriate level through its association with certain Dunkl-type operators; see, for instance, [29, 33, 8, 30, 31]. By contrast, when q is a root of unity of order at least 3, one remains within the same general pattern, now seen in a finite-order, or torsion, regime. Indeed, as long as q is not a root of unity, the values of X along a non-degenerate full-step progression typically lie on an infinite q-geometric configuration. When q has finite order, that configuration closes up in the relevant parametrisations, the image may become finite, and truncation phenomena may occur. If attention is confined to the visible support, it is easy to conclude that one has entered a genuinely different regime. Yet that impression is often deceptive. What may have changed is not the nature of the parametrising mechanism, but only the arithmetic regime in which one observes it. For this reason, finite families arising at roots of unity must be handled with particular care: the finiteness of the support and the presence of truncation are real phenomena, but neither of them, taken by itself, signals the emergence of a genuinely new situation. One may simply be seeing the finite shadow of an already existing mechanism. Seen in this way, the constructions studied in [27, 28] are best understood as explicit manifestations of the same general framework in a torsion regime.

The purpose of the present paper is not merely to reorganise known formulas at the level of presentation, nor to restore to what we take to be their proper place all familiar families that reappear under the mask of a specific problem, a particular notation, or a local point of view. It is, rather, to relocate the study of classical orthogonal polynomials to the level at which their underlying structure can be properly understood, and thereby to encourage a thorough re-examination of the subject. We do so by working not with image sets in isolation but with domains stable under half-step translation and with maps defined on them. This shift in viewpoint may seem modest at first, but its consequences are substantial. It explains why supports that look unfamiliar may nevertheless arise from very classical underlying configurations, why local algebraic relations among neighbouring values should be read as consequences rather than as points of departure, and why torsion phenomena in the q-regime belong to the same general picture rather than to a separate universe. At the same time, this relocation of the problem is inseparable from the way orthogonality itself is understood here. Throughout the paper, orthogonality is not tied in advance to positivity, to the existence of a measure, or to a preferred real support. It is formulated, rather, in the continuous dual of a suitable locally convex space (LCS) of polynomials. This is not an artificial generalisation introduced merely for the sake of abstraction; it is the natural level at which the algebraic content of the theory becomes fully visible. Once orthogonality is placed in that dual-topological setting, one can separate what is genuinely structural from what belongs only to a particular realisation: positivity becomes a special circumstance rather than a defining principle, finite and infinite situations can be treated within the same framework, and distinctions inherited from a narrow measure-theoretic presentation cease to obscure algebraic equivalences. In that sense, the topological point of view adopted here is not ornamental. It is part of the mechanism by which the correct structural picture comes into focus.

The examples discussed above should not be read as isolated curiosities, but rather as representative instances of a much broader phenomenon, one that recurs throughout the literature on the subject: known objects, presented under a different parametrisation or under a different visible support, come to be regarded as new because the level at which the comparison ought to be made has shifted unnoticed. We shall return to these examples later, once the necessary framework has been developed to place them where they properly belong. What matters here is that the accumulation of such cases makes it unmistakably clear that the issue is structural rather than accidental. In the theory of orthogonal polynomials, one may easily be misled by the surface form of a parametrisation, by the visible shape of a support, or by a particularly striking local identity, and thereby mistake a disguised reformulation for a genuinely new phenomenon. One should not begin by asking whether a visible support, or a particular weight arising in a positive orthogonality setting, appears to be new. One should begin by asking from what sets it arises, what class of map gives rise to it, and in what sense the orthogonality is to be understood. Only at that level can one distinguish what is intrinsic from what is merely presentational. Much of what follows is guided by this principle.

Section 2 introduces the class of half-step-invariant sets, for a fixed non-zero complex parameter h, which provide the natural setting for the present theory, and shows that such sets decompose canonically into cosets of the additive subgroup $\frac{h}{2}\mathbb{Z}$. Section 3 studies maps defined on these sets and isolates the basic notion of parametrising map, while Section 4 introduces the stronger notion of admissibility, whose role is to ensure that the symmetric divided-difference calculus closes polynomially and thus to make possible an intrinsic treatment of the geometry encoded by neighbouring values. Section 5 contains the corresponding classification theorem: every admissible map is shown to be necessarily of quadratic, alternating, or q-exponential type, with no other behaviour compatible with admissibility. Section 6 turns to the functional-analytic core of the paper. Orthogonality is formulated in the continuous dual of a suitable LCS of polynomials, and classicality is introduced by duality, through the transposed divided-difference and averaging operators. On that basis, we develop in Section 7 the regularity theory attached to the admissible map types and obtain explicit formulas for the recurrence coefficients of the corresponding classical orthogonal polynomial sequences in the quadratic and q-exponential regimes, including the torsion case of finite order; the alternating case is analysed separately in Section 8, since there the mechanism is governed instead by quadratic substitution and a first-order structural constraint. Section 9 then turns to the associated Nikiforov–Uvarov operator, establishing the precise relation between the notion of classicality in that framework and the one proposed here, and concludes by showing that the Hahn property is the point at which the distinction between the admissible map types becomes structurally unavoidable.

2. Half-step-invariant sets

We begin with a small piece of notation, followed by the simple invariance condition on sets that will be used throughout the paper.

Notation 2.1. *Whenever $E \subseteq \mathbb{C}$ is a set containing 0, we write*

$$E^\times = E \setminus \{0\}.$$

Fix $h \in \mathbb{C}^\times$. In this section, $U \subseteq \mathbb{C}$ denotes a non-empty set satisfying

$$U \pm \frac{h}{2} \subseteq U.$$

No further assumptions are imposed on U; in particular, U need not be discrete, need not be contained in $\mathbb{R}$, and need not carry any presupposed order.

We begin with the basic notion underlying the section.

Definition 2.2 (Half-step-invariant set). *Fix $h \in \mathbb{C}^\times$. A non-empty subset $U \subseteq \mathbb{C}$ is called* half-step-invariant *if*

$$U \pm \frac{h}{2} \subseteq U.$$

The dependence on the fixed parameter h will remain implicit throughout.

The following proposition justifies the terminology introduced in Definition 2.2.

Proposition 2.3. *Fix $h \in \mathbb{C}^\times$, and let $U \subseteq \mathbb{C}$ be a half-step-invariant set. Then*

$$U + \frac{h}{2}\mathbb{Z} = U.$$

In particular, U is a disjoint union of cosets of the additive subgroup $\frac{h}{2}\mathbb{Z}$; that is,

$$U = \bigsqcup_{v \in V} \left(v + \frac{h}{2}\mathbb{Z} \right), \tag{2.1}$$

where $V \subseteq U$ is a set of representatives of the distinct cosets of $\frac{h}{2}\mathbb{Z}$ that meet U.

Proof. By half-step invariance, for every $k \in \mathbb{Z}$ we have $U + \frac{h}{2}k \subseteq U$, hence

$$U + \frac{h}{2}\mathbb{Z} \subseteq U.$$

The reverse inclusion is immediate since $0 \in \mathbb{Z}$, so $U \subseteq U + \frac{h}{2}\mathbb{Z}$. Thus $U + \frac{h}{2}\mathbb{Z} = U$. Define a relation $\sim$ on U by declaring that $u \sim v$ if and only if

$$u - v \in \frac{h}{2}\mathbb{Z}.$$

Then $\sim$ is an equivalence relation. The equivalence class of $u \in U$ is initially

$$\left(u + \frac{h}{2}\mathbb{Z}\right) \cap U.$$

Since $U + \frac{h}{2}\mathbb{Z} = U$, the full coset $u + \frac{h}{2}\mathbb{Z}$ is contained in U, and hence the class is precisely $u + \frac{h}{2}\mathbb{Z}$. These classes partition U. Choosing one representative from each class yields (2.1). ☐

Proposition 2.3 shows that every half-step-invariant set U is a disjoint union of cosets of $\frac{h}{2}\mathbb{Z}$. Equivalently, U may be viewed as a union of bi-infinite half-step arithmetic progressions, each lying in an affine line in $\mathbb{C}$ parallel to h.

The following examples illustrate some possible forms of half-step-invariant sets.

Example 2.4 (Half-step-invariant sets)**.** *We retain the notation of Proposition 2.3. We emphasise that the choice of representatives is not canonical: it depends on selecting one element from each coset contained in $U \subseteq \mathbb{C}$.*

(1) Let

$$U = \frac{h}{2}\mathbb{Z}.$$

Then U consists of a single coset, namely $\frac{h}{2}\mathbb{Z}$ itself. Hence the decomposition (2.1) *holds for any choice of a singleton set*

$$V = \{v\} \subseteq \frac{h}{2}\mathbb{Z}.$$

(2) Let $h = 2$, and fix $a, b \in \mathbb{C}$ such that $a - b \notin \mathbb{Z}$. Define

$$U = \left\{\ldots, a - \frac{3}{2}, b - \frac{3}{2}, a - \frac{1}{2}, b - \frac{1}{2}, a + \frac{1}{2}, b + \frac{1}{2}, a + \frac{3}{2}, b + \frac{3}{2}, \ldots\right\}.$$

This displayed enumeration is purely suggestive; U is a set, and no ordering of its elements is intended. Equivalently,

$$U = \left(a - \frac{1}{2} + \mathbb{Z}\right) \cup \left(b - \frac{1}{2} + \mathbb{Z}\right).$$

Since $a - b \notin \mathbb{Z}$, U is the disjoint union of two distinct cosets. Figure 1 is schematic and serves only to indicate the decomposition of U into two half-step arithmetic progressions. Taking

$$a = \frac{1}{2}, \quad b = \frac{\gamma}{2} + \frac{1}{2},$$

or vice versa, one recovers the set $\mathbb{S}$ considered in the introduction. More general configurations of the same kind may readily be envisaged, as the following case shows.

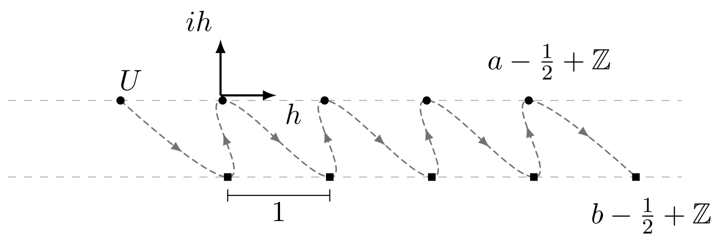

Figure 1. Black discs denote points of the coset $a - \frac{1}{2} + \mathbb{Z}$, and black squares denote points of the coset $b - \frac{1}{2} + \mathbb{Z}$.

(3) Fix $h \in \mathbb{C}^{\times}$ and consider the strip

$$S = \left\{ (x+iy)h : \ 0 \le x < \frac{1}{2}, \ y \in \mathbb{R} \right\}.$$

Let $E \subseteq S$ be non-empty and define

$$U = \bigcup_{v \in E} \left(v + \frac{h}{2}\mathbb{Z} \right).$$

Then U is half-step-invariant. Moreover, the union is disjoint. Indeed, each coset occurring in the union meets S in exactly one point, namely its chosen representative $v \in E$. Existence is immediate from the definition, since

$$v \in \left(v + \frac{h}{2}\mathbb{Z} \right) \cap S$$

for every $v \in E$. For uniqueness, let $v, w \in S$ and assume that $v + \frac{h}{2}\mathbb{Z} = w + \frac{h}{2}\mathbb{Z}$. Then $v - w \in \frac{h}{2}\mathbb{Z}$. On the other hand, $\Re\left(\frac{v}{h}\right), \Re\left(\frac{w}{h}\right) \in \left[0, \frac{1}{2}\right)$, and therefore $\Re(\frac{v-w}{h}) \in (-\frac{1}{2}, \frac{1}{2})$. Since

$$\frac{v-w}{h} \in \frac{1}{2}\mathbb{Z} \subset \mathbb{R},$$

it follows that

$$\frac{v-w}{h} \in \frac{1}{2}\mathbb{Z} \cap \left(-\frac{1}{2}, \frac{1}{2} \right) = \{0\},$$

hence $v = w$. Therefore distinct $v \in E$ lie in distinct cosets, and the sets

$$\left\{ v + \frac{h}{2}\mathbb{Z} : \ v \in E \right\}$$

are pairwise disjoint. The strip S therefore acts as a transversal for the cosets of $\frac{h}{2}\mathbb{Z}$: each coset has exactly one representative in S. This is illustrated schematically in Figure 2. The choice of E determines how U sits inside $\mathbb{C}$. For example, if E is finite, then U is a finite disjoint union of discrete arithmetic progressions; in Figure 2 one has $E = \{v_1, v_2, v_3\}$. Conversely, if E is dense in S, then U is dense in $\mathbb{C}$. Indeed, for every $z \in \mathbb{C}$, there are unique $v \in S$ and $k \in \mathbb{Z}$ such that

$$z = v + \frac{h}{2}k.$$

If E is dense in S, then one may choose a sequence $v_n \in E$ with $v_n \to v$. Hence $v_n + \frac{h}{2}k \in U$ and

$$v_n + \frac{h}{2}k \to z,$$

so U is dense in $\mathbb{C}$.

A concrete dense example is

$$E = \left\{ (r+si)h : \ r, s \in \mathbb{Q}, \ 0 \le r < \frac{1}{2} \right\}.$$

Then E is dense in S, and since $\mathbb{Q} = \left(\mathbb{Q} \cap [0, \frac{1}{2})\right) + \frac{1}{2}\mathbb{Z}$ it follows that $U = \{(p+qi)h : \ p, q \in \mathbb{Q}\}$, which is dense in $\mathbb{C}$. The preceding examples should not, however, be taken to suggest that half-step-invariant sets must essentially resemble a discretisation of $\mathbb{C}$. As the following case shows, they may also arise in a rather different geometric form, in which the underlying coset structure is much less immediately apparent.

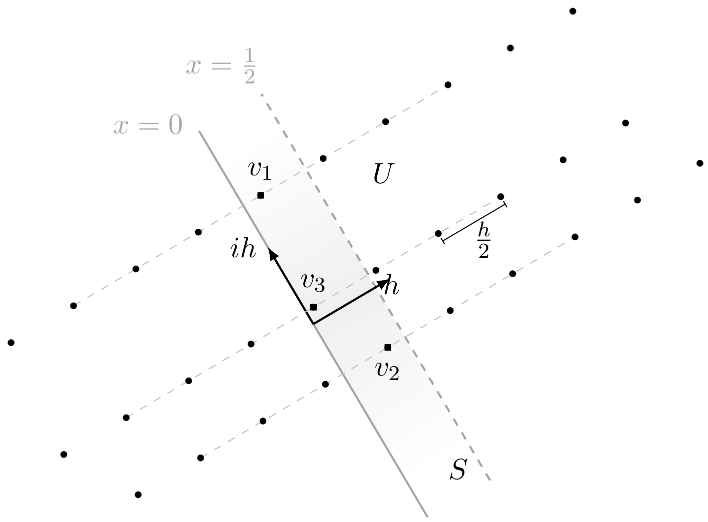

FIGURE 2. The strip S provides one representative for each coset of $\frac{h}{2}\mathbb{Z}$; the figure shows three selected representatives. Black squares denote representatives, black discs denote points in the corresponding arithmetic progressions, and the pale dashed lines denote the affine lines parallel to h passing through these representatives.

(4) *Let $E \subseteq \mathbb{C}^\times$ be non-empty, and define*

$$U = \left\{ z \in \mathbb{C} : \exp\left(\frac{4\pi i}{h} z\right) \in E \right\}.$$

Equivalently, U is the preimage of E under the map

$$z \longmapsto \exp\left(\frac{4\pi i}{h} z\right).$$

Because the complex exponential maps $\mathbb{C}$ onto $\mathbb{C}^\times$, the set U is non-empty whenever $E \neq \varnothing$. Moreover,

$$\exp\left(\frac{4\pi i}{h}\left(z \pm \frac{h}{2}\right)\right) = \exp\left(\frac{4\pi i}{h} z\right),$$

so U is half-step-invariant. By Proposition 2.3, U is therefore a disjoint union of cosets, even though this coset decomposition is not immediately evident from the defining condition. Now specialise to $h = 1$ and take $E = A$, where

$$A = \left\{ w \in \mathbb{C} : \frac{1}{2} < |w| < 2,\ -\frac{\pi}{4} < \operatorname{Arg}(w) < \frac{\pi}{4} \right\},$$

where Arg *denotes the principal argument. Writing $z = x + iy$, the condition* $\exp(4\pi i z) \in A$ *becomes*

$$-\frac{\ln 2}{4\pi} < y < \frac{\ln 2}{4\pi}, \quad x \in \bigcup_{n \in \mathbb{Z}} \left(\frac{n}{2} - \frac{1}{16}, \frac{n}{2} + \frac{1}{16}\right).$$

Thus, in this case,

$$U = \bigsqcup_{n \in \mathbb{Z}} \left(\left(\frac{n}{2} - \frac{1}{16}, \frac{n}{2} + \frac{1}{16}\right) \times \left(-\frac{\ln 2}{4\pi}, \frac{\ln 2}{4\pi}\right) \right).$$

The set U is therefore a periodic array of translates of a single rectangle. This configuration is illustrated schematically in Figure 3: the left panel shows the annular sector A, while the right panel shows its preimage U under $z \mapsto \exp(4\pi i z)$.

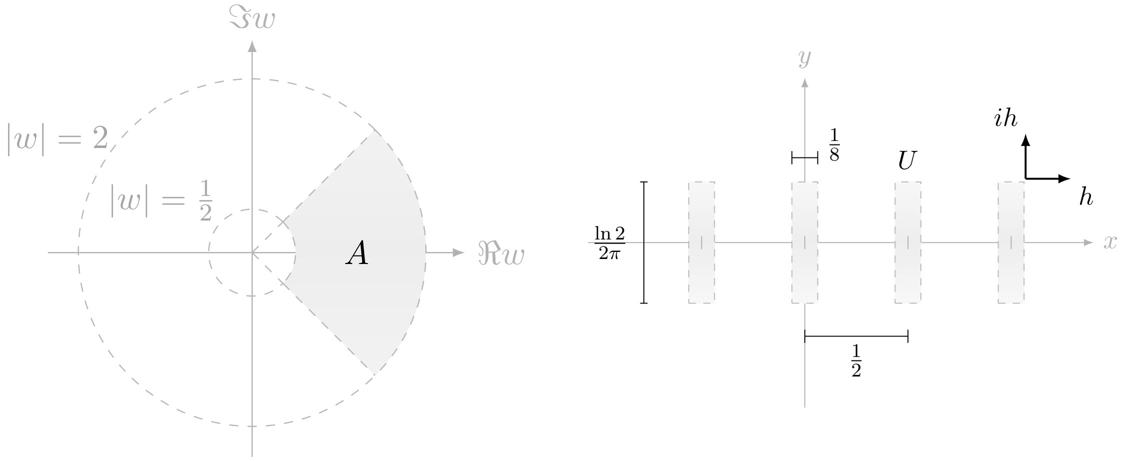

FIGURE 3. For $h = 1$ and $E = A$, the set U is a disjoint union of congruent rectangles.

3. Parametrising maps

Fix $h \in \mathbb{C}^\times$, let $U \subseteq \mathbb{C}$ be a half-step-invariant set, for instance any of those presented in Example 2.4, and let $X : U \to \mathbb{C}$ be a map.

Notation 3.1. *With h, U, and X as above, observe first that, since U is half-step-invariant, it is invariant under translations by $\pm h/2$ and hence also by $\pm h$. Therefore, for every $s \in U$, the points $s \pm h/2$ and $s \pm h$ still belong to U. We may consequently define*

$$Y(s) = X\left(s + \frac{h}{2}\right), \quad Z(s) = X\left(s - \frac{h}{2}\right),$$

$$Y_1(s) = X(s + h), \qquad Z_1(s) = X(s - h).$$

Thus Y and Z denote the half-step neighbours of X, whereas Y_1 and Z_1 denote its full-step neighbours. All identities involving the functions X, Y, Z, Y_1, and Z_1 are understood to hold on U.

We now single out the class of maps on U for which the two half-step neighbours never coincide.

Definition 3.2 (Parametrising map)**.** *Fix $h \in \mathbb{C}^\times$, and let $U \subseteq \mathbb{C}$ be a half-step-invariant set. A* parametrising map *on U is a map $X : U \to \mathbb{C}$ such that $Y - Z$ is nowhere-vanishing on U. This notion is understood relative to the same fixed parameter h that governs the half-step-invariance of U and the definitions of Y and Z. That dependence will remain implicit throughout.*

No condition is imposed in Definition 3.2 on the full-step difference $Y_1 - Z_1$. Without some additional restriction, however, the class of parametrising maps is far too broad.

Example 3.3 (Parametrising maps on the preceding half-step-invariant sets)**.** *The following examples illustrate parametrising maps on the half-step-invariant sets described in Example 2.4.*

(1) Let $h = 2$ and $U = \mathbb{Z}$. Fix $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \frac{1}{2}$, and define

$$X(s) = \left(\alpha - \frac{1}{2}\right) s + \left(\beta - \frac{1}{2}\right)\left(1 - (-1)^s\right) \tag{3.1}$$

for every $s \in U$. Then

$$Y - Z = 2\left(\alpha - \frac{1}{2}\right)$$

throughout U, and hence X is a parametrising map on U. Moreover, setting

$$c = 2\alpha - 1, \quad d = \alpha + 2\beta - \frac{3}{2},$$

we have

$$X(U) = c\,\mathbb{Z} \cup \big(d + c\,\mathbb{Z}\big). \tag{3.2}$$

In particular, if one takes

$$\alpha = \frac{3}{2}, \quad \beta = \frac{\gamma}{2},$$

one recovers

$$X(s) = s + \frac{1}{2}(\gamma - 1)\big(1 - (-1)^s\big),$$

namely, the map discussed in the introduction.

(2) *Let U be as in Example 2.4(2). Fix $c \in \mathbb{C}^\times$ and $d \in \mathbb{C}$, and assume that*

$$b - a \in \left(\frac{d}{c} + \mathbb{Z}\right) \cup \left(-\frac{d}{c} + \mathbb{Z}\right).$$

Then there exists an affine map $X(s) = cs + e$ such that

$$X(U) = c\,\mathbb{Z} \;\cup\; \big(d + c\,\mathbb{Z}\big).$$

Indeed, if $b - a - \frac{d}{c} \in \mathbb{Z}$, one may take

$$X(s) = cs - c\left(a - \frac{1}{2}\right),$$

whereas if $b - a + \frac{d}{c} \in \mathbb{Z}$, one may take

$$X(s) = cs - c\left(a - \frac{1}{2}\right) + d.$$

Then

$$Y - Z = 2c$$

throughout U, and hence X is a parametrising map on U. In either case, the image $X(U)$ is equal to a set of the form displayed in item (1). In particular, taking

$$a = \frac{1}{2}, \quad b = \frac{\gamma + 1}{2}, \quad c = 2, \quad d = \gamma,$$

we have

$$b - a = \frac{\gamma}{2} = \frac{d}{c},$$

and therefore one may choose

$$X(s) = 2s,$$

which is precisely the affine map presented in the introduction as an alternative to the map of item (1).

(3) *Let U be as in Example 2.4(3). Assume that $0 < q < 1$[1], and define*

$$X(s) = q^s, \quad q^s = e^{s \ln q},$$

for every $s \in U$. Then

$$(Y - Z)(s) = q^{s - \frac{h}{2}}\big(q^h - 1\big).$$

[1] The restriction $0 < q < 1$ is imposed here only for simplicity of exposition. It allows one to write $q^s = e^{s \ln q}$ with the ordinary real logarithm, thereby avoiding any discussion of branches.

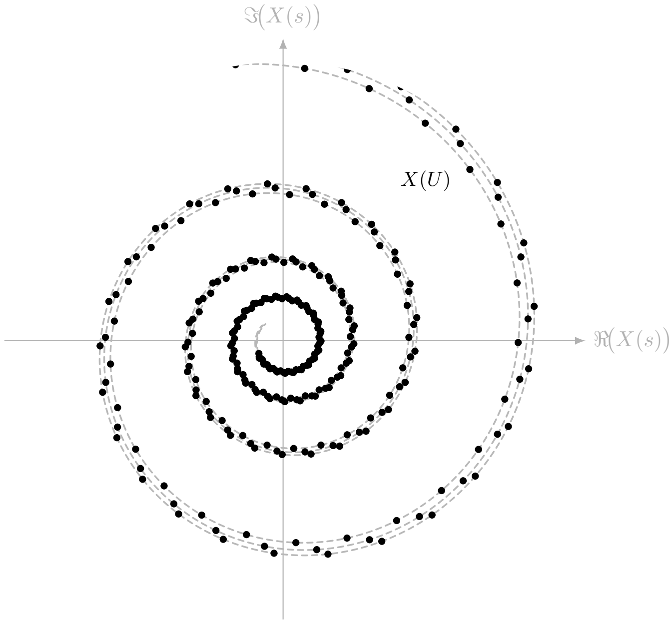

FIGURE 4. The set $X(U)$ for $X(s) = q^s$ with $q = e^{-1/20}$ and $h = 1 + 10i$, where U is the union of the three cosets $v_j + \frac{h}{2}\mathbb{Z}$ determined by the representatives $v_1 = (0.18 + 0.04\,i)h$, $v_2 = (0.38 - 0.03\,i)h$, and $v_3 = (0.08 + 0.09\,i)h$. Each arithmetic progression is mapped onto a discrete subset of a logarithmic spiral; the black points represent the plotted elements of $X(U)$, and the dashed curves are included to indicate the underlying spirals.

Since $q^{s-\frac{h}{2}} \neq 0$ for every $s \in \mathbb{C}$, it follows that $Y - Z$ is nowhere vanishing on U if and only if $q^h \neq 1$. Thus X is a parametrising map on U precisely under this condition. Moreover,

$$X(U) = \bigcup_{v \in E} \left\{ q^s : s \in v + \frac{h}{2}\mathbb{Z} \right\}.$$

Figure 4 is schematic: for each $v \in E$, the coset $v + \frac{h}{2}\mathbb{Z}$ is mapped by $X(s) = q^s$ onto the geometric progression

$$q^v \left(q^{\frac{h}{2}} \right)^{\mathbb{Z}},$$

and $X(U)$ is obtained as the union of these images.

(4) *Let U be as in Example 2.4(4). Define*

$$X(s) = s^2 + \frac{1}{4}\, s,$$

for every $s \in U$. A direct computation gives

$$(Y - Z)(s) = h\left(2s + \frac{1}{4}\right),$$

so, since $h \neq 0$, the function $Y - Z$ has the unique zero $s = -\frac{1}{8}$. Hence X is a parametrising map on U if and only if $-\frac{1}{8} \notin U$. In the particular case

$$U = \bigsqcup_{n \in \mathbb{Z}} \left(\left(\frac{n}{2} - \frac{1}{16}, \frac{n}{2} + \frac{1}{16} \right) \times \left(-\frac{\ln 2}{4\pi}, \frac{\ln 2}{4\pi} \right) \right),$$

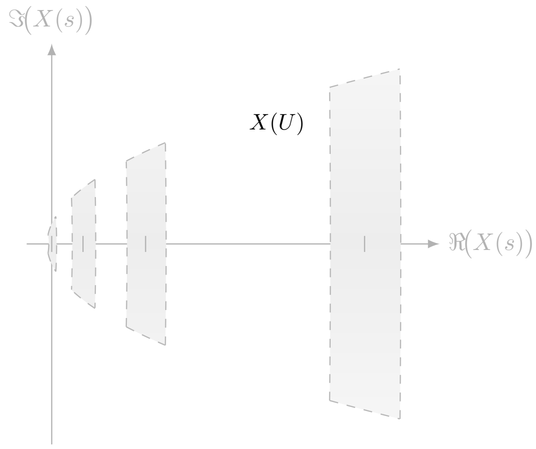

FIGURE 5. The four rectangles appearing in the right-hand panel of Figure 3, mapped into the X-plane by $X(s) = s^2 + \frac{1}{4}\,s$.

one has $-\frac{1}{8} \notin U$. Indeed, if $-\frac{1}{8} \in U$, then for some $n \in \mathbb{Z}$ one would have

$$\left|-\frac{1}{8} - \frac{n}{2}\right| < \frac{1}{16}.$$

However,

$$\left|-\frac{1}{8} - \frac{n}{2}\right| = \frac{|4n+1|}{8} \geq \frac{1}{8} > \frac{1}{16},$$

since $4n+1$ is a non-zero integer. This contradiction shows that $-\frac{1}{8} \notin U$. Consequently, X is a parametrising map on U. Figure 5 shows the image, under X, of the four rectangles displayed in Figure 3. In contrast with the original periodic rectangular decomposition, these images are bounded by quadratic arcs, thereby reflecting the non-affine nature of the parametrisation.

In Examples 3.3(1) and 3.3(2), the parametrising maps are genuinely different, even though their image sets coincide. This may at first suggest that both choices are equally compatible with the theory under consideration here. The admissibility condition introduced in the following section is intended precisely to remove this ambiguity.

4. Admissible maps

The following definition introduces the sole restriction imposed in this work on parametrising maps. It constrains simultaneously the map itself and the half-step-invariant set on which it is defined.

Definition 4.1 (Admissible map[2]). *Fix $h \in \mathbb{C}^\times$, let $U \subseteq \mathbb{C}$ be a half-step-invariant set, and let $X : U \to \mathbb{C}$ be a parametrising map on U. The map X is said to be* admissible on U *if $X(U)$ is infinite*[3] *and, for every polynomial p of degree at least 1, there exists a polynomial Q of degree at most $p-1$ such that*

$$\frac{p(Y) - p(Z)}{Y - Z} = Q(X) \tag{4.1}$$

[2]The term *admissible map* is used here in place of *lattice*, as found in the English-language literature on orthogonal polynomials, since the latter may suggest an algebraic or group-theoretic lattice structure that is not intended here. For that reason, we prefer not to retain the traditional terminology. In the original Russian literature, including [24], one finds instead the word сетка (plural сетки), meaning “grid” or “mesh”. Within the Nikiforov and Uvarov framework, this is arguably the more accurate term.

[3]This guarantees that the polynomial Q in (4.1) is uniquely determined.

throughout U.

Example 4.2 (Admissibility for the preceding parametrising maps)**.** *The following examples determine which of the parametrising maps introduced in Example 3.3 are admissible.*

(1) *In Example 2.4(1), take $h = 2$, so that $U = \mathbb{Z}$. Let $X : U \to \mathbb{C}$ be given by*

$$X(s) = \left(\alpha - \frac{1}{2}\right)s + \left(\beta - \frac{1}{2}\right)\left(1 - (-1)^s\right),$$

for every $s \in U$, where $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq \frac{1}{2}$. Then

$$Y(s) = X(s+1), \quad Z(s) = X(s-1).$$

Specialising p in (4.1) to $p(x) = x^2$, the divided difference reduces to $Y+Z$, so there must exist an affine polynomial Q such that $Q\big(X\big) = Y + Z$. A direct computation shows that

$$(Y+Z)(s) = 2X(s) + (-1)^s(4\beta - 2),$$

for every $s \in \mathbb{Z}$. Since $\alpha \neq \frac{1}{2}$, both $X(2\mathbb{Z})$ and $X(2\mathbb{Z}+1)$ are infinite. Hence the affine polynomial Q satisfies

$$Q(x) = 2x \pm (4\beta - 2)$$

for infinitely many values of x, with the sign $+$ on $X(2\mathbb{Z})$ and the sign $-$ on $X(2\mathbb{Z}+1)$. Since both expressions define the same affine polynomial Q, they must therefore agree identically. Thus $4\beta - 2 = -(4\beta - 2)$, and therefore $\beta = \frac{1}{2}$. It follows that admissibility forces $\beta = \frac{1}{2}$. In that case,

$$X(s) = \left(\alpha - \frac{1}{2}\right)s,$$

which is a non-constant affine map, and hence is admissible by the argument given in the next item.

(2) *Let U be as in Example 2.4(2). Let $X : U \to \mathbb{C}$ be given by*

$$X(s) = cs + e,$$

for every $s \in U$, where $c \in \mathbb{C}^\times$ and $e \in \mathbb{C}$, as in Example 3.3(2). In this case $X(U)$ is infinite, and X is admissible on U. Indeed,

$$Y = X + c, \quad Z = X - c,$$

throughout U, so for every polynomial p of degree at least 1,

$$\frac{p(X+c) - p(X-c)}{2c} = Q(X).$$

To see that Q is a polynomial of degree at most $p-1$, it suffices by linearity to treat monomials. If $p(x) = x^n$ with $n \in \mathbb{N}^\times$, then

$$\frac{(X+c)^n - (X-c)^n}{2c} = \sum_{\substack{j=0 \\ j \text{ odd}}}^{n} \binom{n}{j} X^{n-j} c^{j-1},$$

which is a polynomial in X of degree $n-1$. Hence (4.1) holds throughout U. Since $c \neq 0$, the image $X(U)$ contains the infinite arithmetic progression $X(v + \frac{h}{2}\mathbb{Z})$ for any $v \in U$, and is therefore infinite. The same argument, with c replaced by $c\,h/2$, shows that every non-constant affine map is admissible on any half-step-invariant domain U. In particular, in the situation discussed in the introduction,

$$\mathbb{S} = \mathbb{Z} \cup \left(\frac{\gamma}{2} + \mathbb{Z}\right)$$

is a half-step-invariant set. Therefore $X(s) = 2s$ *is admissible on the whole of* $\mathbb{S}$, *that is, simultaneously on both arithmetic progressions that make up the domain. From this point of view, the sum in* (1.1) *is simply the sum written over those two arithmetic progressions, as indicated schematically by the dashed line in Figure 1. It is here that the structural issue must be stated with some care. Should the aim be merely to recover the set displayed in* (3.2)*, one cannot—within the constraints of admissibility, and hence within any theory that requires it—begin with* $U = \mathbb{Z}$ *and introduce* (3.1) *under the guise of a "bi-lattice". The correct way to formulate the situation is to start from a half-step-invariant set* U *already decomposed into distinct cosets, as in Figure 2, and then map those cosets under the affine transformation. Accordingly, as already observed in the introduction, the relevant decomposition takes place at the level of the half-step-invariant set* U*, which splits into a disjoint union of cosets, and not at the level of the parametrising map, whose form may suggest an interlacing that is not intrinsic to the admissible map structure, as we saw in the preceding example.*

(3) *Let* U *be as in Example 2.4(3), assume that* $0 < q < 1$ *and* $q^h \neq 1$*, and let* $X : U \to \mathbb{C}$ *be given by*

$$X(s) = q^s,$$

for every $s \in U$*, as in Example 3.3(3). Set* $a = q^{h/2}$*. Then*

$$Y = aX, \quad Z = a^{-1}X,$$

throughout U*. Since* $Y - Z = (a - a^{-1})X$*, it follows that*

$$\frac{p(aX) - p(a^{-1}X)}{(a - a^{-1})X} = Q(X).$$

As in the preceding item, to see that the resulting expression is in fact a polynomial in X *of degree at most* $p-1$*, it suffices by linearity to treat monomials. If* $p(x) = x^n$ *with* $n \in \mathbb{N}^\times$*, then pointwise on* U *one has*

$$\frac{p(aX) - p(a^{-1}X)}{(a - a^{-1})X} = \frac{a^n - a^{-n}}{a - a^{-1}} X^{n-1}.$$

Thus, although the quotient is first written with a denominator, it coincides on U *with a genuine polynomial in the indeterminate* X*, whose degree is* $n-1$*. Moreover, since* $a^2 = q^h \neq 1$*, we have* $a \neq a^{-1}$*, and since* $X(s) = q^s \neq 0$ *for every* $s \in U$*, the denominator in the original pointwise expression never vanishes. Hence* (4.1) *holds throughout* U*. Consequently,* X *is admissible on* U *exactly when the remaining requirement in Definition 4.1, namely the infinitude of* $X(U)$*, is also satisfied. Since*

$$U = \bigsqcup_{v \in E} \left(v + \frac{h}{2}\mathbb{Z} \right),$$

it follows that

$$X\left(v + \frac{h}{2}\mathbb{Z} \right) = q^v a^{\mathbb{Z}},$$

and therefore

$$X(U) = \bigcup_{v \in E} q^v a^{\mathbb{Z}}.$$

Thus the infinitude of $X(U)$ *depends on the combined effect of the collection of cosets contained in* U*, and not merely on the behaviour along a single coset. In particular, if* a *is not a root of unity, then each set* $q^v a^{\mathbb{Z}}$ *is infinite, and hence* $X(U)$ *is infinite.*

(4) *Let U be any half-step-invariant set with $-\frac{1}{8} \notin U$, as in Example 2.4(4). Let $X : U \to \mathbb{C}$ be given by*

$$X(s) = s^2 + \frac{1}{4}\,s,$$

for every $s \in U$, as in Example 3.3(4). A direct computation yields

$$Y + Z = 2X + \frac{h^2}{2}, \quad YZ = X^2 - \frac{h^2}{2}\,X + \frac{h^2(4h^2-1)}{64},$$

throughout U, so that $Y+Z$ and YZ are polynomials in X. Again, it suffices by linearity to consider a monomial $p(x) = x^n$ with $n \in \mathbb{N}^\times$. One then has

$$\frac{Y^n - Z^n}{Y - Z} = \sum_{k=0}^{n-1} Y^{n-1-k} Z^k,$$

which is symmetric in Y and Z. It is therefore a polynomial in the elementary symmetric functions $Y+Z$ and YZ. More precisely, the above polynomial is symmetric in Y and Z, homogeneous of total degree $n-1$, and can therefore be written as a linear combination of monomials of the form

$$(Y+Z)^{n-1-2j}(YZ)^j,$$

with j ranging over the integers satisfying $0 \le j \le \lfloor \frac{n-1}{2} \rfloor$. Now $Y+Z$ is affine in X, whereas YZ is quadratic in X. Hence each such monomial has degree at most $(n-1-2j)+2j = n-1$ as a polynomial in X. Hence (4.1) holds throughout U. It remains to verify that $X(U)$ is infinite. Since U is non-empty and half-step-invariant, for every $v \in U$ it contains the whole coset $v + \frac{h}{2}\mathbb{Z}$. On that coset one has

$$X\left(v + k\frac{h}{2}\right) = \left(v + k\frac{h}{2}\right)^2 + \frac{1}{4}\left(v + k\frac{h}{2}\right),$$

which is a non-constant quadratic polynomial in $k \in \mathbb{Z}$, with leading coefficient $h^2/4 \neq 0$. Its set of values is therefore infinite, and hence so is $X(U)$. Thus X is admissible on U. In particular, X is admissible on the half-step-invariant periodic array of rectangles whose visible portion is shown on the right-hand side of Figure 3.

The next example treats an important exponential case that will play a central role below.

Example 4.3. *Let $q \in \mathbb{C}^\times$. Assume that either*

$$q = e^{2\pi i M/\nu}, \quad 1 \le M < \nu, \quad \gcd(M,\nu) = 1, \quad \nu \ge 3,$$

or else

$$|q| \neq 1.$$

Fix a determination of $\log q$, let $h \in \mathbb{C}^\times$, and let $U \subseteq \mathbb{C}$ be a half-step-invariant set. Let $X : U \to \mathbb{C}$ be given by

$$X(s) = \frac{1}{2}\big(q^s + q^{-s}\big), \quad q^s = e^{s \log q},$$

for every $s \in U$. Set $a = q^{h/2}$. Then

$$Y(s) = \frac{1}{2}\big(a\,q^s + a^{-1}q^{-s}\big), \quad Z(s) = \frac{1}{2}\big(a^{-1}q^s + a\,q^{-s}\big),$$

and a direct computation gives

$$(Y - Z)(s) = \frac{1}{2}\,(a - a^{-1})\big(q^s - q^{-s}\big). \tag{4.2}$$

Set

$$K = \frac{\pi i}{\log q}\,\mathbb{Z}.$$

Since $a^2 = q^h$, *it follows from* (4.2) *that* X *is a parametrising map on* U *if and only if*

$$q^h \neq 1, \quad U \cap K = \varnothing.$$

Equivalently, since $q^{2s} = 1$ *if and only if* $q^s = \pm 1$, *and*

$$X(s) = \frac{1}{2}\big(q^s + q^{-s}\big) = \pm 1$$

holds if and only if $q^s = \pm 1$, *this condition may be written as*

$$q^h \neq 1, \quad X(s) \neq \pm 1$$

for all $s \in U$. *Assume henceforth that these conditions are satisfied. Then* $Y + Z$ *and* YZ *are polynomial functions of* X*; indeed,*

$$Y + Z = (a + a^{-1})\, X, \quad YZ = X^2 + \frac{(a - a^{-1})^2}{4}.$$

Consequently, the same argument as in Example 4.2(4) *shows that* X *is admissible on* U *whenever* $X(U)$ *is infinite. We next relate this criterion to the sets introduced in Example 2.4.*

(1) In Example 2.4(1), the coset

$$U = \frac{h}{2}\mathbb{Z}$$

is not suitable in the present setting, since $0 \in U \cap K$, *and hence* $X(0) = 1$. *One therefore replaces* U *by a translated coset*

$$U_\varepsilon = \varepsilon + \frac{h}{2}\mathbb{Z},$$

where $\varepsilon \notin K - \frac{h}{2}\mathbb{Z}$. *Since* $K - \frac{h}{2}\mathbb{Z}$ *is countable, such a choice of* ε *always exists.*

(2) In Example 2.4(2), one argues similarly: it suffices to choose representatives of two distinct cosets such that neither of the corresponding cosets meets K.

(3) In Example 2.4(3), define

$$E' = \left\{ v \in E : \ \left(v + \frac{h}{2}\mathbb{Z}\right) \cap K = \varnothing \right\}, \quad U' = \bigcup_{v \in E'} \left(v + \frac{h}{2}\mathbb{Z}\right).$$

Then $X(s) \neq \pm 1$ *for all* $s \in U'$. *Moreover,* $U' \neq \varnothing$ *whenever*

$$E \not\subseteq K - \frac{h}{2}\mathbb{Z};$$

in particular, this holds whenever E *is uncountable. In the situation represented in Figure 2, passing from* E *to* E' *amounts simply to discarding those representatives whose cosets meet* K.

(4) In Example 2.4(4), with $h = 1$ *and* $E = A$, *the original set meets* K *at* 0. *Indeed, one has* $0 \in K$, *and also* $0 \in U$, *since* $e^{4\pi i \cdot 0} = 1 \in A$. *Thus the corresponding map is not a parametrising map on that original set. One could again restrict to those cosets that avoid* K, *but that configuration will play no role in what follows.*

For admissibility, it remains only to ensure that the image of the chosen set is infinite. To simplify notation, write again U *for any one of the modified sets arising from Example 2.4*(1)–(3). *Since* U *is a union of cosets of* $\frac{h}{2}\mathbb{Z}$, *we have*

$$X(U) = \bigcup_{v \in V} X\left(v + \frac{h}{2}\mathbb{Z}\right),$$

where $V \subseteq U$ is a set of representatives of the cosets contained in U. For each such v,

$$X\left(v + k\frac{h}{2}\right) = \frac{1}{2}\left(a^k q^v + a^{-k} q^{-v}\right),$$

for every $k \in \mathbb{Z}$. If $a = q^{h/2}$ is not a root of unity, then $X\big(v + \frac{h}{2}\mathbb{Z}\big)$ is infinite for every v. Indeed, if

$$\frac{1}{2}\left(a^k q^v + a^{-k} q^{-v}\right) = \zeta,$$

then, upon setting $x = a^k$, one obtains the quadratic equation

$$q^v x^2 - 2\zeta x + q^{-v} = 0.$$

Thus, for every fixed ζ, the quantity $x = a^k$ can take at most two values. Since a is not a root of unity, the map $k \mapsto a^k$ is injective on $\mathbb{Z}$, and therefore only finitely many integers k can give the same value ζ. It follows that $X\big(v + \frac{h}{2}\mathbb{Z}\big)$ is infinite. Consequently, in the situations arising from Examples 2.4(1) and 2.4(2), the corresponding modified domains yield admissible maps whenever $q^h \neq 1$ and $a = q^{h/2}$ is not a root of unity. The same conclusion holds for the restricted set arising from Example 2.4(3), provided $E' \neq \varnothing$. When $a = q^{h/2}$ is a root of unity, the image of each individual coset $v + \frac{h}{2}\mathbb{Z}$ is finite, since, for every $k \in \mathbb{Z}$,

$$X\left(v + k\frac{h}{2}\right) = \frac{1}{2}\left(a^k q^v + a^{-k} q^{-v}\right)$$

depends only on the residue class of k modulo the order of a. This does not, however, preclude admissibility. What matters is the behaviour of the full set U, not merely that of a single arithmetic progression. In that case,

$$X(U) = \bigcup_{v \in V} X\left(v + \frac{h}{2}\mathbb{Z}\right),$$

and $X(U)$ is infinite if and only if infinitely many of the finite sets $X(v + \frac{h}{2}\mathbb{Z})$, are distinct. This possibility is not merely formal, and it already occurs in the setting of Example 2.4(3). Indeed, take $h = 1$, let $q = e^{2\pi i/3}$, and choose $\log q = 2\pi i/3$. Then $a = q^{1/2} = e^{\pi i/3}$ is a root of unity, whereas $q^h = q \neq 1$. Now let

$$E = \{iy :\ y \in \mathbb{R}\} \subset \left\{x + iy :\ 0 \le x < \frac{1}{2},\ y \in \mathbb{R}\right\},$$

and define

$$U = \bigcup_{y \in \mathbb{R}} \left(iy + \frac{1}{2}\mathbb{Z}\right).$$

Since

$$K = \frac{3}{2}\mathbb{Z} \subset \mathbb{R},$$

every coset $iy + \frac{1}{2}\mathbb{Z}$ with $y \neq 0$ avoids K. After removing the single bad coset $\frac{1}{2}\mathbb{Z}$, one obtains a half-step-invariant set U' on which X is parametrising and which still contains infinitely many cosets. For each fixed $y \neq 0$, the set $X(iy + \frac{1}{2}\mathbb{Z})$ is finite, since a has finite order. On the other hand,

$$X(iy) = \frac{1}{2}\left(q^{iy} + q^{-iy}\right) = \cosh\left(\frac{2\pi y}{3}\right),$$

so $X(U')$ is infinite. Thus the torsion case already exhibits the basic phenomenon that will be important later: although the image of each individual coset may be finite, the full image may still be infinite because infinitely many distinct cosets contribute distinct finite pieces.

One may ask from where the map

$$X(s) = \frac{1}{2}\bigl(q^{s} + q^{-s}\bigr)$$

really arises. Its most familiar classical antecedent is the cosine variable

$$\cos\theta = \frac{1}{2}\bigl(e^{i\theta} + e^{-i\theta}\bigr),$$

which already lies behind the Chebyshev polynomials and, more generally, behind the passage from an exponential parametrisation to a variable symmetric under inversion. From this point of view, the map X is more fundamental than any particular family in which it later came to prominence, including the Askey–Wilson polynomials; see [2] and the references therein. For the purposes of the present paper, the historical question, whether for this map or for any other, is secondary. Our concern is not to trace the first appearance of a particular parametrisation, but to explain why such forms arise at all. What matters is that, once admissibility is imposed, the form above emerges as one of the possibilities singled out by the Nikiforov–Uvarov mechanism itself; this is precisely the content of the next section.

5. The form of the admissible maps

The content of this section marks our first contact with the material of the 1983 preprint of Nikiforov and Uvarov [24], closely related to the treatment in their 1984 book [25][4]. Their starting point is the discretisation of the Bochner differential equation: derivatives are replaced by symmetric finite differences, yielding a second-order difference equation that serves both as a discrete analogue of the continuous equation and, in the appropriate regime, as a second-order approximation to it. From the present perspective, that choice should be viewed not as a substantive restriction but rather as an expository normalisation naturally adapted to their way of understanding orthogonality. Once this discretisation has been adopted, Nikiforov and Uvarov ask a structural question: for which underlying discrete domains does the resulting equation retain the expected hypergeometric features, in the sense that the relevant operators preserve polynomiality and the coefficient functions retain the appropriate algebraic form? The following theorem gives the classification of admissible maps, placing the corresponding classification of Nikiforov and Uvarov in the present framework.

Theorem 5.1. *Fix $h \in \mathbb{C}^{\times}$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be an admissible map. Then there exist constants $A, B \in \mathbb{C}$ such that*

$$Y_1 + Z_1 = A\,X + B \tag{5.1}$$

throughout U. In particular, on every full-step arithmetic progression $s_0+h\mathbb{Z} \subseteq U$ contained in U, the values of X satisfy a second-order linear recurrence with constant coefficients, and are therefore determined along that progression by the pair $X(s_0)$ and $Y_1(s_0)$. Moreover, exactly one of the following cases holds.

(i) $A = 2$. *For every $s_0 \in U$ and every $k \in \mathbb{Z}$,*

$$X(s_0 + kh) = a\,k^2 + b_{s_0}\,k + c_{s_0},$$

where

$$a = \frac{B}{2}, \quad b_{s_0} = Y_1(s_0) - X(s_0) - \frac{B}{2}, \quad c_{s_0} = X(s_0).$$

[4]Although the later book written with Suslov [23] gives a refined and very useful exposition of this material, we shall not need to rely on it here. The part relevant to the present paper does not go beyond Section 3.9 of [21, 23], and is already contained in the earlier works of Nikiforov and Uvarov cited above.

(ii) $A = -2$. *Admissibility forces the additional first-order relation*

$$Y_1 + X = \frac{B}{2} \tag{5.2}$$

throughout U, *and consequently, for every* $s_0 \in U$ *and every* $k \in \mathbb{Z}$,

$$X(s_0 + kh) = (-1)^k \, a_{s_0} + b,$$

where

$$a_{s_0} = X(s_0) - \frac{B}{4}, \quad b = \frac{B}{4}.$$

(iii) $A^2 \neq 4$. *Fix* $q \in \mathbb{C}^\times$ *with* $q \neq \pm 1$ *such that* $A = q + q^{-1}$, *and set*

$$c = \frac{B}{2 - A}.$$

Then, for every $s_0 \in U$ *and every* $k \in \mathbb{Z}$,

$$X(s_0 + kh) = a_{s_0} \, q^k + b_{s_0} \, q^{-k} + c,$$

where a_{s_0} *and* b_{s_0} *are given by*

$$a_{s_0} = \frac{Y_1(s_0) - c - (X(s_0) - c) \, q^{-1}}{q - q^{-1}}, \quad b_{s_0} = \frac{(X(s_0) - c) \, q - Y_1(s_0) + c}{q - q^{-1}}.$$

In particular, along each full-step arithmetic progression $s_0 + h\mathbb{Z} \subseteq U$, *the map* X *is necessarily of quadratic type* (i), *alternating type* (ii), *or* q*-exponential type* (iii). *No other dependence of* X *on* s *along a full-step arithmetic progression can be compatible with admissibility. The constants* A *and* B *are global, that is, independent of the choice of* s_0. *By contrast, the coefficients not already fixed globally by* A *and* B *are determined by the initial values along the chosen arithmetic progression and may therefore vary from one arithmetic progression to another.*

The following remark explains the relation with the original Nikiforov–Uvarov recurrence.

Remark 5.2 (Relation with Nikiforov–Uvarov, 1983)**.** *The relation* (5.1) *becomes, upon restriction to a full-step arithmetic progression* $s_0 + h\mathbb{Z} \subseteq U$, *the constant-coefficient recurrence considered by Nikiforov and Uvarov in* [24, (39)]. *Fix* $s_0 \in U$ *and restrict* X *to the full-step arithmetic progression* $s_0 + h\mathbb{Z} \subseteq U$. *If one sets, for every* $k \in \mathbb{Z}$,

$$x_k = X(s_0 + kh),$$

then (5.1) *becomes*

$$x_{k+1} + x_{k-1} = A \, x_k + B,$$

which is exactly the recurrence appearing in [24, (39)].

The following example illustrates the alternating case in Theorem 5.1.

Example 5.3 (Normalised alternating case)**.** *Let* $U \subseteq \mathbb{C}$ *be the dense half-step-invariant set from Example 2.4*(3) *with* $h = 1$, *namely*

$$U = \{p + qi : \; p, q \in \mathbb{Q}\} \subset \mathbb{C}.$$

Let $X : U \to \mathbb{C}$ *be given by*

$$X(s) = e^{\pi i s},$$

for every $s \in U$. *Equivalently, one may write* $X(s) = (-1)^s$, *provided the convention*

$$(-1)^s = e^{\pi i s}$$

is understood. For each fixed $s_0 \in U$, *the restriction of* X *to the full-step arithmetic progression* $s_0 + \mathbb{Z} \subseteq U$ *is given by*

$$X(s_0 + k) = e^{\pi i (s_0 + k)} = (-1)^k X(s_0),$$

so the dependence on the chosen arithmetic progression enters only through the initial value $X(s_0) = e^{\pi i s_0}$*. Moreover, for every* $s \in U$*,*

$$Y(s) = i\,e^{\pi is} = i\,X(s), \qquad Z(s) = -i\,e^{\pi is} = -i\,X(s),$$

$$Y_1(s) = -e^{\pi is} = -X(s), \quad Z_1(s) = -e^{\pi is} = -X(s).$$

Since $Y = iX$ *and* $Z = -iX$*, one has* $Y - Z = 2iX \neq 0$ *throughout* U*. Moreover, writing*

$$p(x) = a(-x^2) + x\,b(-x^2),$$

for polynomials a *and* b*, one finds*

$$\frac{p(Y) - p(Z)}{Y - Z} = b(X^2),$$

so (4.1) *holds with a polynomial in* X *of degree at most* $p - 1$*. Since* $X(U)$ *is infinite, it follows that* X *is admissible on* U*. Furthermore,* (5.1) *and* (5.2) *hold with* $A = -2$ *and* $B = 0$*. Therefore the present map is a normalised realisation of case* (ii) *of Theorem 5.1.*

Figure 6 is schematic. It is drawn for the present case $h = 1$*, so that the half-step arithmetic progressions are horizontal and the transversal strip* S *is the vertical strip*

$$S = \left\{ x + iy : \ 0 \le x < \frac{1}{2}, \ y \in \mathbb{R} \right\}.$$

The black squares represent chosen representatives in S*, while the black discs indicate points in the corresponding half-step arithmetic progressions. The heavier dashed curve is only a schematic device: it indicates how one may pass through representatives of distinct half-step arithmetic progressions, instead of remaining confined to a single progression. It is not meant to be a new geometric object attached to* U*, nor a literal path contained in* U*. This distinction is important in the present alternating case. On a single full-step arithmetic progression* $s_0 + \mathbb{Z}$*, the map* $X(s) = e^{\pi is}$ *takes only the two values* $X(s_0)$ *and* $-X(s_0)$*. Even on a single half-step arithmetic progression* $s_0 + \frac{1}{2}\mathbb{Z}$*, it takes only four values. Thus a restriction to one such progression does not reveal the map as an admissible parametrisation with infinite image. The role of the ambient set* U*, and of the passage through representatives suggested by the figure, is to make visible a domain on which the same formula* $X(s) = e^{\pi is}$ *has infinitely many distinct values. In the present example,*

$$U = \{p + qi : \ p, q \in \mathbb{Q}\},$$

and this is consistent with the fact that U *is dense and that the corresponding set of representatives in* S *is likewise dense. The infinitude of* $X(U)$*, however, is not inferred from the picture, but follows directly from the calculation*

$$X(qi) = e^{-\pi q}, \quad q \in \mathbb{Q},$$

which yields infinitely many distinct values. In fact, for the present choice $h = 1$*, the map* $X(s) = e^{\pi is}$ *is injective on the strip* S*. Indeed, if* $s, t \in S$ *and* $X(s) = X(t)$*, then* $e^{\pi i(s-t)} = 1$*, so* $s - t \in 2\mathbb{Z}$*. Since*

$$\Re(s - t) \in \left(-\frac{1}{2}, \frac{1}{2}\right),$$

it follows that $s = t$*. Thus distinct representatives in* S *give distinct values of* X*. In any case, since* $X(U)$ *is infinite, the polynomial* Q *in* (4.1) *is uniquely determined. By contrast, if one restricts* X *to finitely many half-step arithmetic progressions, then only finitely many values can occur. Indeed, on the arithmetic progression* $s_0 + \frac{1}{2}\mathbb{Z}$ *one has*

$$X\left(s_0 + \frac{k}{2}\right) = e^{\pi i(s_0 + k/2)} = i^k X(s_0),$$

so X takes exactly four values there. In particular, on the arithmetic progression $s_0 + \mathbb{Z}$ we have

$$X(s_0 + k) = (-1)^k X(s_0),$$

so exactly two values occur. This helps explain why, in the framework considered by Nikiforov and Uvarov, this parametrising map is naturally set aside: their analysis is, in effect, confined to a single arithmetic progression, where only this two-point behaviour is visible; see [24, p. 19]. *This observation concerns the admissible parametrisation itself. A functional representation, when introduced, may of course be supported on a subset of U chosen for that purpose, and need not be tied to the particular schematic traversal displayed in the figure.*

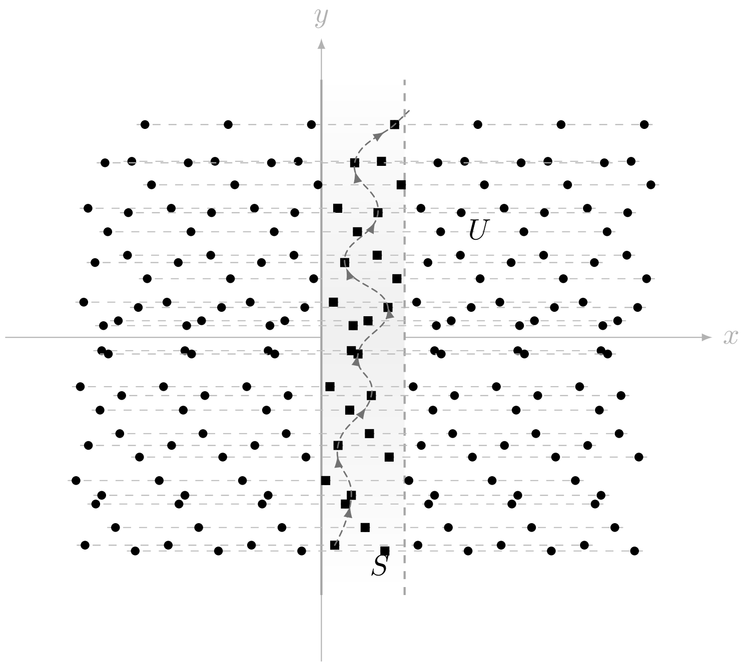

FIGURE 6. The heavier dashed curve indicates a traversal through representatives (moving upward through S), suggesting how one ranges over infinitely many distinct arithmetic progressions inside the dense set U.

The next example illustrates the same alternating mechanism on a non-discrete half-step-invariant set.

Example 5.4 (Normalised alternating case revisited)**.** *Let $U \subseteq \mathbb{C}$ be the half-step-invariant set from Example 2.4*(4) *with $h = 1$, namely*

$$U = \bigsqcup_{k\in\mathbb{Z}} \left(\left(\frac{k}{2} - \frac{1}{16}, \frac{k}{2} + \frac{1}{16} \right) \times \left(-\frac{\ln 2}{4\pi}, \frac{\ln 2}{4\pi} \right) \right) \subset \mathbb{C},$$

where we identify (x, y) with $x + iy$. Let $X : U \to \mathbb{C}$ be given by

$$X(s) = e^{\pi i s},$$

for every $s \in U$. As in Example 5.3, one has

$$Y(s) = iX(s), \quad Z(s) = -iX(s),$$

for every $s \in U$, and the same decomposition argument shows that (4.1) *holds with a polynomial in X of degree at most $p - 1$. Thus, provided $X(U)$ is infinite, the map X is* *admissible on U. This remaining condition is indeed satisfied. For*

$$y \in \left(-\frac{\ln 2}{4\pi}, \frac{\ln 2}{4\pi}\right)$$

one has $iy \in U$, and

$$X(iy) = e^{\pi i(iy)} = e^{-\pi y} \in \left(2^{-1/4}, 2^{1/4}\right),$$

so $X(U)$ contains an open real interval.

Figure 7 is schematic. It is drawn for the present case $h = 1$, so that the half-step arithmetic progressions are horizontal. The black squares represent chosen representatives, while the black discs indicate points in the corresponding half-step arithmetic progressions. As in Figure 6, the heavier dashed curve should not be interpreted as a literal path in U, nor as an additional geometric structure attached to U. It is a schematic device indicating a possible traversal through representatives of distinct horizontal arithmetic progressions, instead of remaining confined to a single one. This distinction is again relevant in the alternating case. On a single full-step arithmetic progression the map $X(s) = e^{\pi i s}$ takes only two values, and on a single half-step arithmetic progression it takes only four. The figure is meant to suggest how the rectangular set U allows one to pass through representatives belonging to distinct horizontal arithmetic progressions, so that the same formula $X(s) = e^{\pi i s}$ is seen on a domain with infinitely many X-values. In the present rectangular model, one is not obliged to choose representatives in different rectangles. Several representatives may be selected within a single rectangle, provided that they lie on distinct horizontal arithmetic progressions. Indeed, even a single rectangle meets infinitely many distinct horizontal arithmetic progressions. Moreover, $X(s) = e^{\pi i s}$ is injective on each individual rectangle occurring in U. For example, on the central rectangle

$$\left(-\frac{1}{16}, \frac{1}{16}\right) \times \left(-\frac{\ln 2}{4\pi}, \frac{\ln 2}{4\pi}\right),$$

if s, t belong to this rectangle and $X(s) = X(t)$, then $e^{\pi i(s-t)} = 1$, so $s - t \in 2\mathbb{Z}$. Since $\Re(s-t) \in (-\frac{1}{8}, \frac{1}{8})$, it follows that $s = t$. The same argument applies, after translation, to every other rectangle in U. Thus distinct representatives chosen within a single rectangle give distinct values of X. If representatives are chosen in different rectangles, distinctness of the corresponding X-values is no longer automatic; however, no such global injectivity is needed here, since it has already been shown directly that $X(U)$ is infinite. This observation concerns the admissible parametrisation itself. A functional representation, when introduced, may be supported on a subset of U chosen for that purpose, and need not coincide with the particular schematic traversal displayed in the figure.

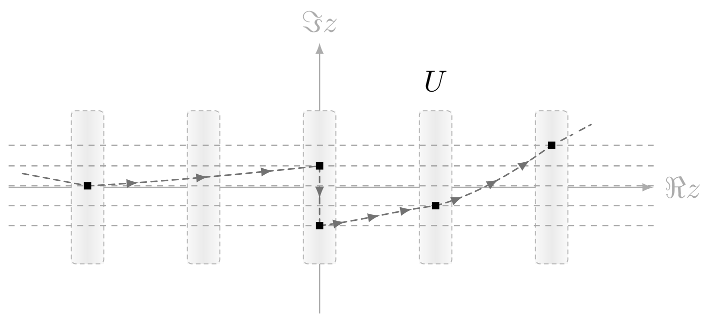

FIGURE 7. The heavier dashed curve indicates a traversal through successive representatives, including more than one chosen within a single rectangle, and is intended to suggest how one may range over infinitely many distinct horizontal arithmetic progressions inside U.

Proof of Theorem 5.1. Step 1: By admissibility, for every polynomial p of degree at least 1, there exists a unique polynomial Q, with $\deg Q \le \deg p - 1$, such that

$$\frac{p(Y)-p(Z)}{Y-Z} = Q(X)$$

throughout U. Since X is a parametrising map, $Y \neq Z$ throughout U. Taking $p(x) = x^2$, we obtain

$$Y + Z = \alpha X + \beta \tag{5.3}$$

for suitable $\alpha, \beta \in \mathbb{C}$.

Step 2: Evaluating the identity (5.3) at $s + \frac{h}{2}$ and $s - \frac{h}{2}$, we obtain

$$X + Y_1 = \alpha Y + \beta, \quad X + Z_1 = \alpha Z + \beta.$$

Adding these identities and using (5.3), we get

$$Y_1 + Z_1 = (\alpha^2 - 2)X + (\alpha + 2)\beta.$$

Hence (5.1) holds throughout U, with

$$A = \alpha^2 - 2, \quad B = (\alpha + 2)\beta.$$

Step 3: Fix $s_0 \in U$, and, for every $k \in \mathbb{Z}$, set

$$x_k = X(s_0 + kh).$$

Then (5.1) restricts to

$$x_{k+1} - A\,x_k + x_{k-1} = B. \tag{5.4}$$

This is an elementary constant-coefficient recurrence, and its closed-form solutions are recorded in [24, p. 20]; see Remark 5.2.

Step 4: Case (i): $A = 2$. Then (5.4) becomes

$$x_{k+1} - 2x_k + x_{k-1} = B.$$

Let $d_k = x_{k+1} - x_k$. Then $d_k - d_{k-1} = B$, and therefore

$$d_k = d_0 + Bk,$$

where

$$d_0 = x_1 - x_0 = Y_1(s_0) - X(s_0).$$

It follows that the quadratic polynomial

$$u_k = \frac{B}{2}\,k^2 + \left(x_1 - x_0 - \frac{B}{2}\right)k + x_0$$

satisfies

$$u_{k+1} - 2u_k + u_{k-1} = B,$$

together with the initial conditions $u_0 = x_0$ and $u_1 = x_1$. By uniqueness for the recurrence (5.4), we conclude that $x_k = u_k$ for all $k \in \mathbb{Z}$, and therefore the expression for x_k is exactly the one stated in Theorem 5.1*(i)*.

Case (ii): $A = -2$. Since $A = \alpha^2 - 2$, the condition $A = -2$ forces $\alpha = 0$. Thus (5.3) reads

$$Y + Z = \beta,$$

and from $B = (\alpha + 2)\beta$ we obtain $B = 2\beta$. Replacing s by $s + \frac{h}{2}$ gives

$$X + Y_1 = \beta = \frac{B}{2},$$

which is (5.2). Evaluating at $s = s_0 + kh$ yields

$$x_{k+1} + x_k = \frac{B}{2},$$

so $x_{k+1} = \frac{B}{2} - x_k$, whence $x_{k+2} = x_k$. Writing

$$x_k = \frac{B}{4} + u_k,$$

we obtain

$$u_{k+1} = -u_k,$$

so that $u_k = (-1)^k u_0$. Hence the corresponding formula for x_k is precisely that stated in Theorem 5.1*(ii)*.

Case (iii): $A^2 \neq 4$. Choose $q \in \mathbb{C}^\times$, $q \neq \pm 1$, with $A = q + q^{-1}$, and set

$$c = \frac{B}{2 - A}, \quad y_k = x_k - c.$$

Then (5.4) becomes the homogeneous recurrence

$$y_{k+1} - A\, y_k + y_{k-1} = 0,$$

whose characteristic polynomial $r^2 - Ar + 1$ has distinct roots q and q^{-1}. Hence

$$y_k = a_{s_0}\, q^k + b_{s_0}\, q^{-k},$$

so

$$x_k = c + a_{s_0}\, q^k + b_{s_0}\, q^{-k}.$$

Moreover,

$$y_0 = x_0 - c = X(s_0) - c, \quad y_1 = x_1 - c = Y_1(s_0) - c,$$

so a_{s_0} and b_{s_0} are uniquely determined by

$$a_{s_0} + b_{s_0} = y_0, \quad a_{s_0} q + b_{s_0} q^{-1} = y_1,$$

and solving gives the stated formulas for a_{s_0} and b_{s_0}. This is Theorem 5.1*(iii)*.

□

Remark 5.5. *It is worth stressing that the proof of Theorem 5.1 uses only the admissibility condition for the single test polynomial*

$$p(x) = x^2.$$

Indeed, this already forces the affine relation

$$Y + Z = \alpha X + \beta,$$

from which the global full-step relation

$$Y_1 + Z_1 = AX + B$$

follows immediately. The subsequent classification is then obtained by solving the corresponding constant-coefficient recurrence along each full-step arithmetic progression. In that sense, the quadratic, alternating, and q-exponential behaviours are already forced at the lowest non-trivial level of the admissibility condition; higher-degree instances of admissibility yield further algebraic consequences, as will be seen in the next remark, but no new admissible maps.

Remark 5.6 (Magnus's conic)**.** *Assume the hypotheses of Theorem 5.1, and retain the notation introduced above. The proof of Theorem 5.1 already shows that $Y + Z$ is an affine function of X. Thus there exist constants $B, D \in \mathbb{C}$, not to be confused with the constants in Theorem 5.1, such that*

$$Y + Z = -2BX - 2D.$$

At that point, the essential structural information is already in place. Passing to divided differences of higher degree does not produce new admissible behaviours, but only further algebraic consequences of what has already been obtained. Indeed, taking $p(t) = t^3$ in Definition 4.1 shows that

$$\frac{Y^3 - Z^3}{Y - Z}$$

is a polynomial function of X of degree at most 2. Since

$$\frac{Y^3 - Z^3}{Y - Z} = (Y + Z)^2 - YZ,$$

and $Y + Z$ is already known to be affine in X, it follows that YZ is a polynomial function of X of degree at most 2. Thus there exist constants $C, E, F \in \mathbb{C}$ such that

$$YZ = CX^2 + 2EX + F.$$

Fix $s \in U$, and set

$$x = X(s), \quad y = Y(s), \quad z = Z(s).$$

By Viète's formulae, the numbers y and z are the roots of the quadratic equation

$$u^2 + (2Bx + 2D)u + Cx^2 + 2Ex + F = 0.$$

This is precisely the form used by Magnus in [15, (1.2), p. 262][16, (5), p. 254]. *The point, however, is that this conic is secondary rather than fundamental. It records one consequence of the local half-step structure, but neither determines the structure of U nor fixes the set of attained values $X(U)$.*

Remark 5.7. *Neither case* (i) *nor case* (ii) *of Theorem 5.1 should be understood as arising by direct substitution of $q = 1$ or $q = -1$ into the formula of case* (iii). *Indeed, in Step 4 of the proof, the representation*

$$x_k = c + a_{s_0} q^k + b_{s_0} q^{-k}$$

is obtained under the assumption that the characteristic polynomial

$$r^2 - Ar + 1$$

has two distinct roots q and q^{-1}, equivalently $A^2 \neq 4$. In particular, the coefficients a_{s_0} and b_{s_0} involve the denominator $q - q^{-1}$, so the parametrisation in case (iii) *is not defined when $q = \pm 1$. The relation between the three cases is instead one of degeneration at the level of the characteristic equation. When $A = 2$, we have*

$$r^2 - Ar + 1 = (r - 1)^2,$$

so the two characteristic roots coalesce at $r = 1$. The corresponding homogeneous recurrence then has basis 1 and k, and the inhomogeneous recurrence yields the quadratic behaviour described in case (i). *When $A = -2$, one has*

$$r^2 - Ar + 1 = (r + 1)^2,$$

so the double root is $r = -1$. At the level of the second-order recurrence alone, the general solution is

$$x_k = \frac{B}{4} + (u + vk)(-1)^k,$$

where $u, v \in \mathbb{C}$. On the other hand, admissibility yields the additional first-order relation

$$x_{k+1} + x_k = \frac{B}{2},$$

which is the progression-wise form of (5.2). *Substituting the above expression for x_k into this identity forces $v = 0$, and hence*

$$x_k = \frac{B}{4} + u(-1)^k,$$

which is precisely the alternating form stated in case (ii)*. Thus cases* (i) *and* (ii) *arise when the two characteristic roots merge, rather than by direct substitution into the formula of case* (iii)*.*

Remark 5.8. *The alternatives in Theorem 5.1 are global. Indeed, the constants A and B are determined on all of U, and therefore the same full-step recurrence holds on every arithmetic progression $s_0 + h\mathbb{Z} \subseteq U$. Consequently, only one of the three cases in the theorem can occur, and it occurs simultaneously on all such progressions. In particular, one cannot have quadratic behaviour on one full-step arithmetic progression and q-exponential behaviour on another. What may vary from one progression to another are only those coefficients which are determined by the initial values of X on the progression in question. Thus, in case* (iii)*, once a parameter $q \in \mathbb{C}^\times$ satisfying*

$$A = q + q^{-1}$$

has been fixed, the same parameter q governs the description on every full-step arithmetic progression, whereas the coefficients a_{s_0} and b_{s_0} may depend on s_0. This distinction is particularly relevant in the alternating case and, more generally, in torsion q-exponential situations. In those regimes, the restriction of X to a single full-step arithmetic progression may have finite image. Hence, whenever the full image $X(U)$ is infinite, that infinitude cannot be inferred from the behaviour along one progression alone; it must come from the contribution of sufficiently many distinct full-step progressions in U.

Remark 5.9 (Non-constant affine map)**.** *The affine situation is already rigid at the level of Theorem 5.1. Indeed, if an admissible map is non-constant affine on a full-step arithmetic progression $s_0 + h\mathbb{Z} \subseteq U$, then the corresponding global case must be case* (i)*, and the quadratic coefficient must vanish, that is, $B = 0$. In that case the formula in the theorem reduces to*

$$X(s_0 + kh) = \big(Y_1(s_0) - X(s_0)\big)k + X(s_0),$$

so the coefficient

$$Y_1(s_0) - X(s_0) = X(s_0 + h) - X(s_0)$$

is exactly the increment of the image along one full step of the underlying arithmetic progression. In other words, it is precisely the common difference of the affine parametrisation on that full-step progression. Since the constants A and B are global, and since in the affine case one has $A = 2$ and $B = 0$, every full-step arithmetic progression in the domain is mapped onto an arithmetic progression, possibly degenerate. However, Theorem 5.1 alone does not imply that the corresponding common difference is the same on all such progressions; a priori, the quantity

$$Y_1(s_0) - X(s_0)$$

may depend on the chosen progression $s_0 + h\mathbb{Z}$. In particular, in the situation discussed in the introduction, one takes $h = 2$, so that

$$\mathbb{S} = \mathbb{Z} \cup \Big(\frac{\gamma}{2} + \mathbb{Z}\Big)$$

is a union of two half-step arithmetic progressions, that is, of two cosets of $\frac{h}{2}\mathbb{Z} = \mathbb{Z}$. For the admissible affine parametrisation

$$X(s) = 2s$$

one has

$$Y - X = 2, \quad Y_1 - X = 4.$$

The two arithmetic progressions are therefore not being treated by two separate admissible parametrisations: the single map $X(s) = 2s$ is admissible on their union $\mathbb{S}$, making explicit the structural reparametrisation described in the Introduction. Thus the two half-step progressions are mapped with the same common difference 2, while the corresponding full-step *progressions are mapped with common difference* 4. *A complete treatment of the affine case, together with the classification of the corresponding classical orthogonal polynomial families, was given in* [5]. *Unlike the affine case, however, the genuinely quadratic case is not governed by a quantity directly interpretable as a slope. There the step of the arithmetic progression merely fixes the discrete scale, while the actual structural datum is the constant second difference*

$$X(s_0 + (k+2)h) - 2X(s_0 + (k+1)h) + X(s_0 + kh) = B.$$

6. Classicality and orthogonality à la Maroni

The material presented in this section is drawn principally from the work of Maroni [17, 18, 19, 20, 7], but has been adapted to the present setting and made self-contained. The underlying facts concerning topological vector spaces are standard and may be found in almost any textbook on the subject. Although Maroni did not consider the structural objects studied in the present paper in the form adopted here, the notions of orthogonality and classicality used below are directly indebted to his functional point of view. A word should also be added about a terminology sometimes encountered in the literature. Maroni's approach is occasionally described as a "formal algebraic approach". The adjective "algebraic" is justified by the aim of isolating the intrinsic polynomial structure of orthogonal polynomial systems, up to the limits of their existence as algebraic objects; this is the point of view discussed in [5]. The adjective "formal", however, is misleading and should be avoided here. As we shall see below, Maroni's framework is functional-analytic in its very formulation: its basic objects are continuous linear functionals on a LCS of polynomials, and its identities are identities in the continuous dual of that space. Thus what is sometimes called "formal" is not an analytically ungrounded symbolic calculus, but a precise analytic setting designed to separate the intrinsic algebraic structure of orthogonal polynomials from the accidental features of any particular measure-theoretic representation.

For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ denote the vector space of all complex polynomials of degree at most n. Since $\mathcal{P}_n$ is finite-dimensional, it carries a unique Hausdorff locally convex vector-space topology; equivalently, after choosing a basis, one may identify it with $\mathbb{C}^{n+1}$ endowed with its usual Euclidean topology. For $n \le m$, the inclusion $\iota_{n,m} : \mathcal{P}_n \hookrightarrow \mathcal{P}_m$ is a continuous linear embedding, and its range is closed. Set

$$\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n.$$

We equip $\mathcal{P}$ with the locally convex inductive-limit topology with respect to the canonical inclusions $\iota_n : \mathcal{P}_n \to \mathcal{P}$. Thus

$$\mathcal{P} = \varinjlim \mathcal{P}_n$$

as a strict countable inductive limit of finite-dimensional spaces. In particular, $\mathcal{P}$ is Hausdorff and locally convex. A subset $\mathcal{B} \subset \mathcal{P}$ is bounded if and only if there exists m such that $\mathcal{B} \subset \mathcal{P}_m$ and $\mathcal{B}$ is bounded in the finite-dimensional space $\mathcal{P}_m$. Likewise, if E is a LCS, a linear map $T : \mathcal{P} \to E$ is continuous if and only if each restriction $T|_{\mathcal{P}_n} : \mathcal{P}_n \to E$ is continuous. Let

$$\mathcal{P}' = \mathcal{L}(\mathcal{P}, \mathbb{C})$$

denote the continuous dual of $\mathcal{P}$. We use the canonical pairing

$$\langle \cdot, \cdot \rangle : \mathcal{P}' \times \mathcal{P} \to \mathbb{C}, \quad \langle \mathbf{u}, p \rangle = \mathbf{u}(p).$$

Since $\mathcal{P}$ is Hausdorff and locally convex, this pairing separates points: if $p \in \mathcal{P}$ is non-zero, then there exists $\mathbf{u} \in \mathcal{P}'$ such that

$$\langle \mathbf{u}, p \rangle \neq 0.$$

Unless explicitly stated otherwise, $\mathcal{P}'$ is endowed with the weak topology $\sigma(\mathcal{P}', \mathcal{P})$, that is, the coarsest topology for which all maps

$$\mathbf{u} \longmapsto \langle \mathbf{u}, p \rangle,$$

where $p \in \mathcal{P}$, are continuous.

If $p_0 \in \mathcal{P}$ is fixed, the multiplication operator

$$M_{p_0} : \mathcal{P} \to \mathcal{P}, \quad q \longmapsto p_0\, q$$

is continuous. Indeed, if $\deg p_0 = d$, then $M_{p_0}(\mathcal{P}_n) \subseteq \mathcal{P}_{n+d}$, so each restriction $M_{p_0}|_{\mathcal{P}_n} : \mathcal{P}_n \to \mathcal{P}_{n+d}$ is continuous, and the continuity criterion for the inductive limit yields continuity of M_{p_0} on $\mathcal{P}$. More generally, if $T : \mathcal{P} \to \mathcal{P}$ is linear and there exists $d \geq 0$ such that

$$T(\mathcal{P}_n) \subseteq \mathcal{P}_{n+d},$$

then T is continuous. Whenever $T : \mathcal{P} \to \mathcal{P}$ is continuous, its transpose $T' : \mathcal{P}' \to \mathcal{P}'$ is well defined by

$$\langle T'\mathbf{u}, p \rangle = \langle \mathbf{u}, Tp \rangle$$

for every $\mathbf{u} \in \mathcal{P}'$ and every $p \in \mathcal{P}$. In particular, for $p_0 \in \mathcal{P}$ we write

$$p_0\mathbf{u} = M'_{p_0}\mathbf{u},$$

so that

$$\langle p_0\mathbf{u}, q \rangle = \langle \mathbf{u}, p_0\, q \rangle,$$

for every $q \in \mathcal{P}$. An identity in $\mathcal{P}'$ is always understood in the functional sense: for $\mathbf{u}, \mathbf{v} \in \mathcal{P}'$,

$$\mathbf{u} = \mathbf{v}$$

means that $\langle \mathbf{u}, p \rangle = \langle \mathbf{v}, p \rangle$ for all $p \in \mathcal{P}$. Since the pairing separates points, this is equivalent to equality as elements of $\mathcal{P}'$. Finally, if $T : \mathcal{P} \to \mathcal{P}$ is continuous, then T' is $\sigma(\mathcal{P}', \mathcal{P})$-continuous. Indeed, for each fixed $p \in \mathcal{P}$,

$$\mathbf{u} \longmapsto \langle T'\mathbf{u}, p \rangle = \langle \mathbf{u}, Tp \rangle$$

is one of the defining weak-coordinate maps. Accordingly, all duality statements involving transposes of continuous linear operators on $\mathcal{P}$ are to be understood in this strictly functional sense.

Notation 6.1. *Throughout this paper, whenever a polynomial sequence $(P_n)_{n \in I} \subset \mathcal{P}$ is indexed by a set $I \subseteq \mathbb{N}$, it is understood that P_n has degree n for every $n \in I$.*

Definition 6.2 (Orthogonality)**.** *Let $\mathbf{u} \in \mathcal{P}'$, and let $I \subseteq \mathbb{N}$ be such that $0 \in I$. A family of polynomials $(P_n)_{n \in I}$ is said to be* orthogonal with respect to $\mathbf{u}$ *if*

$$\langle \mathbf{u}, P_n P_m \rangle = 0$$

for every $m, n \in I$ with $m < n$, and

$$\langle \mathbf{u}, P_n^2 \rangle \neq 0$$

for every $n \in I$. We say that $\mathbf{u}$ is regular *if there exists an orthogonal polynomial sequence $(P_n)_{n \in I}$ with respect to $\mathbf{u}$, where either $I = \mathbb{N}$ or $I = \{0, 1, \ldots, N\}$ for some $N \in \mathbb{N}$. When it is necessary to specify the finite length explicitly, we shall say that $\mathbf{u}$ is* regular of order $N+1$ *if there exists an orthogonal polynomial sequence $P_0, P_1, \ldots, P_N$ with respect to $\mathbf{u}$. Thus the order counts the number of polynomials, not the largest degree.*

At this point, the role of admissibility becomes more concrete. Its purpose is precisely to ensure that the symmetric divided difference of a polynomial, computed from the neighbouring values Y and Z, is again a polynomial in the base variable X. Thus admissibility is exactly the condition that allows one to pass from the pointwise quotient in (4.1) to a well-defined operator on $\mathcal{P}$. The next definition isolates that operator.

Definition 6.3 (Divided-difference operator)**.** *Fix $h \in \mathbb{C}^{\times}$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be an admissible map. The symmetric divided-difference operator associated with X is the linear map $D : \mathcal{P} \to \mathcal{P}$ which assigns to each $p \in \mathcal{P}$ the unique polynomial $Dp \in \mathcal{P}$ satisfying*

$$(Dp)(X) = \frac{p(Y) - p(Z)}{Y - Z}$$

throughout U. The dependence of D on the underlying admissible map X is understood and will remain implicit unless explicit notation is needed. The distributional transpose of D, hereafter simply called its transpose, is denoted by $\mathbf{D} : \mathcal{P}' \to \mathcal{P}'$, and is defined by

$$\langle \mathbf{D}\mathbf{u}, p\rangle = -\langle \mathbf{u}, Dp\rangle.$$

Thus

$$\mathbf{D} = -D',$$

where D' denotes the ordinary transpose of D.

Definition 6.4 (Averaging operator)**.** *Fix $h \in \mathbb{C}^{\times}$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be an admissible map. The symmetric averaging operator associated with X is the linear map $S : \mathcal{P} \to \mathcal{P}$ which assigns to each $p \in \mathcal{P}$ the unique polynomial $Sp \in \mathcal{P}$ satisfying*

$$(Sp)(X) = \frac{p(Y) + p(Z)}{2}$$

throughout U. The dependence of S on the underlying admissible map X is understood and will remain implicit unless explicit notation is needed. The transpose of S is denoted by $\mathbf{S} : \mathcal{P}' \to \mathcal{P}'$, and is defined by

$$\langle \mathbf{S}\mathbf{u}, p\rangle = \langle \mathbf{u}, Sp\rangle,$$

for every $\mathbf{u} \in \mathcal{P}'$ and every $p \in \mathcal{P}$.

Remark 6.5. *The operator D is well defined directly from Definition 4.1. Indeed, if* $\deg p \geq 1$, *admissibility gives a polynomial Q, uniquely determined because $X(U)$ is infinite, such that*

$$Q(X) = \frac{p(Y) - p(Z)}{Y - Z}$$

throughout U, and we set $Dp = Q$. If p is constant, then $p(Y) - p(Z) = 0$, and we set $Dp = 0$. The operator S is also well defined. By Remark 5.6, the quantities $Y + Z$ and YZ are polynomial functions of X. Since

$$\frac{p(Y) + p(Z)}{2}$$

is symmetric in Y and Z, it is a polynomial in the elementary symmetric functions $Y + Z$ and YZ. Hence it is a polynomial function of X. Since $X(U)$ is infinite, this polynomial is uniquely determined, and we denote it by Sp.

The operators D and S are continuous on $\mathcal{P}$. Indeed, by admissibility, for every $n \in \mathbb{N}^{\times}$ one has $D(\mathcal{P}_n) \subseteq \mathcal{P}_{n-1}$, while $D(\mathcal{P}_0) = \{0\}$. Moreover, $S(\mathcal{P}_n) \subseteq \mathcal{P}_n$ for every $n \in \mathbb{N}$. For $n \in \mathbb{N}^{\times}$, the restrictions $D|_{\mathcal{P}_n} : \mathcal{P}_n \to \mathcal{P}_{n-1}$ are linear maps between finite-dimensional spaces, and for every $n \in \mathbb{N}$, the restrictions $S|_{\mathcal{P}_n} : \mathcal{P}_n \to \mathcal{P}_n$ are likewise linear maps between finite-dimensional spaces. Hence all these restrictions are continuous. By the continuity criterion for the inductive-limit topology on $\mathcal{P}$, both D and S are therefore continuous. Consequently, the dual operators

$$\mathbf{D} = -D', \quad \mathbf{S} = S',$$

are well defined and $\sigma(\mathcal{P}', \mathcal{P})$-continuous. In particular, finite compositions involving multiplication by fixed polynomials and the operators D and S are again continuous linear endomorphisms of $\mathcal{P}$.

Definition 6.6 (Classicality). *Fix $h \in \mathbb{C}$. If $h \neq 0$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be an admissible map on U. Let $D, S : \mathcal{P} \to \mathcal{P}$ be the associated divided-difference and averaging operators. If $h = 0$, set*

$$D = \frac{d}{dx}, \quad S = \mathrm{id}_{\mathcal{P}}.$$

In either case, let $\mathbf{D} = -D'$ and $\mathbf{S} = S'$ denote the corresponding dual operators on $\mathcal{P}'$. A functional $\mathbf{u} \in \mathcal{P}'$ is said to be classical *if it is regular of order at least 3 and there exist polynomials ϕ and ψ, where ϕ has degree at most 2 and ψ has degree at most 1, not both identically zero, such that*

$$\mathbf{D}(\phi\, \mathbf{u}) = \mathbf{S}(\psi\, \mathbf{u}).$$

The requirement, in the preceding definition, that $\mathbf{u}$ be regular of order at least 3, rather than merely of order at least 2, is imposed in order to exclude certain degenerate situations that will be described in the following remark.

Remark 6.7. *In the case $h = 0$, one has $\mathbf{D}(\phi\, \mathbf{u}) = \psi\, \mathbf{u}$. We record what happens in the two degenerate cases. Assume first that $\phi = 0$. Then*

$$\psi\, \mathbf{u} = 0.$$

Since ϕ and ψ are not both identically zero, one has $\psi \neq 0$. If ψ is a non-zero constant, then $\mathbf{u} = 0$, which is impossible for a regular functional. Hence $\deg \psi = 1$, and we may write $\psi(x) = a(x - \tau)$, with $a \in \mathbb{C}^\times$ and $\tau \in \mathbb{C}$. Thus

$$(x - \tau)\mathbf{u} = 0.$$

Equivalently,

$$\langle \mathbf{u}, p \rangle = p(\tau)\langle \mathbf{u}, 1 \rangle$$

for every $p \in \mathcal{P}$. Hence $\mathbf{u}$ is a non-zero scalar multiple of the evaluation functional at τ, and such a functional cannot be regular beyond order 1. For if P_1 were orthogonal to $P_0 = 1$, then

$$0 = \langle \mathbf{u}, P_1 \rangle = P_1(\tau)\langle \mathbf{u}, 1 \rangle,$$

so $P_1(\tau) = 0$, and consequently

$$\langle \mathbf{u}, P_1^2 \rangle = P_1(\tau)^2 \langle \mathbf{u}, 1 \rangle = 0.$$

Assume next that $\psi = 0$. Then $\phi \neq 0$ and

$$\mathbf{D}(\phi\, \mathbf{u}) = 0.$$

Since $D = \frac{d}{dx}$ is surjective on $\mathcal{P}$, it follows that $\phi\, \mathbf{u} = 0$. Indeed, given $q \in \mathcal{P}$, choose $p \in \mathcal{P}$ such that $Dp = q$. Then

$$0 = \langle \mathbf{D}(\phi\, \mathbf{u}), p \rangle = -\langle \phi\, \mathbf{u}, q \rangle.$$

Thus $\mathbf{u}$ vanishes on the ideal $\phi\mathcal{P}$, and therefore factors through the quotient $\mathcal{P}/\phi\mathcal{P}$. Since $\deg \phi \leq 2$, this quotient has dimension at most 2. Consequently, $\mathbf{u}$ cannot be regular of order greater than 2. To see this, if $P_0, \ldots, P_N$ were orthogonal with respect to $\mathbf{u}$, then their classes in $\mathcal{P}/\phi\mathcal{P}$ would be linearly independent: whenever

$$\sum_{j=0}^{N} c_j P_j \in \phi\mathcal{P},$$

one has, for every $m = 0, \ldots, N$,

$$0 = \left\langle \mathbf{u}, \left(\sum_{j=0}^{N} c_j P_j \right) P_m \right\rangle = c_m \langle \mathbf{u}, P_m^2 \rangle,$$

and hence $c_m = 0$. Therefore

$$N + 1 \leq \dim(\mathcal{P}/\phi\mathcal{P}) \leq 2.$$

Thus, in the continuous case, degenerate choices of ϕ and ψ cannot give classical functionals in the sense of Definition 6.6. More precisely, when $\phi = 0$, regularity is possible only up to order 1, whereas, when $\psi = 0$, regularity is possible only up to order 2.

7. Regularity and recurrence coefficients

In this section we refine the known necessary and sufficient conditions for the regularity of classical functionals in the q-exponential and quadratic cases, and derive the corresponding recurrence coefficients. We treat first the q-exponential case when q is not a root of unity, then the quadratic case as its degenerate counterpart, and finally indicate how part of the same picture persists in the finite torsion regime when q is a root of unity.

7.1. **q-exponential map: q is not a root of unity.** At the level of ideas, the proof of the following theorem is not essentially different from that of [6, Theorem 4.1] and [7, Theorem 9.2]. Nevertheless, the argument given there is not written in a form that makes entirely transparent how finite orthogonal polynomial sequences are to be included; moreover, it assumes unnecessarily that $q > 0$, and it does not make explicit how the decomposition of the underlying half-step-invariant sets enters the argument.

Theorem 7.1. *Fix $h \in \mathbb{C}^\times$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be an admissible map. Let $D, S : \mathcal{P} \to \mathcal{P}$ be the associated divided-difference and averaging operators, with transposes $\mathbf{D}, \mathbf{S} : \mathcal{P}' \to \mathcal{P}'$. Let $\mathbf{u} \in \mathcal{P}'$ be such that $\langle \mathbf{u}, 1\rangle \neq 0$, and assume that there exist polynomials ϕ and ψ, where ϕ has degree at most 2 and ψ has degree at most 1, not both identically zero, such that*

$$\mathbf{D}(\phi\,\mathbf{u}) = \mathbf{S}(\psi\,\mathbf{u})$$

in $\mathcal{P}'$. Write

$$\phi(x) = \phi_2 x^2 + \phi_1 x + \phi_0, \quad \psi(x) = \psi_1 x + \psi_0.$$

Fix $s_0 \in U$, and assume that, on the full-step arithmetic progression $s_0 + h\mathbb{Z} \subseteq U$, there exist $a_{s_0}, b_{s_0}, c \in \mathbb{C}$ and $q \in \mathbb{C}^\times$, with q not a root of unity, such that

$$X(s_0 + kh) = a_{s_0} q^k + b_{s_0} q^{-k} + c,$$

for every $k \in \mathbb{Z}$. Fix a choice of $q^{1/2}$, and, for every $n \in \mathbb{Z}$, define

$$\alpha_n = \frac{q^{n/2} + q^{-n/2}}{2}, \quad \gamma_n = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$

Write also $\alpha = \alpha_1$. Set

$$d_n = \phi_2\,\gamma_n + \psi_1\,\alpha_n, \quad e_n = (2\phi_2 c + \phi_1)\gamma_n + (\psi_1 c + \psi_0)\alpha_n,$$

and define

$$\phi^{[n]}(x) = \big(\psi_1(\alpha^2 - 1)\gamma_{2n} + \phi_2\alpha_{2n}\big)\big((x-c)^2 - 2a_{s_0}b_{s_0}\big)$$
$$+ \big(\phi'(c)\alpha_n + \psi(c)(\alpha^2-1)\gamma_n\big)(x-c) + \phi(c) + 2\phi_2 a_{s_0} b_{s_0}.$$

Let $I \subseteq \mathbb{N}$ be either $I = \mathbb{N}$, or $I = \{0, 1, \dots, N+1\}$ for some $N \in \mathbb{N}$. In the finite case set

$$J_I = \{0, 1, \dots, 2N+1\}, \quad K_I = \{0, 1, \dots, N\},$$

whereas in the infinite case set

$$J_I = K_I = \mathbb{N}.$$

Then there exists a monic orthogonal polynomial sequence $(P_n)_{n\in I}$ with respect to $\mathbf{u}$ if and only if

$$d_j \neq 0,$$

for every $j \in J_I$ and

$$\phi^{[n]}\left(c - \frac{e_n}{d_{2n}}\right) \neq 0,$$

for every $n \in K_I$. Whenever this holds, the corresponding monic orthogonal polynomial sequence is uniquely determined by $P_{-1} = 0$, $P_0 = 1$, and by the recurrence

$$P_{n+1}(x) = (x - B_n)P_n(x) - C_nP_{n-1}(x),$$

for every $n \in K_I$, with the convention that the term involving C_0P_{-1} is void. The coefficients are given by

$$B_0 = c - \frac{\gamma_1 e_0}{d_0}, \quad C_1 = -\frac{\gamma_1}{d_1}\,\phi^{[0]}\left(c - \frac{e_0}{d_0}\right),$$

and, for every $n \in K_I^\times$,

$$B_n = c + \frac{\gamma_n e_{n-1}}{d_{2n-2}} - \frac{\gamma_{n+1}e_n}{d_{2n}},$$

$$C_{n+1} = -\frac{\gamma_{n+1}d_{n-1}}{d_{2n-1}d_{2n+1}}\,\phi^{[n]}\left(c - \frac{e_n}{d_{2n}}\right).$$

In the finite case, the last coefficient C_{N+1} is the terminal norm coefficient; it is not used to construct a polynomial P_{N+2}.

Proof. Fix $s_0 \in U$, and abbreviate $a = a_{s_0}$ and $b = b_{s_0}$. Since q is not a root of unity, one has $\gamma_n \neq 0$ for every $n \in \mathbb{Z}^\times$, so all normalisations below are well defined. We first treat the case where $I = \{0, 1, \ldots, N+1\}$ for some $N \in \mathbb{N}$. The case $I = \mathbb{N}$ will then follow by applying the same argument to arbitrary finite initial segments.

Step 1: Define

$$\psi^{[n]}(x) = d_{2n}(x - c) + e_n,$$

where

$$d_n = \phi_2\gamma_n + \psi_1\alpha_n, \quad e_n = (2\phi_2 c + \phi_1)\gamma_n + (\psi_1 c + \psi_0)\alpha_n.$$

Also define $\phi^{[n]}$ by

$$\begin{aligned}\phi^{[n]}(x) = {}& \big(\psi_1(\alpha^2 - 1)\gamma_{2n} + \phi_2\alpha_{2n}\big)\big((x-c)^2 - 2ab\big) \\ & + \big(\phi'(c)\alpha_n + \psi(c)(\alpha^2-1)\gamma_n\big)(x - c) + \phi(c) + 2\phi_2 ab. \end{aligned} \tag{7.1}$$

These are exactly the transformed coefficients appearing in [7, Proposition 9.2], written in the notation of the present paper.

Step 2: Define recursively functionals $\mathbf{u}^{[n]} \in \mathcal{P}'$ by

$$\mathbf{u}^{[0]} = \mathbf{u},$$

and, for every $n \in \mathbb{N}$,

$$\mathbf{u}^{[n+1]} = \mathbf{D}\Big((\alpha^2 - 1)\big((x-c)^2 - 2ab\big)\,\psi^{[n]}\mathbf{u}^{[n]}\Big) - \mathbf{S}\big(\phi^{[n]}\mathbf{u}^{[n]}\big). \tag{7.2}$$

For $n = 0$, the identity

$$\mathbf{D}\big(\phi^{[0]}\mathbf{u}^{[0]}\big) = \mathbf{S}\big(\psi^{[0]}\mathbf{u}^{[0]}\big)$$

is precisely the assumed relation, since

$$\mathbf{u}^{[0]} = \mathbf{u}, \quad \phi^{[0]} = \phi, \quad \psi^{[0]} = \psi.$$

The transformation formula of [7, Proposition 9.2], rewritten in the present notation and used together with the definition of $\mathbf{u}^{[n+1]}$, shows that the validity of the identity at level n implies the validity of the corresponding identity at level $n+1$. Hence, by induction,

$$\mathbf{D}\big(\phi^{[n]}\mathbf{u}^{[n]}\big) = \mathbf{S}\big(\psi^{[n]}\mathbf{u}^{[n]}\big), \tag{7.3}$$

for every $n \in \mathbb{N}$.

Step 3: Assume that $d_j \neq 0$ for every $j \in J_I$, and that

$$\phi^{[n]}\left(c - \frac{e_n}{d_{2n}}\right) \neq 0$$

for every $n \in K_I$. We claim that, for each $n \in I$, there exists a polynomial R_n such that

$$R_n\,\mathbf{u} = \mathbf{D}^n \mathbf{u}^{[n]}. \tag{7.4}$$

Moreover, these polynomials satisfy

$$R_{-1} = 0, \quad R_0 = 1, \quad R_1(x) = -\alpha\,\psi^{[0]}(x) = -\alpha\big(d_0(x-c) + e_0\big),$$

and, for every $n \in K_I^{\times}$,

$$R_{n+1}(x) = (a_n x - s_n)R_n(x) - t_n R_{n-1}(x), \tag{7.5}$$

where

$$a_n = -\alpha\,\frac{d_{2n} d_{2n-1}}{d_{n-1}}, \tag{7.6}$$

$$s_n = a_n\left(c + \frac{\gamma_n e_{n-1}}{d_{2n-2}} - \frac{\gamma_{n+1} e_n}{d_{2n}}\right), \tag{7.7}$$

$$t_n = a_n\,\frac{\alpha\gamma_n d_{2n-2}}{d_{2n-1}}\,\phi^{[n-1]}\left(c - \frac{e_{n-1}}{d_{2n-2}}\right). \tag{7.8}$$

This is the q-exponential Rodrigues construction proved in [7, Theorem 9.1], rewritten in the notation of the present paper. What matters here is simply that the construction is inductive: its initial step requires only the non-vanishing of d_0, and the passage from level n to level $n+1$ uses only the non-vanishing assumptions up to that level, namely $d_j \neq 0$ for every $j \in \{0, 1, \ldots, 2n+1\}$, and

$$\phi^{[j]}\left(c - \frac{e_j}{d_{2j}}\right) \neq 0$$

for every $j \in \{0, 1, \ldots, n\}$. Consequently, the same inductive construction remains valid in the present finite setting. In particular, under the assumptions above, it yields the existence of $(R_n)_{n\in I}$. We now normalise these polynomials so as to obtain a monic family.

Step 4: For every $n \in I$, set

$$k_0 = 1, \quad k_n = (-\alpha)^{-n}\prod_{j=1}^{n} d_{n+j-2}^{-1},$$

and define

$$P_n = k_n R_n.$$

By the leading-coefficient computation contained in [7, Theorem 9.1], this choice of k_n makes P_n monic. Multiplying (7.5) by k_{n+1}, and using (7.6)–(7.8), one obtains

$$P_{n+1}(x) = (x - B_n)P_n(x) - C_n P_{n-1}(x),$$

for $n = 0, 1, \ldots, N$, where

$$B_0 = c - \frac{\gamma_1 e_0}{d_0}, \quad C_1 = -\frac{\gamma_1}{d_1}\,\phi^{[0]}\left(c - \frac{e_0}{d_0}\right),$$

and, for every $n \in K_I^{\times}$,

$$B_n = c + \frac{\gamma_n e_{n-1}}{d_{2n-2}} - \frac{\gamma_{n+1} e_n}{d_{2n}},$$

$$C_{n+1} = -\frac{\gamma_{n+1} d_{n-1}}{d_{2n-1} d_{2n+1}} \, \phi^{[n]}\left(c - \frac{e_n}{d_{2n}}\right).$$

Under the present assumptions, all these coefficients are well defined. Moreover, $C_n \neq 0$ for every $n \in I^{\times}$.

Step 5: Let $0 \leq m < n \leq N+1$. Since $\deg P_m = m < n$, one has $D^n P_m = 0$. Using (7.4) and the definition of the transpose $\mathbf{D}$, we obtain

$$\begin{aligned}\langle \mathbf{u}, P_n P_m \rangle &= k_n \langle \mathbf{u}, R_n P_m \rangle = k_n \langle R_n \mathbf{u}, P_m \rangle = k_n \langle \mathbf{D}^n \mathbf{u}^{[n]}, P_m \rangle \\ &= (-1)^n k_n \langle \mathbf{u}^{[n]}, D^n P_m \rangle = 0.\end{aligned}$$

Thus

$$\langle \mathbf{u}, P_n P_m \rangle = 0$$

whenever $0 \leq m < n \leq N+1$. It remains to show that the squared norms do not vanish. Since $P_0 = 1$, we have

$$\langle \mathbf{u}, P_0^2 \rangle = \langle \mathbf{u}, 1 \rangle \neq 0.$$

Now let $n = 1, \ldots, N$. Pairing

$$P_n(x) = (x - B_{n-1}) P_{n-1}(x) - C_{n-1} P_{n-2}(x)$$

with P_n, and using orthogonality, gives

$$\langle \mathbf{u}, P_n^2 \rangle = \langle \mathbf{u}, x P_{n-1} P_n \rangle.$$

Pairing

$$P_{n+1}(x) = (x - B_n) P_n(x) - C_n P_{n-1}(x)$$

with P_{n-1}, and again using orthogonality, yields

$$\langle \mathbf{u}, x P_n P_{n-1} \rangle = C_n \langle \mathbf{u}, P_{n-1}^2 \rangle.$$

Hence

$$\langle \mathbf{u}, P_n^2 \rangle = C_n \langle \mathbf{u}, P_{n-1}^2 \rangle.$$

The preceding recurrence argument gives the non-vanishing of the squared norms at every non-terminal level for which the corresponding recurrence relation is used inside the finite segment. The terminal norm is obtained from the same norm computation which gives the terminal coefficient C_{N+1}. In the present notation this gives

$$\langle \mathbf{u}, P_{N+1}^2 \rangle = C_{N+1} \langle \mathbf{u}, P_N^2 \rangle.$$

Since the terminal non-vanishing condition gives $C_{N+1} \neq 0$, the terminal norm is non-zero as well. Consequently,

$$\langle \mathbf{u}, P_n^2 \rangle \neq 0$$

for $n = 0, 1, \ldots, N+1$. Hence $(P_n)_{n \in I}$ is a monic orthogonal polynomial sequence with respect to $\mathbf{u}$. This proves the sufficiency of the stated non-vanishing conditions in the case $I = \{0, 1, \ldots, N+1\}$.

Step 6: Assume now that there exists a monic orthogonal polynomial sequence $(P_n)_{n \in I}$ with respect to $\mathbf{u}$, where $I = \{0, 1, \ldots, N+1\}$. For integers $k, j \in \mathbb{N}$ with $j + k \leq N+1$, define

$$P_j^{[k]}(x) = \frac{D^k P_{j+k}(x)}{\prod_{\ell=1}^{k} \gamma_{j+\ell}}. \tag{7.9}$$

Since q is not a root of unity, one has $\gamma_{j+\ell} \neq 0$ for every index appearing in the product, so this is well defined. Moreover, D lowers degree by one, and its action on the leading term is governed by the coefficient γ_n; hence each $P_j^{[k]}$ is monic of degree j.

Step 7: Let $\phi_2^{[k]}$ denote the quadratic coefficient of $\phi^{[k]}$, and $\psi_1^{[k]}$ the linear coefficient of $\psi^{[k]}$. By (7.1) and the definition of $\psi^{[k]}$,

$$\phi_2^{[k]} = \psi_1(\alpha^2-1)\gamma_{2k} + \phi_2\alpha_{2k}, \quad \psi_1^{[k]} = \phi_2\gamma_{2k} + \psi_1\alpha_{2k}.$$

Define

$$d_n^{[k]} = \phi_2^{[k]}\gamma_n + \psi_1^{[k]}\alpha_n.$$

A direct computation from the definitions of α_n and γ_n gives

$$\alpha_{2k}\gamma_n + \gamma_{2k}\alpha_n = \gamma_{n+2k}, \quad \alpha_{2k}\alpha_n + (\alpha^2-1)\gamma_{2k}\gamma_n = \alpha_{n+2k},$$

and therefore

$$d_n^{[k]} = \phi_2\gamma_{n+2k} + \psi_1\alpha_{n+2k} = d_{n+2k}. \tag{7.10}$$

Step 8: We now use the norm identity proved in the course of the proof of [7, Lemma 9.4]. Rewritten in the present notation, it states that whenever the quantities involved are defined,

$$\big\langle \mathbf{u}^{[k+1]}, P_n^{[k+1]}P_m^{[k+1]}\big\rangle = \alpha\,\frac{d_n^{[k]}}{\gamma_{n+1}}\,\big\langle \mathbf{u}^{[k]}, (P_{n+1}^{[k]})^2\big\rangle\delta_{n,m}. \tag{7.11}$$

The proof of this identity in [7, Lemma 9.4] involves only finitely many degrees: for fixed k and n, it uses only the transformed relation at level k, the derived polynomials up to index $n+1$, and the orthogonality relations for polynomials whose indices do not exceed $n+k+1$. In particular, no passage to an infinite family is used at that point. Therefore the same computation remains valid in the present finite setting whenever

$$n+k+1 \leq N+1.$$

Step 9: We claim that, for $k = 0, 1, \ldots, N$, the sequence $P_0^{[k]}, P_1^{[k]}, \ldots, P_{N+1-k}^{[k]}$ is orthogonal with respect to $\mathbf{u}^{[k]}$, and that

$$\big\langle \mathbf{u}^{[k]}, (P_n^{[k]})^2\big\rangle \neq 0$$

for $n = 0, 1, \ldots, N+1-k$. For $k = 0$, this is precisely the assumption that $(P_n)_{n\in I}$ is orthogonal with respect to $\mathbf{u}$. Assume it holds for some $k \leq N-1$. The local form of the argument in [7, Lemma 9.4], applied at level k, shows that the sequence $P_0^{[k+1]}, P_1^{[k+1]}, \ldots, P_{N-k}^{[k+1]}$ is orthogonal with respect to $\mathbf{u}^{[k+1]}$. The same computation gives, for $0 \leq n \leq N-k-1$,

$$\big\langle \mathbf{u}^{[k+1]}, (P_n^{[k+1]})^2\big\rangle = \alpha\,\frac{d_n^{[k]}}{\gamma_{n+1}}\,\big\langle \mathbf{u}^{[k]}, (P_{n+1}^{[k]})^2\big\rangle.$$

Since the same local argument identifies the derived family as the monic orthogonal polynomial sequence associated with $\mathbf{u}^{[k+1]}$, the squared norms on the left are non-zero. Since the squared norms at level k are non-zero by the induction hypothesis, and since $\alpha \neq 0$ and $\gamma_{n+1} \neq 0$, the identity forces

$$d_n^{[k]} \neq 0$$

for $0 \leq n \leq N-k-1$. This proves the induction step.

Step 10: Fix $n \in K_I$. By Step 9, the sequence $P_0^{[n]}, P_1^{[n]}, \ldots, P_{N+1-n}^{[n]}$ is orthogonal with respect to $\mathbf{u}^{[n]}$, with non-zero squared norms. Hence its first recurrence coefficients $B_0^{[n]}$ and $C_1^{[n]}$ are well defined, with

$$C_1^{[n]} \neq 0.$$

Applied to the transformed pair $(\phi^{[n]}, \psi^{[n]})$ and to the orthogonal polynomial sequence associated with $\mathbf{u}^{[n]}$, the computation carried out in the proof of [7, Theorem 9.2] yields

$$B_0^{[n]} = c - \frac{\gamma_1 e_n}{d_0^{[n]}},$$

and

$$C_1^{[n]} = -\frac{\gamma_1}{d_1^{[n]}}\,\phi^{[n]}\left(c - \frac{e_n}{d_0^{[n]}}\right). \tag{7.12}$$

Using (7.10), we have

$$d_0^{[n]} = d_{2n}, \quad d_1^{[n]} = d_{2n+1}.$$

Since the displayed formula for $B_0^{[n]}$ is well defined, it follows that $d_{2n} \neq 0$. Since the displayed formula for $C_1^{[n]}$ is well defined and $C_1^{[n]} \neq 0$, it follows that

$$d_{2n+1} \neq 0, \quad \phi^{[n]}\left(c - \frac{e_n}{d_{2n}}\right) \neq 0.$$

As n is arbitrary in K_I, we conclude that $d_j \neq 0$ for every $j \in J_I$, and

$$\phi^{[n]}\left(c - \frac{e_n}{d_{2n}}\right) \neq 0$$

for every $n \in K_I$. This proves the necessity of the stated non-vanishing conditions when $I = \{0, 1, \ldots, N+1\}$.

This establishes the theorem in the finite case. Indeed, under the stated non-vanishing conditions, Step 4 yields a monic family satisfying the recurrence with the explicit coefficients displayed in the statement. Conversely, Step 10 shows that the stated non-vanishing conditions hold, and therefore Step 4 may be applied to the same data. The monic family thereby obtained satisfies the same orthogonality conditions as the given one, and by the standard uniqueness of the monic orthogonal polynomial sequence indexed by I, the two families coincide. Hence the recurrence coefficients are necessarily the coefficients displayed in the statement.

Step 11: The case $I = \mathbb{N}$ follows by applying the preceding finite argument to arbitrary finite initial segments. Since the non-vanishing conditions are then assumed for all relevant indices, the construction produces polynomials of every degree, and the necessity argument applies to every finite truncation. This completes the proof. □

Remark 7.2. *In the q-exponential case, the classification theorem allows the coefficients a_{s_0} and b_{s_0} to depend on the full-step arithmetic progression $s_0 + h\mathbb{Z} \subseteq U$. However, once a single regular functional $\mathbf{u}$ is fixed, this freedom is not completely arbitrary. Assume that the same admissible structure (U, X, h), together with the same regular functional $\mathbf{u}$, permits the application of Theorem 7.1 after restricting X to two full-step arithmetic progressions*

$$s_0 + h\mathbb{Z} \subseteq U, \quad s_0' + h\mathbb{Z} \subseteq U.$$

Set

$$\Pi_{s_0} = a_{s_0} b_{s_0}.$$

Since the monic orthogonal polynomial sequence associated with $\mathbf{u}$ is unique, the recurrence coefficient C_2 must be independent of the chosen progression. Taking $n = 1$ in Theorem 7.1, one obtains

$$C_2 = -\frac{\gamma_2 d_0}{d_1 d_3}\,\phi^{[1]}\left(c - \frac{e_1}{d_2}\right).$$

The only dependence on the chosen full-step progression in this expression comes through Π_{s_0}. *Indeed,*

$$\phi^{[1]}(x) = \big(\psi_1(\alpha^2-1)\gamma_2 + \phi_2\alpha_2\big)\big((x-c)^2 - 2\Pi_{s_0}\big)$$
$$+ \big(\phi'(c)\alpha_1 + \psi(c)(\alpha^2-1)\gamma_1\big)(x-c) + \phi(c) + 2\phi_2\Pi_{s_0}.$$

Thus the coefficient multiplying Π_{s_0} *in* $\phi^{[1]}$ *is*

$$-2\big(\psi_1(\alpha^2-1)\gamma_2 + \phi_2\alpha_2\big) + 2\phi_2.$$

Using

$$\gamma_2 = 2\alpha, \quad \alpha_2 = 2\alpha^2 - 1,$$

this coefficient becomes

$$-4(\alpha^2-1)(\phi_2 + \alpha\psi_1) = -4(\alpha^2-1)d_1.$$

Consequently, the coefficient multiplying Π_{s_0} *in* C_2 *is*

$$-\frac{\gamma_2 d_0}{d_1 d_3}\big(-4(\alpha^2-1)d_1\big) = \frac{4\gamma_2 d_0(\alpha^2-1)}{d_3}.$$

Under the regularity assumptions, one has $d_0 \neq 0$ *and* $d_3 \neq 0$. *Moreover, in the present* q*-exponential regime* $q \neq \pm 1$, *and hence*

$$\gamma_2 \neq 0, \quad \alpha^2 - 1 \neq 0.$$

Therefore C_2 *depends on* Π_{s_0} *through a non-zero linear coefficient. Since* C_2 *is a global recurrence coefficient, it cannot depend on the chosen full-step progression. It follows that*

$$\Pi_{s_0} = \Pi_{s_0'}.$$

Thus, although the individual coefficients a_{s_0} *and* b_{s_0} *may depend on the chosen progression, their product* $a_{s_0}b_{s_0}$ *is forced by the global orthogonal polynomial sequence. In particular, the argument does not imply that* a_{s_0} *and* b_{s_0} *are separately independent of the progression. The equality* $a_{s_0}b_{s_0} = a_{s_0'}b_{s_0'}$ *is therefore a necessary compatibility condition imposed by the existence of a single regular functional* $\mathbf{u}$ *governing the whole construction. It rules out mutually incompatible local descriptions, but it should not be read as a sufficient condition for assembling those local descriptions into a single functional and a single orthogonal polynomial sequence.*

The role of U *should be understood in this operator-theoretic sense. The functional* $\mathbf{u}$ *is an element of* $\mathcal{P}'$, *not a functional on* U. *The set* U, *together with* X *and* h, *enters through the definition of the operators* D *and* S. *It becomes visible at the level of support only when* $\mathbf{u}$ *is represented by a measure, or by a discrete sum, involving the points* $X(s)$. *This is the situation alluded to in the introduction: different full-step progressions in the parameter set may contribute to one and the same representing support, without producing different orthogonal polynomial sequences.*

Remark 7.3. *In the* q*-exponential case, the regularity criterion of Theorem 7.1 shows that the degenerate possibilities* $\phi = 0$ *and* $\psi = 0$, *although not excluded a priori by Definition 6.6, cannot occur in any regular situation covered by that theorem. Indeed, if there exists a monic orthogonal polynomial sequence* $(P_n)_{n\in I}$ *with respect to* $\mathbf{u}$, *then the theorem gives in particular*

$$d_0 \neq 0.$$

Since

$$d_0 = \phi_2\gamma_0 + \psi_1\alpha_0 = \psi_1,$$

it follows that $\psi_1 \neq 0$. *Hence* $\deg\psi = 1$. *Writing*

$$\psi(x) = \psi_1\left(x + \frac{\psi_0}{\psi_1}\right),$$

one also has $e_0 = \psi_1 c + \psi_0$, *and therefore*
$$c - \frac{e_0}{d_0} = -\frac{\psi_0}{\psi_1}.$$
For $n = 0$, *the transformed polynomial* $\phi^{[0]}$ *reduces to* ϕ, *since*
$$\phi^{[0]}(x) = \phi_2(x-c)^2 + \phi'(c)(x-c) + \phi(c) = \phi(x).$$
Thus the regularity criterion gives
$$0 \neq \phi^{[0]}\left(c - \frac{e_0}{d_0}\right) = \phi\left(-\frac{\psi_0}{\psi_1}\right).$$
In particular, ϕ *is not the zero polynomial. Therefore, in the* q*-exponential case, any classical functional in the sense of Definition 6.6 to which Theorem 7.1 applies necessarily satisfies*
$$\deg\psi = 1, \quad \phi\left(-\frac{\psi_0}{\psi_1}\right) \neq 0.$$
Consequently, neither $\phi = 0$ *nor* $\psi = 0$ *can occur.*

7.2. **Quadratic map.** Although the following theorem may, at a purely formal level, be regarded as the degeneration $q \to 1$ of Theorem 7.1, the argument is not obtained by a mere passage to the limit in the stated formulas. Indeed, the parameters governing the q-quadratic description along the arithmetic progression do not remain stable as $q \to 1$, and the transformed quantities governing regularity appear only after a renormalised limiting process and a careful analysis of the cancellations involved. For this reason, and since the cancellation mechanism leading to the quadratic coefficients is not usually written out in detail, we provide the derivation in full.

Theorem 7.4. *Fix* $h \in \mathbb{C}^\times$, *let* $U \subseteq \mathbb{C}$ *be half-step-invariant, and let* $X : U \to \mathbb{C}$ *be an admissible map. Let* $D, S : \mathcal{P} \to \mathcal{P}$ *be the associated divided-difference and averaging operators, with transposes* $\mathbf{D}, \mathbf{S} : \mathcal{P}' \to \mathcal{P}'$. *Let* $\mathbf{u} \in \mathcal{P}'$ *be such that* $\langle \mathbf{u}, 1\rangle \neq 0$, *and assume that there exist polynomials* ϕ *and* ψ, *where* ϕ *has degree at most* 2 *and* ψ *has degree at most* 1, *not both identically zero, such that*
$$\mathbf{D}(\phi\,\mathbf{u}) = \mathbf{S}(\psi\,\mathbf{u})$$
in $\mathcal{P}'$. *Write*
$$\phi(x) = \phi_2 x^2 + \phi_1 x + \phi_0, \quad \psi(x) = \psi_1 x + \psi_0.$$
Fix $s_0 \in U$, *and assume that, on the arithmetic progression* $s_0 + h\mathbb{Z} \subseteq U$, *there exist* $a, b_{s_0}, c_{s_0} \in \mathbb{C}$ *such that*
$$X(s_0 + kh) = ak^2 + b_{s_0}k + c_{s_0},$$
for every $k \in \mathbb{Z}$. *Set*
$$d_n = \phi_2 n + \psi_1, \quad e_n = \phi_1 n + \psi_0 + \frac{1}{2}\,a\psi_1 n^2,$$
and define
$$\phi^{[n]}(x) = \phi_2 x^2 + \left(\phi_1 + \frac{3}{2}\,an\,d_n\right)x$$
$$+ \phi\left(\frac{1}{4}\,an^2\right) + \frac{1}{2}\,an\,\psi\left(\frac{1}{4}\,an^2\right) + \frac{n}{4}\big(b_{s_0}^2 - 4ac_{s_0}\big)d_n.$$
Let $I \subseteq \mathbb{N}$ *be either* $I = \mathbb{N}$, *or* $I = \{0, 1, \ldots, N+1\}$ *for some* $N \in \mathbb{N}$. *In the finite case set*
$$J_I = \{0, 1, \ldots, 2N+1\}, \quad K_I = \{0, 1, \ldots, N\},$$
whereas in the infinite case set
$$J_I = K_I = \mathbb{N}.$$

Then there exists a monic orthogonal polynomial sequence $(P_n)_{n\in I}$ with respect to $\mathbf{u}$ if and only if

$$d_j \neq 0,$$

for every $j \in J_I$ and

$$\phi^{[n]}\left(-\frac{1}{4}\,an^2 - \frac{e_n}{d_{2n}}\right) \neq 0,$$

for every $n \in K_I$. Whenever this holds, the corresponding monic orthogonal polynomial sequence is uniquely determined by $P_{-1} = 0$, $P_0 = 1$, and by the recurrence

$$P_{n+1}(x) = (x - B_n)P_n(x) - C_nP_{n-1}(x),$$

for every $n \in K_I$, with the convention that the term involving C_0P_{-1} is void. The coefficients are given by

$$B_0 = -\frac{e_0}{d_0}, \quad C_1 = -\frac{1}{d_1}\,\phi^{[0]}\left(-\frac{e_0}{d_0}\right),$$

and, for every $n \in K_I^\times$,

$$B_n = \frac{n\,e_{n-1}}{d_{2n-2}} - \frac{(n+1)e_n}{d_{2n}} - \frac{1}{2}\,an(n-1),$$

$$C_{n+1} = -\frac{(n+1)d_{n-1}}{d_{2n-1}d_{2n+1}}\,\phi^{[n]}\left(-\frac{1}{4}\,an^2 - \frac{e_n}{d_{2n}}\right).$$

In the finite case, the last coefficient C_{N+1} is the terminal norm coefficient; it is not used to construct a polynomial P_{N+2}.

Proof. We follow the same scheme as in the proof of Theorem 7.1. We first identify the quadratic analogues of the transformed quantities $\psi^{[n]}$, $\phi^{[n]}$, and of the recurrence coefficients B_n, C_{n+1}. Once these have been identified, the remainder of the argument is the same as in the q-exponential case.

Fix $s_0 \in U$, and abbreviate $b = b_{s_0}$ and $c = c_{s_0}$. Thus, on the arithmetic progression $s_0 + h\mathbb{Z} \subseteq U$, one has

$$X(s_0 + kh) = ak^2 + bk + c.$$

We first treat the case where $I = \{0, 1, \dots, N+1\}$ for some $N \in \mathbb{N}$. The case $I = \mathbb{N}$ will then follow by applying the same argument to arbitrary finite initial segments.

Step 1: For $\varepsilon \neq 0$, set

$$q_\varepsilon = e^\varepsilon, \quad a_\varepsilon = \frac{a}{\varepsilon^2} + \frac{b}{2\varepsilon}, \quad b_\varepsilon = \frac{a}{\varepsilon^2} - \frac{b}{2\varepsilon}, \quad c_\varepsilon = c - \frac{2a}{\varepsilon^2},$$

and define

$$X_\varepsilon(s_0 + kh) = a_\varepsilon q_\varepsilon^k + b_\varepsilon q_\varepsilon^{-k} + c_\varepsilon,$$

for every $k \in \mathbb{Z}$. Using the expansions

$$e^{k\varepsilon} = 1 + k\varepsilon + \frac{k^2\varepsilon^2}{2} + O(\varepsilon^3), \quad e^{-k\varepsilon} = 1 - k\varepsilon + \frac{k^2\varepsilon^2}{2} + O(\varepsilon^3),$$

we obtain

$$X_\varepsilon(s_0 + kh) = \left(\frac{a}{\varepsilon^2} + \frac{b}{2\varepsilon}\right)\left(1 + k\varepsilon + \frac{k^2\varepsilon^2}{2} + O(\varepsilon^3)\right)$$

$$+ \left(\frac{a}{\varepsilon^2} - \frac{b}{2\varepsilon}\right)\left(1 - k\varepsilon + \frac{k^2\varepsilon^2}{2} + O(\varepsilon^3)\right) + c - \frac{2a}{\varepsilon^2}.$$

The terms of order ε^{-2} and ε^{-1} cancel, and the terms of order ε cancel as well. Hence

$$X_\varepsilon(s_0 + kh) = ak^2 + bk + c + O(\varepsilon^2),$$

for each fixed $k \in \mathbb{Z}$. Therefore

$$X_\varepsilon(s_0 + kh) \longrightarrow X(s_0 + kh)$$

as $\varepsilon \to 0$.

Step 2: For the parametrising map X_ε, the quantities appearing in Theorem 7.1 are

$$\alpha_n^{(\varepsilon)} = \frac{q_\varepsilon^{n/2} + q_\varepsilon^{-n/2}}{2} = \cosh\Big(\frac{n\varepsilon}{2}\Big), \quad \gamma_n^{(\varepsilon)} = \frac{q_\varepsilon^{n/2} - q_\varepsilon^{-n/2}}{q_\varepsilon^{1/2} - q_\varepsilon^{-1/2}} = \frac{\sinh\Big(\frac{n\varepsilon}{2}\Big)}{\sinh\Big(\frac{\varepsilon}{2}\Big)}.$$

Expanding at $\varepsilon = 0$, one finds

$$\alpha_n^{(\varepsilon)} = 1 + \frac{n^2\varepsilon^2}{8} + O(\varepsilon^4), \quad \gamma_n^{(\varepsilon)} = n + \frac{n(n^2-1)\varepsilon^2}{24} + O(\varepsilon^4).$$

In particular,

$$\alpha_n^{(\varepsilon)} \longrightarrow 1, \quad \gamma_n^{(\varepsilon)} \longrightarrow n,$$

for every fixed $n \in \mathbb{Z}$. Accordingly, the quantities

$$d_n^{(\varepsilon)} = \phi_2 \gamma_n^{(\varepsilon)} + \psi_1 \alpha_n^{(\varepsilon)}$$

satisfy

$$d_n^{(\varepsilon)} \longrightarrow \phi_2 n + \psi_1 = d_n.$$

For the q-exponential theorem, the transformed affine factor is

$$\psi_\varepsilon^{[n]}(x) = d_{2n}^{(\varepsilon)}(x - c_\varepsilon) + e_n^{(\varepsilon)},$$

where

$$e_n^{(\varepsilon)} = (2\phi_2 c_\varepsilon + \phi_1)\gamma_n^{(\varepsilon)} + (\psi_1 c_\varepsilon + \psi_0)\alpha_n^{(\varepsilon)}.$$

The quantity $e_n^{(\varepsilon)}$ diverges termwise, since

$$c_\varepsilon = c - \frac{2a}{\varepsilon^2}.$$

What matters is not $e_n^{(\varepsilon)}$ itself, but the root of $\psi_\varepsilon^{[n]}$, namely

$$c_\varepsilon - \frac{e_n^{(\varepsilon)}}{d_{2n}^{(\varepsilon)}}.$$

Using the expansions above and collecting the singular terms, one checks that

$$c_\varepsilon - \frac{e_n^{(\varepsilon)}}{d_{2n}^{(\varepsilon)}} = -\frac{1}{4}\,an^2 - \frac{e_n}{d_{2n}} + O(\varepsilon^2),$$

where

$$e_n = \phi_1 n + \psi_0 + \frac{1}{2}\,a\psi_1 n^2.$$

Thus

$$c_\varepsilon - \frac{e_n^{(\varepsilon)}}{d_{2n}^{(\varepsilon)}} \longrightarrow -\frac{1}{4}\,an^2 - \frac{e_n}{d_{2n}}.$$

This leads us to define, in the quadratic case,

$$\psi^{[n]}(x) = d_{2n}\left(x + \frac{1}{4}\,an^2\right) + e_n,$$

so that its unique root is exactly the limiting value just obtained, namely

$$-\frac{1}{4}\,an^2 - \frac{e_n}{d_{2n}}.$$

Step 3: For the q-exponential map X_ε, the transformed polynomial from Theorem 7.1 is

$$\phi_\varepsilon^{[n]}(x) = \big(\psi_1((\alpha^{(\varepsilon)})^2-1)\gamma_{2n}^{(\varepsilon)} + \phi_2\alpha_{2n}^{(\varepsilon)}\big)\big((x-c_\varepsilon)^2 - 2a_\varepsilon b_\varepsilon\big)$$
$$+ \big(\phi'(c_\varepsilon)\alpha_n^{(\varepsilon)} + \psi(c_\varepsilon)((\alpha^{(\varepsilon)})^2-1)\gamma_n^{(\varepsilon)}\big)(x-c_\varepsilon) + \phi(c_\varepsilon) + 2\phi_2 a_\varepsilon b_\varepsilon,$$

where

$$\alpha^{(\varepsilon)} = \alpha_1^{(\varepsilon)}.$$

We rewrite this as

$$\phi_\varepsilon^{[n]}(x) = A_\varepsilon\big((x-c_\varepsilon)^2 - 2a_\varepsilon b_\varepsilon\big) + B_\varepsilon(x-c_\varepsilon) + C_\varepsilon,$$

where

$$A_\varepsilon = \psi_1((\alpha^{(\varepsilon)})^2-1)\gamma_{2n}^{(\varepsilon)} + \phi_2\alpha_{2n}^{(\varepsilon)}, \quad B_\varepsilon = \phi'(c_\varepsilon)\alpha_n^{(\varepsilon)} + \psi(c_\varepsilon)((\alpha^{(\varepsilon)})^2-1)\gamma_n^{(\varepsilon)},$$
$$C_\varepsilon = \phi(c_\varepsilon) + 2\phi_2 a_\varepsilon b_\varepsilon.$$

From Step 2, one has

$$(\alpha^{(\varepsilon)})^2 - 1 = \frac{\varepsilon^2}{4} + O(\varepsilon^4), \quad \gamma_{2n}^{(\varepsilon)} = 2n + O(\varepsilon^2), \quad \alpha_{2n}^{(\varepsilon)} = 1 + \frac{n^2\varepsilon^2}{2} + O(\varepsilon^4),$$

and therefore

$$\begin{aligned} A_\varepsilon &= \psi_1\big((\alpha^{(\varepsilon)})^2-1\big)\gamma_{2n}^{(\varepsilon)} + \phi_2\alpha_{2n}^{(\varepsilon)} \\ &= \psi_1\left(\frac{\varepsilon^2}{4} + O(\varepsilon^4)\right)\big(2n + O(\varepsilon^2)\big) + \phi_2\left(1 + \frac{n^2\varepsilon^2}{2} + O(\varepsilon^4)\right) \\ &= \phi_2 + \frac{1}{2}\big(\phi_2 n^2 + \psi_1 n\big)\varepsilon^2 + O(\varepsilon^4). \end{aligned}$$

Likewise,

$$2a_\varepsilon b_\varepsilon = 2\left(\frac{a}{\varepsilon^2} + \frac{b}{2\varepsilon}\right)\left(\frac{a}{\varepsilon^2} - \frac{b}{2\varepsilon}\right) = \frac{2a^2}{\varepsilon^4} - \frac{b^2}{2\varepsilon^2},$$

and

$$x - c_\varepsilon = x - c + \frac{2a}{\varepsilon^2}.$$

Substituting these expressions into $\phi_\varepsilon^{[n]}(x)$, we expand each of the three contributions $A_\varepsilon\big((x-c_\varepsilon)^2 - 2a_\varepsilon b_\varepsilon\big)$, $B_\varepsilon(x-c_\varepsilon)$, and C_ε separately. First,

$$(x-c_\varepsilon)^2 = (x-c)^2 + \frac{4a}{\varepsilon^2}(x-c) + \frac{4a^2}{\varepsilon^4},$$

whereas

$$2a_\varepsilon b_\varepsilon = \frac{2a^2}{\varepsilon^4} - \frac{b^2}{2\varepsilon^2}.$$

Hence

$$(x-c_\varepsilon)^2 - 2a_\varepsilon b_\varepsilon = (x-c)^2 + \frac{4a}{\varepsilon^2}(x-c) + \frac{2a^2}{\varepsilon^4} + \frac{b^2}{2\varepsilon^2}.$$

Multiplying by

$$A_\varepsilon = \phi_2 + \frac{1}{2}(\phi_2 n^2 + \psi_1 n)\varepsilon^2 + O(\varepsilon^4),$$

we obtain

$$A_\varepsilon\big((x-c_\varepsilon)^2-2a_\varepsilon b_\varepsilon\big)=\phi_2(x-c)^2+\frac{4a\phi_2}{\varepsilon^2}(x-c)+\frac{2a^2\phi_2}{\varepsilon^4}+\frac{b^2\phi_2}{2\varepsilon^2}$$
$$+2a(\phi_2n^2+\psi_1n)(x-c)+a^2(\phi_2n^2+\psi_1n)\varepsilon^{-2}$$
$$+\frac{1}{4}b^2(\phi_2n^2+\psi_1n)+O(\varepsilon^2).$$

Next, using

$$\phi'(c_\varepsilon)=2\phi_2c_\varepsilon+\phi_1=-\frac{4a\phi_2}{\varepsilon^2}+2\phi_2c+\phi_1,$$

and

$$\psi(c_\varepsilon)=\psi_1c_\varepsilon+\psi_0=-\frac{2a\psi_1}{\varepsilon^2}+\psi_1c+\psi_0,$$

together with

$$\alpha_n^{(\varepsilon)}=1+\frac{n^2\varepsilon^2}{8}+O(\varepsilon^4),\quad (\alpha^{(\varepsilon)})^2-1=\frac{\varepsilon^2}{4}+O(\varepsilon^4),\quad \gamma_n^{(\varepsilon)}=n+O(\varepsilon^2),$$

we get

$$B_\varepsilon=-\frac{4a\phi_2}{\varepsilon^2}+\big(2\phi_2c+\phi_1\big)-\frac{an^2\phi_2}{2}-\frac{an\psi_1}{2}+O(\varepsilon^2).$$

Since

$$x-c_\varepsilon=(x-c)+\frac{2a}{\varepsilon^2},$$

it follows that

$$B_\varepsilon(x-c_\varepsilon)=-\frac{4a\phi_2}{\varepsilon^2}(x-c)-\frac{8a^2\phi_2}{\varepsilon^4}$$
$$+\Big(2\phi_2c+\phi_1-\frac{an^2\phi_2}{2}-\frac{an\psi_1}{2}\Big)(x-c)$$
$$+\frac{2a}{\varepsilon^2}\Big(2\phi_2c+\phi_1-\frac{an^2\phi_2}{2}-\frac{an\psi_1}{2}\Big)+O(\varepsilon^2).$$

Finally,

$$C_\varepsilon=\phi(c_\varepsilon)+2\phi_2a_\varepsilon b_\varepsilon=\phi_2c_\varepsilon^2+\phi_1c_\varepsilon+\phi_0+\frac{2a^2\phi_2}{\varepsilon^4}-\frac{b^2\phi_2}{2\varepsilon^2}.$$

Since

$$c_\varepsilon=c-\frac{2a}{\varepsilon^2},$$

this becomes

$$C_\varepsilon=\frac{6a^2\phi_2}{\varepsilon^4}-\frac{(4ac\phi_2+2a\phi_1+b^2\phi_2/2)}{\varepsilon^2}+\phi(c)+O(\varepsilon^2).$$

Adding the three contributions, the coefficients of ε^{-4} and ε^{-2} cancel identically, and the remaining finite part is precisely

$$\phi^{[n]}(x)=\phi_2x^2+\Big(\phi_1+\frac{3}{2}an\,d_n\Big)x+\phi\Big(\frac{1}{4}an^2\Big)+\frac{1}{2}an\,\psi\Big(\frac{1}{4}an^2\Big)+\frac{n}{4}\big(b^2-4ac\big)d_n.$$

Therefore

$$\phi_\varepsilon^{[n]}(x)=\phi^{[n]}(x)+O(\varepsilon^2)$$

coefficientwise.

Step 4: We next pass to the limit in the explicit recurrence coefficients from Theorem 7.1. For $n = 0$, that theorem gives

$$B_0^{(\varepsilon)} = c_\varepsilon - \frac{\gamma_1^{(\varepsilon)} e_0^{(\varepsilon)}}{d_0^{(\varepsilon)}}.$$

Since $\gamma_1^{(\varepsilon)} = 1$, the same cancellation as in Step 2 shows that

$$B_0^{(\varepsilon)} \longrightarrow -\frac{e_0}{d_0} = B_0.$$

For every $n \in K_I^\times$,

$$B_n^{(\varepsilon)} = c_\varepsilon + \frac{\gamma_n^{(\varepsilon)} e_{n-1}^{(\varepsilon)}}{d_{2n-2}^{(\varepsilon)}} - \frac{\gamma_{n+1}^{(\varepsilon)} e_n^{(\varepsilon)}}{d_{2n}^{(\varepsilon)}}.$$

Substituting the expansions from Steps 2 and 3, and once more cancelling the singular terms, one obtains

$$B_n^{(\varepsilon)} \longrightarrow \frac{n\, e_{n-1}}{d_{2n-2}} - \frac{(n+1)e_n}{d_{2n}} - \frac{1}{2}\, an(n-1) = B_n.$$

Similarly,

$$C_{n+1}^{(\varepsilon)} = -\frac{\gamma_{n+1}^{(\varepsilon)} d_{n-1}^{(\varepsilon)}}{d_{2n-1}^{(\varepsilon)} d_{2n+1}^{(\varepsilon)}}\, \phi_\varepsilon^{[n]}\left(c_\varepsilon - \frac{e_n^{(\varepsilon)}}{d_{2n}^{(\varepsilon)}}\right).$$

We have

$$\gamma_{n+1}^{(\varepsilon)} \to n+1, \quad d_m^{(\varepsilon)} \to d_m,$$

and, by the preceding computation,

$$c_\varepsilon - \frac{e_n^{(\varepsilon)}}{d_{2n}^{(\varepsilon)}} \longrightarrow -\frac{1}{4}\, an^2 - \frac{e_n}{d_{2n}}.$$

Moreover,

$$\phi_\varepsilon^{[n]}(x) \to \phi^{[n]}(x)$$

coefficientwise. Hence

$$C_{n+1}^{(\varepsilon)} \longrightarrow -\frac{(n+1)d_{n-1}}{d_{2n-1}d_{2n+1}}\, \phi^{[n]}\left(-\frac{1}{4}\, an^2 - \frac{e_n}{d_{2n}}\right) = C_{n+1}.$$

Step 5: At this point, the quadratic analogues of the explicit quantities occurring in the proof of Theorem 7.1 have been identified. More precisely, the role of

$$\psi^{[n]}(x) = d_{2n}(x-c) + e_n$$

there is now played by

$$\psi^{[n]}(x) = d_{2n}\left(x + \frac{1}{4}\, an^2\right) + e_n,$$

the role of the transformed polynomial is played by the polynomial $\phi^{[n]}$ obtained in Step 3, the role played there by γ_n is here played by n, and the recurrence coefficients are those computed in Step 4. The identities used in the Rodrigues construction and in the norm computation are polynomial identities in the coefficients of the transformed pair $(\phi^{[n]}, \psi^{[n]})$, in the quantities α_j, γ_j, and in the recurrence coefficients. The preceding steps identify the coefficientwise limits of all these quantities as $q_\varepsilon \to 1$, with

$$\alpha_j^{(\varepsilon)} \to 1, \quad \gamma_j^{(\varepsilon)} \to j.$$

Consequently the same algebraic identities hold in the quadratic case, with γ_j replaced by j. This is precisely the limiting mechanism behind the quadratic-lattice regularity theorem stated in [7, Theorem 9.3]; here it has been written out in the notation of the present admissible-map framework. Thus the inductive construction of the polynomials, the orthogonality argument, and the norm computation are identical to those in the proof of Theorem 7.1, after the substitutions displayed above. In particular, for the finite initial segment $I = \{0, 1, \ldots, N+1\}$, one obtains that there exists a monic orthogonal polynomial sequence $(P_n)_{n\in I}$ with respect to $\mathbf{u}$ if and only if $d_j \neq 0$ for every $j \in J_I$, and

$$\phi^{[n]}\left(-\frac{1}{4}\,an^2 - \frac{e_n}{d_{2n}}\right) \neq 0$$

for every $n \in K_I$. Moreover, whenever this holds, the corresponding monic orthogonal polynomial sequence is determined by $P_{-1} = 0$, $P_0 = 1$, and by the recurrence

$$P_{n+1}(x) = (x - B_n)P_n(x) - C_n P_{n-1}(x),$$

for every $n \in K_I$, with the coefficients B_n and C_{n+1} given in the statement. □

Remark 7.5. *Assume that the same admissible structure (U, X, h), together with the same regular functional $\mathbf{u}$, permits the application of Theorem 7.4 after restricting X to two full-step arithmetic progressions*

$$s_0 + h\mathbb{Z} \subseteq U, \quad s_0' + h\mathbb{Z} \subseteq U.$$

Since the monic orthogonal polynomial sequence associated with a regular functional is unique, the recurrence coefficient C_2 must be independent of the chosen progression. We claim that this already forces the discriminant-like quantity

$$\Delta_{s_0} = b_{s_0}^2 - 4ac_{s_0}$$

to be independent of the chosen progression. Indeed, the formula for C_2 in Theorem 7.4 gives

$$C_2 = -\frac{2d_0}{d_1 d_3}\,\phi^{[1]}\left(-\frac{1}{4}\,a - \frac{e_1}{d_2}\right).$$

In this expression the only dependence on the chosen full-step progression comes from the final term in $\phi^{[1]}$, namely

$$\frac{1}{4}\big(b_{s_0}^2 - 4ac_{s_0}\big)d_1.$$

Therefore the coefficient of Δ_{s_0} in C_2 is

$$-\frac{2d_0}{d_1 d_3} \cdot \frac{1}{4}d_1 = -\frac{d_0}{2d_3}.$$

By the regularity assumptions of Theorem 7.4, the quantities d_0 and d_3 are non-zero. Hence this coefficient is non-zero. Since C_2 is a global recurrence coefficient, it has the same value when computed from either progression. Consequently,

$$\Delta_{s_0} = \Delta_{s_0'}.$$

Here the coefficient a is already global, whereas b_{s_0} and c_{s_0} may a priori depend on the chosen arithmetic progression. Thus the equality of the discriminant-like quantities is a necessary compatibility condition imposed by the existence of a single global regular functional $\mathbf{u}$.

As in Remark 7.2, this statement should be understood in the operator-theoretic sense. The functional $\mathbf{u}$ is an element of $\mathcal{P}'$, not a functional on U. The different full-step progressions enter through the same admissible structure (U, X, h), and hence through the same operators D and S. They may become visible at the level of support only when $\mathbf{u}$ is represented by a measure, or by a discrete sum, involving the points $X(s)$.

Remark 7.6. *In the quadratic case, the regularity criterion of Theorem 7.4 likewise shows that the degenerate possibilities $\phi = 0$ and $\psi = 0$, although not excluded a priori by Definition 6.6, cannot occur in any regular situation covered by that theorem. Indeed, if there* *exists a monic orthogonal polynomial sequence* $(P_n)_{n\in I}$ *with respect to* $\mathbf{u}$*, then the theorem gives in particular*

$$d_0 \neq 0.$$

Since

$$d_0 = \phi_2 \cdot 0 + \psi_1 = \psi_1,$$

it follows that $\psi_1 \neq 0$*. Hence* $\deg \psi = 1$*. Writing*

$$\psi(x) = \psi_1 \left(x + \frac{\psi_0}{\psi_1} \right),$$

one has $e_0 = \psi_0$*, and therefore*

$$-\frac{e_0}{d_0} = -\frac{\psi_0}{\psi_1}.$$

Moreover, for $n = 0$*, the transformed polynomial* $\phi^{[0]}$ *is exactly* ϕ*, since*

$$\phi^{[0]}(x) = \phi_2 x^2 + \phi_1 x + \phi_0 = \phi(x).$$

Hence the regularity criterion yields

$$0 \neq \phi^{[0]} \left(-\frac{e_0}{d_0} \right) = \phi \left(-\frac{\psi_0}{\psi_1} \right).$$

In particular, ϕ *is not the zero polynomial. Therefore, in the quadratic case, any classical functional in the sense of Definition 6.6 to which Theorem 7.4 applies necessarily satisfies*

$$\deg \psi = 1, \quad \phi \left(-\frac{\psi_0}{\psi_1} \right) \neq 0.$$

Consequently, neither $\phi = 0$ *nor* $\psi = 0$ *can occur.*

7.3. **q-exponential map: q is a root of unity.** The following corollary gives the finite torsion version of the q-exponential regularity criterion. In [6, 7] the parameter q is assumed to be a positive real number. Thus the finite torsion regime considered in this subsection is neither treated nor discussed there, and becomes accessible only after the finite version of the preceding regularity criteria has been made explicit. As noted above, the q-exponential argument extends to arbitrary complex values of q provided that the denominators which occur in the finite construction do not vanish. In particular, the infinite case is covered when $q \in \mathbb{C}^\times$ is not a root of unity. The degenerate cases $q = 1$ and $q = -1$ are treated separately: the former corresponds to the quadratic case considered in Theorem 7.4, whereas the latter belongs to the alternating regime discussed later. For finite truncations, however, certain torsion values of q are still admissible: if the order of q is large enough relative to the truncation level, the same denominator check remains valid.

Corollary 7.7. *Retain the notation of Theorem 7.1, except for the assumption that* q *is not a root of unity. Assume instead that* q *is a primitive* ν*-th root of unity, with* $\nu \geq 3$*, and let* $I = \{0, 1, \ldots, N+1\}$ *for some* $N \in \mathbb{N}$ *such that*

$$N + 1 < \nu.$$

Then, for this finite initial segment, the conclusions of Theorem 7.1 remain valid, with the same non-vanishing conditions and the same formulas for the recurrence coefficients.

Proof. We revisit the proof of Theorem 7.1 and check where the assumption that q is not a root of unity is used. Apart from the non-vanishing conditions explicitly imposed in Theorem 7.1, namely the non-vanishing of the relevant quantities d_j and of the relevant values of $\phi^{[n]}$, the only additional quantities that must be non-zero for the proof to make sense are α and the finitely many factors γ_m, with $m \in \{1, \ldots, N+1\}$. Indeed, $\alpha \neq 0$ is needed in the Rodrigues construction and in the normalisation of the polynomials. As for the factors γ_m, they occur in denominators only in the definition of the derived polynomials and in the norm identity. In those places the indices satisfy

$$j + k \leq N + 1, \quad n + k + 1 \leq N + 1.$$

Hence only factors γ_m with $m \in \{1, \ldots, N+1\}$ can occur in denominators. No denominator involving γ_m with $m > N + 1$ appears in the finite argument. Since α has already been fixed, the condition $\alpha = 0$ would imply $q = -1$, which is impossible because q is a primitive ν-th root of unity with $\nu \geq 3$. Fix a choice of $r = q^{1/2}$. Since $q \neq 1$, one has

$$r - r^{-1} \neq 0,$$

and

$$\gamma_m = \frac{r^m - r^{-m}}{r - r^{-1}}.$$

The equality $\gamma_m = 0$ is therefore equivalent to $r^{2m} = 1$. Since $r^2 = q$, this is equivalent to $q^m = 1$. If $m = 1, \ldots, N + 1$, then $m < \nu$, because $0 \leq N \leq \nu - 2$. Since q has exact order ν, it follows that

$$q^m \neq 1$$

for every $m = 1, \ldots, N + 1$, and therefore

$$\gamma_m \neq 0$$

for every $m = 1, \ldots, N+1$. Thus every additional denominator appearing in the finite proof of Theorem 7.1 remains non-zero under the present hypotheses, once the non-vanishing conditions of that theorem are imposed on the finite initial segment $I = \{0, 1, \ldots, N + 1\}$. Consequently, the same finite proof applies for this choice of I, and all the conclusions of Theorem 7.1 remain valid on that segment. ☐

Remark 7.8. *The observation made in Remark 7.2 remains valid in the finite torsion regime covered by Corollary 7.7. Indeed, the proof only uses the formula for C_2 and the non-vanishing of the factors appearing in it. Thus, for any two full-step arithmetic progressions to which the same admissible structure (U, X, h), together with the same regular functional $\mathbf{u}$, permits the application of the finite torsion criterion, one has*

$$a_{s_0} b_{s_0} = a_{s_0'} b_{s_0'}.$$

Remark 7.9. *In the finite torsion q-exponential regime covered by Corollary 7.7, the degenerate possibilities $\phi = 0$ and $\psi = 0$, although not excluded a priori by Definition 6.6, cannot occur for a classical functional. Indeed, by Definition 6.6, $\mathbf{u}$ is regular of order at least 3. Hence, in the finite torsion setting under consideration, we may work on a finite initial segment $I = \{0, 1, \ldots, N + 1\}$ with $N \in \mathbb{N}^\times$ and $N + 1 < \nu$, for instance $N = 1$. Whenever the hypotheses of Corollary 7.7 are satisfied for such a segment, the corollary gives the same non-vanishing conditions as in Theorem 7.1. Therefore, by the same argument used in the non-torsion case, neither $\phi = 0$ nor $\psi = 0$ can occur.*

The next proposition records a standard finite discrete orthogonality consequence associated with the simple zeros of the truncating polynomial P_{N+1}. The reader should keep in mind that this is a rather special situation within the present theory. Nevertheless, since even this restricted case gives rise to apparently new consequences, it is useful to pause and treat it explicitly.

Proposition 7.10. *Retain the hypotheses and notation of Corollary 7.7, and set*

$$h_n = \langle \mathbf{u}, P_n^2 \rangle,$$

for $n = 0, 1, \ldots, N$. Assume, in addition, that the polynomial P_{N+1} has $N+1$ simple zeros $x_0, x_1, \ldots, x_N$. Then the following statements hold.

(a)
$$P_N(x_s) \neq 0$$
for every $s = 0, 1, \ldots, N$.

(b) *The Christoffel numbers*
$$\lambda_s = \frac{h_N}{P_N(x_s)\, P'_{N+1}(x_s)},$$
for every $s = 0, 1, \ldots, N$, *are well defined.*

(c) *For every polynomial* p *with* $\deg p \leq 2N+1$, *one has*
$$\langle \mathbf{u}, p\rangle = \sum_{s=0}^{N} \lambda_s\, p(x_s).$$

(d) *In particular,*
$$\sum_{s=0}^{N} \lambda_s\, P_n(x_s) P_m(x_s) = h_n\, \delta_{nm}$$
for every $0 \leq n, m \leq N$.

Proof. This is the standard finite Christoffel–Darboux argument. If $N \in \mathbb{N}^\times$, then $C_N \neq 0$, and hence the consecutive polynomials P_N and P_{N+1} have no common zero. If $N = 0$, the same conclusion is immediate because $P_0 = 1$. Thus $P_N(x_s) \neq 0$, and the numbers λ_s are well defined because the zeros x_s are simple. The Gaussian quadrature formula
$$\langle \mathbf{u}, p\rangle = \sum_{s=0}^{N} \lambda_s p(x_s), \quad \deg p \leq 2N+1,$$
then follows by the usual Lagrange interpolation proof, using the orthogonality of $P_0, \ldots, P_N$ and the fact that P_{N+1} vanishes at the nodes $x_0, \ldots, x_N$. Taking $p = P_n P_m$ gives the displayed discrete orthogonality relation. □

The following example illustrates the preceding root-of-unity theory for the q-exponential map.

Example 7.11 (q-exponential map for q a root of unity)**.** *Let*
$$q = e^{2\pi i M/\nu}, \quad 1 \leq M < \nu, \quad \gcd(M, \nu) = 1, \quad \nu \geq 3,$$
and fix once and for all a choice of $q^{1/2}$ *and of* $\sqrt{AB}$, *where* $A, B \in \mathbb{C}^\times$. *Consider a full-step arithmetic progression* $s_0 + h\mathbb{Z} \subseteq U$ *on which*
$$X(s_0 + kh) = Aq^{-k} + Bq^{k} + C,$$
with $C \in \mathbb{C}$. *Let* $\mathbf{u} \in \mathcal{P}'$ *be such that* $\langle \mathbf{u}, 1\rangle \neq 0$, *and satisfy*
$$\mathbf{D}(\phi\, \mathbf{u}) = \mathbf{S}(\psi\, \mathbf{u}),$$
where
$$\phi(x) = 2(1+abcd)(x-C)^2 - 2\sqrt{AB}\,(a+b+c+d+abc+abd+acd+bcd)(x-C)$$
$$+\, 4AB\,(ab+ac+ad+bc+bd+cd-abcd-1),$$
$$\psi(x) = \frac{4q^{1/2}}{q-1}\Big((abcd-1)(x-C) + \sqrt{AB}\,(a+b+c+d-abc-abd-acd-bcd)\Big),$$
with $a, b, c, d \in \mathbb{C}$. *Set*
$$g = abcd.$$

For this case, the quantity d_n appearing in Theorem 7.1 is

$$d_n = -\frac{4}{q^{1/2} - q^{-1/2}}\, q^{-n/2}(1 - gq^n).$$

By Corollary 7.7, for each N with $0 \le N \le \nu - 2$, $\mathbf{u}$ is regular of order $N + 2$ if and only if

$$1 - gq^j \neq 0$$

for every $j = 0, 1, \ldots, 2N + 1$, and

$$AB(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - bcq^n)(1 - bdq^n)(1 - cdq^n) \neq 0$$

for $n = 0, 1, \ldots, N$. Moreover, the recurrence coefficients furnished by Theorem 7.1 are as follows. The coefficient B_0 is given by the corresponding initial formula in Theorem 7.1. For $n \in \mathbb{N}^\times$, one has

$$\begin{aligned} B_n = C + 2\sqrt{AB}\,\Bigg(a + \frac{1}{a} &- \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - gq^{n-1})}{a(1 - gq^{2n-1})(1 - gq^{2n})} \\ &- \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - gq^{2n-1})(1 - gq^{2n-2})} \Bigg), \end{aligned}$$

with the usual limiting interpretation when $a = 0$, and

$$\begin{aligned} C_{n+1} &= AB(1 - q^{n+1})(1 - gq^{n-1}) \\ &\quad \times \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - bcq^n)(1 - bdq^n)(1 - cdq^n)}{(1 - gq^{2n-1})(1 - gq^{2n})^2(1 - gq^{2n+1})}. \end{aligned}$$

Assume now that $\mathbf{u}$ is regular of order ν. In particular, the non-vanishing conditions above force

$$g \notin \{q^k : \ k = 0, 1, \ldots, \nu - 1\}.$$

In order to use the standard root-of-unity formula for the next Askey–Wilson polynomial, assume in addition that

$$a, b, c, d \in \mathbb{C}^\times, \quad ab, ac, ad, bc, bd, cd \notin \{q^k : \ k = 0, 1, \ldots, \nu - 1\}.$$

Let $P_0, P_1, \ldots, P_{\nu-1}$ be the corresponding monic orthogonal polynomial sequence. In particular, $C_n \neq 0$ for $n = 1, \ldots, \nu - 1$. These coefficients are precisely the recurrence coefficients of the monic Askey–Wilson polynomial sequence. Hence, by uniqueness of the monic polynomial sequence determined by the three-term recurrence, one has

$$P_n(x) = 2^n (AB)^{n/2} Q_n\left(\frac{x - C}{2\sqrt{AB}}; a, b, c, d \,\Big|\, q\right),$$

for $n = 0, 1, \ldots, \nu - 1$, where $(Q_n)_{n \in \mathbb{N}}$ denotes the monic Askey–Wilson polynomial sequence; see [12, (14.1.5)]. *In particular, if one writes*

$$x = C + \sqrt{AB}\,(t + t^{-1}),$$

then

$$\frac{x - C}{2\sqrt{AB}} = \frac{t + t^{-1}}{2}.$$

The denominators involving powers of g in the displayed rational factors are non-zero under the preceding non-vanishing assumptions. Moreover,

$$C_\nu = 0,$$

since the formula for C_ν contains the factor

$$1 - q^\nu = 0.$$

Thus the sequence truncates at degree $\nu-1$. Define P_ν by the same explicit Askey–Wilson formula. For the monic Askey–Wilson polynomial of degree ν one has

$$Q_\nu\left(\frac{t+t^{-1}}{2}; a, b, c, d \,\Big|\, q\right) = 2^{-\nu}\big(t^\nu + t^{-\nu} - \mathcal{E}_\nu\big),$$

where

$$\mathcal{E}_\nu = \frac{a^\nu + b^\nu + c^\nu + d^\nu - (abc)^\nu - (abd)^\nu - (acd)^\nu - (bcd)^\nu}{1-g^\nu}.$$

Substituting this into the preceding relation for P_ν, one obtains

$$\begin{aligned} P_\nu(x) &= 2^\nu (AB)^{\nu/2} Q_\nu\left(\frac{x-C}{2\sqrt{AB}}; a, b, c, d \,\Big|\, q\right) \\ &= (AB)^{\nu/2}\left(t^\nu + t^{-\nu} - \mathcal{E}_\nu\right). \end{aligned}$$

Assume, in addition, that

$$\mathcal{E}_\nu \neq \pm 2.$$

Then the zeros of P_ν are simple. More precisely, if r is chosen so that

$$r^\nu = \frac{\mathcal{E}_\nu}{2} + \sqrt{\frac{\mathcal{E}_\nu^2}{4} - 1},$$

then the zeros of P_ν are

$$\xi_s = C + \sqrt{AB}\,(rq^s + r^{-1}q^{-s}),$$

for every $s = 0, 1, \ldots, \nu-1$. Thus P_ν, although not part of the regular family of non-zero norm, is the next polynomial generated by the same monic three-term recurrence. The terminal coefficient satisfies

$$C_\nu = 0,$$

whereas $C_n \neq 0$ for every $n \in \{1, \ldots, \nu-1\}$. Consequently, P_ν is the truncating polynomial for the finite recurrence system associated with $P_0, P_1, \ldots, P_{\nu-1}$. Together with the assumption that P_ν has ν simple zeros, the usual Christoffel–Darboux and Lagrange interpolation argument yields a discrete orthogonality formula on the support $\{\xi_0, \xi_1, \ldots, \xi_{\nu-1}\}$. Writing

$$h_n = \langle \mathbf{u}, P_n^2\rangle,$$

for $n = 0, 1, \ldots, \nu-1$, one obtains

$$\sum_{s=0}^{\nu-1} \lambda_s\, P_n(\xi_s) P_m(\xi_s) = h_n\, \delta_{nm}$$

for every $0 \le n, m \le \nu-1$, where

$$\lambda_s = \frac{h_{\nu-1}}{P_{\nu-1}(\xi_s)\, P_\nu'(\xi_s)}.$$

To compute $P_\nu'(\xi_s)$, differentiate the identity

$$P_\nu(x) = (AB)^{\nu/2}\big(t^\nu + t^{-\nu} - \mathcal{E}_\nu\big), \quad x = C + \sqrt{AB}\,(t + t^{-1}).$$

One has

$$P_\nu'(x) = \frac{dP_\nu/dt}{dx/dt} = \nu (AB)^{(\nu-1)/2} \frac{t^\nu - t^{-\nu}}{t - t^{-1}}.$$

Evaluating at $t = rq^s$, one finds

$$P_\nu'(\xi_s) = \nu (AB)^{(\nu-1)/2} \frac{r^\nu - r^{-\nu}}{rq^s - r^{-1}q^{-s}}.$$

Under the additional non-vanishing assumptions just imposed, the explicit root-of-unity Askey–Wilson weights apply. In particular, the ratios λ_{s+1}/λ_s satisfy

$$\mathcal{A}_{s+1}\lambda_{s+1} = \mathcal{C}_s \lambda_s,$$

where

$$\mathcal{A}_s = gq^{-1}\frac{(1-rq^s/a)(1-rq^s/b)(1-rq^s/c)(1-rq^s/d)}{(1-r^2q^{2s-1})(1-r^2q^{2s})},$$

$$\mathcal{C}_s = \frac{(1-arq^s)(1-brq^s)(1-crq^s)(1-drq^s)}{(1-r^2q^{2s})(1-r^2q^{2s+1})}.$$

Therefore

$$\frac{\lambda_{s+1}}{\lambda_s} = \frac{q}{g}\frac{1-r^2q^{2s+2}}{1-r^2q^{2s}}\frac{(1-arq^s)(1-brq^s)(1-crq^s)(1-drq^s)}{(1-rq^{s+1}/a)(1-rq^{s+1}/b)(1-rq^{s+1}/c)(1-rq^{s+1}/d)}.$$

Iterating from $s = 0$, one obtains

$$\lambda_s = \lambda_0\left(\frac{q}{g}\right)^s \frac{(1-r^2q^{2s})(ar, br, cr, dr; q)_s}{(1-r^2)(qr/a, qr/b, qr/c, qr/d; q)_s},$$

for every $s = 0, 1, \ldots, \nu - 1$. Thus the finite root-of-unity orthogonality formula appears here as a special instance of the general framework developed above, subject to the additional requirement that the truncating polynomial have simple zeros.

Remark 7.12. *The Askey–Wilson root-of-unity case treated in* [28] *is a direct specialisation of the present framework. The q-ultraspherical results in* [27] *may be viewed as arising from a further specialisation of the same root-of-unity Askey–Wilson setting, although* [27] *imposes additional positivity and representation-theoretic restrictions. The advantage of the present approach is that it separates the structural part of the argument from the explicit family-specific calculations. In the setting adopted here, finite orthogonality is first obtained at the general level from Corollary 7.7. The additional requirement that the truncating polynomial have simple zeros enters only afterwards, through Proposition 7.10, where one obtains a discrete orthogonality formula supported on those zeros, with Christoffel numbers given by the usual finite orthogonality formula. This separation is not made in* [28]*, whose argument is organised around the four-parameter Askey–Wilson family itself. There, the simplicity of the truncating polynomial and the genericity conditions ensuring that the recurrence coefficients are well defined are imposed before deriving the explicit orthogonality measure. In the present framework, by contrast, the structural existence of a finite orthogonal family is kept distinct from the subsequent problem of identifying the support and evaluating the weights.*

The latter step is no longer purely structural. It requires an explicit analysis of the truncating polynomial and may impose further genericity assumptions. In the four-parameter Askey–Wilson case, for instance, the simplicity condition becomes

$$\mathcal{E}_\nu \neq \pm 2,$$

which excludes a non-trivial algebraic locus in parameter space. Together with the non-vanishing conditions for the recurrence coefficients, this hypothesis gives distinct support points and hence a finite discrete orthogonality formula. The remaining task is the explicit evaluation of those points and of the corresponding weights. Thus the present framework explains why such root-of-unity orthogonality formulas appear, which part of them is structural, and which part depends on explicit family-specific calculations. The results of [28, 27] *should therefore be understood as particular manifestations of a more general structural picture.*

8. Normalised alternating map

Although Definition 6.6 is formally uniform across all admissible map types, its degenerate subcases behave differently in the three regimes. In the quadratic and q-exponential settings, regularity forces $\deg\psi = 1$, whereas in the alternating case additional degenerate phenomena may still occur. We now turn to a distinguished globally normalised subcase of case *(ii)* in Theorem 5.1, namely the alternating map

$$X(s) = e^{\pi i s/h}.$$

This map is not merely formal: as observed above, there are natural half-step-invariant sets on which it is admissible. In practice, such sets often arise in simple forms. For instance, in Example 2.4*(3)*, take $E = h(\mathbb{Q} \cap [0, \frac{1}{2}))$. Then one obtains

$$U = \bigsqcup_{v \in E} \left(v + \frac{h}{2}\mathbb{Z} \right) = h\mathbb{Q}.$$

In this case all representatives lie on the affine line $h\mathbb{R}$, although they still determine infinitely many distinct cosets of $\frac{h}{2}\mathbb{Z}$. For a fixed $s_0 \in E$, on the corresponding full-step arithmetic progression $s_0 + h\mathbb{Z}$, one has

$$X(s_0 + kh) = e^{\pi i (s_0 + kh)/h} = (-1)^k X(s_0),$$

which is precisely the alternating form in Theorem 5.1*(ii)*, with

$$a_{s_0} = X(s_0), \quad b = 0.$$

Thus the coefficient multiplying $(-1)^k$ is allowed to depend on the chosen full-step progression, exactly as the theorem states. In the present example, this dependence is genuinely nontrivial. Indeed, if $s_0, s_0' \in E$ and $X(s_0) = X(s_0')$, then

$$e^{\pi i (s_0 - s_0')/h} = 1,$$

so that

$$\frac{s_0 - s_0'}{h} \in 2\mathbb{Z}.$$

But, since $s_0, s_0' \in E = h\big(\mathbb{Q} \cap [0, \frac{1}{2})\big)$, one has

$$\frac{s_0 - s_0'}{h} \in \mathbb{Q} \cap \left(-\frac{1}{2}, \frac{1}{2} \right),$$

and the only even integer in that interval is 0. Hence $s_0 = s_0'$. Therefore the map $s_0 \mapsto X(s_0)$ is injective on E. Since E is infinite, it follows that $X(E)$, and therefore also $X(h\mathbb{Q})$, is infinite.

We shall not pursue here the full range of alternating admissible maps allowed by Theorem 5.1. The normalised case already contains the structural mechanism that distinguishes the alternating regime from the quadratic and q-exponential ones. Accordingly, throughout the remainder of this section, let $h \in \mathbb{C}^{\times}$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be an admissible normalised alternating map on U. Then, for every polynomial p,

$$(Dp)(X) = \frac{p(iX) - p(-iX)}{2i\,X}, \quad (Sp)(X) = \frac{p(iX) + p(-iX)}{2},$$

throughout U. This is the key structural feature of the normalised alternating case. In the quadratic and q-exponential regimes, the operators D and S are governed by neighbouring values of the parametrising map within the ordinary polynomial calculus. Here, by contrast, the half-step neighbours are simply iX and $-iX$, and the corresponding calculus factors through the decomposition of $\mathcal{P}$ induced by the quadratic substitution. For this reason, the alternating case is better treated as a separate structural regime rather than as a naive $q \to -1$ limit.

The following proposition makes this decomposition explicit.

Proposition 8.1. *Fix $h \in \mathbb{C}^\times$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be the admissible map on U defined by*

$$X(s) = e^{\pi i s/h}$$

for every $s \in U$. Let $D, S : \mathcal{P} \to \mathcal{P}$ be the associated divided-difference and averaging operators. Then every polynomial p admits a unique decomposition

$$p(x) = a(-x^2) + x\,b(-x^2),$$

where a and b are polynomials. For these uniquely determined polynomials a and b, one has

$$(Dp)(X) = b(X^2), \quad (Sp)(X) = a(X^2)$$

throughout U.

Proof. Fix $p \in \mathcal{P}$. By decomposition into even and odd parts, there exist unique polynomials a and b such that $p(x) = a(-x^2) + x\,b(-x^2)$. Evaluating at $x = X$, one obtains

$$p(X) = a(-X^2) + X\,b(-X^2)$$

throughout U. Now

$$p(iX) = a(X^2) + iX\,b(X^2),$$

since $(iX)^2 = -X^2$. Likewise,

$$p(-iX) = a(X^2) - iX\,b(X^2).$$

Hence

$$p(iX) - p(-iX) = 2iX\,b(X^2),$$

and therefore

$$(Dp)(X) = b(X^2).$$

Similarly,

$$(Sp)(X) = a(X^2).$$

This proves the stated formulas. □

Definition 8.2 (Quadratic substitution operator)**.** *The quadratic substitution operator is the linear map $\sigma : \mathcal{P} \to \mathcal{P}$ defined by*

$$(\sigma p)(x) = p(x^2)$$

for every $p \in \mathcal{P}$. The transpose of σ is denoted by $\boldsymbol{\sigma} : \mathcal{P}' \to \mathcal{P}'$ and defined by

$$\langle \boldsymbol{\sigma}\mathbf{u}, p\rangle = \langle \mathbf{u}, \sigma p\rangle$$

for every $p \in \mathcal{P}$ and every $\mathbf{u} \in \mathcal{P}'$.

The operator σ is continuous on $\mathcal{P}$. Indeed, for every $n \in \mathbb{N}$ one has $\sigma(\mathcal{P}_n) \subseteq \mathcal{P}_{2n}$. Since each restriction

$$\sigma|_{\mathcal{P}_n} : \mathcal{P}_n \to \mathcal{P}_{2n}$$

is linear between finite-dimensional spaces, it is continuous. By the continuity criterion for the inductive-limit topology on $\mathcal{P}$, σ is therefore continuous. Hence its transpose $\boldsymbol{\sigma}$ is well defined and $\sigma(\mathcal{P}', \mathcal{P})$-continuous.

Proposition 8.1 shows that $\mathcal{P}$ splits, in the alternating case, according to the quadratic substitution. The operators S and D detect the two components of the even–odd decomposition written in the form

$$p(x) = a(-x^2) + x\,b(-x^2).$$

More generally, for every fixed $\tau \in \mathbb{C}$, one also has the direct-sum decomposition

$$\mathcal{P} = \sigma(\mathcal{P}) \oplus (x - \tau)\sigma(\mathcal{P}).$$

Thus σ records the passage from the original polynomial variable x to the quadratic variable x^2, while its transpose $\boldsymbol{\sigma}$ records the corresponding quadratic component on the dual side. We now translate this decomposition into the corresponding dual statement. The next proposition makes the resulting annihilation conditions explicit.

Proposition 8.3. *Fix $h \in \mathbb{C}^\times$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be the admissible map on U defined by*

$$X(s) = e^{\pi i s/h}$$

for every $s \in U$. Let $D, S : \mathcal{P} \to \mathcal{P}$ be the associated divided-difference and averaging operators, with transposes $\mathbf{D}, \mathbf{S} : \mathcal{P}' \to \mathcal{P}'$. Let $\mathbf{u} \in \mathcal{P}'$, and let ϕ and ψ be polynomials, not both identically zero, such that ϕ has degree at most 2 and ψ has degree at most 1. Then the following conditions are equivalent:

(i)

$$\mathbf{D}(\phi\, \mathbf{u}) = \mathbf{S}(\psi\, \mathbf{u})$$

in $\mathcal{P}'$.

(ii)

$$\langle \mathbf{u}, \phi(x)\, p(x^2)\rangle = 0, \quad \langle \mathbf{u}, \psi(x)\, p(x^2)\rangle = 0,$$

for every $p \in \mathcal{P}$.

Proof. Assume *(i)*, and let $p \in \mathcal{P}$. First take

$$p_o(x) = x\, p(-x^2).$$

By Proposition 8.1, the decomposition of p_o is obtained with even part equal to 0 and odd part equal to p. Hence

$$(Dp_o)(X) = p(X^2), \quad (Sp_o)(X) = 0$$

throughout U. Since X is admissible, $X(U)$ is infinite. Hence, as both sides are polynomials in the free variable x, these identities imply

$$Dp_o(x) = p(x^2), \quad Sp_o(x) = 0$$

as polynomial identities. Therefore

$$0 = \big\langle \mathbf{D}(\phi\, \mathbf{u}) - \mathbf{S}(\psi\, \mathbf{u}), p_o\big\rangle = -\langle \mathbf{u}, \phi(x)\, p(x^2)\rangle.$$

Thus

$$\langle \mathbf{u}, \phi(x)\, p(x^2)\rangle = 0$$

for every $p \in \mathcal{P}$. Next take

$$p_e(x) = p(-x^2).$$

By Proposition 8.1, the decomposition of p_e has odd part equal to 0 and even part equal to p. Hence

$$(Dp_e)(X) = 0, \quad (Sp_e)(X) = p(X^2)$$

throughout U. Again, since X is admissible, $X(U)$ is infinite, and therefore these are polynomial identities:

$$Dp_e(x) = 0, \quad Sp_e(x) = p(x^2).$$

Hence

$$0 = \big\langle \mathbf{D}(\phi\, \mathbf{u}) - \mathbf{S}(\psi\, \mathbf{u}), p_e\big\rangle = -\langle \mathbf{u}, \psi(x)\, p(x^2)\rangle.$$

Thus *(ii)* follows.

Conversely, assume *(ii)*, and let $q \in \mathcal{P}$. By Proposition 8.1, there exist polynomials a and b such that

$$(Dq)(X) = b(X^2), \quad (Sq)(X) = a(X^2)$$

throughout U. Since X is admissible, $X(U)$ is infinite, and this gives the polynomial identities

$$Dq(x) = b(x^2), \quad Sq(x) = a(x^2).$$

Using *(ii)*, we obtain

$$\langle \mathbf{u}, \phi(x)Dq(x)\rangle = 0, \quad \langle \mathbf{u}, \psi(x)Sq(x)\rangle = 0.$$

Therefore

$$\big\langle \mathbf{D}(\phi\, \mathbf{u}) - \mathbf{S}(\psi\, \mathbf{u}), q\big\rangle = -\langle \mathbf{u}, \phi(x)Dq(x)\rangle - \langle \mathbf{u}, \psi(x)Sq(x)\rangle = 0.$$

Since this holds for every $q \in \mathcal{P}$, the two functionals agree on $\mathcal{P}$. Hence *(i)* follows. □

Before turning to the non-degenerate case $\deg\psi = 1$, we record a phenomenon peculiar to the alternating map. In the quadratic and q-exponential regimes, the regularity criteria force

$$d_0 = \psi_1 \neq 0,$$

and hence $\deg\psi = 1$. Thus the cases $\psi = 0$ and ψ constant are automatically excluded there in the regular situation. For the normalised alternating map this is no longer automatic. By Proposition 8.3, the structural equation is equivalent to

$$\langle \mathbf{u}, \phi(x)\, p(x^2)\rangle = 0, \quad \langle \mathbf{u}, \psi(x)\, p(x^2)\rangle = 0$$

for every $p \in \mathcal{P}$.

If $\psi = 0$, the condition reduces to

$$\langle \mathbf{u}, \phi(x)\, p(x^2)\rangle = 0,$$

that is, $\mathbf{u}$ vanishes on $\phi(x)\mathcal{P}(x^2)$. Writing the even and odd components of $\mathbf{u}$ as

$$\langle \mathbf{v}, p\rangle = \langle \mathbf{u}, p(x^2)\rangle, \quad \langle \mathbf{w}, p\rangle = \langle \mathbf{u}, xp(x^2)\rangle,$$

and writing

$$\phi(x) = \phi_2 x^2 + \phi_1 x + \phi_0,$$

this is equivalent to

$$\phi_1 \mathbf{w} = -(\phi_2 x + \phi_0)\mathbf{v}$$

in $\mathcal{P}'$. Hence, if $\phi_1 \neq 0$, the odd component $\mathbf{w}$ is determined by the even component $\mathbf{v}$. If $\phi_1 = 0$, the condition reduces to

$$(\phi_2 x + \phi_0)\mathbf{v} = 0$$

and imposes no restriction on $\mathbf{w}$. Thus the case $\psi = 0$ is not empty; it may contain regular functionals, but it is governed only by this relation between the even and odd components. Concrete examples can be produced, although they are not especially revealing, since in the alternating setting the same regular functional may also satisfy the equation for nonzero choices of ψ. We do not pursue this degenerate case further, nor do we modify the definition of classicality so as to exclude it, since that would artificially remove part of the alternating phenomenon.

If ψ is a non-zero constant, then the second annihilation condition gives

$$\langle \mathbf{u}, p(x^2)\rangle = 0$$

for every $p \in \mathcal{P}$. In particular, $\langle \mathbf{u}, 1\rangle = 0$, so this case is incompatible with regularity.

We now pass to the non-degenerate case. Thus ψ is assumed to have degree 1, and, after multiplication of the structural equation by a non-zero constant, we may write

$$\psi(x) = x - \tau$$

for a uniquely determined $\tau \in \mathbb{C}$. This is the case in which the quadratic substitution yields the reconstruction theorem below.

Definition 8.4 (Alternating splitting operator)**.** *Fix $\tau \in \mathbb{C}$. The alternating splitting operator $J_\tau : \mathcal{P} \to \mathcal{P}$ is defined by $J_\tau p = a$, where a is determined by the unique decomposition*

$$p(x) = a(x^2) + (x - \tau)b(x^2)$$

of $p \in \mathcal{P}$. The transpose of J_τ is denoted by $\mathbf{J}_\tau : \mathcal{P}' \to \mathcal{P}'$ and defined by

$$\langle \mathbf{J}_\tau \mathbf{v}, p \rangle = \langle \mathbf{v}, J_\tau p \rangle,$$

for every $p \in \mathcal{P}$ and every $\mathbf{v} \in \mathcal{P}'$.

The operator J_τ is continuous on $\mathcal{P}$. Indeed, if $p \in \mathcal{P}_n$, then in the decomposition $p(x) = a(x^2) + (x - \tau)b(x^2)$ one has $\deg a \leq \left\lfloor \frac{n}{2} \right\rfloor$, and, for every $n \in \mathbb{N}^\times$, $\deg b \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, while $b = 0$ when $n = 0$. Hence

$$J_\tau(\mathcal{P}_n) \subseteq \mathcal{P}_{\lfloor n/2 \rfloor}.$$

The restrictions $J_\tau|_{\mathcal{P}_n} : \mathcal{P}_n \to \mathcal{P}_{\lfloor n/2 \rfloor}$ are linear maps between finite-dimensional spaces, and therefore continuous. By the continuity criterion for the inductive-limit topology on $\mathcal{P}$, J_τ is continuous. Consequently, its transpose $\mathbf{J}_\tau$ is well defined and continuous for the weak topology $\sigma(\mathcal{P}', \mathcal{P})$.

Proposition 8.5. *Fix $\tau \in \mathbb{C}$, and let $\mathbf{v} \in \mathcal{P}'$. If $\mathbf{u} = \mathbf{J}_\tau \mathbf{v}$, then*

$$\langle \mathbf{u}, (x - \tau) r(x^2) \rangle = 0$$

for every $r \in \mathcal{P}$, and $\boldsymbol{\sigma}\mathbf{u} = \mathbf{v}$. Conversely, if $\mathbf{u} \in \mathcal{P}'$ satisfies

$$\langle \mathbf{u}, (x - \tau) r(x^2) \rangle = 0$$

for every $r \in \mathcal{P}$, then $\mathbf{u} = \mathbf{J}_\tau(\boldsymbol{\sigma}\mathbf{u})$. If, in addition, $\langle \mathbf{u}, 1 \rangle \neq 0$, then τ is uniquely determined by $\mathbf{u}$, and

$$\tau = \frac{\langle \mathbf{u}, x \rangle}{\langle \mathbf{u}, 1 \rangle}.$$

Proof. The decomposition

$$p(x) = a(x^2) + (x - \tau)b(x^2)$$

is unique for every $p \in \mathcal{P}$, so the definition of $\mathbf{J}_\tau \mathbf{v}$ is well posed. Let $\mathbf{v} \in \mathcal{P}'$, and set $\mathbf{u} = \mathbf{J}_\tau \mathbf{v}$. Applying the definition to polynomials of the form

$$p(x) = (x - \tau) r(x^2),$$

for which $a = 0$, gives

$$\langle \mathbf{u}, (x - \tau) r(x^2) \rangle = 0$$

for every $r \in \mathcal{P}$. Applying it to polynomials of the form

$$p(x) = r(x^2),$$

for which $a = r$, gives

$$\langle \mathbf{u}, r(x^2) \rangle = \langle \mathbf{v}, r \rangle$$

for every $r \in \mathcal{P}$. Hence $\boldsymbol{\sigma}\mathbf{u} = \mathbf{v}$.

Conversely, suppose that $\mathbf{u} \in \mathcal{P}'$ satisfies

$$\langle \mathbf{u}, (x - \tau) r(x^2) \rangle = 0$$

for every $r \in \mathcal{P}$. If

$$p(x) = a(x^2) + (x - \tau)b(x^2),$$

then

$$\langle \mathbf{u}, p \rangle = \langle \mathbf{u}, a(x^2) \rangle = \langle \boldsymbol{\sigma}\mathbf{u}, a \rangle.$$

By the definition of $\mathbf{J}_\tau$, this is precisely $\mathbf{u} = \mathbf{J}_\tau(\boldsymbol{\sigma}\mathbf{u})$. Finally, taking $r = 1$ gives $\langle \mathbf{u}, x - \tau \rangle = 0$. If $\langle \mathbf{u}, 1 \rangle \neq 0$, then

$$\tau = \frac{\langle \mathbf{u}, x \rangle}{\langle \mathbf{u}, 1 \rangle},$$

so τ is uniquely determined by $\mathbf{u}$. □

The preceding proposition shows that, once the annihilation condition

$$\langle \mathbf{u}, (x-\tau)r(x^2)\rangle = 0$$

is imposed, the functional $\mathbf{u}$ is completely recovered from its quadratic component $\boldsymbol{\sigma}\mathbf{u}$. Thus, in the non-degenerate alternating case, the orthogonality problem is reduced to an orthogonality problem in the quadratic variable. The next theorem gives the precise reconstruction.

Theorem 8.6. *Fix $h \in \mathbb{C}^\times$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be the admissible map on U defined by*

$$X(s) = e^{\pi i s/h}$$

for every $s \in U$. Let $D, S : \mathcal{P} \to \mathcal{P}$ be the associated divided-difference and averaging operators, with transposes $\mathbf{D}, \mathbf{S} : \mathcal{P}' \to \mathcal{P}'$. Let $\mathbf{u} \in \mathcal{P}'$ be such that $\langle \mathbf{u}, 1\rangle \neq 0$.

Then the following statements are equivalent:

(i) *There exist polynomials ϕ and ψ, where ϕ has degree at most 2 and ψ has degree 1, such that*

$$\mathbf{D}(\phi\,\mathbf{u}) = \mathbf{S}(\psi\,\mathbf{u})$$

in $\mathcal{P}'$.

(ii) *There exists $\tau \in \mathbb{C}$ such that*

$$\langle \mathbf{u}, (x-\tau)r(x^2)\rangle = 0$$

for every $r \in \mathcal{P}$.

(iii) *There exist $\tau \in \mathbb{C}$ and $\mathbf{v} \in \mathcal{P}'$ such that*

$$\mathbf{u} = \mathbf{J}_\tau \mathbf{v}.$$

In that case, τ and $\mathbf{v}$ are uniquely determined by $\mathbf{u}$, and are given by

$$\tau = \frac{\langle \mathbf{u}, x\rangle}{\langle \mathbf{u}, 1\rangle}, \quad \mathbf{v} = \boldsymbol{\sigma}\mathbf{u}.$$

Assume henceforth that these equivalent conditions hold. Let $I \subseteq \mathbb{N}$ be either $I = \mathbb{N}$, or $I = \{0, 1, \dots, 2N+2\}$ for some $N \in \mathbb{N}$. In the finite case set

$$J_I = \{0, 1, \dots, N+1\}, \quad K_I = \{0, 1, \dots, N\},$$

whereas in the infinite case set

$$J_I = K_I = \mathbb{N}.$$

Then the following statements are equivalent:

(iv) *There exists a monic orthogonal polynomial sequence $(R_n)_{n\in J_I}$ with respect to $\mathbf{v}$ such that*

$$R_n(\tau^2) \neq 0$$

for every $n \in J_I$.

(v) *There exists a monic orthogonal polynomial sequence $(P_n)_{n\in I}$ with respect to $\mathbf{u}$.*

Whenever these conditions hold, the functional $(x-\tau^2)\mathbf{v}$ admits a monic orthogonal polynomial sequence $(S_n)_{n\in K_I}$ with respect to $(x-\tau^2)\mathbf{v}$, where, for every $n \in K_I$,

$$S_n(x) = \frac{R_{n+1}(x) - \dfrac{R_{n+1}(\tau^2)}{R_n(\tau^2)}\,R_n(x)}{x-\tau^2}.$$

Moreover, the monic orthogonal polynomial sequence with respect to $\mathbf{u}$ is given, for every $n \in K_I$, by

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x-\tau)S_n(x^2),$$

together with

$$P_{2N+2}(x) = R_{N+1}(x^2)$$

in the finite case.

Proof. The implication *(i)* $\Rightarrow$ *(ii)* follows immediately from Proposition 8.3, since the condition $\deg \psi = 1$ means precisely that

$$\psi(x) = c(x - \tau)$$

for some $c \in \mathbb{C}^\times$ and some $\tau \in \mathbb{C}$. Dividing by c, one obtains

$$\langle \mathbf{u}, (x - \tau) r(x^2) \rangle = 0$$

for every $r \in \mathcal{P}$. Conversely, if *(ii)* holds, then Proposition 8.3 applies with

$$\phi = 0, \quad \psi = x - \tau,$$

and yields *(i)*. The equivalence of *(ii)* and *(iii)* is exactly Proposition 8.5. That proposition also gives

$$\tau = \frac{\langle \mathbf{u}, x \rangle}{\langle \mathbf{u}, 1 \rangle}, \quad \mathbf{v} = \boldsymbol{\sigma} \mathbf{u},$$

and hence the uniqueness of τ and $\mathbf{v}$. Assume henceforth that these equivalent conditions hold, and set $a = \tau^2$.

We treat the finite case. Thus let $I = \{0, 1, \ldots, 2N + 2\}$ for some $N \in \mathbb{N}$. The case $I = \mathbb{N}$ is obtained by the same argument, with the finite index ranges removed; the precise point at which this extension is made will be indicated at the end of the proof.

Step 1: Assume that $\mathbf{v}$ admits a monic orthogonal polynomial sequence $R_0, R_1, \ldots, R_{N+1}$ with respect to $\mathbf{v}$, and that

$$R_n(a) \neq 0$$

for $n \in \{0, 1, \ldots, N + 1\}$. Since the distinguished factor in the original variable is $x - \tau$, the corresponding distinguished point in the quadratic variable is $a = \tau^2$. Accordingly, the relevant transformed functional is $(x - a)\mathbf{v}$, and by the Christoffel criterion for multiplication by $x - a$, the functional $(x - a)\mathbf{v}$ admits a monic orthogonal polynomial sequence $S_0, S_1, \ldots, S_N$ with respect to $(x - a)\mathbf{v}$. Moreover, for $n = 0, 1, \ldots, N$, one has

$$S_n(x) = \frac{R_{n+1}(x) - \dfrac{R_{n+1}(a)}{R_n(a)} R_n(x)}{x - a}.$$

Step 2: Since $\mathbf{u} = \mathbf{J}_\tau \mathbf{v}$, Proposition 8.5 gives

$$\langle \mathbf{u}, r(x^2) \rangle = \langle \mathbf{v}, r \rangle, \quad \langle \mathbf{u}, (x - \tau) r(x^2) \rangle = 0$$

for every $r \in \mathcal{P}$. Moreover, for every $r \in \mathcal{P}$,

$$\langle \mathbf{u}, (x - \tau)^2 r(x^2) \rangle = \langle \mathbf{u}, (x^2 - a) r(x^2) \rangle = \langle (x - a)\mathbf{v}, r \rangle.$$

Indeed,

$$(x - \tau)^2 - (x^2 - a) = -2\tau(x - \tau),$$

and therefore

$$\big((x - \tau)^2 - (x^2 - a)\big) r(x^2) = -2\tau(x - \tau) r(x^2),$$

whose pairing with $\mathbf{u}$ vanishes by the defining annihilation condition. For every $n = 0, 1, \ldots, N$, define

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x - \tau) S_n(x^2),$$

and also set

$$P_{2N+2}(x) = R_{N+1}(x^2).$$

Using the preceding identities, we obtain

$$\langle \mathbf{u}, P_{2n}P_{2m}\rangle = \langle \mathbf{v}, R_n R_m\rangle,$$

$$\langle \mathbf{u}, P_{2n}P_{2m+1}\rangle = 0,$$

$$\langle \mathbf{u}, P_{2n+1}P_{2m+1}\rangle = \langle (x-a)\mathbf{v}, S_n S_m\rangle.$$

Moreover,

$$\langle \mathbf{u}, P_{2N+2}P_{2m+1}\rangle = 0,$$

and

$$\langle \mathbf{u}, P_{2N+2}P_{2m}\rangle = \langle \mathbf{v}, R_{N+1}R_m\rangle = 0.$$

Hence

$$\langle \mathbf{u}, P_n P_m\rangle = 0$$

whenever $0 \leq m < n \leq 2N+2$. Moreover, for $n = 0, 1, \ldots, N$, one has

$$\langle \mathbf{u}, P_{2n}^2\rangle = \langle \mathbf{v}, R_n^2\rangle \neq 0,$$

and

$$\langle \mathbf{u}, P_{2n+1}^2\rangle = \langle (x-a)\mathbf{v}, S_n^2\rangle \neq 0.$$

Finally,

$$\langle \mathbf{u}, P_{2N+2}^2\rangle = \langle \mathbf{v}, R_{N+1}^2\rangle \neq 0.$$

Thus $P_0, P_1, \ldots, P_{2N+2}$ is a monic orthogonal polynomial sequence with respect to $\mathbf{u}$. This proves that *(iv)* implies *(v)* in the finite case. The displayed formula for S_n is already known from Step 1, and the displayed formulas for the polynomials P_n hold by construction.

Step 3: Assume now that there exists a monic orthogonal polynomial sequence $P_0, P_1, \ldots, P_{2N+2}$ with respect to $\mathbf{u}$. Since $P_0, P_1, \ldots, P_{2N+2}$ is a monic orthogonal polynomial sequence with respect to $\mathbf{u}$, it follows that, for each $m = 0, 1, \ldots, N+1$, the bilinear form induced by $\mathbf{u}$ is non-degenerate on $\mathcal{P}_{2m}$, and, for each $m = 0, 1, \ldots, N$, it is also non-degenerate on $\mathcal{P}_{2m+1}$. Indeed, with respect to the basis $P_0, P_1, \ldots, P_{2m}$ (resp. $P_0, P_1, \ldots, P_{2m+1}$), the Gram matrix of the bilinear form induced by $\mathbf{u}$ on $\mathcal{P}_{2m}$ (resp. $\mathcal{P}_{2m+1}$) is diagonal with non-zero diagonal entries. Now, for each $m = 0, 1, \ldots, N+1$, one has the orthogonal decomposition

$$\mathcal{P}_{2m} = \sigma(\mathcal{P}_m) \oplus (x-\tau)\sigma(\mathcal{P}_{m-1}),$$

with the convention that $\sigma(\mathcal{P}_{-1}) = \{0\}$, and for each $m = 0, 1, \ldots, N$,

$$\mathcal{P}_{2m+1} = \sigma(\mathcal{P}_m) \oplus (x-\tau)\sigma(\mathcal{P}_m).$$

As above, these summands are orthogonal with respect to the bilinear form induced by $\mathbf{u}$. Since the whole form is non-degenerate on $\mathcal{P}_{2m}$ and $\mathcal{P}_{2m+1}$, respectively, the restrictions to the summands are non-degenerate as well.

The restriction of the bilinear form induced by $\mathbf{u}$ to $\sigma(\mathcal{P}_m)$ is precisely the bilinear form induced by $\mathbf{v}$ on $\mathcal{P}_m$. Hence, for every $m = 0, 1, \ldots, N+1$, that form is non-degenerate on $\mathcal{P}_m$. Likewise, the restriction of the bilinear form induced by $\mathbf{u}$ to $(x-\tau)\sigma(\mathcal{P}_m)$ is precisely the bilinear form induced by $(x-a)\mathbf{v}$ on $\mathcal{P}_m$, where $a = \tau^2$. Hence, for every $m = 0, 1, \ldots, N$, that form is non-degenerate on $\mathcal{P}_m$. Therefore $\mathbf{v}$ is regular of order $N+2$, while $(x-a)\mathbf{v}$ is regular of order $N+1$. Equivalently, $\mathbf{v}$ admits a monic orthogonal family $R_0, R_1, \ldots, R_{N+1}$ and $(x-a)\mathbf{v}$ admits a monic orthogonal polynomial sequence $S_0, S_1, \ldots, S_N$. For each $n = 0, 1, \ldots, N+1$, the polynomial $R_n(x^2)$ is monic of degree $2n$. Let $q \in \mathcal{P}$ have degree at most $2n-1$. By the direct-sum decomposition underlying the definition of J_τ, we may write

$$q(x) = A(x^2) + (x-\tau)B(x^2),$$

with $\deg A \le n-1$. Hence

$$\langle \mathbf{u}, R_n(x^2)q(x)\rangle = \langle \mathbf{v}, R_n A\rangle = 0.$$

Thus $R_n(x^2)$ is orthogonal to every polynomial of degree at most $2n-1$. Hence, by uniqueness of the monic orthogonal polynomial of degree $2n$,

$$P_{2n}(x) = R_n(x^2)$$

for $n = 0, 1, \ldots, N+1$.

Step 4: Let $n = 0, 1, \ldots, N$. Since P_{2n+1} is monic of degree $2n+1$, its unique decomposition associated with J_τ has the form

$$P_{2n+1}(x) = A_n(x^2) + (x-\tau)S_n(x^2),$$

where $A_n, S_n \in \mathcal{P}$. If $Q \in \mathcal{P}$ has degree at most n, then $\deg Q(x^2) \le 2n$, so orthogonality gives

$$0 = \langle \mathbf{u}, P_{2n+1}(x)Q(x^2)\rangle = \langle \mathbf{v}, A_n Q\rangle.$$

Since $R_0, R_1, \ldots, R_n$ are orthogonal with respect to $\mathbf{v}$, the bilinear form induced by $\mathbf{v}$ is non-degenerate on $\mathcal{P}_n$. Hence $A_n = 0$. It follows that

$$P_{2n+1}(x) = (x-\tau)S_n(x^2)$$

for $n = 0, 1, \ldots, N$, with S_n monic.

Step 5: Let $n = 0, 1, \ldots, N$, and let $Q \in \mathcal{P}$ have degree at most $n-1$. Then

$$\deg\big((x-\tau)Q(x^2)\big) \le 2n-1,$$

and therefore

$$0 = \langle \mathbf{u}, P_{2n+1}(x)(x-\tau)Q(x^2)\rangle = \langle (x-a)\mathbf{v}, S_n Q\rangle.$$

Thus S_n is orthogonal, with respect to $(x-a)\mathbf{v}$, to every polynomial of degree at most $n-1$. Moreover,

$$0 \ne \langle \mathbf{u}, P_{2n+1}^2\rangle = \langle (x-a)\mathbf{v}, S_n^2\rangle.$$

Hence $S_0, S_1, \ldots, S_N$ is the monic orthogonal polynomial sequence with respect to $(x-a)\mathbf{v}$. In particular, $(x-a)\mathbf{v}$ admits monic polynomials of degrees $0, 1, \ldots, N$ orthogonal with respect to it. Applying the Christoffel criterion, we conclude that

$$R_n(a) \ne 0$$

for $n \in \{0, 1, \ldots, N+1\}$, and that

$$S_n(x) = \frac{R_{n+1}(x) - \dfrac{R_{n+1}(a)}{R_n(a)}\, R_n(x)}{x-a}$$

for $n \in \{0, 1, \ldots, N\}$. This proves that *(v)* implies *(iv)* in the finite case. This establishes the theorem in the case $I = \{0, 1, \ldots, 2N+2\}$.

Step 6: We now turn to the case $I = \mathbb{N}$. Assume that $\mathbf{v}$ admits a monic orthogonal polynomial sequence $(R_n)_{n\in\mathbb{N}}$ with respect to $\mathbf{v}$. If

$$R_n(\tau^2) \ne 0$$

for every $n \in \mathbb{N}$, then for each fixed $N \in \mathbb{N}$ the finite family $R_0, R_1, \ldots, R_{N+1}$ satisfies the finite case already proved. Hence there exist monic polynomials $P_0, P_1, \ldots, P_{2N+2}$ orthogonal with respect to $\mathbf{u}$, where

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x-\tau)S_n(x^2)$$

for $n = 0, 1, \ldots, N$, and

$$P_{2N+2}(x) = R_{N+1}(x^2),$$

with

$$S_n(x) = \frac{R_{n+1}(x) - \dfrac{R_{n+1}(\tau^2)}{R_n(\tau^2)} R_n(x)}{x-\tau^2}$$

for $n = 0, 1, \ldots, N$. For each fixed n, the formulas

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x-\tau)S_n(x^2),$$

and

$$S_n(x) = \frac{R_{n+1}(x) - \dfrac{R_{n+1}(\tau^2)}{R_n(\tau^2)} R_n(x)}{x-\tau^2}$$

do not depend on the truncation level N. Hence the finite families obtained for different values of N are compatible. Therefore there exist monic polynomials $(P_n)_{n\in\mathbb{N}}$ orthogonal with respect to $\mathbf{u}$, and monic polynomials $(S_n)_{n\in\mathbb{N}}$ orthogonal with respect to $(x-\tau^2)\mathbf{v}$, with the stated formulas. Thus *(iv)* implies *(v)* when $I = \mathbb{N}$.

Conversely, assume that there exists a monic orthogonal polynomial sequence $P_0, P_1, P_2, \ldots$ with respect to $\mathbf{u}$. Then, for each $N \in \mathbb{N}$, the finite subfamily $P_0, P_1, \ldots, P_{2N+2}$ satisfies the finite case already proved. Hence, for each N, the functional $\mathbf{v}$ is regular of order $N+2$. Since N is arbitrary, $\mathbf{v}$ admits a monic orthogonal polynomial sequence $(R_n)_{n\in\mathbb{N}}$. By uniqueness of the monic orthogonal polynomial of each degree, the finite polynomials obtained from the finite case coincide with the corresponding initial segment of this global sequence. Applying the finite case once more, we obtain

$$R_n(\tau^2) \neq 0$$

for $n = 0, 1, \ldots, N+1$. Since N is arbitrary, it follows that

$$R_n(\tau^2) \neq 0$$

for every $n \in \mathbb{N}$. Thus *(v)* implies *(iv)* when $I = \mathbb{N}$. Finally, the formulas for S_n, P_{2n}, and P_{2n+1} in the case $I = \mathbb{N}$ follow, for each fixed n, by applying the finite case with $N \geq n$. □

Remark 8.7. *Theorem 8.6 shows that, in the non-degenerate case, the alternating theory is not primitive: it is reconstructed from an ordinary orthogonality problem in the quadratic variable, together with a single linear annihilation condition. More precisely, once τ is fixed, the functional $\mathbf{u}$ is completely determined by $\mathbf{v} = \boldsymbol{\sigma}\mathbf{u}$ and by*

$$\langle \mathbf{u}, (x-\tau)r(x^2)\rangle = 0$$

for every $r \in \mathcal{P}$. Thus the normalised alternating case is structurally rigid, but in a manner entirely different from the quadratic and q-exponential regimes.

Example 8.8 (Jacobi family for the normalised alternating map)**.** *Fix $h \in \mathbb{C}^\times$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be the admissible map given by*

$$X(s) = e^{\pi i s/h}$$

for every $s \in U$. Fix $\alpha, \beta \in \mathbb{C}$ such that

$$-\alpha,\ -\beta,\ -(\alpha+\beta+1) \notin \mathbb{N}^\times,$$

and let

$$\mathbf{v}_{\alpha,\beta} \in \mathcal{P}'$$

be the shifted Jacobi functional associated with the monic shifted Jacobi polynomial sequence with parameters (α, β); see [5]*. Let $(R_n)_{n\in\mathbb{N}}$ denote the corresponding monic orthogonal polynomial sequence. Thus*

$$R_n(t) = (-1)^n \frac{(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\, {}_2F_1\left(\begin{matrix} -n,\ n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; t\right).$$

We now choose $\tau = 1$*, and define*

$$\mathbf{u}_{\alpha,\beta} = \mathbf{J}_1 \mathbf{v}_{\alpha,\beta}.$$

Since

$$R_n(1) = \frac{(\beta+1)_n}{(n+\alpha+\beta+1)_n},$$

the assumptions on α *and* β *imply that*

$$R_n(1) \neq 0$$

for every $n \in \mathbb{N}$*. Hence Theorem 8.6 applies. Moreover, by Proposition 8.5,*

$$\langle \mathbf{u}_{\alpha,\beta}, (x-1)P(x^2) \rangle = 0$$

for every $P \in \mathcal{P}$*, and*

$$\boldsymbol{\sigma}\mathbf{u}_{\alpha,\beta} = \mathbf{v}_{\alpha,\beta}.$$

Now

$$(x-\tau^2)\mathbf{v}_{\alpha,\beta} = (x-1)\mathbf{v}_{\alpha,\beta},$$

and, up to a non-zero scalar factor, this is precisely the shifted Jacobi functional with parameters $(\alpha, \beta+1)$*. Therefore its monic orthogonal polynomial sequence* $(S_n)_{n\in\mathbb{N}}$ *is given by*

$$S_n(t) = (-1)^n \frac{(\alpha+1)_n}{(n+\alpha+\beta+2)_n}\, {}_2F_1\left(\begin{matrix} -n,\ n+\alpha+\beta+2 \\ \alpha+1 \end{matrix}; t\right).$$

Theorem 8.6 now yields the monic orthogonal polynomial sequence $(P_n)_{n\in\mathbb{N}}$ *of* $\mathbf{u}_{\alpha,\beta}$*. For*

$$\varepsilon_n = \frac{1-(-1)^n}{2}, \quad m_n = \left\lfloor \frac{n}{2} \right\rfloor,$$

one has

$$P_n(x) = (-1)^{m_n}(x-1)^{\varepsilon_n} \frac{(\alpha+1)_{m_n}}{(m_n+\alpha+\beta+1+\varepsilon_n)_{m_n}}\, {}_2F_1\left(\begin{matrix} -m_n,\ m_n+\alpha+\beta+1+\varepsilon_n \\ \alpha+1 \end{matrix}; x^2\right).$$

Example 8.9 (Hermite family for the normalised alternating map)**.** *Retain the normalised alternating map* X *introduced in Example 8.8. Let* $\mathbf{v} \in \mathcal{P}'$ *be the Laguerre functional with parameter* $-\frac{1}{2}$*, and let* $(R_n)_{n\in\mathbb{N}}$ *denote its monic orthogonal polynomial sequence. Thus*

$$R_n(t) = (-1)^n \left(\frac{1}{2}\right)_n {}_1F_1\left(\begin{matrix} -n \\ \frac{1}{2} \end{matrix}; t\right).$$

We now choose $\tau = 0$*, and define*

$$\mathbf{u} = \mathbf{J}_0 \mathbf{v}.$$

Since

$$R_n(0) = (-1)^n \left(\frac{1}{2}\right)_n,$$

one has

$$R_n(0) \neq 0$$

for every $n \in \mathbb{N}$*. Hence Theorem 8.6 applies. Moreover, by Proposition 8.5,*

$$\langle \mathbf{u}, x\, P(x^2) \rangle = 0$$

for every $P \in \mathcal{P}$*, and*

$$\boldsymbol{\sigma}\mathbf{u} = \mathbf{v}.$$

Now

$$(x-\tau^2)\mathbf{v} = x\,\mathbf{v},$$

and this is precisely the Laguerre functional with parameter $\frac{1}{2}$. Therefore its monic orthogonal polynomial sequence $(S_n)_{n\in\mathbb{N}}$ is given by

$$S_n(t) = (-1)^n \left(\frac{3}{2}\right)_n {}_1F_1\left(\begin{matrix} -n \\ \frac{3}{2} \end{matrix}; t\right).$$

Theorem 8.6 now yields the monic orthogonal polynomial sequence $(P_n)_{n\in\mathbb{N}}$ of $\mathbf{u}$. For

$$\varepsilon_n = \frac{1-(-1)^n}{2}, \quad m_n = \left\lfloor \frac{n}{2} \right\rfloor,$$

one has

$$P_n(x) = (-1)^{m_n} x^{\varepsilon_n} \left(\frac{1}{2}+\varepsilon_n\right)_{m_n} {}_1F_1\left(\begin{matrix} -m_n \\ \frac{1}{2}+\varepsilon_n \end{matrix}; x^2\right).$$

More precisely, if H_n denotes the classical Hermite polynomial normalised by

$$H_n(x) = 2^n x^n + \cdots,$$

then

$$P_n = 2^{-n} H_n.$$

In particular, the same annihilation condition

$$\langle \mathbf{u}, x\, P(x^2)\rangle = 0$$

has two interpretations through Proposition 8.3. Taking $\phi = 0$ and $\psi = x$, one obtains the non-degenerate structural equation

$$\mathbf{S}(x\,\mathbf{u}) = 0.$$

Taking instead $\phi = x$, $\psi = 0$, one obtains the degenerate structural equation

$$\mathbf{D}(x\,\mathbf{u}) = 0.$$

Thus the Hermite family occupies a distinguished position in the alternating theory: it belongs simultaneously to the non-degenerate case $\psi(x) = x$ and to the degenerate case $\psi = 0$. In this sense, it sits exactly at the point where the two descriptions meet.

The preceding theorem suggests a different reading of the so-called -1 families. In [8], these families are described through Dunkl-type operators involving reflections and as $q \to -1$ limits of certain q-polynomials. Such descriptions are useful, and in many concrete situations indispensable, but they do not identify the structural mechanism isolated here. In the present framework, this mechanism is the normalised alternating map together with the quadratic-substitution decomposition of $\mathcal{P}$. The $q \to -1$ limit should therefore be viewed as one realisation of the mechanism, not as its source.

We now illustrate this point with representative examples. It is convenient to isolate the reconstruction mechanism obtained above in a form that may be applied directly to concrete families.

Corollary 8.10. *Fix $\tau \in \mathbb{C}$, and let $\mathbf{v} \in \mathcal{P}'$ be regular, with monic orthogonal polynomial sequence $R_0, R_1, \ldots, R_{N+1}$. Assume that*

$$R_n(\tau^2) \neq 0$$

for $n = 0, 1, \ldots, N+1$. Then there exists a unique functional $\mathbf{u} \in \mathcal{P}'$ such that

$$\boldsymbol{\sigma}\mathbf{u} = \mathbf{v}, \quad \langle \mathbf{u}, (x-\tau)P(x^2)\rangle = 0$$

for every $P \in \mathcal{P}$, and $\mathbf{u}$ is regular of order $2N+3$. More precisely, if

$$S_n(x) = \frac{R_{n+1}(x) - \dfrac{R_{n+1}(\tau^2)}{R_n(\tau^2)}\, R_n(x)}{x - \tau^2},$$

for $n = 0, 1, \dots, N$, then $S_0, S_1, \dots, S_N$ is the monic orthogonal polynomial sequence associated with

$$(x - \tau^2)\mathbf{v},$$

and the monic orthogonal polynomial sequence of $\mathbf{u}$ is given by

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x - \tau)S_n(x^2),$$

for $n = 0, 1, \dots, N$, together with

$$P_{2N+2}(x) = R_{N+1}(x^2).$$

Proof. Set

$$\mathbf{u} = \mathbf{J}_\tau \mathbf{v}.$$

By Proposition 8.5, this is the unique functional satisfying

$$\boldsymbol{\sigma}\mathbf{u} = \mathbf{v}, \quad \langle \mathbf{u}, (x - \tau)P(x^2)\rangle = 0$$

for every $P \in \mathcal{P}$. Since $\mathbf{v}$ is regular, one has $\langle \mathbf{v}, 1\rangle \neq 0$. Hence

$$\langle \mathbf{u}, 1\rangle = \langle \mathbf{u}, \sigma(1)\rangle = \langle \boldsymbol{\sigma}\mathbf{u}, 1\rangle = \langle \mathbf{v}, 1\rangle \neq 0.$$

Therefore $\mathbf{u}$ satisfies condition *(ii)* of Theorem 8.6, with associated functional

$$\mathbf{v} = \boldsymbol{\sigma}\mathbf{u}.$$

Now apply Theorem 8.6 with $I = \{0, 1, \dots, 2N + 2\}$. Then

$$J_I = \{0, 1, \dots, N + 1\}, \quad K_I = \{0, 1, \dots, N\},$$

so the non-vanishing assumption

$$R_n(\tau^2) \neq 0$$

for $n = 0, 1, \dots, N + 1$ is precisely condition *(iv)* in that theorem. It follows that there exists a monic orthogonal polynomial sequence $P_0, P_1, \dots, P_{2N+2}$ with respect to $\mathbf{u}$. In particular, $\mathbf{u}$ is regular of order $2N + 3$. The same theorem also shows that $S_0, S_1, \dots, S_N$ is the monic orthogonal polynomial sequence of

$$(x - \tau^2)\mathbf{v},$$

and that the monic orthogonal polynomial sequence of $\mathbf{u}$ is given by

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x - \tau)S_n(x^2),$$

for $n = 0, 1, \dots, N$, together with

$$P_{2N+2}(x) = R_{N+1}(x^2).$$

This completes the proof. □

Example 8.11 (Complementary Bannai–Ito polynomials)**.** *Fix parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, and set*

$$g = \alpha + \beta - \gamma - \delta.$$

Assume that $\mathbf{v} \in \mathcal{P}'$ is a regular functional whose monic orthogonal polynomial sequence $(R_n)_{n\in\mathbb{N}}$ satisfies $R_{-1} = 0$, $R_0 = 1$, and the recurrence

$$R_1(t) + \big(\beta^2 - a_0 + c_0\big)R_0(t) = t\, R_0(t),$$

together with, for $n \in \mathbb{N}^\times$, the relation

$$R_{n+1}(t) + \big(\beta^2 - a_n + c_n\big)R_n(t) - a_{n-1}c_n\, R_{n-1}(t) = t\, R_n(t), \tag{8.1}$$

where

$$a_n = \frac{(n+g+1)(n+\alpha+\beta+1)\big(n+\beta-\gamma+\frac{1}{2}\big)\big(n+\beta-\delta+\frac{1}{2}\big)}{(2n+g+1)(2n+g+2)},$$

$$c_n = -\frac{n(n-\gamma-\delta)\big(n+\alpha-\gamma+\frac{1}{2}\big)\big(n+\alpha-\delta+\frac{1}{2}\big)}{(2n+g)(2n+g+1)}.$$

Assume that all denominators occurring in the definitions of a_n and c_n are non-zero for the indices under consideration. In the infinite case, assume moreover that

$$R_n(\beta^2) \neq 0$$

for every $n \in \mathbb{N}$. In a finite truncation, the same assumption is required only for the indices which occur in the construction. We now write out the infinite case. Set

$$\mathbf{u} = \mathbf{J}_\beta \mathbf{v}.$$

By Proposition 8.5, this is the unique functional satisfying

$$\boldsymbol{\sigma}\mathbf{u} = \mathbf{v}, \quad \langle \mathbf{u}, (x-\beta)P(x^2)\rangle = 0$$

for every $P \in \mathcal{P}$. We now apply Theorem 8.6 with

$$\tau = \beta.$$

In the infinite case, since $\mathbf{v}$ is regular and

$$R_n(\beta^2) \neq 0$$

for every $n \in \mathbb{N}$, the functional $\mathbf{u}$ is regular and its monic orthogonal polynomial sequence $(P_n)_{n\in\mathbb{N}}$ is given by

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x-\beta)S_n(x^2),$$

where

$$S_n(t) = \frac{R_{n+1}(t) - \dfrac{R_{n+1}(\beta^2)}{R_n(\beta^2)}\, R_n(t)}{t-\beta^2},$$

for every $n \in \mathbb{N}$.

We first compute the quotient

$$\frac{R_{n+1}(\beta^2)}{R_n(\beta^2)}.$$

Since $c_0 = 0$, the initial recurrence gives

$$R_1(t) + (\beta^2 - a_0)R_0(t) = tR_0(t),$$

and therefore

$$R_1(t) = t - \beta^2 + a_0.$$

Evaluating at $t = \beta^2$, we obtain

$$R_1(\beta^2) = a_0 = a_0 R_0(\beta^2).$$

Assume inductively that

$$R_n(\beta^2) = a_{n-1}R_{n-1}(\beta^2).$$

Evaluating (8.1) *at $t = \beta^2$, we obtain*

$$R_{n+1}(\beta^2) = (a_n - c_n)R_n(\beta^2) + a_{n-1}c_n R_{n-1}(\beta^2).$$

Using the induction hypothesis, this becomes

$$R_{n+1}(\beta^2) = (a_n - c_n)R_n(\beta^2) + c_n R_n(\beta^2) = a_n R_n(\beta^2).$$

Hence

$$\frac{R_{n+1}(\beta^2)}{R_n(\beta^2)} = a_n$$

for every $n \in \mathbb{N}$. Consequently,

$$S_n(t) = \frac{R_{n+1}(t) - a_n R_n(t)}{t - \beta^2}$$

for every $n \in \mathbb{N}$. Since

$$R_{n+1}(\beta^2) - a_n R_n(\beta^2) = 0,$$

the numerator is divisible by $t - \beta^2$, and therefore S_n is indeed a polynomial. Substituting this expression into the definition of P_{2n+1}, we obtain

$$P_{2n+1}(x) = (x - \beta)\,\frac{R_{n+1}(x^2) - a_n R_n(x^2)}{x^2 - \beta^2} = \frac{R_{n+1}(x^2) - a_n R_n(x^2)}{x + \beta}.$$

This identity is legitimate because the numerator vanishes at $x = -\beta$, being equal there to

$$R_{n+1}(\beta^2) - a_n R_n(\beta^2) = 0.$$

In particular, $P_0(x) = 1$, $P_1(x) = x - \beta$. We claim that

$$P_{n+1}(x) + (-1)^n \beta\, P_n(x) + \tau_n P_{n-1}(x) = x P_n(x), \tag{8.2}$$

where

$$\tau_{2n} = c_n, \quad \tau_{2n+1} = -a_n.$$

Indeed, from the preceding identity one has

$$R_{n+1}(x^2) = (x + \beta) P_{2n+1}(x) + a_n P_{2n}(x).$$

For $n = 0$, the recurrence gives directly

$$x P_0(x) = P_1(x) + \beta P_0(x).$$

Now let $n \in \mathbb{N}^\times$. Replacing n by $n - 1$, one finds

$$P_{2n-1}(x) = \frac{R_n(x^2) - a_{n-1} R_{n-1}(x^2)}{x + \beta},$$

and hence

$$a_{n-1} R_{n-1}(x^2) = R_n(x^2) - (x + \beta) P_{2n-1}(x) = P_{2n}(x) - (x + \beta) P_{2n-1}(x).$$

Substituting these identities into (8.1) with $t = x^2$, we get

$$\begin{aligned} x^2 P_{2n}(x) &= R_{n+1}(x^2) + (\beta^2 - a_n + c_n) R_n(x^2) - a_{n-1} c_n R_{n-1}(x^2) \\ &= \big((x + \beta) P_{2n+1}(x) + a_n P_{2n}(x)\big) + (\beta^2 - a_n + c_n) P_{2n}(x) \\ &\quad - c_n\big(P_{2n}(x) - (x + \beta) P_{2n-1}(x)\big). \end{aligned}$$

The terms involving a_n and c_n cancel, and one obtains

$$x^2 P_{2n}(x) = (x + \beta) P_{2n+1}(x) + \beta^2 P_{2n}(x) + c_n (x + \beta) P_{2n-1}(x).$$

Therefore

$$(x + \beta)(x - \beta) P_{2n}(x) = (x + \beta)\big(P_{2n+1}(x) + c_n P_{2n-1}(x)\big).$$

Since both sides are polynomials and $x + \beta$ is a common factor, it follows that

$$(x - \beta) P_{2n}(x) = P_{2n+1}(x) + c_n P_{2n-1}(x),$$

that is,

$$x P_{2n}(x) = P_{2n+1}(x) + \beta P_{2n}(x) + c_n P_{2n-1}(x).$$

This is exactly (8.2) for even index, since

$$\tau_{2n} = c_n.$$

For odd index, multiplying the formula for P_{2n+1} by $x + \beta$ yields

$$(x + \beta) P_{2n+1}(x) = R_{n+1}(x^2) - a_n R_n(x^2) = P_{2n+2}(x) - a_n P_{2n}(x).$$

Thus

$$xP_{2n+1}(x) = P_{2n+2}(x) - \beta P_{2n+1}(x) - a_n P_{2n}(x),$$

that is,

$$P_{2n+2}(x) - \beta P_{2n+1}(x) - a_n P_{2n}(x) = xP_{2n+1}(x).$$

This is exactly (8.2) *for odd index, since*

$$\tau_{2n+1} = -a_n.$$

Therefore the monic polynomial sequence $(P_n)_{n\in\mathbb{N}}$ *satisfies the recurrence* (8.2)*, with*

$$\tau_{2n} = c_n, \quad \tau_{2n+1} = -a_n.$$

By the definitions of a_n *and* c_n*, this means explicitly*

$$\tau_{2n} = -\frac{n(n-\gamma-\delta)\big(n+\alpha-\gamma+\frac{1}{2}\big)\big(n+\alpha-\delta+\frac{1}{2}\big)}{(2n+g)(2n+g+1)},$$

$$\tau_{2n+1} = -\frac{(n+g+1)(n+\alpha+\beta+1)\big(n+\beta-\gamma+\frac{1}{2}\big)\big(n+\beta-\delta+\frac{1}{2}\big)}{(2n+g+1)(2n+g+2)}.$$

Together with $P_0(x) = 1$ *and* $P_1(x) = x - \beta$*, this is precisely the standard recurrence relation for the monic complementary Bannai–Ito polynomials. More explicitly, under the parameter identification* $\rho_1 = \alpha$*,* $\rho_2 = \beta$*,* $r_1 = \gamma$*, and* $r_2 = \delta$*, the recurrence obtained above is exactly the monic complementary Bannai–Ito recurrence displayed in* [8, (3.4), (3.5)].

Remark 8.12. *In* [8]*, the complementary Bannai–Ito polynomials are introduced through their recurrence and orthogonality relations, and their even–odd splitting is written out explicitly. They are then shown to satisfy a Dunkl-type bispectral problem. The point of the preceding construction is not merely that the complementary Bannai–Ito recurrence is recovered from the alternating pullback. Rather, it shows that the even–odd splitting is the expected functional manifestation of the alternating mechanism.*

Example 8.13 (Dual (-1)-Hahn polynomials: the even-N case)**.** *We now illustrate the alternating mechanism with the dual* (-1)*-Hahn polynomials, restricting ourselves to the case where* N *is even. This is already sufficient to make the structural point completely transparent. Let* N *be even, and let* $R_0, R_1, \ldots, R_N$ *denote the monic dual* (-1)*-Hahn polynomial sequence. By* [30, (3.6), (3.7)]*, one has* $R_{-1} = 0$*,* $R_0 = 1$*, and, for* $n \in \{0, 1, \ldots, N-1\}$*, one has*

$$R_{n+1}(t) + b_n R_n(t) + u_n R_{n-1}(t) = t\, R_n(t),$$

where

$$u_n = \begin{cases} 4n(\alpha - n), & \text{if } n \text{ is even}, \\ 4(N-n+1)(n+\beta-N-1), & \text{if } n \text{ is odd}, \end{cases}$$

$$b_n = \begin{cases} 2N+1-\alpha-\beta, & \text{if } n \text{ is even}, \\ -2N-3+\alpha+\beta, & \text{if } n \text{ is odd}. \end{cases}$$

Set

$$\tau = 2N + 2 - \alpha - \beta.$$

Then

$$b_n = (-1)^n \tau - 1.$$

Define the shifted monic sequence

$$\widehat{R}_n(t) = R_n(t-1).$$

It follows immediately that $\widehat{R}_{-1} = 0$, $\widehat{R}_0 = 1$, *and*

$$\widehat{R}_{n+1}(t) + (-1)^n \tau\, \widehat{R}_n(t) + u_n \widehat{R}_{n-1}(t) = t\, \widehat{R}_n(t). \tag{8.3}$$

Indeed, substituting $t-1$ *for* t *in the recurrence for* R_n, *one obtains*

$$\widehat{R}_{n+1}(t) + \big(b_n + 1\big)\widehat{R}_n(t) + u_n \widehat{R}_{n-1}(t) = t\, \widehat{R}_n(t),$$

and the identity $b_n + 1 = (-1)^n \tau$ *yields* (8.3). *Thus, after the elementary shift* $t \mapsto t-1$, *the even-*N *dual* (-1)*-Hahn family falls exactly into the alternating recurrence pattern considered above. This is precisely the recurrence-level shadow of the normalised alternating pullback. At the recurrence level, the shifted family* $\widehat{R}_0, \widehat{R}_1, \ldots, \widehat{R}_N$ *has exactly the alternating form of the full monic sequence in the present framework. Its even and odd subsequences therefore play the roles, respectively, of the quadratic base family and of its Christoffel transform. Equivalently, this is the recurrence-level shadow of the functional mechanism*

$$\mathbf{u} = \mathbf{J}_\tau \mathbf{v}, \quad \boldsymbol{\sigma}\mathbf{u} = \mathbf{v}, \quad \langle \mathbf{u}, (x-\tau) p(x^2) \rangle = 0,$$

with $\sigma p = p(x^2)$. *Thus the even members of the full family must be polynomials in* t^2, *while the odd members must contain the distinguished factor* $t - \tau$. *Write*

$$N = 2M.$$

Consequently, the recurrence-level alternating splitting applies here and yields monic polynomial sequences $P_0, P_1, \ldots, P_M$ *and* $Q_0, Q_1, \ldots, Q_{M-1}$. *More precisely, for* $n = 0, 1, \ldots, M$, *the even part is described by*

$$\widehat{R}_{2n}(t) = P_n(t^2),$$

whereas, for $n = 0, 1, \ldots, M-1$, *the odd part is described by*

$$\widehat{R}_{2n+1}(t) = (t-\tau) Q_n(t^2).$$

The corresponding quadratic recurrence relations are then the following. For $n = 0$, *the last term is absent and one has*

$$P_1(x) + \big(u_0 + u_1 + \tau^2\big) P_0(x) = x\, P_0(x).$$

For $n \in \{1, \ldots, M-1\}$, *one has*

$$P_{n+1}(x) + \big(u_{2n} + u_{2n+1} + \tau^2\big) P_n(x) + u_{2n} u_{2n-1} P_{n-1}(x) = x\, P_n(x).$$

Similarly, one has

$$Q_{n+1}(x) + \big(u_{2n+2} + u_{2n+1} + \tau^2\big) Q_n(x) + u_{2n} u_{2n+1} Q_{n-1}(x) = x\, Q_n(x)$$

for $n = 0, 1, \ldots, M-2$, *with the convention* $Q_{-1} = 0$. *Finally, for* $n = 0, 1, \ldots, M-1$, *the Christoffel relation is*

$$Q_n(x) = \frac{P_{n+1}(x) + u_{2n+1} P_n(x)}{x - \tau^2}. \tag{8.4}$$

Now

$$u_{2n} = 8n(\alpha - 2n), \quad u_{2n+1} = 4(N-2n)(2n + \beta - N),$$

so the preceding formulas agree with the quadratic recurrence relations displayed in [30, (4.2)–(4.5)]. *Moreover, the explicit hypergeometric formulas in* [30, (4.6), (4.7)] *identify* P_n *and* Q_n *as ordinary dual Hahn polynomials, after the corresponding identification of the quadratic variable.*

In the even-N *case, the decomposition exhibited in* [30] *is naturally read as an instance of the alternating splitting mechanism. What appears there through direct manipulation of the recurrence is seen here as the expected manifestation of the quadratic-substitution pattern underlying the alternating case. The odd-*N *case leads, after the corresponding modification of the coefficients, to the same conclusion. We do not repeat it here, since, for the present purpose, the even-*N *case already makes the mechanism entirely clear.*

The examples discussed above also clarify the precise role of the families considered in [30] and [8]. From the viewpoint developed here, these families should not be regarded as isolated exceptional objects. Nor is their nature adequately explained by saying that they arise as $q \to -1$ limits, or that they are governed by Dunkl-type operators. Such statements may be computationally correct, but they do not by themselves provide the relevant structural explanation. The same applies, mutatis mutandis, to the Bannai–Ito polynomials [29], to the big (-1)-Jacobi polynomials [34], to the (-1)-Meixner–Pollaczek polynomials [26], and to other families appearing in the literature.

The point is not to deny the validity of the calculations leading to Dunkl-type operators or to $q \to -1$ limits. The point is that such calculations should not be mistaken for the conceptual origin of the phenomenon. The alternating case is not merely the formal substitution $q = -1$ inside a q-exponential construction, just as the quadratic case is not merely the value $q = 1$ of that construction. It is a distinct admissible regime. This returns us to the principle suggested at the outset by the Erdős anecdote: a correct formal description is not enough; one must identify the structure that governs the phenomenon.

Perhaps the clearest way to see this is to recall the much-cited article *A "missing" family of classical orthogonal polynomials* [32]. The title is revealing. In the usual classifications, the structural possibilities explicitly encoded are essentially the quadratic and q-exponential ones. From that standpoint the family described in [32] may appear to be "missing". From the viewpoint developed here, however, it is not missing at all; the theory itself tells us that it was being sought in a place where it could not reside. It is a concrete manifestation of the alternating mechanism, and could not be expected to occupy a place in a classification which does not admit that mechanism. As the preceding sections show, the case "$q = -1$" belongs to a different admissible reality: the alternating regime.

9. The Nikiforov–Uvarov equation

The next theorem, which is independent of any regularity assumption, makes clear the direct connection between the developments obtained so far and the Nikiforov–Uvarov framework.

Theorem 9.1. *Fix $h \in \mathbb{C}^\times$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be an admissible map on U. Let $D, S : \mathcal{P} \to \mathcal{P}$ be the associated divided-difference and averaging operators, with transposes $\mathbf{D}, \mathbf{S} : \mathcal{P}' \to \mathcal{P}'$. Let $\mathbf{u} \in \mathcal{P}'$, let ϕ and ψ be polynomials, and define*

$$L = \phi\, D^2 + \psi\, SD.$$

Then the following statements hold.

(i) *If $\mathbf{D}(\phi\, \mathbf{u}) = \mathbf{S}(\psi\, \mathbf{u})$ in $\mathcal{P}'$, then*

$$\langle \mathbf{u}, (Lp)\, r\rangle = \langle \mathbf{u}, p\, (Lr)\rangle$$

for every $p, r \in \mathcal{P}$.

(ii) *Assume, in addition, that $D(\mathcal{P}_{n+1}) = \mathcal{P}_n$ for every $n \in \mathbb{N}$. If $\langle \mathbf{u}, (Lp)\, r\rangle = \langle \mathbf{u}, p\, (Lr)\rangle$ for every $p, r \in \mathcal{P}$, then*

$$\mathbf{D}(\phi\, \mathbf{u}) = \mathbf{S}(\psi\, \mathbf{u})$$

in $\mathcal{P}'$.

(iii) *Let $N \in \mathbb{N}^\times$. Assume that $D(\mathcal{P}_{n+1}) = \mathcal{P}_n$ for $n = 0, 1, \ldots, 2N-1$. If*

$$\langle \mathbf{u}, (Lp)\, r\rangle = \langle \mathbf{u}, p\, (Lr)\rangle$$

for every $p \in \mathcal{P}$ and every $r \in \mathcal{P}_{2N}$, then

$$\big\langle \mathbf{D}(\phi\, \mathbf{u}) - \mathbf{S}(\psi\, \mathbf{u}),\, v\big\rangle = 0$$

for every $v \in \mathcal{P}_{2N-1}$.

Moreover, for every function $f : U \to \mathbb{C}$, set

$$(\Delta f)(s) = f(s+h) - f(s), \quad (\nabla f)(s) = f(s) - f(s-h).$$

Then, for every $p \in \mathcal{P}$,

$$(Lp)(X(s)) = \phi(X(s))\frac{1}{Y(s)-Z(s)}\,\Delta\left(\frac{\nabla(p\circ X)}{X-Z_1}\right)(s)$$
$$+\frac{\psi(X(s))}{2}\left(\frac{\Delta(p\circ X)(s)}{Y_1(s)-X(s)}+\frac{\nabla(p\circ X)(s)}{X(s)-Z_1(s)}\right)$$

for every $s \in U$[5].

Proof. By Definition 6.3, evaluating the defining identity at $s+\frac{h}{2}$ and $s-\frac{h}{2}$ gives

$$(Dp)(Y(s)) = \frac{\Delta(p\circ X)(s)}{Y_1(s)-X(s)}, \quad (Dp)(Z(s)) = \frac{\nabla(p\circ X)(s)}{X(s)-Z_1(s)}.$$

Hence

$$(D^2p)(X(s)) = \frac{1}{Y(s)-Z(s)}\,\Delta\left(\frac{\nabla(p\circ X)}{X-Z_1}\right)(s).$$

Likewise,

$$(SDp)(X(s)) = \frac{1}{2}\left(\frac{\Delta(p\circ X)(s)}{Y_1(s)-X(s)}+\frac{\nabla(p\circ X)(s)}{X(s)-Z_1(s)}\right).$$

This establishes the displayed formula for L. Now let $r \in \mathcal{P}$, and define

$$w = (Dp)(Sr) - (Sp)(Dr).$$

A direct computation gives

$$w(X) = \frac{p(Y)\,r(Z) - p(Z)\,r(Y)}{Y-Z}.$$

Evaluating at $s+\frac{h}{2}$ and $s-\frac{h}{2}$, one obtains

$$w(Y) = \frac{p(Y_1)\,r(X) - p(X)\,r(Y_1)}{Y_1-X}, \quad w(Z) = \frac{p(X)\,r(Z_1) - p(Z_1)\,r(X)}{X-Z_1}.$$

Hence

$$\begin{aligned}(Dw)(X) &= \frac{1}{Y-Z}\left(r(X)\frac{p(Y_1)-p(X)}{Y_1-X} - r(X)\frac{p(X)-p(Z_1)}{X-Z_1}\right)\\ &\quad - \frac{1}{Y-Z}\left(p(X)\frac{r(Y_1)-r(X)}{Y_1-X} - p(X)\frac{r(X)-r(Z_1)}{X-Z_1}\right)\\ &= (D^2p)(X)\,r(X) - p(X)(D^2r)(X).\end{aligned}$$

Therefore

$$Dw = (D^2p)\,r - p\,(D^2r).$$

[5]The denominators which occur in this pointwise expression are non-zero by admissibility. Indeed, $Y(s)-Z(s)$ is the admissible denominator at s, while

$$Y_1(s)-X(s) = X(s+h)-X(s) = Y\left(s+\frac{h}{2}\right) - Z\left(s+\frac{h}{2}\right)$$

and

$$X(s)-Z_1(s) = X(s)-X(s-h) = Y\left(s-\frac{h}{2}\right) - Z\left(s-\frac{h}{2}\right).$$

Since U is half-step-invariant, the shifted points $s+\frac{h}{2}$ and $s-\frac{h}{2}$ still belong to U.

Likewise,

$$Sw = (SDp)\, r - p\, (SDr).$$

Therefore

$$\begin{aligned}(Lp)\, r - p\, (Lr) &= \phi\big((D^2p)r - p(D^2r)\big) + \psi\big((SDp)r - p(SDr)\big)\\ &= \phi\, Dw + \psi\, Sw.\end{aligned}$$

Assume first that

$$\mathbf{D}(\phi\, \mathbf{u}) = \mathbf{S}(\psi\, \mathbf{u}).$$

Then, for every $v \in \mathcal{P}$,

$$0 = \big\langle \mathbf{D}(\phi\, \mathbf{u}) - \mathbf{S}(\psi\, \mathbf{u}),\, v\big\rangle = -\langle \mathbf{u}, \phi\, Dv + \psi\, Sv\rangle. \tag{9.1}$$

Applying (9.1) with $v = w$, we obtain $\langle \mathbf{u}, (Lp)\, r\rangle = \langle \mathbf{u}, p\, (Lr)\rangle$. This proves *(i)*. Conversely, assume that

$$\langle \mathbf{u}, (Lp)\, r\rangle = \langle \mathbf{u}, p\, (Lr)\rangle$$

for every $p, r \in \mathcal{P}$, and that $D(\mathcal{P}_{n+1}) = \mathcal{P}_n$ for every $n \in \mathbb{N}$. Taking $p = 1$, and using $L1 = 0$, we obtain $\langle \mathbf{u}, Lr\rangle = 0$. Thus

$$\langle \mathbf{u}, \phi\, D^2r + \psi\, SDr\rangle = 0.$$

Using the definitions of $\mathbf{D}$ and $\mathbf{S}$, this becomes

$$\big\langle \mathbf{S}(\psi\, \mathbf{u}) - \mathbf{D}(\phi\, \mathbf{u}), Dr\big\rangle = 0.$$

Now let $p \in \mathcal{P}_n$. By hypothesis, there exists $r \in \mathcal{P}_{n+1}$ such that $Dr = p$. Hence

$$\big\langle \mathbf{S}(\psi\, \mathbf{u}) - \mathbf{D}(\phi\, \mathbf{u}), p\big\rangle = 0$$

for every $p \in \mathcal{P}$. Since the pairing between $\mathcal{P}'$ and $\mathcal{P}$ separates points, condition *(ii)* follows.

Finally, assume the hypotheses of *(iii)*. Taking again $p = 1$, we obtain $\langle \mathbf{u}, Lr\rangle = 0$ for every $r \in \mathcal{P}_{2N}$. Hence

$$\big\langle \mathbf{S}(\psi\, \mathbf{u}) - \mathbf{D}(\phi\, \mathbf{u}), Dr\big\rangle = 0$$

for every $r \in \mathcal{P}_{2N}$. Now let $v \in \mathcal{P}_{2N-1}$. By the hypothesis $D(\mathcal{P}_{n+1}) = \mathcal{P}_n$ for $n = 0, 1, \ldots, 2N-1$, there exists $r \in \mathcal{P}_{2N}$ such that $Dr = v$. Therefore $\big\langle \mathbf{S}(\psi\, \mathbf{u}) - \mathbf{D}(\phi\, \mathbf{u}), v\big\rangle = 0$ for every $v \in \mathcal{P}_{2N-1}$. Equivalently,

$$\big\langle \mathbf{D}(\phi\, \mathbf{u}) - \mathbf{S}(\psi\, \mathbf{u}),\, v\big\rangle = 0$$

for every $v \in \mathcal{P}_{2N-1}$. □

Remark 9.2. *The additional surjectivity hypothesis in parts* (ii) *and* (iii) *is automatic in the quadratic case, but not in general in the q-exponential one, and it fails in the alternating case. Indeed, in the quadratic case one has*

$$D(x^{n+1}) - (n+1)x^n \in \mathcal{P}_{n-1},$$

and therefore

$$D(\mathcal{P}_{n+1}) = \mathcal{P}_n$$

for every $n \in \mathbb{N}$. In the q-exponential case one has

$$D(x^{n+1}) - \gamma_{n+1}x^n \in \mathcal{P}_{n-1},$$

where the numbers γ_n are those introduced in Theorem 7.1. Hence

$$D(\mathcal{P}_{n+1}) = \mathcal{P}_n$$

for every $n \in \mathbb{N}$ if and only if $\gamma_{n+1} \neq 0$ for every $n \in \mathbb{N}$. In particular, this holds whenever q is not a root of unity. If q is a root of unity, then $\gamma_m = 0$ for some $m \geq 1$, so the above surjectivity-by-degree condition fails from that level onward, even though a finite theory may still persist below the first vanishing index. In that case, part (iii) *of Theorem 9.1 may still* *be applicable up to the corresponding finite level, whereas part* (ii) *need not apply globally. Finally, by Proposition 8.1, in the normalised alternating case one has* $D(\mathcal{P}) = \mathcal{P}(x^2)$. *Hence,*

$$D(\mathcal{P}_{n+1}) \subseteq \mathcal{P}(x^2) \cap \mathcal{P}_n,$$

whereas $\mathcal{P}_n$ *contains odd polynomials. It follows that*

$$D(\mathcal{P}_{n+1}) \neq \mathcal{P}_n$$

for every $n \in \mathbb{N}^\times$. *Thus the implication asserted in part* (ii) *of Theorem 9.1 applies automatically in the quadratic case, applies in the q-exponential case precisely under the non-vanishing condition stated above, and does not apply in the normalised alternating case; in the torsion q-exponential situation, however, the finite-level variant given by part* (iii) *may still remain available.*

In the alternating case, the converse direction in Theorem 9.1 fails in general, since D is not surjective on $\mathcal{P}$. When ψ is of degree 1, however, formal symmetry still recovers the first-order structural condition appearing in Definition 6.6. The following corollary makes this precise.

Corollary 9.3. *Retain the hypotheses and notation of Theorem 9.1. Assume, in addition, that* X *is the normalised alternating map given by*

$$X(s) = e^{\pi i s/h}$$

for every $s \in U$, *and that* ψ *is of degree* 1. *If*

$$\langle \mathbf{u}, (Lp)\, r\rangle = \langle \mathbf{u}, p\,(Lr)\rangle$$

for every $p, r \in \mathcal{P}$, *then there exist* $c \in \mathbb{C}^\times$ *and* $\tau \in \mathbb{C}$ *such that*

$$\psi(x) = c(x-\tau),$$

and

$$\mathbf{S}\big((x-\tau)\mathbf{u}\big) = 0$$

in $\mathcal{P}'$. *Equivalently,*

$$\langle \mathbf{u}, (x-\tau)p(x^2)\rangle = 0$$

for every $p \in \mathcal{P}$.

Proof. By hypothesis, $\langle \mathbf{u}, (Lp)\, r\rangle = \langle \mathbf{u}, p\,(Lr)\rangle$ for every $p, r \in \mathcal{P}$. Taking $p = 1$, and using $L1 = 0$, we obtain $\langle \mathbf{u}, Lr\rangle = 0$. Hence

$$\big\langle \mathbf{S}(\psi\,\mathbf{u}) - \mathbf{D}(\phi\,\mathbf{u}), Dr\big\rangle = 0.$$

Now assume that X is the normalised alternating map. By Proposition 8.1, one has

$$D(\mathcal{P}) = \mathcal{P}(x^2).$$

Therefore

$$\boldsymbol{\sigma}\big(\mathbf{S}(\psi\,\mathbf{u}) - \mathbf{D}(\phi\,\mathbf{u})\big) = 0.$$

Moreover,

$$\boldsymbol{\sigma}\mathbf{D}(\phi\,\mathbf{u}) = 0.$$

Indeed, for every $p \in \mathcal{P}$,

$$\big\langle \boldsymbol{\sigma}\mathbf{D}(\phi\,\mathbf{u}), p\big\rangle = \big\langle \mathbf{D}(\phi\,\mathbf{u}), p(x^2)\big\rangle = -\big\langle \phi\,\mathbf{u}, D(p(x^2))\big\rangle = 0,$$

since $D(p(x^2)) = 0$. Thus

$$\boldsymbol{\sigma}\mathbf{S}(\psi\,\mathbf{u}) = 0.$$

Since ψ is of degree 1, there exist $c \in \mathbb{C}^\times$ and $\tau \in \mathbb{C}$ such that

$$\psi(x) = c(x-\tau).$$

It follows that

$$\boldsymbol{\sigma}\mathbf{S}\big((x-\tau)\mathbf{u}\big)=0.$$

By definition of $\boldsymbol{\sigma}$ and $\mathbf{S}$, this is equivalent to

$$\langle \mathbf{u},(x-\tau)p(-x^2)\rangle=0.$$

Replacing $p(x)$ by $p(-x)$, we obtain

$$\langle \mathbf{u},(x-\tau)p(x^2)\rangle=0$$

for every $p\in\mathcal{P}$. Since, in the normalised alternating case, one has $S(\mathcal{P})=\mathcal{P}(x^2)$, the last identity is equivalent to

$$\mathbf{S}\big((x-\tau)\mathbf{u}\big)=0$$

in $\mathcal{P}'$. This proves the conclusion. □

Corollary 9.4. *Retain the hypotheses and notation of Theorem 9.1, and assume that $\mathbf{u}\in\mathcal{P}'$ is classical. Let $(P_n)_{n\in I}$ be the monic orthogonal polynomial sequence with respect to $\mathbf{u}$, where either $I=\mathbb{N}$ or $I=\{0,1,\dots,N\}$ for some $N\geq 2$. Then, for every $n\in I$, there exists $\lambda_n\in\mathbb{C}$ such that*

$$LP_n=\lambda_nP_n.$$

Proof. By Definition 6.6, there exist polynomials ϕ and ψ, where ϕ has degree at most 2 and ψ has degree at most 1, such that

$$\mathbf{D}(\phi\,\mathbf{u})=\mathbf{S}(\psi\,\mathbf{u}).$$

Hence, by Theorem 9.1*(i)*, the operator L is formally symmetric with respect to $\mathbf{u}$, that is,

$$\langle \mathbf{u},(Lp)\,r\rangle=\langle \mathbf{u},p\,(Lr)\rangle$$

for every $p,r\in\mathcal{P}$. Since ϕ and ψ have degrees at most 2 and 1, respectively, one has

$$\phi\,D^2(\mathcal{P}_n)\subseteq\mathcal{P}_n,\quad \psi\,SD(\mathcal{P}_n)\subseteq\mathcal{P}_n,$$

and therefore

$$L(\mathcal{P}_n)\subseteq\mathcal{P}_n$$

for every $n\in\mathbb{N}$. In particular,

$$LP_n\in\mathcal{P}_n.$$

If $n=0$, then $P_0=1$ and $L1=0$. Hence $LP_0=0$, so one may take $\lambda_0=0$. Assume now that $n\in I^\times$, and let $r\in\mathcal{P}_{n-1}$. Then, by formal symmetry,

$$\langle \mathbf{u},LP_n\,r\rangle=\langle \mathbf{u},P_n\,(Lr)\rangle.$$

Since $Lr\in\mathcal{P}_{n-1}$ and P_n is orthogonal to $\mathcal{P}_{n-1}$, it follows that

$$\langle \mathbf{u},LP_n\,r\rangle=0.$$

Thus $LP_n\in\mathcal{P}_n$ is orthogonal to $\mathcal{P}_{n-1}$. Since $\mathbf{u}$ is regular and P_n is the monic orthogonal polynomial of degree n, there exists $\lambda_n\in\mathbb{C}$ such that

$$LP_n=\lambda_nP_n,$$

which is precisely the desired conclusion. □

Remark 9.5. *Let λ_n be as in Corollary 9.4. Then*

$$\lambda_0=0.$$

Moreover, for each $n\in I^\times$, λ_n is the leading coefficient of $L(x^n)$. Indeed, since P_n is monic, one has

$$P_n-x^n\in\mathcal{P}_{n-1},$$

and since $L(\mathcal{P}_{n-1})\subseteq\mathcal{P}_{n-1}$, it follows that

$$LP_n-L(x^n)\in\mathcal{P}_{n-1}.$$

Hence the coefficient of x^n in LP_n coincides with that in $L(x^n)$. Since

$$LP_n = \lambda_n P_n$$

and P_n is monic, this coefficient is precisely λ_n. Consequently,

$$\lambda_n \neq 0$$

if and only if $L(x^n)$ is of degree n. In the quadratic case, since

$$D(x^n) = nx^{n-1} + \cdots, \quad S(x^m) = x^m + \cdots,$$

one obtains

$$L(x^n) = n\big(\phi_2(n-1) + \psi_1\big)x^n + \cdots.$$

Thus

$$\lambda_n = n\, d_{n-1}.$$

In the q-exponential case, since

$$D(x^n) = \gamma_n x^{n-1} + \cdots, \quad S(x^m) = \alpha_m x^m + \cdots,$$

one obtains

$$L(x^n) = \gamma_n\big(\phi_2\gamma_{n-1} + \psi_1\alpha_{n-1}\big)x^n + \cdots.$$

Thus

$$\lambda_n = \gamma_n d_{n-1}.$$

Consequently, the possible vanishing of λ_n is governed by the same factors that appear in the regularity criteria established above.

As a final point, we record one structural feature of particular significance: the point at which the distinction between quadratic and q-exponential maps, on the one hand, and the normalised alternating map, on the other, becomes intrinsic and ineliminable, while at the same time marking a decisive departure from the ordinary theory, where D is ordinary differentiation and $S = \mathrm{id}_{\mathcal{P}}$. This is the Hahn property. The classical form of this phenomenon goes back to Hahn's work [9], and to his later note on higher derivatives of orthogonal polynomials [10]. In modern terminology, the Hahn property asserts that, in the ordinary differential setting, an orthogonal polynomial sequence whose derivative sequence is again orthogonal must belong to the classical families.

Proposition 9.6. *Fix $h \in \mathbb{C}^\times$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let $X : U \to \mathbb{C}$ be an admissible map. Let $D, S : \mathcal{P} \to \mathcal{P}$ be the associated divided-difference and averaging operators, with transposes $\mathbf{D}, \mathbf{S} : \mathcal{P}' \to \mathcal{P}'$. Let $I \subseteq \mathbb{N}$ be either $I = \mathbb{N}$, or $I = \{0, 1, \ldots, N+1\}$ for some $N \geq 2$. Let $\mathbf{u} \in \mathcal{P}'$ be regular, with monic orthogonal polynomial sequence $(P_n)_{n\in I}$. In the finite case, for every $k \in \{0, 1, \ldots, N+1\}$, set $I_k = \{0, 1, \ldots, N+1-k\}$, whereas in the infinite case, for every $k \in \mathbb{N}$, set $I_k = \mathbb{N}$. Assume that X is of quadratic type or of q-exponential type with q not a root of unity, in the sense of Theorem 5.1. Set*

$$\kappa_{n,0} = 1,$$

and, for every $k \in \mathbb{N}^\times$, set

$$\kappa_{n,k} = \prod_{j=1}^{k} \gamma_{n+j}, \quad Q_n^{[k]} = \kappa_{n,k}^{-1} D^k P_{n+k},$$

where, in the q-exponential case, the numbers γ_n are those introduced in Theorem 7.1, whereas in the quadratic case one has $\gamma_n = n$. If there exist polynomials ϕ and ψ, where ϕ has degree at most 2 and ψ has degree at most 1, not both identically zero, such that

$$\mathbf{D}(\phi\, \mathbf{u}) = \mathbf{S}(\psi\, \mathbf{u}),$$

then there exist $k \in \mathbb{N}^\times$ and a classical functional $\mathbf{u}^{[k]} \in \mathcal{P}'$ such that the family $(Q_n^{[k]})_{n\in I_k}$ is the monic orthogonal polynomial sequence with respect to $\mathbf{u}^{[k]}$.

Proof. Assume that there exist polynomials ϕ and ψ, where ϕ has degree at most 2 and ψ has degree at most 1, not both identically zero, such that $\mathbf{D}(\phi\,\mathbf{u}) = \mathbf{S}(\psi\,\mathbf{u})$. If X is of quadratic type, we apply Theorem 7.4. If X is of q-exponential type, with q not a root of unity, we apply Theorem 7.1. In either case, the construction given in the proof of the corresponding theorem shows that the first derived functional $\mathbf{u}^{[1]}$ is classical, and that the derived family $(Q_n^{[1]})_{n\in I_1}$ is the monic orthogonal polynomial sequence with respect to $\mathbf{u}^{[1]}$. Hence the conclusion holds with $k = 1$. □

Remark 9.7. *In the q-exponential case, the assumption that q is not a root of unity is used to ensure that the normalising factors $\kappa_{n,k}$ which occur in the definition of the derived families do not vanish for any relevant indices. Equivalently, the factors of the form γ_m which enter these normalisations remain non-zero throughout the construction. Hence the derived families are well defined at every order. If q is a root of unity, this mechanism can break down once one of the indices involved in the normalising factors reaches a vanishing value. In that case, one cannot expect the derived families to be well defined at all orders. Nevertheless, finite versions of the construction may still survive, provided that all normalising factors appearing up to the prescribed truncation level are non-zero. Thus, in the infinite case, the global conclusion of Proposition 9.6 may fail at roots of unity, whereas in the finite case the same construction remains valid up to any truncation level for which the relevant normalisations are non-vanishing.*

The reader should not underestimate the difficulty of proving the converse implication in Proposition 9.6, although the implication is trivial in certain cases, such as affine maps or purely q-exponential ones. For Maroni, the notion of classicality was always identified with Bochner's theorem in functional form, rather than with Hahn's property, even though in his setting the equivalence is immediate. In the present framework, the analogous role is played by the Nikiforov–Uvarov equation formulated in the same functional-analytic language.

10. Conclusions

At the first conference on orthogonal polynomials, held in Bar-le-Duc in 1984, Andrews and Askey wrote, in Section 4 of a particularly interesting article [1], under the deliberately resonant title *The final set of classical orthogonal polynomials*, the following words: "This section is titled in a very strong way, and we hope that someone will come along and prove we are wrong, just as has happened to everyone else who has tried to characterise the classical polynomials." The notion of classicality proposed here is not intended to suggest that there was any error in that legacy, nor in any of the developments that followed it. Rather, it seeks to embrace that legacy, together with many other results that have remained apparently outside that framework owing to the technical restrictions imposed by earlier definitions. Its aim is to gather them and to place them within a setting sufficiently broad for their internal unity to become visible, and from which Bochner's result may once again be viewed in continuity with the historical tradition to which this problem belongs. This viewpoint, moreover, cannot be separated from half-step-invariant sets or from the form of admissible maps, for it is precisely through them that one gains access to what, in our view, constitutes the true rigidity associated with classicality: a rigidity often neither visible nor necessary at first sight, until one attempts to dig deeply into a particular problem and suddenly comes up against it with full force. What is perhaps most striking, and what the preceding pages have sought to lay fully bare, is that the conception of classicality advocated here may, in substance, be viewed as a return to the original work of Nikiforov and Uvarov in 1983, though now set within a firmer and more illuminating framework through its reformulation in Maroni's functional-analytic language. After all these pages, we may, in the end, have said very little that is genuinely new. If there is any novelty here, it lies perhaps less in the results themselves than in a way of understanding, gathering, and rearticulating what was already there.

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References

1. G. E. Andrews and R. Askey, *Classical orthogonal polynomials*, Polynômes orthogonaux et applications. (French) [Orthogonal polynomials and applications]. Proceedings of the Laguerre symposium held at Bar-le-Duc, October 15–18, 1984. (Berlin) (C. Brezinski, A. Draux, A. P. Magnus, P. Maroni, and A. Ronveaux, eds.), Lecture Notes in Math., vol. 1171, Springer-Verlag, 1985, pp. 36–62.
2. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, vol. 54, Mem. Amer. Math. Soc., 1985.
3. N. M. Atakishiyev, M. Rahman, and S. K. Suslov, *On classical orthogonal polynomials*, Constr. Approx. **11** (1995), 181–126.
4. K. Castillo and G. Gordillo-Núñez, *On recurrence coefficients of classical orthogonal polynomials*, Lett. Math. Phys. **115** (2025), 132.
5. K. Castillo and G. Gordillo-Núñez, *Yet another characterisation of classical orthogonal polynomials?*, arXiv preprint arXiv:2603.08543 (2026).
6. K. Castillo, D. Mbouna, and J. Petronilho, *On the functional equation for classical orthogonal polynomials on lattices*, J. Math. Anal. Appl. **515** (2022), 126390.
7. K. Castillo and J. Petronilho, *A first course on orthogonal polynomials: Classical orthogonal polynomials and related topics*, CRC Press, Chapman & Hall, 2025.
8. V. X. Genest, L. Vinet, and A. Zhedanov, *Bispectrality of the Complementary Bannai–Ito Polynomials*, SIGMA **9** (2013), 018.
9. W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen. (German) [On the Jacobi polynomials and two related classes of polynomials]*, Math. Z. **39** (1935), 634–638.
10. ———, *Über höhere ableitungen von orthogonalpolynomen*, Math. Z. **43** (1938), 101.
11. P. Hoffman, *The Man Who Loved Only Numbers: The Story of Paul Erdős and the Search for Mathematical Truth*, Hyperion, New York, 1998.
12. R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer Monographs in Mathematics, 2010.
13. J.-M. Lemay, L. Vinet, and A. Zhedanov, *The para-Racah polynomials*, J. Math. Anal. Appl. **438** (2016), no. 2, 565–577.
14. ———, *A q-generalization of the para-Racah polynomials*, J. Math. Anal. Appl. **462** (2018), no. 1, 323–336.
15. A. P. Magnus, *Associated Askey–Wilson polynomials as Laguerre–Hahn orthogonal polynomials*, Orthogonal Polynomials and Their Applications (M. Alfaro, J. S. Dehesa, F. J. Marcellán, J. L. Rubio de Francia, and J. Vinuesa, eds.), Lecture Notes in Mathematics, vol. 1329, Springer-Verlag, Berlin, 1988, Proceedings of the conference held in Segovia, 1986, pp. 261–278.
16. ———, *Special nonuniform lattice (snul) orthogonal polynomials on discrete dense sets of points*, J. Comput. Appl. Math. **65** (1995), 253–265.
17. P. Maroni, *Sur quelques espaces de distributions qui sont des formes linéaires sur l'espace vectoriel des polynômes. (French) [Some distribution spaces that are linear forms on the vector space of polynomials]*, Polynômes orthogonaux et applications. (French) [Orthogonal polynomials and applications]. Proceedings of the Laguerre symposium held at Bar-le-Duc, October 15–18, 1984. (Berlin) (C. Brezinski, A. Draux, A. P. Magnus, P. Maroni, and A. Ronveaux, eds.), Lecture Notes in Math., vol. 1171, Springer-Verlag, 1985, pp. 184–194.
18. ———, *Le calcul des formes linéaires et les polynômes orthogonaux semi-classiques. (French) [Calculation of linear forms and semiclassical orthogonal polynomials]*, Orthogonal polynomials and their applications. Proceedings of the Second International Symposium on Orthogonal Polynomials and Their Applications held at Segovia, September 22–27, 1986 (M. Alfaro, J. S. Dehesa, F. J. Marcellán,

J. L. Rubio de Francia, and J. Vinuesa, eds.), Lecture Notes in Math., vol. 1329, Springer-Verlag, 1988, pp. 279–290.

19. ———, *Une théorie algébrique des polynômes orthogonaux. Applications aux polynômes orthogonaux semiclassiques. (French) [An algebraic theory of orthogonal polynomials. Application to semiclassical orthogonal polynomials]*, Orthogonal polynomials and their applications. Proceedings of the Third International Symposium held in Erice, June 1-8, 1990 (Basel) (C. Brezinski, L. Gori, and A. Ronveaux, eds.), vol. 9, IMACS Ann. Comput. Appl. Math., no. 1-4, J. C. Baltzer A.G., 1991, pp. 95–130.
20. ———, *Variations autour des polynômes orthogonaux classiques. (French. English summary) [Variations on classical orthogonal polynomials]*, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), no. 5, 209–212.
21. A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Классические ортогональные полиномы дискретной переменной [Classical Orthogonal Polynomials of a Discrete Variable]*, Nauka, Moscow, 1985, In Russian.
22. ———, *Классические ортогональные полиномы дискретной переменной на неравномерных сетках [Classical Orthogonal Polynomials of a Discrete Variable on Nonuniform Lattices]*, Doklady Akademii Nauk SSSR **291** (1986), no. 5, 1056–1059, In Russian.
23. A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer Series in Computational Physics, Springer-Verlag, Berlin, 1991, Translated from the Russian.
24. A. F. Nikiforov and V. B. Uvarov, *Классические ортогональные полиномы дискретной переменной на неравномерных сетках [Classical Orthogonal Polynomials of a Discrete Variable on Nonuniform Lattices]*, Tech. Report Препринт No. 17, Keldysh Institute of Applied Mathematics, USSR Academy of Sciences, 1983, In Russian. Available at `https://keldysh.ru/papers/1983/prep1983_17.pdf`.
25. ———, *Специальные функции математической физики [Special Functions of Mathematical Physics]*, Nauka, Moscow, 1984, In Russian.
26. J. Pelletier, L. Vinet, and A. Zhedanov, *Continuous -1 hypergeometric orthogonal polynomials*, Stud. Appl. Math. **153** (2024), no. 3, e12728.
27. V. Spiridonov and A. Zhedanov, *q-ultraspherical polynomials for q a root of unity*, Lett. Math. Phys. **37** (1996), no. 2, 173–180.
28. ———, *Zeros and orthogonality of the Askey–Wilson polynomials for q a root of unity*, Duke Math. J. **89** (1997), no. 2, 283–305.
29. S. Tsujimoto, L. Vinet, and A. Zhedanov, *Dunkl shift operators and Bannai–Ito polynomials*, Adv. Math. **229** (2012), no. 4, 2123–2158.
30. ———, *Dual -1 Hahn polynomials: "classical" polynomials beyond the Leonard duality*, Proc. Amer. Math. Soc. **141** (2013), no. 3, 959–970.
31. G. X. Vincent, L. Vinet, and A. Zhedanov, *A "Continuous" Limit of the Complementary Bannai–Ito Polynomials: Chihara Polynomials*, SIGMA **10** (2014), 038.
32. L. Vinet and A. Zhedanov, *A "missing" family of classical orthogonal polynomials*, J. Phys. A: Math. Theor. **44** (2011), 085201.
33. ———, *A limit $q = -1$ for the big q-Jacobi polynomials*, Trans. Amer. Math. Soc. **364** (2012), no. 10, 5491–5507.
34. ———, *Para-Krawtchouk polynomials on a bi-lattice and a quantum spin chain with perfect state transfer*, J. Phys. A **45** (2012), no. 26, 265304, 11 pp.

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