

THE MANUFACTURE OF EXAMPLES: THE (-1) -CLASSICAL ORTHOGONAL POLYNOMIALS

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ABSTRACT. The (-1) -Jacobi, Bannai–Ito, and (-1) -Meixner–Pollaczek polynomials are studied in [Trans. Amer. Math. Soc. 364 (2012), 5491–5507], [Adv. Math. 229 (2012), 2123–2158], and [Stud. Appl. Math. 153 (2024), e12728], respectively, through polynomial eigenfunctions of first-order Dunkl operators. The purpose of the present note is to show that these families are not isolated phenomena, but particular instances of a single alternating mechanism which is most naturally formulated at the level of orthogonality functionals, transpose operators, and structural identities. This functional-analytic point of view also leads to an explicit algorithm which, starting from an ordinary orthogonal polynomial sequence in the quadratic variable, systematises the construction of such (-1) -classical families.

1. INTRODUCTION

Despite the antiquity of the subject and its long-established place in analysis, orthogonal polynomial sequences advertised as “new classical families” continue to appear in the literature with rather surprising regularity. Of course, much depends on what is meant by “classical”. If “classical” is understood in the sense of the *NIST Handbook of Mathematical Functions* (2010), of the corresponding version of the *NIST Digital Library of Mathematical Functions*, and of much of the literature on the subject, then the appearance of new families is not altogether surprising. Indeed, within that NIST-inherited view of the subject, the underlying conception of orthogonality is excessively restrictive, and therefore unable to accommodate the algebraic freedom of these allegedly classical families, even to the point of presenting as distinct families which are algebraically identical, or which are obtained from known families by affine transformations [2]; moreover, many families are excluded by definition before they even come into being, for structural reasons which are not properly justified, such as those associated with admissible alternating maps [1]. It is precisely these latter families that will occupy us in the present work. The (-1) -Jacobi, Bannai–Ito, and (-1) -Meixner–Pollaczek polynomials provide particularly instructive examples of this phenomenon. They are introduced through first-order Dunkl operators, or through limiting and spectral-transform arguments. Their construction, however, is in fact governed by a simple alternating mechanism directly connected with the pioneering work of Nikiforov and Uvarov [4, 5], although this mechanism was not developed there; see [1, Section 8].

The (-1) -classical families, a term that we shall use for the families mentioned above, could never belong to the NIST scheme, itself an heir to the Askey scheme. Although we shall not reopen that question here, [1, Theorem 5.1], where the mutually exclusive cases (i), (ii), and (iii) are displayed, makes the structural reason transparent. Indeed, what is usually meant by the Askey scheme consists of families arising from situations related to cases (i) and (iii) of that theorem, corresponding, respectively, to admissible quadratic

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and q -exponential maps. The (-1) -classical families considered in the literature belong instead to case (ii), the alternating case. Their absence from the Askey scheme is therefore not a curiosity, but a direct consequence of the general theory.

We shall proceed, in a self-contained manner, to explain how families of this kind may be generated within the alternating framework, starting from an orthogonal polynomial sequence. The examples mentioned above will then appear as particular instances of a general construction. In practical terms, rather than producing examples one by one through ad hoc devices, we shall describe a systematic procedure for generating them, while at the same time exposing the structural reason for their existence.

The material that follows is drawn from [1, Section 8] and adapted to make the present note self-contained. Let $h \in \mathbb{C}^\times$, and let $U \subseteq \mathbb{C}$ be a non-empty set such that

$$U \pm \frac{h}{2} \subseteq U.$$

Such a set will be called *half-step-invariant*. A simple example is

$$U = h\mathbb{Q},$$

which is plainly stable under translations by $\pm h/2$. We consider the normalised alternating map

$$X : U \rightarrow \mathbb{C}, \quad X(s) = e^{\pi i s/h}.$$

In this case,

$$X\left(s + \frac{h}{2}\right) = iX(s), \quad X\left(s - \frac{h}{2}\right) = -iX(s),$$

so the two half-step neighbours of $X(s)$ are obtained simply by multiplication by $\pm i$. Note, for instance, that $X(h\mathbb{Q})$ is infinite. Whenever U is half-step-invariant and $X(U)$ is infinite, we shall say that the normalised alternating map X is *admissible* on U . The formulas that follow are first obtained pointwise on U , but admissibility ensures that they can be read as identities in the polynomial variable X . Indeed, since $X(U)$ is infinite, any two polynomials that agree on $X(U)$ must agree identically. Without this, one would have only relations valid on a finite image, and hence no intrinsic polynomial calculus.

If $\mathcal{P}$ denotes the vector space of complex polynomials in one variable, then every polynomial $p \in \mathcal{P}$ admits a unique decomposition of the form

$$p(x) = a(-x^2) + x b(-x^2), \quad a, b \in \mathcal{P}.$$

Substituting iX and $-iX$ into this decomposition gives

$$p(iX) = a(X^2) + iX b(X^2), \quad p(-iX) = a(X^2) - iX b(X^2),$$

and hence

$$\frac{p(iX) - p(-iX)}{2iX} = b(X^2), \quad \frac{p(iX) + p(-iX)}{2} = a(X^2).$$

By admissibility, these are not merely pointwise identities on the image of U , but genuine polynomial identities. Thus the alternating divided-difference and averaging procedures do not generate arbitrary new expressions: they simply separate the odd and even components of p relative to the quadratic substitution $x \mapsto -x^2$. This is the structural feature from which the subsequent analysis will emerge.

Accordingly, to the fixed admissible normalised alternating map X on U we attach linear operators

$$D, S : \mathcal{P} \rightarrow \mathcal{P}$$

defined by

$$(Dp)(X) = \frac{p(iX) - p(-iX)}{2iX}, \quad (Sp)(X) = \frac{p(iX) + p(-iX)}{2}, \quad p \in \mathcal{P}.$$

By admissibility, these identities define Dp and Sp uniquely as polynomials in the variable X . In particular,

$$(Dp)(X) = b(X^2), \quad (Sp)(X) = a(X^2).$$

We regard $\mathcal{P}$ as a locally convex space endowed with the strict countable inductive-limit topology arising from the filtration by the finite-dimensional spaces $\mathcal{P}_n$ of polynomials of degree at most n . We write $\mathcal{P}'$ for its continuous dual, endowed with the weak topology $\sigma(\mathcal{P}', \mathcal{P})$. Since the operators D and S attached to the fixed normalised alternating map X are continuous, their transposes are well defined, and we obtain weakly continuous operators

$$\mathbf{D}, \mathbf{S} : \mathcal{P}' \rightarrow \mathcal{P}'$$

characterised by

$$\langle \mathbf{D}\mathbf{u}, p \rangle = -\langle \mathbf{u}, Dp \rangle, \quad \langle \mathbf{S}\mathbf{u}, p \rangle = \langle \mathbf{u}, Sp \rangle,$$

for every $\mathbf{u} \in \mathcal{P}'$ and every $p \in \mathcal{P}$, see [1, Section 6]. We now specialise the general notion of orthogonality and regularity given in [1, Definition 6.2].

Definition 1.1 (A Particular Regime of Orthogonality). *Let $\mathbf{u} \in \mathcal{P}'$. A polynomial sequence $(P_n)_{n \in \mathbb{N}}$ is said to be orthogonal with respect to $\mathbf{u}$ if*

$$\langle \mathbf{u}, P_n P_m \rangle = 0$$

for every $m, n \in \mathbb{N}$ with $m < n$, and

$$\langle \mathbf{u}, P_n^2 \rangle \neq 0$$

for every $n \in \mathbb{N}$. We say that $\mathbf{u}$ is regular if there exists an orthogonal polynomial sequence $(P_n)_{n \in \mathbb{N}}$ with respect to $\mathbf{u}$.

We now specialise the notion of classicality given in [1, Definition 6.6] to the non-degenerate normalised alternating setting.

Definition 1.2 ((-1)-Classicality). *Fix $h \in \mathbb{C}^\times$, let $U \subseteq \mathbb{C}$ be half-step-invariant, and let*

$$X : U \rightarrow \mathbb{C}, \quad X(s) = e^{\pi i s/h},$$

be an admissible normalised alternating map on U . Let

$$D, S : \mathcal{P} \rightarrow \mathcal{P}$$

be the divided-difference and averaging operators associated with this fixed map X , and let

$$\mathbf{D}, \mathbf{S} : \mathcal{P}' \rightarrow \mathcal{P}'$$

denote their transposes. A functional $\mathbf{u} \in \mathcal{P}'$ is said to be classical for the normalised alternating map X if there exists a monic orthogonal polynomial sequence with respect to $\mathbf{u}$, and if there exist polynomials $\phi, \psi \in \mathcal{P}$, with

$$\deg \phi \leq 2, \quad \deg \psi = 1,$$

such that

$$\mathbf{D}(\phi \mathbf{u}) = \mathbf{S}(\psi \mathbf{u})$$

in $\mathcal{P}'$.

Let

$$\sigma : \mathcal{P} \rightarrow \mathcal{P}, \quad (\sigma p)(x) = p(x^2),$$

be the quadratic substitution operator. Since σ is continuous, its transpose is well defined, and we obtain a weakly continuous operator

$$\sigma : \mathcal{P}' \rightarrow \mathcal{P}'$$

characterised by

$$\langle \sigma \mathbf{u}, p \rangle = \langle \mathbf{u}, \sigma p \rangle,$$

for every $\mathbf{u} \in \mathcal{P}'$ and every $p \in \mathcal{P}$. If $\mathbf{u} \in \mathcal{P}'$ is (-1) -classical, [1, Theorem 8.6] yields

$$\mathbf{S}((x - \tau)\mathbf{u}) = 0.$$

Equivalently,

$$\langle \mathbf{u}, (x - \tau)p(x^2) \rangle = 0$$

for every $p \in \mathcal{P}$. Thus, in the normalised alternating case, classicality is governed by a first-order annihilation condition in the quadratic variable. This single condition is the whole mechanism. In particular, it will be enough to reduce, in a direct and transparent way, constructions which otherwise tend to appear in the literature as long and rather specialised arguments.

Proposition 1.3. *Let $\mathbf{v} \in \mathcal{P}'$ admit a monic orthogonal polynomial sequence $(R_n)_{n \in \mathbb{N}}$, and let $\tau \in \mathbb{C}$. Assume that*

$$R_n(\tau^2) \neq 0$$

for every $n \in \mathbb{N}$. Then there exists a unique functional $\mathbf{u} \in \mathcal{P}'$ such that

$$\sigma \mathbf{u} = \mathbf{v}, \quad \langle \mathbf{u}, (x - \tau)p(x^2) \rangle = 0$$

for every $p \in \mathcal{P}$. Moreover, if

$$S_n(x) = \frac{R_{n+1}(x) - \frac{R_{n+1}(\tau^2)}{R_n(\tau^2)} R_n(x)}{x - \tau^2},$$

and $(P_n)_{n \in \mathbb{N}}$ is defined by

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x - \tau)S_n(x^2),$$

then $(P_n)_{n \in \mathbb{N}}$ is the monic orthogonal polynomial sequence with respect to $\mathbf{u}$.

Proof. Every polynomial $f \in \mathcal{P}$ can be written uniquely in the form

$$f(x) = p(x^2) + (x - \tau)q(x^2), \quad p, q \in \mathcal{P}.$$

We define $\mathbf{u} \in \mathcal{P}'$ by

$$\langle \mathbf{u}, p(x^2) + (x - \tau)q(x^2) \rangle = \langle \mathbf{v}, p \rangle.$$

This is well defined by the uniqueness of the preceding decomposition. It immediately gives

$$\sigma \mathbf{u} = \mathbf{v}, \quad \langle \mathbf{u}, (x - \tau)q(x^2) \rangle = 0,$$

and uniqueness follows from the same decomposition. It remains only to identify the orthogonal polynomials. By the Christoffel formula, the condition $R_n(\tau^2) \neq 0$ implies that

$$S_n(y) = \frac{R_{n+1}(y) - \frac{R_{n+1}(\tau^2)}{R_n(\tau^2)} R_n(y)}{y - \tau^2}$$

is the monic orthogonal polynomial sequence with respect to $(y - \tau^2)\mathbf{v}$. Now set

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x - \tau)S_n(x^2).$$

These polynomials are monic and have degrees $2n$ and $2n + 1$, respectively. We check orthogonality. First,

$$\langle \mathbf{u}, P_{2n}P_{2m+1} \rangle = \langle \mathbf{u}, R_n(x^2)(x - \tau)S_m(x^2) \rangle = 0$$

by the defining annihilation condition for $\mathbf{u}$. Secondly,

$$\langle \mathbf{u}, P_{2n}P_{2m} \rangle = \langle \mathbf{u}, R_n(x^2)R_m(x^2) \rangle = \langle \mathbf{v}, R_nR_m \rangle,$$

which vanishes for $n \neq m$. Finally,

$$\langle \mathbf{u}, P_{2n+1}P_{2m+1} \rangle = \langle \mathbf{u}, (x - \tau)^2 S_n(x^2)S_m(x^2) \rangle.$$

Since

$$(x - \tau)^2 = (x^2 - \tau^2) - 2\tau(x - \tau),$$

and since every term containing the factor $(x - \tau)$ is annihilated by $\mathbf{u}$, we obtain

$$\langle \mathbf{u}, P_{2n+1}P_{2m+1} \rangle = \langle \mathbf{u}, (x^2 - \tau^2)S_n(x^2)S_m(x^2) \rangle.$$

Hence

$$\langle \mathbf{u}, P_{2n+1}P_{2m+1} \rangle = \langle \mathbf{v}, (y - \tau^2)S_n(y)S_m(y) \rangle,$$

which vanishes for $n \neq m$, since $(S_n)_{n \in \mathbb{N}}$ is orthogonal with respect to $(y - \tau^2)\mathbf{v}$. Thus $(P_n)_{n \in \mathbb{N}}$ is the monic orthogonal polynomial sequence with respect to $\mathbf{u}$. $\square$

Proposition 1.3 clarifies the scope of the continuous -1 scheme described in [6]. That scheme organises a substantial collection of families by means of $q \rightarrow -1$ limits, specialisations, and explicit recurrence calculations. However, once the alternating pullback mechanism is isolated, these families are no longer obtained one by one through carefully chosen limits from already known families. Rather, they arise systematically from ordinary orthogonal polynomial systems in the quadratic variable. The role of $q \rightarrow -1$ is therefore not accidental, but neither is it the source of the phenomenon. It is a limiting manifestation of the alternating case, whose intrinsic structure is already determined by the quadratic decomposition of polynomials.

In [1, Examples 8.11 and 8.12], we showed that the so-called complementary Bannai–Ito polynomials and the dual (-1) -Hahn polynomials are nothing more than particular instances of the preceding construction. Several families encountered in the literature admit the same interpretation, although their true identity is sometimes concealed behind a second door.

2. EXAMPLES

What follows consists of elementary calculations whose sole purpose is to make explicit the rather special nature of the cases treated in [8], [7], and [6].

2.1. Big (-1) -Jacobi polynomials. The construction described in [8] is a particular instance of Algorithm 1. More precisely, the family considered there arises in two successive steps. First, one applies the alternating pullback construction to a suitable regular Jacobi family with the choice $\tau = 1$. This yields the intermediate family denoted $(R_n)_{n \in \mathbb{N}}$ in [8, Section 3]. Secondly, one applies the Geronimus transform recorded in [8, (4.1), (4.2)]. The resulting family is precisely the big (-1) -Jacobi family, and the explicit formulas [8, (2.24), (2.25)] are then recovered by direct calculation.

Throughout this example we assume

$$0 < \lambda < 1,$$

and we choose complex parameters $a, b \in \mathbb{C}$ so that the Jacobi family considered below forms a monic orthogonal polynomial sequence. In the present case, this is equivalent to requiring that the denominators appearing in the recurrence coefficients do not vanish; see [2, Section 1]. The reader should keep in mind that no positivity assumption is imposed here: regularity is meant in the functional-analytic sense. In the parametrisation used later, this amounts to choosing $c, d \in \mathbb{C}$ so that the same non-vanishing conditions hold after the substitutions

$$a = \frac{c-1}{2}, \quad b = \frac{d+1}{2}.$$

We begin with the monic shifted Jacobi polynomial sequence

$$(\tilde{R}_n)_{n \in \mathbb{N}},$$

Algorithm 1 A Procedure for Generating (-1) -Classical Orthogonal Polynomials

Require: A regular functional $\mathbf{v} \in \mathcal{P}'$, its monic orthogonal polynomial sequence

$$(R_n)_{n \in \mathbb{N}},$$

and a parameter $\tau \in \mathbb{C}$ such that

$$R_n(\tau^2) \neq 0.$$

Ensure: The monic orthogonal polynomial sequence

$$(P_n)_{n \in \mathbb{N}}$$

associated with the functional $\mathbf{u}$ determined by

$$\sigma \mathbf{u} = \mathbf{v}, \quad \langle \mathbf{u}, (x - \tau)p(x^2) \rangle = 0$$

for every $p \in \mathcal{P}$.

1: Construct the monic orthogonal polynomial sequence

$$(S_n)_{n \in \mathbb{N}}$$

by

$$S_n(y) = \frac{R_{n+1}(y) - \frac{R_{n+1}(\tau^2)}{R_n(\tau^2)} R_n(y)}{y - \tau^2}.$$

2: Construct the monic orthogonal polynomial sequence

$$(P_n)_{n \in \mathbb{N}}$$

associated with $\mathbf{u}$ by

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x - \tau)S_n(x^2).$$

3: Return the sequence

$$(P_n)_{n \in \mathbb{N}},$$

which is the alternating family generated from

$$(R_n)_{n \in \mathbb{N}}.$$

given by

$$\tilde{R}_n(y) = (-1)^n \frac{(a+1)_n}{(n+a+b+1)_n} {}_2F_1 \left(\begin{matrix} -n, & n+a+b+1 \\ & a+1 \end{matrix} ; y \right).$$

As in [8, (3.13)], we perform the affine change of variable

$$y = \frac{1-t}{1-\lambda^2},$$

and define

$$R_n(t) = (\lambda^2 - 1)^n \tilde{R}_n \left(\frac{1-t}{1-\lambda^2} \right).$$

Since

$$(\lambda^2 - 1)^n (-1)^n = (1 - \lambda^2)^n,$$

this becomes

$$R_n(t) = (1 - \lambda^2)^n \frac{(a+1)_n}{(n+a+b+1)_n} {}_2F_1 \left(\begin{matrix} -n, & n+a+b+1 \\ & a+1 \end{matrix} ; \frac{1-t}{1-\lambda^2} \right).$$

This is exactly the monic Jacobi-type family used in [8, (3.13)]. We now choose

$$\tau = 1.$$

To apply Algorithm 1, we must verify that

$$R_n(1) \neq 0$$

for every $n \in \mathbb{N}$. But

$$R_n(1) = (1 - \lambda^2)^n \frac{(a+1)_n}{(n+a+b+1)_n},$$

because the hypergeometric term is evaluated at 0, and therefore equals 1. Thus the assumptions of Algorithm 1 are satisfied, so that this family provides precisely the input required for the algorithm.

Set

$$S_n(t) = \frac{R_{n+1}(t) - \frac{R_{n+1}(1)}{R_n(1)} R_n(t)}{t - 1}.$$

This is precisely the companion family introduced in [8, (3.14)]. From the explicit expression for $R_n(1)$, we obtain

$$\frac{R_{n+1}(1)}{R_n(1)} = (1 - \lambda^2) \frac{(a+n+1)(a+b+n+1)}{(2n+a+b+1)(2n+a+b+2)}.$$

This is exactly the relabelled form of the coefficient appearing in [8, (3.16)]. Substituting this into the Christoffel formula and simplifying, one obtains the explicit expression

$$S_n(t) = (1 - \lambda^2)^n \frac{(a+2)_n}{(n+a+b+2)_n} {}_2F_1 \left(\begin{matrix} -n, & n+a+b+2 \\ & a+2 \end{matrix} ; \frac{1-t}{1-\lambda^2} \right),$$

which coincides, after relabelling the parameters, with [8, (3.15)]. We now apply Algorithm 1. Define

$$P_{2m}(x) = R_m(x^2), \quad P_{2m+1}(x) = (x-1)S_m(x^2).$$

We now rewrite these formulas in the parameters used in [8, Section 4], namely

$$a = \frac{c-1}{2}, \quad b = \frac{d+1}{2}, \quad z = \frac{1-x^2}{1-\lambda^2}.$$

Then

$$P_{2m}(x) = (1 - \lambda^2)^m \frac{\left(\frac{c+1}{2}\right)_m}{\left(\frac{2m+c+d+2}{2}\right)_m} {}_2F_1 \left(\begin{matrix} -m, & \frac{2m+c+d+2}{2} \\ & \frac{c+1}{2} \end{matrix} ; z \right)$$

$$P_{2m+1}(x) = -(1-x)(1 - \lambda^2)^m \frac{\left(\frac{c+3}{2}\right)_m}{\left(\frac{2m+c+d+4}{2}\right)_m} {}_2F_1 \left(\begin{matrix} -m, & \frac{2m+c+d+4}{2} \\ & \frac{c+3}{2} \end{matrix} ; z \right).$$

These are exactly the corresponding formulas in [8, (3.21), (3.22)], after relabelling the parameters.

Up to this point, nothing essentially new has been used: the construction is just a direct application of Algorithm 1. This application produces the family $(P_n)_{n \in \mathbb{N}}$, which corresponds to the family denoted by $(S_n)_{n \in \mathbb{N}}$ in [8, Section 3]. It is important, however, not to identify this family with the big (-1) -Jacobi polynomials themselves. The latter are obtained from this first output by one additional inverse step, namely a Geronimus inversion. At the level of orthogonality functionals, this means the following. Let $\mathbf{w}$ be the functional with respect to which $(P_n)_{n \in \mathbb{N}}$ is orthogonal, and let $\mathbf{u}$ be the functional with

respect to which $(P_n^{(-1)})_{n \in \mathbb{N}}$ is orthogonal. In the notation of [8, Section 4], the Geronimus point is $\mu = -c$; in the present notation this becomes $\mu = -\lambda$. Hence

$$(x + \lambda)\mathbf{u} = \mathbf{w}.$$

Consequently, $\mathbf{u}$ is determined only up to the addition of a multiple of $\delta_{-\lambda}$, since

$$(x + \lambda)\delta_{-\lambda} = 0.$$

Thus the Dirac mass is not an ad hoc feature of the example, but the unavoidable ambiguity inherent in the Geronimus inversion; see, for instance, [3, Proposition 1.4]. In the present case, [8, (4.1), (4.2)] writes the big (-1) -Jacobi polynomials as

$$P_n^{(-1)}(x) = P_n(x) - g_n P_{n-1}(x),$$

where

$$g_n = \begin{cases} \frac{(1 - \lambda)n}{2n + c + d}, & n \text{ even}, \\ -\frac{(1 + \lambda)(n + c)}{2n + c + d}, & n \text{ odd}. \end{cases}$$

We now check, explicitly and separately in the even and odd cases, that this formula gives precisely [8, (2.24), (2.25)].

Even case. Assume $m \geq 1$, so that $n = 2m \geq 2$. Then

$$P_{2m}^{(-1)}(x) = P_{2m}(x) - g_{2m} P_{2m-1}(x),$$

with

$$g_{2m} = \frac{(1 - \lambda)2m}{4m + c + d}.$$

The case $m = 0$ is immediate, since $P_0^{(-1)}(x) = 1$. Hence

$$\begin{aligned} P_{2m}^{(-1)}(x) &= (1 - \lambda^2)^m \frac{\left(\frac{c+1}{2}\right)_m}{\left(\frac{2m+c+d+2}{2}\right)_m} {}_2F_1\left(-m, \frac{2m+c+d+2}{\frac{c+1}{2}}; z\right) \\ &\quad + \frac{(1 - \lambda)2m}{4m + c + d} (1 - x)(1 - \lambda^2)^{m-1} \frac{\left(\frac{c+3}{2}\right)_{m-1}}{\left(\frac{2m+c+d+2}{2}\right)_{m-1}} \\ &\quad \times {}_2F_1\left(1 - m, \frac{2m+c+d+2}{\frac{c+3}{2}}; z\right). \end{aligned}$$

We now simplify the coefficient in the second term. First,

$$\left(\frac{c+1}{2}\right)_m = \frac{c+1}{2} \left(\frac{c+3}{2}\right)_{m-1}.$$

Secondly,

$$\left(\frac{2m+c+d+2}{2}\right)_m = \left(\frac{2m+c+d+2}{2}\right)_{m-1} \left(\frac{4m+c+d}{2}\right).$$

Therefore

$$\begin{aligned} & (1 - \lambda^2)^m \frac{\left(\frac{c+1}{2}\right)_m}{\left(\frac{2m+c+d+2}{2}\right)_m} \\ &= (1 - \lambda^2)^m \frac{(c+1) \left(\frac{c+3}{2}\right)_{m-1}}{(4m+c+d) \left(\frac{2m+c+d+2}{2}\right)_{m-1}}. \end{aligned}$$

Hence

$$(1 - \lambda^2)^{m-1} \frac{\left(\frac{c+3}{2}\right)_{m-1}}{\left(\frac{2m+c+d+2}{2}\right)_{m-1}} = \frac{4m+c+d}{(1 - \lambda^2)(c+1)} (1 - \lambda^2)^m \frac{\left(\frac{c+1}{2}\right)_m}{\left(\frac{2m+c+d+2}{2}\right)_m}.$$

Multiplying by g_{2m} , we obtain

$$\begin{aligned} & \frac{(1 - \lambda)2m}{4m+c+d} (1 - \lambda^2)^{m-1} \frac{\left(\frac{c+3}{2}\right)_{m-1}}{\left(\frac{2m+c+d+2}{2}\right)_{m-1}} \\ &= \frac{2m}{(1 + \lambda)(c+1)} (1 - \lambda^2)^m \frac{\left(\frac{c+1}{2}\right)_m}{\left(\frac{2m+c+d+2}{2}\right)_m}. \end{aligned}$$

Thus

$$\begin{aligned} \kappa_{2m}^{-1} P_{2m}^{(-1)}(x) &= {}_2F_1 \left(-m, \frac{2m+c+d+2}{\frac{c+1}{2}}; z \right) \\ &+ \frac{2m(1-x)}{(1+\lambda)(c+1)} {}_2F_1 \left(1-m, \frac{2m+c+d+2}{\frac{c+3}{2}}; z \right), \end{aligned}$$

where

$$\kappa_{2m} = (1 - \lambda^2)^m \frac{\left(\frac{c+1}{2}\right)_m}{\left(\frac{2m+c+d+2}{2}\right)_m}.$$

Replacing $2m$ by n , we obtain exactly [8, (2.24), (2.26)].

Odd case. Proceeding exactly as in the preceding case, one obtains

$$\begin{aligned} \kappa_{2m+1}^{-1} P_{2m+1}^{(-1)}(x) &= {}_2F_1 \left(-m, \frac{2m+c+d+2}{\frac{c+1}{2}}; z \right) \\ &\quad - \frac{(2m+c+d+2)(1-x)}{(1+\lambda)(c+1)} {}_2F_1 \left(-m, \frac{2m+c+d+4}{\frac{c+3}{2}}; z \right), \end{aligned}$$

where

$$\kappa_{2m+1} = (1+\lambda)(1-\lambda^2)^m \frac{\left(\frac{c+1}{2}\right)_{m+1}}{\left(\frac{2m+c+d+2}{2}\right)_{m+1}}.$$

Replacing $2m+1$ by n , this is exactly [8, (2.25), (2.26)]. Thus the example should not be read as an isolated ad hoc construction, but as a concrete manifestation of a more general structural mechanism.

2.2. Bannai–Ito polynomials. We now turn to the Bannai–Ito case. Here the first alternating step need not be reconstructed from scratch, since that part has already been carried out in [1, Example 8.11]. In the notation adopted here, the output of Algorithm 1 is the intermediate family

$$(P_n)_{n \in \mathbb{N}},$$

which coincides with the complementary Bannai–Ito family. The point to stress, exactly as in the big (-1) -Jacobi case, is that the Bannai–Ito polynomials do not coincide with this first output. Rather, they are obtained from it by one further inverse step. In [7, Section 5], the complementary Bannai–Ito polynomials are described as Christoffel transforms of the Bannai–Ito polynomials. Thus, in the present notation, the inverse relation takes the form

$$B_n(x) = P_n(x) - g_n P_{n-1}(x),$$

where

$$g_n = \begin{cases} -\frac{n(n-2c-2d)}{4(n-c-d+a+b)}, & n \text{ even}, \\ -\frac{(n-2d+2b)(n-2c+2b)}{4(n-c-d+a+b)}, & n \text{ odd}. \end{cases}$$

This is exactly the Geronimus inverse of the Christoffel step described in [7, Section 5], rewritten in the present notation, with the intermediate family $(P_n)_{n \in \mathbb{N}}$ identified with the complementary Bannai–Ito family and the final family $(B_n)_{n \in \mathbb{N}}$ identified with the Bannai–Ito family itself.

We now write explicitly the intermediate family produced by the algorithm. In the notation adopted here, the complementary Bannai–Ito family takes the form

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x-b)S_n(x^2),$$

where

$$R_n(x^2) = \kappa_n^{(1)} {}_4F_3 \left(\begin{matrix} -n, n+a+b-c-d+1, b+x, b-x \\ a+b+1, b-c+\frac{1}{2}, b-d+\frac{1}{2} \end{matrix}; 1 \right),$$

and

$$S_n(x^2) = \kappa_n^{(2)} {}_4F_3 \left(\begin{matrix} -n, n+a+b-c-d+2, b+1+x, b+1-x \\ a+b+2, b-c+\frac{3}{2}, b-d+\frac{3}{2} \end{matrix}; 1 \right).$$

The monic normalising factors are

$$\kappa_n^{(1)} = \frac{(1+a+b)_n \left(b-c+\frac{1}{2}\right)_n \left(b-d+\frac{1}{2}\right)_n}{(n+a+b-c-d+1)_n},$$

$$\kappa_n^{(2)} = \frac{(2+a+b)_n \left(b-c+\frac{3}{2}\right)_n \left(b-d+\frac{3}{2}\right)_n}{(n+a+b-c-d+2)_n}.$$

These are exactly the formulas obtained in [7, (5.18), (5.19), (5.20)], rewritten in the present notation and with the monic normalising factors chosen accordingly. We now verify, in direct parallel with the preceding example, that the Geronimus formula produces the Bannai–Ito family from this intermediate output.

Even case. Assume $m \geq 1$, so that $n = 2m \geq 2$. Then

$$B_{2m}(x) = P_{2m}(x) - g_{2m}P_{2m-1}(x).$$

The case $m = 0$ is immediate, since $B_0(x) = P_0(x) = 1$. Using the formulas above, we obtain

$$P_{2m}(x) = R_m(x^2), \quad P_{2m-1}(x) = (x-b)S_{m-1}(x^2),$$

and therefore

$$B_{2m}(x) = R_m(x^2) - g_{2m}(x-b)S_{m-1}(x^2).$$

Since

$$g_{2m} = -\frac{2m(2m-2c-2d)}{4(2m-c-d+a+b)} = -\frac{m(m-c-d)}{2m-c-d+a+b},$$

it follows that

$$\begin{aligned} B_{2m}(x) &= \kappa_m^{(1)} {}_4F_3 \left(\begin{matrix} -m, m+a+b-c-d+1, b+x, b-x \\ a+b+1, b-c+\frac{1}{2}, b-d+\frac{1}{2} \end{matrix} ; 1 \right) \\ &\quad + \frac{m(m-c-d)}{2m-c-d+a+b} (x-b) \kappa_{m-1}^{(2)} \\ &\quad \times {}_4F_3 \left(\begin{matrix} 1-m, m+a+b-c-d+1, b+1+x, b+1-x \\ a+b+2, b-c+\frac{3}{2}, b-d+\frac{3}{2} \end{matrix} ; 1 \right). \end{aligned}$$

Replacing $2m$ by n , this is exactly what follows from [7, (5.20), (5.21)] after substituting the explicit expressions in [7, (5.18), (5.19)].

Odd case. Proceeding exactly as in the preceding case, one obtains

$$\begin{aligned} B_{2m+1}(x) &= (x-b) \kappa_m^{(2)} {}_4F_3 \left(\begin{matrix} -m, m+a+b-c-d+2, b+1+x, b+1-x \\ a+b+2, b-c+\frac{3}{2}, b-d+\frac{3}{2} \end{matrix} ; 1 \right) \\ &\quad + \frac{(2m+1-2d+2b)(2m+1-2c+2b)}{4(2m+1-c-d+a+b)} \kappa_m^{(1)} \\ &\quad \times {}_4F_3 \left(\begin{matrix} -m, m+a+b-c-d+1, b+x, b-x \\ a+b+1, b-c+\frac{1}{2}, b-d+\frac{1}{2} \end{matrix} ; 1 \right). \end{aligned}$$

Replacing $2m+1$ by n , this is exactly what follows from [7, (5.20), (5.21)] after substituting the explicit expressions in [7, (5.18), (5.19)]. Thus the Bannai–Ito family is recovered from

the output of Algorithm 1 by one further Geronimus transform, exactly as in the big (-1) -Jacobi case.

2.3. The (-1) -Meixner–Pollaczek polynomials. We finish with the (-1) -Meixner–Pollaczek polynomials, since in this case the reduction is immediate. The input for Algorithm 1 is the Laguerre functional shifted by γ^2 . More precisely, let $\mathbf{v}$ be the regular functional whose monic orthogonal polynomial sequence is

$$R_n(y) = (-1)^n \left(\alpha + \frac{1}{2} \right)_n {}_1F_1 \left(\begin{matrix} -n \\ \alpha + \frac{1}{2} \end{matrix}; y - \gamma^2 \right).$$

Thus $(R_n)_{n \in \mathbb{N}}$ is simply the monic Laguerre family in the variable $y - \gamma^2$, with parameter $\alpha - \frac{1}{2}$.

We now apply Algorithm 1 with

$$\tau = \gamma.$$

The required non-vanishing condition is

$$R_n(\gamma^2) = (-1)^n \left(\alpha + \frac{1}{2} \right)_n \neq 0,$$

under the usual regularity assumptions on the Laguerre functional. The Christoffel transform of $\mathbf{v}$ at the point γ^2 has monic orthogonal polynomial sequence

$$S_n(y) = (-1)^n \left(\alpha + \frac{3}{2} \right)_n {}_1F_1 \left(\begin{matrix} -n \\ \alpha + \frac{3}{2} \end{matrix}; y - \gamma^2 \right).$$

Therefore the algorithm gives

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x - \gamma)S_n(x^2).$$

That is,

$$P_{2n}(x) = (-1)^n \left(\alpha + \frac{1}{2} \right)_n {}_1F_1 \left(\begin{matrix} -n \\ \alpha + \frac{1}{2} \end{matrix}; x^2 - \gamma^2 \right),$$

and

$$P_{2n+1}(x) = (-1)^n \left(\alpha + \frac{3}{2} \right)_n (x - \gamma) {}_1F_1 \left(\begin{matrix} -n \\ \alpha + \frac{3}{2} \end{matrix}; x^2 - \gamma^2 \right).$$

These are precisely the (-1) -Meixner–Pollaczek polynomials displayed in [6]. Thus there is no separate construction to be discovered here: one feeds the shifted Laguerre functional into the alternating algorithm, takes $\tau = \gamma$, and the family appears at once.

3. CONCLUSION

In conclusion, the point of the preceding construction is not merely to enlarge the already extensive list of (-1) -classical families. Rather, it is to identify the mechanism which makes such families possible. Once the alternating structure is recognised, the apparently specialised formulae are seen to arise from ordinary orthogonal polynomial systems in the quadratic variable, together with a Christoffel transform and a quadratic pullback. Thus the essential question, when one encounters a so-called (-1) -classical family, is no longer whether it lies outside the Askey scheme, nor whether it can be obtained through a carefully arranged limiting procedure. The decisive question is instead which sequence $(R_n)_{n \in \mathbb{N}}$ generates it through the alternating construction. After that identification, the remaining work consists of explicit, sometimes lengthy, but fundamentally elementary calculations.

This also leads to a more critical reading of the diagram presented in [6, Figure A1] under the title “Scheme of continuous (-1) -hypergeometric orthogonal polynomials”. The word “scheme” suggests more than a catalogue of formulae or a network of limiting relations: it suggests an organising principle. Yet, from the point of view developed here, the diagram records primarily the external genealogy of the families. It tells us how certain

entries may be reached by $q \rightarrow -1$ limits, specialisations, or spectral transformations, but it does not identify the structural reason why those entries exist in the first place. The distinction is important. A limiting diagram may explain how one named family degenerates into another, but it does not by itself explain why the same even-odd pattern, the same quadratic dependence, and the same Christoffel–Geronimus mechanism keep reappearing. These features are not accidental consequences of hypergeometric manipulation. They are symptoms of the underlying alternating case. In the normalised alternating setting, every polynomial splits according to

$$p(x) = p_0(x^2) + (x - \tau)p_1(x^2),$$

and this elementary decomposition already contains the mechanism behind the families under consideration. The even subsequence comes from an ordinary orthogonal polynomial sequence in the variable $y = x^2$, while the odd subsequence is obtained from its Christoffel companion at $y = \tau^2$:

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = (x - \tau)S_n(x^2).$$

Thus the structure precedes the hypergeometric realisation. In this sense, the continuous (-1) -scheme of [6, Figure A1] is best understood as a phenomenological chart rather than as a structural classification. It may have its uses, but it remains attached to the names of particular families, to specific limiting procedures, and to explicit hypergeometric representations. The present approach reverses the perspective, placing the conceptual mechanism before the catalogue of some visible instances.

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DATA AVAILABILITY STATEMENT

No data were generated or analysed in the course of this study.

CONFLICT OF INTEREST STATEMENT

The authors declare that they have no conflict of interest.

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