HIGHER HÖLDER REGULARITY FOR DEGENERATE ELLIPTIC PDES WITH DATA IN MORREY SPACES

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Abstract. We establish sharp local $C^{1,\alpha}-$regularity for weak solutions to degenerate elliptic equations of $p-$Laplacian type with data in Morrey spaces. The proof relies on the Fefferman–Phong inequality and standard tools from regularity theory for nonlinear PDEs.

1. Introduction

This note addresses the issue of higher regularity for degenerate elliptic PDEs of the form

$$-\operatorname{div} A(x,u,Du) = B(x,u,Du) \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. The functions A and B satisfy suitable structural assumptions modeled on the $p-$Poisson equation

$$-\Delta_p u = f, \tag{1.2}$$

where $\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$ is the $p-$Laplacian operator. The novelty is that f enjoys a modest degree of integrability, namely it is merely an L^1-function satisfying a decay condition on its integrals over balls, *i.e.*, it belongs to a suitable Morrey space. More precisely, we will assume

$$f \in L^{1,\lambda}(\Omega), \qquad n-1<\lambda<n. \tag{1.3}$$

The choice of a Morrey space is motivated by the fact that, in some special cases, this condition is also necessary for regularity (see [6] and [21]).

The study of regularity for equations of this type has a rich and long history. After the pioneering De Giorgi–Nash–Moser theory, the work of Ladyzhenskaya and Uraltseva (see [17], and also [12]) established the local $C^{1,\alpha}-$regularity of weak solutions for the homogeneous equation, and was later extended to equations with a quite general right-hand side satisfying suitable integrability assumptions (see, for example, [4, 26]). The study of regularity under less restrictive integrability assumptions was pioneered by Morrey in [19] and advanced through contributions from various authors (see [5], [18] and references therein). Under assumption (1.3), weak solutions

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of (1.2) are known to be locally of class $C^{0,\alpha}$, for some $\alpha \in (0,1)$ (see [5, Theorem 2.5], [27, Theorem 3], [9], [8] and [6]). Moreover, they are differentiable under several different assumptions on f (see, *e.g.*, [18] and [2, 11]).

Our main result establishes higher Hölder regularity, namely the $C^{1,\alpha}$ local regularity, for

$$\alpha = \begin{cases} \min\left(\gamma, \lambda + 1 - \frac{2n}{p}\right), & \frac{2n}{\lambda+1} < p \leq 2; \\ \\ \min\left(\gamma, \dfrac{\lambda+1-n}{p-1}\right), & 2 \leq p < n. \end{cases}$$

Here, $\gamma \in (0,1)$ is the Hölder regularity exponent for the gradient of the solution to the homogeneous equation having u as boundary data. Such a result is known in the linear case $p = 2$. It is shown in [5, Theorem 2.8] that if u is a solution to $Lu = f$, where L is a linear uniformly elliptic operator with smooth coefficients, then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$, provided (1.3) holds. Still, the proof heavily relies on properties of the Green function that are not available for the nonlinear degenerate case.

Our analysis relies heavily on the powerful Fefferman–Phong inequality (see Theorem 2.1), a weighted Poincaré-type inequality instrumental in estimating the term containing the right-hand side. This inequality dates back to the 1980s [13, 14], and it was used to understand the distribution of eigenvalues of self-adjoint operators. A simplified proof of the Fefferman–Phong inequality from [13] was given in [3] under an L^r assumption on the right-hand side. For other variants of the Fefferman–Phong inequality, see [10], treating the case of general metric spaces with a doubling measure, and [16], containing the extension to the fractional setting.

The note is structured as follows. Section 2 overviews the main properties of the function spaces used in our analysis. To ensure clarity, in Section 3, we write a detailed proof of the main result for the model equation (1.2). In Section 4, we sketch the proof for the general case of (1.1).

2. Morrey spaces and Stummel classes

This section provides a brief overview of Morrey spaces and Stummel classes. We refer the reader to [20, 22, 23] for a detailed and comprehensive treatment. Throughout the note, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary.

Definition 2.1. *Let $1 \leq p < \infty$ and $\lambda \geq 0$. The Morrey space $L^{p,\lambda}(\Omega)$ is the set of functions $f \in L^p(\Omega)$ for which*

$$\|f\|_{L^{p,\lambda}(\Omega)} := \sup_{\substack{x\in\Omega \\ 0<r<\operatorname{diam}(\Omega)}} \left(\frac{1}{r^\lambda} \int_{\Omega\cap B_r(x)} |f(y)|^p \, dy \right)^{1/p} < \infty. \tag{2.1}$$

The space $L^{p,\lambda}(\Omega)$ is a Banach space (a Hilbert space if $p = 2$) under the norm (2.1).

Remark 2.1. *For any $1 \le p < \infty$, $L^{p,0}(\Omega) = L^p(\Omega)$ and $L^{p,n}(\Omega) = L^\infty(\Omega)$. Furthermore, if $\lambda > n$, then $L^{p,\lambda}(\Omega) = \{0\}$ (see* [20, Theorem 5.5.1]*).*

Morrey spaces enjoy several embedding properties; for the following result, see [20, Theorem 5.5.1].

Proposition 2.1. *If $1 \le p \le q < \infty$ and $\lambda, \mu \in [0, n]$ are such that*

$$\frac{n-\mu}{q} \le \frac{n-\lambda}{p},$$

then

$$L^{q,\mu}(\Omega) \subseteq L^{p,\lambda}(\Omega).$$

Moreover, there exists a constant $c > 0$ such that

$$\|f\|_{L^{p,\lambda}(\Omega)} \le c\,\|f\|_{L^{q,\mu}(\Omega)}.$$

Next, we recall the definition of Stummel-Kato classes (see, for example, [1, 15, 25]), which are strictly related to Morrey spaces.

Definition 2.2. *Let $f \in L^1(\Omega)$, and $1 \le p < n$. Set*

$$\eta(r) = \eta_p(f, r) := \sup_{x\in\Omega} \int_{\Omega\cap B_r(x)} \frac{|f(y)|}{|x-y|^{n-p}}\, dy\,. \tag{2.2}$$

We say that f belongs to the Stummel-Kato class $S_p(\Omega)$ if the Stummel-Kato modulus (2.2) *is finite and, moreover,*

$$\lim_{r\downarrow 0} \eta(r) = 0.$$

In case $\eta(r)$ is just bounded around zero, we say that $f \in \tilde{S}_p(\Omega)$.

For the proof of the following result, we refer to [22] or [5].

Lemma 2.1. *If $1 \le p < n$ and $n - p < \lambda < n$, then*

$$L^{1,\lambda}(\Omega) \subset S_p(\Omega) \subset \tilde{S}_p(\Omega) \subset L^{1,n-p}(\Omega).$$

Moreover

$$\eta(r) \le c\, r^{\lambda-n+p} \|f\|_{L^{1,\lambda}(\Omega)},$$

for a constant $c > 0$ independent of f and r.

We also recall the Fefferman–Phong inequality, a weighted Poincaré type inequality fundamental in our analysis (see, *e.g.*, [10, Corollary 3.4] and [7]).

Theorem 2.1. *If $1 \le p < n$ and $f \in \tilde{S}_p(\Omega)$, then there exists a constant $c > 0$, depending only on p and n, such that, for every ball $B \subset \Omega$ with radius $r > 0$, one has*

$$\int_B |f||\varphi|^p\, dx \le c\, \eta(2r) \int_B |D\varphi|^p\, dx,$$

for all $\varphi \in W_0^{1,p}(B)$.

As a consequence of Theorem 2.1 and Lemma 2.1, we obtain the following corollary, which plays a crucial role in proving our main result.

Corollary 2.1. *If* $1 \leq p < n$, $n-p < \lambda < n$, *and* $f \in L^{1,\lambda}(\Omega)$, *then there exists a constant* $c > 0$, *depending only on* p, λ *and* n, *such that for every ball* $B \subset \Omega$ *with radius* $r > 0$, *one has*

$$\int_B |f||\varphi|^p \, dx \leq c\, r^{\lambda-n+p} \|f\|_{L^{1,\lambda}(\Omega)} \int_B |D\varphi|^p \, dx, \tag{2.3}$$

for all $\varphi \in W_0^{1,p}(B)$.

We close this section by recalling the celebrated Campanato-Meyers characterization of Hölder spaces (see [20]). We denote with $(u)_r$ the integral average of u over a ball with radius r.

Theorem 2.2. *Let* $1 \leq p < \infty$ *and* $0 < \gamma \leq 1$. *If there exists* $c > 0$ *such that*

$$⨍_B |u-(u)_r|^p \, dx \leq c\, r^{\gamma p},$$

for every ball $B \subset \Omega$ *with radius* $r > 0$, *then* $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$.

3. Higher Hölder regularity

In this section, we establish our main result. As noted earlier, for clarity, we first provide a detailed proof for the model equation (1.2), postponing the treatment of the general case (1.1) to Section 4.

We start with the definition of a weak solution.

Definition 3.1. *Let* $1 < p < n$, $n-1 < \lambda < n$ *and* $f \in L^{1,\lambda}(\Omega)$. *A function* $u \in W^{1,p}(\Omega)$ *is a weak solution to* (1.2) *if*

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = \int_\Omega f\varphi \, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{3.1}$$

We first show that the right-hand side of (3.1) is finite. Let B be a ball with radius $r > 0$, contained in Ω. For $\varphi \in C_0^\infty(B)$, the finiteness of the integral follows from the Fefferman–Phong inequality. Indeed, using Hölder's inequality, (2.1) and (2.3), we obtain

$$\begin{aligned}
\int_B |f||\varphi| \, dx &\leq \left(\int_B |f| \, dx\right)^{1/p'} \left(\int_B |f||\varphi|^p \, dx\right)^{1/p} \\
&\leq c\, \|f\|^{\frac{1}{p'}}_{L^{1,\lambda}(B)} r^{\frac{\lambda}{p'}} \|f\|^{\frac{1}{p}}_{L^{1,\lambda}(B)} r^{\frac{\lambda-n+p}{p}} \left(\int_B |D\varphi|^p \, dx\right)^{1/p} \\
&\leq c\, r^{\lambda+1-\frac{n}{p}} \|f\|_{L^{1,\lambda}(\Omega)} \|\varphi\|_{W^{1,p}(\Omega)} < \infty.
\end{aligned}$$

Since $\lambda + 1 - \frac{n}{p} > 0$ and $r \leq \operatorname{diam}(\Omega)$, we get

$$\int_B |f||\varphi|\,dx \leq c\,\|f\|_{L^{1,\lambda}(\Omega)}\|\varphi\|_{W^{1,p}(\Omega)}, \quad \forall \varphi \in C_0^\infty(B), \tag{3.2}$$

where $c > 0$ depends only on p, n, λ and $\operatorname{diam}(\Omega)$. Next, we aim to extend (3.2) from B to Ω. Indeed, if $\varphi \in C_0^\infty(\Omega)$, then we can cover the $\operatorname{supp}\varphi$ with finitely many balls $\{B_i\}_{i=1}^m \subset \Omega$, and construct a partition of unity $\{\eta_i\}_{i=1}^m \in C_0^\infty(B_i)$, with $\eta_i \in [0,1]$, such that, in $\operatorname{supp}\varphi$, one has

$$\sum_{i=1}^m \eta_i = 1.$$

Observe that if $\varphi_i := \varphi\eta_i \in C_0^\infty(B_i)$, $1 \leq i \leq m$, then

$$\|\varphi_i\|_{W^{1,p}(\Omega)} \leq C_i\|\varphi\|_{W^{1,p}(\Omega)}, \tag{3.3}$$

where the constant $C_i > 0$ depends only on η_i. Furthermore, in $\operatorname{supp}\varphi$, we have

$$\varphi = \varphi\sum_{i=1}^m \eta_i = \sum_{i=1}^m \varphi_i.$$

Combining the latter with (3.2) and (3.3), we get

$$\begin{aligned}\int_\Omega |f||\varphi|\,dx &\leq \sum_{i=1}^m \int_{B_i} |f||\varphi_i|\,dx \\ &\leq c\,\|f\|_{L^{1,\lambda}(\Omega)}\sum_{i=1}^m \|\varphi_i\|_{W^{1,p}(\Omega)} \\ &\leq C\,\|f\|_{L^{1,\lambda}(\Omega)}\|\varphi\|_{W^{1,p}(\Omega)},\end{aligned}$$

where $C > 0$ is a constant depending only on c and $\sum_{i=1}^m C_i$. Thus,

$$\int_\Omega |f||\varphi|\,dx \leq c\,\|f\|_{L^{1,\lambda}(\Omega)}\|\varphi\|_{W^{1,p}(\Omega)}, \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{3.4}$$

Observe now that (3.4) remains true for test functions in $W_0^{1,p}(\Omega)$. Indeed, if $\varphi \in W_0^{1,p}(\Omega)$ and $\varphi_i \in C_0^\infty(\Omega)$ is such that $\varphi_i \to \varphi$ in $W^{1,p}(\Omega)$, as $i \to \infty$, then, up to a subsequence (still denoted by φ_i), one has $\varphi_i \to \varphi$, a.e. in Ω. By (3.4),

$$\int_\Omega |f||\varphi_i|\,dx \leq c\,\|f\|_{L^{1,\lambda}(\Omega)}\|\varphi_i\|_{W^{1,p}(\Omega)}. \tag{3.5}$$

Since $\|\varphi_i\|_{W^{1,p}(\Omega)} \to \|\varphi\|_{W^{1,p}(\Omega)}$, employing Fatou's lemma, (3.5) yields

$$\int_\Omega |f||\varphi|\,dx \leq \liminf_{i\to\infty}\int_\Omega |f||\varphi_i|\,dx \leq c\,\|f\|_{L^{1,\lambda}(\Omega)}\|\varphi\|_{W^{1,p}(\Omega)} < \infty.$$

Remark 3.1. *We stress the low integrability assumption on f, which is merely an L^1-function. In fact, our results still hold if f is replaced by a measure. A measure μ in Ω is said to belong to the Morrey space of measures $ML^{1,\lambda}(\mathbb{R}^n)$ if there exists a constant c such that*

$$|\mu(B_r(x))| \leq c\, r^{\lambda},$$

for any $x \in \mathbb{R}^n$ and $r > 0$.

To proceed, we first revisit the following lemma from [17] (see also [4, 12, 26]), which will be used in proving our main result.

Lemma 3.1. *Let $1 < p < \infty$ and $u \in W^{1,p}(\Omega)$. There exists $\gamma \in (0,1)$ such that the problem*

$$\begin{cases} -\operatorname{div}(|Dv|^{p-2}Dv) = 0 & \text{in } \Omega \\ v - u \in W_0^{1,p}(\Omega) \end{cases}$$

has a unique solution $v \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$.

We are now ready to prove our main result.

Theorem 3.1. *Let $n-1 < \lambda < n$, $f \in L^{1,\lambda}(\Omega)$, and $\gamma \in (0,1)$ be as in Lemma 3.1. If u is a weak solution of (1.2), then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, where*

$$\alpha := \begin{cases} \min\left(\gamma, \lambda + 1 - \frac{2n}{p}\right), & \frac{2n}{\lambda+1} < p \leq 2; \\ \\ \min\left(\gamma, \dfrac{\lambda+1-n}{p-1}\right), & 2 \leq p < n. \end{cases} \tag{3.6}$$

Proof. Aiming to use Theorem 2.2, we take any ball $B \subset \Omega$ with radius $r > 0$, such that $B^{\varepsilon} \subset \Omega$, where B^{ε} is the concentric ball with radius $r + \varepsilon$. Taking the solution v of

$$\begin{cases} -\operatorname{div}(|Dv|^{p-2}Dv) = 0 & \text{in } B^{\varepsilon} \\ v - u \in W_0^{1,p}(B^{\varepsilon}), \end{cases} \tag{3.7}$$

we estimate

$$\begin{aligned} \fint_B |Du - (Du)_r|^p\,dx &\leq C\left(\fint_B |Du - Dv|^p\,dx + \fint_B |Dv - (Dv)_r|^p\,dx \right. \\ &\qquad \left. + \fint_B |(Dv)_r - (Du)_r|^p\,dx\right) \\ &=: C(I_1 + I_2 + I_3), \end{aligned} \tag{3.8}$$

for a constant $C > 0$, depending only on p. To estimate I_2, observe that by Lemma 3.1, $v \in C^{1,\gamma}_{\mathrm{loc}}(B^{\varepsilon})$, and the Campanato characterization of Hölder spaces yields

$$I_2 = \fint_B |Dv - (Dv)_r|^p\,dx \leq C r^{\gamma p}. \tag{3.9}$$

Furthermore, using Hölder's inequality, we bound

$$I_3 = \fint_B |(Dv)_r - (Du)_r|^p \, dx \le \fint_B |Du - Dv|^p \, dx = I_1. \tag{3.10}$$

To estimate I_1, we take $u - v \in W_0^{1,p}(B^\varepsilon)$ as a test function both in (1.2) and in (3.7), to obtain

$$\int_{B^\varepsilon} |Du|^{p-2} Du \cdot D(u-v) \, dx = \int_{B^\varepsilon} f(u-v) \, dx$$

and

$$\int_{B^\varepsilon} |Dv|^{p-2} Dv \cdot D(u-v) \, dx = 0.$$

Therefore,

$$\int_{B^\varepsilon} \left(|Du|^{p-2} Du - |Dv|^{p-2} Dv \right) \cdot D(u-v) \, dx = \int_{B^\varepsilon} f(u-v) \, dx. \tag{3.11}$$

We now discriminate between the degenerate and singular cases.

Case 1. For $2 \le p < n$, we use the well-known inequality,

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta \right) \cdot (\xi - \eta) \ge C|\xi - \eta|^p, \quad \forall\, \xi, \eta \in \mathbb{R}^n,$$

for a constant $C > 0$, depending only on p. Combined with (3.11), it gives

$$\begin{aligned}\int_B |Du - Dv|^p \, dx &\le \int_{B^\varepsilon} |Du - Dv|^p \, dx \\ &\le \frac{1}{C} \int_{B^\varepsilon} \left(|Du|^{p-2} Du - |Dv|^{p-2} Dv \right) \cdot D(u-v) \, dx \\ &= \frac{1}{C} \int_{B^\varepsilon} f(u-v) \, dx.\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\int_B |Du - Dv|^p \, dx \le \frac{1}{C} \int_B f(u-v) \, dx.$$

Using now the Hölder and Fefferman–Phong inequalities, we estimate

$$\begin{aligned}\int_B |f||u-v| \, dx &\le \left(\int_B |f| \, dx \right)^{1/p'} \left(\int_B |f||u-v|^p \, dx \right)^{1/p} \\ &\le c\, \|f\|_{L^{1,\lambda}(\Omega)}^{\frac{1}{p'}} r^{\frac{\lambda}{p'}} \|f\|_{L^{1,\lambda}(\Omega)}^{\frac{1}{p}} r^{\frac{\lambda+p-n}{p}} \left(\int_B |Du - Dv|^p \, dx \right)^{1/p} \\ &= c\, r^{\lambda+1-\frac{n}{p}} \|f\|_{L^{1,\lambda}(\Omega)} \left(\int_B |Du - Dv|^p \, dx \right)^{\frac{1}{p}}.\end{aligned} \tag{3.12}$$

Hence, we obtain

$$\left(\int_B |Du - Dv|^p \, dx \right)^{\frac{1}{p'}} \le C r^{\lambda+1-\frac{n}{p}} \|f\|_{L^{1,\lambda}(\Omega)},$$

and thus

$$
\begin{aligned}
I_1 &= \fint_B |Du - Dv|^p \, dx \\
&\leq C r^{(\lambda+1-\frac{n}{p})p'-n} \|f\|^{p'}_{L^{1,\lambda}(\Omega)} \\
&= C r^{\frac{(\lambda+1-n)p}{p-1}} \|f\|^{\frac{p}{p-1}}_{L^{1,\lambda}(\Omega)}. \qquad (3.13)
\end{aligned}
$$

Combining (3.8), (3.9), (3.10) and (3.13), we reach

$$
\begin{aligned}
\fint_B |Du - (Du)_r|^p \, dx &\leq C(I_1 + I_2) \\
&\leq C\left(r^{\frac{(\lambda+1-n)p}{p-1}} + r^{\gamma p}\right) \\
&\leq C r^{\alpha p},
\end{aligned}
$$

where $C > 0$ depends only on p, n and $\|f\|_{L^{1,\lambda}(\Omega)}$, and α is defined by (3.6). Recalling Theorem 2.2, we conclude that $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

Case 2. For $\frac{2n}{1+\lambda} < p < 2$, using Hölder's inequality with $2/p$ and $2/(2-p)$, we estimate

$$
\begin{aligned}
&\int_B |Du - Dv|^p \, dx \leq \int_{B^\varepsilon} |Du - Dv|^p \, dx \\
&= \int_{B^\varepsilon} \left[(|Du| + |Dv|)^{p-2} |Du - Dv|^2\right]^{\frac{p}{2}} (|Du| + |Dv|)^{(2-p)\frac{p}{2}} \, dx \qquad (3.14) \\
&\leq \left[\int_{B^\varepsilon} (|Du| + |Dv|)^{p-2} |Du - Dv|^2 \, dx\right]^{\frac{p}{2}} \left[\int_{B^\varepsilon} (|Du| + |Dv|)^p \, dx\right]^{\frac{2-p}{2}}.
\end{aligned}
$$

To bound the first term on the right-hand side of (3.14), we use the well-known inequality

$$
\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \geq c_p \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p}}, \quad \forall \xi, \eta \in \mathbb{R}^n,
$$

where $c_p > 0$ is a constant depending only on p. Additionally, taking $u - v$ as a test function for (3.7) and using Hölder's inequality, we get

$$
\int_{B^\varepsilon} |Dv|^p \, dx \leq \int_{B^\varepsilon} |Du|^p \, dx.
$$

Thus, letting $\varepsilon \to 0$, (3.14) leads to

$$
\int_B |Du - Dv|^p \, dx \leq C \left[\int_B f(u - v) \, dx\right]^{\frac{p}{2}}, \qquad (3.15)
$$

where $C > 0$ depends only on p and $\|Du\|_{L^p(B)}$. Combining (3.15) with (3.12), we arrive at

$$
\fint_B |Du - Dv|^p \, dx \leq C \, r^{(\lambda+1-\frac{n}{p})p-n} \|f\|^p_{L^{1,\lambda}(\Omega)}.
$$

Note that since $p > \frac{2n}{1+\lambda}$, then $(\lambda + 1 - \frac{n}{p})p - n > 0$. As before,

$$
\begin{aligned}
⨍_B |Du - (Du)_r|^p \, dx &\leq C(I_1 + I_2) \\
&\leq C\left(r^{(\lambda+1-\frac{n}{p})p-n} + r^{\gamma p}\right) \\
&\leq Cr^{\alpha p}.
\end{aligned}
$$

□

Remark 3.2. *Observe that, for $p \geq 2$, we always have*

$$\frac{\lambda + 1 - n}{p - 1} \in (0, 1),$$

while, for $\frac{2n}{\lambda+1} < p \leq 2$, we have

$$\lambda + 1 - \frac{2n}{p} \in (0, 1).$$

Remark 3.3. *The Serrin's functions (cf. [24])*

$$u(x) = |x|^\gamma, \quad 0 < \gamma \leq 1,$$

which are not more regular than Lipschitz continuous, solve (1.2) in B_1, with

$$f = \gamma^{p-1} [(\gamma - 1)(p - 1) - 1 + n] \, |x|^{(\gamma-1)(p-1)-1}.$$

Since $f \in L^{1,\lambda}(B_1)$, for $\lambda = (\gamma - 1)(p - 1) - 1 + n \leq n - 1$, we conclude that our assumption $\lambda > n - 1$ cannot be relaxed.

4. A class of quasilinear equations

As commented earlier, the sharp $C^{1,\alpha}_{\text{loc}}$ regularity result of the previous section holds for a more general class of quasilinear operators. In this section, we state the general regularity result and sketch its proof. Let

$$A(x, \mu, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

be differentiable and satisfy the structural assumptions

$$
\begin{cases}
\displaystyle\sum_{i,j=1}^{n} \frac{\partial A_j}{\partial \xi_i}(x, \mu, \xi) \cdot \zeta_i \zeta_j \geq C_1 (\kappa + |\xi|)^{p-2} |\zeta|^2 \\
\displaystyle\sum_{i,j=1}^{n} \left| \frac{\partial A_j}{\partial \xi_i}(x, \mu, \xi) \right| \leq C_2 (\kappa + |\xi|)^{p-2} \\
\displaystyle\sum_{i,j=1}^{n} \left| \frac{\partial A_j}{\partial x_i}(x, \mu, \xi) \right| + \left| \frac{\partial A_j}{\partial \mu}(x, \mu, \xi) \right| \leq C_3 (\kappa + |\xi|)^{p-2} |\xi|,
\end{cases}
\tag{4.1}
$$

for any $\zeta \in \mathbb{R}^n$ and for some $\kappa \in [0, 1]$, for positive constants C_1, C_2, C_3. The right-hand side in (1.1) is assumed to satisfy

$$|B(x, \mu, \xi)| \leq h(x) |\xi|^{p-1} + g(x) |\mu|^{p-1} + f(x), \tag{4.2}$$

where

$$h \in L^{p,\lambda}(\Omega), \quad g, f \in L^{1,\lambda}(\Omega), \quad n-1 < \lambda < n. \tag{4.3}$$

Solutions to (1.1) are understood in the weak sense according to the following definition.

Definition 4.1. *A function $u \in W^{1,p}(\Omega)$ is called a weak solution to (1.1) if*

$$\int_\Omega A(x,u,Du)\cdot D\varphi\,dx = \int_\Omega B(x,u,Du)\varphi\,dx, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

As before, the definition is justified by the Fefferman–Phong inequality. Furthermore, we recall an analog of Lemma 3.1 (see [17] and [4, 26], for example).

Lemma 4.1. *If $p > 1$ and $u \in W^{1,p}(\Omega)$, then the problem*

$$\begin{cases} -\operatorname{div} A(x,u,Du) = 0 & \text{in } \Omega \\ v - u \in W_0^{1,p}(\Omega) \end{cases} \tag{4.4}$$

has a unique solution $v \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$, for a certain $\gamma \in (0,1)$.

As in Section 3, Lemma 4.1 leads to the following regularity result.

Theorem 4.1. *Let (4.1)-(4.3) hold, and $\gamma \in (0,1)$ be as in Lemma 4.1. If u is a weak solution of (1.1), then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, where α is defined by (3.6).*

Sketch of the proof. For $2 \le p < n$, we have (cf. [26, Lemma 1])

$$(A(x,\mu,\xi) - A(x,\mu,\eta))\cdot(\xi-\eta) \ge C|\xi-\eta|^p,$$

for a constant $C > 0$, depending on p, n and the C_i in (4.1). Combined with (4.2), as in the proof of Theorem 3.1, this leads to (3.13), where $v \in C^{1,\gamma}_{\mathrm{loc}}(B^\varepsilon)$ is the unique solution of (4.4) in B^ε. The rest of the proof follows analogously.

If $\frac{2n}{\lambda+1} < p < 2$, we use

$$(A(x,\mu,\xi) - A(x,\mu,\eta))\cdot(\xi-\eta) \ge c\frac{|\xi-\eta|^2}{(\kappa+|\xi|+|\eta|)^{2-p}}, \quad \forall \xi,\eta \in \mathbb{R}^n,$$

where $c > 0$ is a constant depending only on p and the structural constants. The rest of the proof proceeds identically. □

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