GRADIENT REGULARITY FOR A CLASS OF SINGULAR OR DEGENERATE ELLIPTIC EQUATIONS

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Abstract. We study the regularity of viscosity solutions to a broad class of singular or degenerate elliptic equations governed by the normalized $p-$Laplacian. We establish sharp Hölder continuity for the gradient, obtain improved regularity when p is close to 2, and derive Sobolev estimates in the homogeneous case.

Contents

1. Introduction

We investigate the regularity theory for viscosity solutions to a class of singular or degenerate elliptic equations involving the normalized $p-$Laplace operator, of the form

$$-\Phi\left(x,|Du|\right)\Delta_p^{\mathrm{N}}u=f(x)\quad \text{in}\quad B_1\subset\mathbb{R}^d, \tag{1.1}$$

with a singularity or degeneracy law $\Phi(\cdot,\cdot)$ covering many important cases in the literature, such as power type and variable exponent degeneracies. We establish sharp Hölder regularity for the gradient of viscosity solutions, and, when p is close to 2 and the equation is homogeneous, obtain an improved regularity result and Sobolev estimates.

Date: June 3, 2026.
2020 *Mathematics Subject Classification.* 35B65, 35J60, 35J70, 35J75, 35D40.
Key words and phrases. Singular or degenerate elliptic equations, sharp regularity, viscosity solutions.

For $p \in (1, +\infty)$, the normalized $p-$Laplace operator is defined as

$$\begin{aligned}\Delta_p^{\mathrm{N}} u &= |Du|^{2-p}\Delta_p u = \Delta u + (p-2)\Delta_\infty^{\mathrm{N}} u \\ &= \Delta u + (p-2)\left\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \right\rangle,\end{aligned}$$

where $\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$ denotes the standard $p-$Laplace operator. This formulation shows that the normalized $p-$Laplace operator can be interpreted as a uniformly elliptic operator in nondivergence form, with ellipticity constants $\min\{p-1, 1\}$ and $\max\{p-1, 1\}$. However, due to the gradient dependence and the singular behavior at points where $Du = 0$, classical $C^{1,\alpha}$ regularity results (such as those in [12, 13]) are not directly applicable.

The operator introduced in (1.1) arises in a wide range of mathematical contexts, including stochastic processes and differential geometry. For example, it admits a geometric interpretation in connection with the mean curvature flow, as discussed in [17], and a probabilistic formulation via noisy tug-of-war games with running pay-off, as explored in [26]. Due to these diverse applications, the normalized $p-$Laplace operator has been the focus of considerable research over the years. Using tools from game theory, the authors in [24] obtained new regularity results for the equation $\Delta_p^{\mathrm{N}} u = 0$, which were later extended in [28] to include right-hand sides f that are positive and bounded. A treatment based on PDE techniques can be found in [14]. We also point out that parabolic counterparts of these regularity results have been developed in [7, 16, 21, 25].

The regularity of the gradient for the viscosity solutions to

$$-\Delta_p^{\mathrm{N}} u = f \quad \text{in } B_1 \tag{1.2}$$

was studied in [4], where the authors showed that viscosity solutions to (1.2) belong to $C^{1,\alpha}(B_{1/2})$, for some $\alpha \in (0,1)$, with appropriate estimates. The authors also derived estimates in terms of the $L^q(B_1)$ norm of f, provided $p > 2$ and $q > \max\{2, d, p/2\}$. In the borderline case, it was shown in [8] that if $f \in L(n,1)$, then any $W^{2,q}-$viscosity solution to (1.2) is differentiable. Moreover, when $f \in L^q$, with $q > d$, the solutions enjoy $C^{1,\alpha}_{\mathrm{loc}}(B_1)$ regularity for some $\alpha \in (0,1)$.

In recent years, considerable attention has also been devoted to degenerate or singular models of the form

$$-|Du|^\gamma \Delta_p^{\mathrm{N}} u = f(x) \quad \text{in } B_1, \tag{1.3}$$

where $\gamma > -1$. This formulation includes both the standard $p-$Laplace operator (when $\gamma = p-2$) and the normalized $p-$Laplace operator (when $\gamma = 0$) as special cases. Concerning regularity theory for viscosity solutions to (1.3), it was first shown in [10] that radial solutions belong to $C^{1,\alpha}$. This result was later extended in [5], where the authors proved local $C^{1,\alpha}$ regularity for all viscosity solutions, assuming the standard condition $f \in C(B_1) \cap L^\infty(B_1)$. Furthermore, for $\gamma \in [-1, 0)$ and $p \in \left(1, 3 + \left(\frac{2}{d} - 2\right)\right)$, solutions were shown to be locally in $W^{2,2}(\Omega)$. For the degenerate regime

$\gamma > 0$, local $W^{2,2}$ estimates were also established under the assumption that $|p-2+\gamma|$ is sufficiently small.

In addition, equations featuring Hamiltonian terms and strong absorption effects have also been investigated. In [9], the authors considered

$$|\nabla u|^{\theta}\left(\Delta_p^{\mathrm{N}} u + \langle \mathcal{B}(x), \nabla u\rangle\, \varrho(x)|\nabla u|^{\theta}\right) = f(x) \quad \text{in } B_1,$$

proving the existence of viscosity solutions as well as gradient bounds. In a different direction, the authors in [1] focused on the so-called dead-core problem,

$$-|\nabla u|^{\theta}\Delta_p^{\mathrm{N}} u = \mathfrak{a}(x)u_+^m(x) \quad \text{in } B_1,$$

where $\mathfrak{a}(\cdot)$ is a suitable bounded and continuous function, and $m \in [0, \theta+1)$. In particular, they demonstrated that solutions are of class $C^{1,\frac{1}{1+\theta}}$ across the free boundary $\partial\{u=0\} \cap B_{1/2}$. Additional properties, such as nondegeneracy and uniform positive density of the free boundary, were also obtained.

In a very recent work [2], the first and second authors worked with equations of the form

$$-\sigma(|Du|)\Delta_p^{\mathrm{N}} u = f(x) \quad \text{in} \quad B_1,$$

where $\sigma(\cdot)$ is a general modulus of continuity, such that $\sigma^{-1}(\cdot)$ is Dini continuous and $f \in L^{\infty}(B_1)$. Under such conditions, it was shown that viscosity solutions are locally of class C^1.

A variant involving nonhomogeneous degeneracies was analyzed in [29], where the authors considered equations of the form

$$-\left[|Du|^{\alpha(x,u)} + a(x)|Du|^{\beta(x,u)}\right]\Delta_p^{\mathrm{N}} u = f(x) \quad \text{in } B_1,$$

under appropriate assumptions on the variable exponents $0 \le \alpha(\cdot), \beta(\cdot)$ and the continuous coefficient $0 \le a(\cdot) \in C(B_1)$. This work established local $C^{1,\alpha}$ regularity of viscosity solutions.

For very general classes of degeneracies, we refer to [6] in the context of fully nonlinear elliptic equations, and to [18] for the fully nonlinear nonlocal elliptic case. In [6], the authors study fully nonlinear elliptic equations of the form

$$\Phi(x, |Du|)\, F(D^2u) = f(x) \quad \text{in } B_1,$$

where $f \in L^{\infty}(B_1)$. They prove that viscosity solutions of this equation are locally of class $C^{1,\alpha}$, where the exponent α is as in (2.5). In their setting, the constant α_0 corresponds to the Hölder regularity of the gradient of viscosity solutions to the homogeneous equation

$$F(D^2u) = 0 \quad \text{in } B_1.$$

Analogous results were obtained in [18], where the authors investigate viscosity solutions to the nonlocal equation

$$\Phi(x, |Du|)\, \mathcal{I}_{\sigma}(u, x) = f(x) \quad \text{in } B_1,$$

where $\mathcal{I}_\sigma$ denotes a suitable fully nonlinear nonlocal operator and $f \in L^\infty(B_1)$. Their results mirror those in the local case, but are established specifically in the degenerate setting.

We emphasize that the degeneracy function $\Phi(\cdot,\cdot)$ considered in these works is quite general. Both [6] and [18] operate under similar assumptions (see also conditions $A1$ and $A2$ described below), in which Φ encompasses a broad class of growth conditions. Examples of such degeneracies include:

(1) $\gamma-$**growth**: $\Phi(x,t) = t^\gamma$, with $-1 < \gamma$, as studied in [5];

(2) **Variable** $\gamma(x)-$**growth**: $\Phi(x,t) = t^{\gamma(x)}$, with $-1 < \gamma(x)$;

(3) **Double phase growth**: $\Phi(x,t) = t^\gamma + a(x)\, t^\sigma$, with $-1 < \gamma \leq \sigma$ and $a(\cdot) \in C(B_1)$;

(4) **Variable exponent double phase**: $\Phi(x,t) = t^{\gamma(x)} + a(x)\, t^{\sigma(x)}$, with $-1 < \gamma(x) \leq \sigma(x)$, $a(\cdot) \in C(B_1)$, in the spirit of [29], where only the degenerate case was analyzed;

(5) **Borderline double phase case**: $\Phi(x,t) = t^\gamma + t^\gamma \log(1+t)$, with $-1 < \gamma \leq \sigma$, $a(\cdot) \in C(B_1)$;

(6) **Multi-phase growth**: $\Phi(x,t) = t^\gamma + a(x)\, t^\sigma + b(x)\, t^\eta$, with $-1 < \gamma \leq \sigma, \eta$, and $a(\cdot), b(\cdot) \in C(B_1)$.

The main goal of this paper is to establish sharp regularity results for viscosity solutions to (1.1) assuming $f \in L^\infty$. We begin by proving that such solutions are locally of class $C^{1,\alpha}$, where α is given by (2.5), together with appropriate quantitative estimates. This is the content of Theorem 2.1. We recall that the constant $\alpha_0 \in (0,1)$ denotes the Hölder exponent for the gradient of viscosity solutions to the homogeneous $p-$Laplace equation,

$$-\Delta_p u = 0 \quad \text{in } B_1, \tag{1.4}$$

which are locally of class C^{1,α_0}. Moreover, there exists a constant $C > 0$ such that

$$\|u\|_{C^{1,\alpha_0}(B_{1/2})} \leq C. \tag{1.5}$$

As we explain below, in general, α_0 acts as an upper bound for the regularity of solutions to (1.1), and it may be arbitrarily small depending on p.

In our second main result, we study the regularity of solutions to (1.4) for p close to 2, and prove that for any given $q \in [1, +\infty)$, they belong to $W^{2,q}(B_{1/2})$. In particular, this implies that $\alpha_0(q)$ may be taken arbitrarily close to 1. Heuristically, as $p \to 2$, the (normalized) $p-$Laplacian converges to the Laplace operator, and $\alpha_0 \to 1$. This result is stated in Theorem 2.2.

The remainder of the manuscript is organized as follows. In Section 2, we collect auxiliary results and state our main theorems. A compactness result and other crucial technical tools are proven in Section 3. Section 4 is devoted to the proof of Theorem 2.1. Finally, in Section 5, we establish the Sobolev regularity for p close to 2.

2. Preliminaries

2.1. **Notation and assumptions.** We denote by $B_r(x_0)$ the ball of radius r and centered at x_0. We use both $\langle \eta, \xi \rangle$ and $\eta \cdot \xi$ to denote the inner product of vectors $\eta, \xi \in \mathbb{R}^d$. Moreover, we say that a constant is universal if it only depends on the ellipticity constants of the operator and the dimension d.

We now record our standing assumptions. We need the definition of an almost increasing and an almost decreasing function.

Definition 2.1. *Let $v : [0, \infty) \to [0, \infty)$. We say that v is almost increasing if there is a constant $M \geq 1$ such that*

$$v(t_1) \leq M v(t_2) \quad \textit{for} \quad 0 \leq t_1 < t_2. \tag{2.1}$$

Similarly, we say that v is almost decreasing if there is a constant $M \geq 1$ such that

$$v(t_2) \leq M v(t_1) \quad \textit{for} \quad 0 \leq t_1 < t_2. \tag{2.2}$$

We begin with the conditions on Φ that are used throughout the paper.

A1 (Degeneracy law). *The function $\Phi : B_1 \times [0, \infty) \to [0, \infty)$ is continuous and there exists a constant $B \geq 1$ such that*

$$\frac{1}{B} \leq \Phi(x, 1) \leq B, \quad \forall x \in B_1. \tag{2.3}$$

A2 (Growth property). *There exist constants $-1 < \gamma \leq \mu$ such that, for every $x \in B_1$, the map*

$$t \mapsto \frac{\Phi(x, t)}{t^\gamma}$$

is almost increasing, and the map

$$t \mapsto \frac{\Phi(x, t)}{t^\mu}$$

is almost decreasing, with constant $M \geq 1$.

We proceed with the hypothesis on the source term f.

A3. *The source term is such that $f \in L^\infty(B_1) \cap C(B_1)$.*

2.2. **Definition of viscosity solution and auxiliary results.** We now introduce the definition of viscosity solution to (1.1). When $\gamma > 0$, the operator in (1.1) is continuous, and the standard notion of viscosity solution applies. However, for $\gamma \leq 0$, the operator is not well defined on the set $\{Du = 0\}$: it exhibits a bounded discontinuity when $\gamma = 0$ and becomes highly singular when $\gamma < 0$. For this reason, we adopt an adapted notion of viscosity solution proposed in [11]. Alternatively, one could use the relaxed version of viscosity solutions introduced in [15]. In fact, they are equivalent, as shown in [5, Proposition 2.4].

Definition 2.2 (Viscosity solution). *Let $1 < p < \infty$ and $f \in C(B_1) \cap L^\infty(B_1)$. We say that $u \in C(B_1)$ is a viscosity subsolution to (1.1) if for every $x_0 \in B_1$, either there exists η such that u is constant in $B_\eta(x_0)$ and*

$0 \le f(x)$ for all $x \in B_\eta(x_0)$, or for any $\varphi \in C^2(B_1)$ such that $u - \varphi$ has a local maximum at x_0 and $D\varphi(x_0) \neq 0$, we have

$$-\Phi(x_0, |D\varphi(x_0)|)\Delta_p^N \varphi(x_0) \le f(x_0).$$

Similarly, we say that $u \in C(B_1)$ is a viscosity supersolution to (1.1) if for every $x_0 \in B_1$, either there exists η such that u is constant in $B_\eta(x_0)$ and $0 \ge f(x)$ for all $x \in B_\eta(x_0)$, or for any $\varphi \in C^2(B_1)$ such that $u - \varphi$ has a local minimum at x_0 and $D\varphi(x_0) \neq 0$, we have

$$-\Phi(x_0, |D\varphi(x_0)|)\Delta_p^N \varphi(x_0) \ge f(x_0).$$

We say that a function $u \in C(B_1)$ is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution. Finally, $u \in C(B_1)$ is a normalized viscosity solution if $\sup_{B_1} |u| \le 1$.

Next, we state a lemma that will lead to the compactness of solutions (see [15]).

Lemma 2.1. *Let B_1 be a unit ball in $\mathbb{R}^d$. Let $\Omega \subset B_1$, $u, v \in C(B_1)$ and φ be twice continuously differentiable in $\Omega \times \Omega$. Define*

$$w(x, y) \coloneqq u(x) + v(y) \quad \forall (x, y) \in \Omega \times \Omega,$$

and assume that $w - \varphi$ attains the maximum at $(x_0, y_0) \in \Omega \times \Omega$. Then for each $\varepsilon > 0$, there exists $X, Y \in \mathcal{S}(d)$ such that

$$(D_x\varphi(x_0, y_0), X) \in \overline{\mathcal{J}}^{2,+} u(x_0), \quad (D_y\varphi(x_0, y_0), Y) \in \overline{\mathcal{J}}^{2,+} v(y_0).$$

Moreover, the block diagonal matrix with entries X and Y satisfies

$$-\left(\frac{1}{\varepsilon} + \|B\|\right) I \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le B + \varepsilon B^2,$$

where $B = D^2\varphi(x_0, y_0) \in \mathcal{S}(d)$.

In the singular case $-1 < \gamma < 0$, we observe that solutions to (1.1) are also solutions to an equivalent equation. This characterization will help us to get better compactness properties in this scenario.

Proposition 2.1. *Assume that $-1 < \gamma < 0$, and that $A1 - A3$ hold true. If u is a viscosity solution to (1.1), then u solves*

$$|Du|^{-\gamma}\Phi(x, |Du|)\Delta_p^{\mathrm{N}} u = |Du|^{-\gamma} f(x) \quad \text{in} \quad B_1, \tag{2.4}$$

in the viscosity sense.

Proof. Let us take a test function $\varphi \in C^2(B_1)$ such that $u - \varphi$ has a local maximum at $x_0 \in B_1$. If the gradient of our test function does not vanish at the point x_0, we have $|D\varphi(x_0)|^{-\gamma} > 0$, and

$$-\Phi(x_0, |D\varphi(x_0)|)\Delta_p^{\mathrm{N}} \varphi(x_0) \le f(x_0).$$

Hence,

$$-|D\varphi(x_0)|^{-\gamma}\Phi(x_0, |D\varphi(x_0)|)\Delta_p^N \varphi(x_0) \le |D\varphi(x_0)|^{-\gamma} f(x_0),$$

and, as a consequence, u is a subsolution to (2.4).

Then, let us focus our attention on the case where x_0 is a critical point of φ, *i.e.*, $D\varphi(x_0) = 0$. In this context, two scenarios are possible. For the first one, we assume that the $D^2\varphi(x_0)$ is invertible, and consequently, the point x_0 is a locally isolated critical point of φ. Thus, for every $\varepsilon > 0$ there exist a small $\rho_0 \geq \rho$ and a sequence $x_\rho \to x_0$ such that $D\varphi(x_\rho) \neq 0$ and

$$-\Phi(x_\rho, |D\varphi(x_\rho)|)\Delta_p^{\mathrm{N}}\varphi(x_\rho) \leq f(x_\rho) + \varepsilon.$$

As a consequence, $|D\varphi(x_\rho)|^{-\gamma} > 0$ and

$$-|D\varphi(x_\rho)|^{-\gamma}\Phi(x_\rho, |D\varphi(x_\rho)|)\Delta_p^{\mathrm{N}}\varphi(x_\rho) \leq |D\varphi(x_\rho)|^{-\gamma}\left(f(x_\rho) + \varepsilon\right).$$

Letting $\varepsilon, \rho \to 0$, we obtain

$$0 = -|D\varphi(x_0)|^{-\gamma}\Phi(x_0, |D\varphi(x_0)|)\Delta_p^{\mathrm{N}}\varphi(x_0) \leq |D\varphi(x_0)|^{-\gamma}f(x_0) = 0.$$

If the $D^2\varphi(x_0)$ is not invertible, we can take a semi-positive definite matrix $M \in \mathcal{S}(d)$ such that, for every $\delta > 0$, we have that $D^2\varphi(x_0) - \delta M$ is invertible. Let us define the auxiliary function

$$\varphi_\delta(x) \coloneqq \varphi(x) - \delta\langle M(x - x_0), x - x_0\rangle,$$

and apply the same argument as in the first scenario with φ replaced by φ_δ to obtain

$$0 = -|D\varphi_\delta(x_0)|^{-\gamma}\Phi(x_0, |D\varphi_\delta(x_0)|)\Delta_p^{\mathrm{N}}\varphi_\delta(x_0) \leq |D\varphi_\delta(x_0)|^{-\gamma}f(x_0) = 0.$$

Thereby, letting $\delta \to 0$, we obtain the desired result. Thus, in both scenarios we have that u is a subsolution to (2.4).

The procedure to show that u is a supersolution is analogous, and we will omit it here. Therefore, u is a viscosity solution to (2.4). □

2.3. **Scaling properties.** Across this manuscript, we will suppose that certain quantities satisfy a smallness regime. This assumption is not restrictive, *i.e.*, we can assume without loss of generality that u is a normalized viscosity solution, the $L^\infty-$norm of the source term is arbitrarily small, and that $B = 1$. In fact, for any $\varepsilon > 0$, the function

$$v(x) := \frac{u(x)}{K}$$

is such that v is a viscosity solution to

$$-\tilde{\Phi}(x, |Du|)\Delta_p^{\mathrm{N}}u = \tilde{f}(x) \quad \text{in} \quad B_1,$$

where

$$\tilde{\Phi}(x, t) := \frac{\Phi(x, Kt)}{\Phi(x, K)} \quad \text{and} \quad \tilde{f}(x) := \frac{f(x)}{\Phi(x, K)K}.$$

Observe that $\tilde{\Phi}(\cdot, \cdot)$ satisfies $A1$ with $B = 1$, and $A2$ with the same constant $M \geq 1$. Moreover, $A2$ also gives

$$\|\tilde{f}\|_{L^\infty(B_1)} \leq \frac{1}{\Phi(x, K)K}\|f\|_{L^\infty(B_1)} \leq \frac{BM}{K^{1+\gamma}}\|f\|_{L^\infty(B_1)}.$$

Hence, recalling that $\gamma > -1$ and taking

$$K := 2\left[\|u\|_{L^\infty(B_1)} + \left(\varepsilon^{-1}BM\|f\|_{L^\infty(B_1)}\right)^{\frac{1}{1+\gamma}}\right],$$

we obtain that

$$\|v\|_{L^\infty(B_1)} \leq 1 \quad \text{and} \quad \|\tilde{f}\|_{L^\infty(B_1)} \leq \varepsilon.$$

2.4. **Main results.** We are now ready to state the main results of our paper. We start with the sharp Hölder regularity for the gradient of viscosity solutions to (1.1). Recall that $\alpha_0 \in (0,1)$ accounts for the Hölder regularity for the gradient of viscosity solutions to $\Delta_p u = 0$.

Theorem 2.1. *Let $u \in C(B_1)$ be a bounded viscosity solution to* (1.1). *Suppose A1-A3 are in force. Then $u \in C^{1,\alpha}_{loc}(B_1)$, and there exists a constant $C > 0$ such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C = C\left(d, p, B, \alpha, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}\right),$$

where

$$\alpha = \begin{cases} \min\left\{\alpha_0^-, \frac{1}{1+\mu}\right\} & \text{if } \gamma \geq 0 \\ \min\left\{\alpha_0^-, \frac{1}{1+\mu-\gamma}\right\} & \text{if } -1 < \gamma < 0. \end{cases} \tag{2.5}$$

Here, α_0^- denotes any number $\beta \in (0, \alpha_0)$.

Notice that, in particular, Theorem 2.1 shows that viscosity solutions to (1.1) in the singular $p-$Laplacian case, *i.e.*, with $\Phi(x,t) = t^{p-2}$, for $p \in (1,2)$, are almost as smooth as viscosity solutions to the homogeneous equation (1.4) since we can take $\gamma = \mu = p - 2$.

It is important to note that α_0 can be arbitrarily small in general. As a result, even when γ and μ are close to zero, or even equal to zero, the regularity of the solutions remains bounded above by α_0. In the next result, we study the homogeneous p-Laplace equation and show that α_0 can be taken arbitrarily close to 1, provided p is sufficiently close to 2.

Theorem 2.2. *Let $u \in C(B_1)$ be a viscosity solution to* (1.4). *Given $q \in [1, \infty)$, there exists a constant $\tilde{\varepsilon} := \tilde{\varepsilon}(q) > 0$, such that if*

$$|p - 2| \leq \tilde{\varepsilon},$$

then $u \in W^{2,q}_{loc}(B_1)$, and there exists a universal constant C such that

$$\|u\|_{W^{2,q}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

Remark 2.1. *The proof can be adapted (cf. [5]) to show that viscosity solutions to*

$$-\Delta_p u = f(x) \quad \text{in } B_1,$$

where $f \in L^q(B_1)$, are of class $W^{2,q}$, as long as $|p-2| \leq \tilde{\varepsilon}(q)$.

A direct application of Theorem 2.2 is an improvement of the regularity estimates provided by Theorem 2.1. In particular, this result generalizes [27, Theorem 1].

Corollary 2.1. *Let $u \in C(B_1)$ be a bounded viscosity solution to* (1.1)*. Suppose that A1-A3 are in force and let*

$$\alpha = \begin{cases} \frac{1}{1+\mu} & \text{if } \gamma \geq 0 \\ \frac{1}{1+\mu-\gamma} & \text{if } -1 < \gamma < 0. \end{cases}$$

Given $\beta \in (0, \alpha]$, there exists a constant $\tilde{\varepsilon} := \tilde{\varepsilon}(\beta) > 0$ such that if

$$|p-2| \leq \tilde{\varepsilon},$$

then $u \in C^{1,\beta}_{loc}(B_1)$, and there exists a constant $C > 0$ such that

$$\|u\|_{C^{1,\beta}(B_{1/2})} \leq C = C\left(d, p, B, \alpha, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}\right).$$

Remark 2.2. *Although Corollary 2.1 improves the regularity of solutions when p is close to 2, it still imposes the constraint that $\tilde{\varepsilon}$ depends on the exponent β. In particular, we cannot allow β to depend on the parameter p, as this would cause $\tilde{\varepsilon}$ to also depend on p, creating a circular dependence.*

3. Compactness of the solutions

In this section, we prove the compactness of the solution to (1.1). Since our arguments involve scaled functions, we will need to deal with a slightly more general equation. Specifically, we consider the following auxiliary problem

$$-\Phi(x, |Du+\xi|)\left[\Delta u + (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|}\right\rangle\right] = f \quad \text{in} \quad B_1, \tag{3.1}$$

where ξ is an arbitrary vector in $\mathbb{R}^d$. To obtain the compactness result independent of ξ, we will analyse separately the degenerate and singular cases. We begin by providing the desired compactness in the case where the norm of ξ is sufficiently large.

Lemma 3.1. *Let $u \in C(B_1)$ be a normalized viscosity solution to* (3.1)*, and assume that $A1 - A3$ are in force. Suppose further that $\gamma \geq 0$. There exists a constant $\Gamma > 1$, such that if $|\xi| > \Gamma$, then $u \in C^{\alpha}_{loc}(B_1)$, for some universal $\alpha \in (0,1)$. Moreover, there exists a constant $C > 0$ such that*

$$\|u\|_{C^{\alpha}_{loc}(B_1)} \leq C.$$

Proof. Let us begin by considering a modulus of continuity $\omega : [0, \infty) \to [0, \infty)$ defined by $\omega(t) := t^\alpha$, for a fixed value of $\alpha \in (0,1)$. For $r \in (0, 9/10)$ and $z \in B_{r/2}$ arbitrary, define the constant

$$\mathbf{S} := \sup_{x,y \in B_r} \left(u(x) - u(y) - J_1\omega(|x-y|) - J_2\left(|x-z|^2 + |y-z|^2\right)\right),$$

with J_1 and J_2 positive constants. If $\mathbf{S} \leq 0$, we obtain the desired result, *i.e.*, we find a positive universal constant such that

$$|u(x) - u(y)| \leq C|x-y|^\alpha.$$

We will argue by contradiction. Let us suppose that for all positive constants J_1 and J_2, we can find a point $z_0 \in B_{r/2}$ such that $\mathbf{S} > 0$.

Step 1: *The choice of the appropriate value of the constant J_2.*

We proceed by constructing two auxiliary functions

$$\psi(x, y) := J_1\omega(|x - y|) + J_2 \left(|x - z_0|^2 + |y - z_0|^2\right),$$

and

$$\varphi(x, y) := u(x) - u(y) - \psi(x, y).$$

Let $(x_0, y_0) \in \overline{B}_r \times \overline{B}_r$ be a point where the maximum of the function φ is achieved, and note that if $x_0 = y_0$ we get that $\mathbf{S} \leq 0$, independently of the choice of $J_2 > 0$. Therefore, we will assume without loss of generality that $|x_0 - y_0| > 0$. Since

$$\varphi(x_0, y_0) = \mathbf{S} > 0$$

and $\|u\|_{L^\infty(B_1)} \leq 1$, we have

$$\psi(x_0, y_0) < u(x_0) - u(y_0) \leq 2.$$

In particular, this implies

$$J_2 \left(|x_0 - z_0|^2 + |y_0 - z_0|^2\right) \leq 2.$$

Now, we choose the constant $J_2 := (32/r)^2$, ensuring that (x_0, y_0) are interior points of $B_r \times B_r$. In fact, note that with this choice of J_2 we have

$$\left(\frac{32}{r}\right)^2 |x_0 - z_0|^2 \leq 2 \qquad \text{and} \qquad \left(\frac{32}{r}\right)^2 |y_0 - z_0|^2 \leq 2,$$

and, as a consequence,

$$|x_0 - z_0| \leq \frac{r}{16} \qquad \text{and} \qquad |y_0 - z_0| \leq \frac{r}{16}. \tag{3.2}$$

Step 2: *Applying Lemma 2.1.*

We compute

$$D_x\psi(x_0, y_0) = J_1\omega'(|x_0 - y_0|)\frac{x_0 - y_0}{|x_0 - y_0|} + 2J_2(x_0 - z_0),$$

and

$$-D_y\psi(x_0, y_0) = J_1\omega'(|x_0 - y_0|)\frac{x_0 - y_0}{|x_0 - y_0|} - 2J_2(y_0 - z_0).$$

Then, Lemma 2.1 ensures the existence of matrices $X, Y \in \mathcal{S}(d)$, such that for each $\varepsilon > 0$, we have

$$-\left(\frac{1}{\varepsilon} + \|B\|\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} B + \varepsilon B^2 & -B - \varepsilon B^2 \\ -B - \varepsilon B^2 & B + \varepsilon B^2 \end{pmatrix}, \tag{3.3}$$

where B is given by

$$B := J_1\alpha|x_0 - y_0|^{\alpha-2}\left(I + (\alpha - 2)\frac{x_0 - y_0}{|x_0 - y_0|} \otimes \frac{x_0 - y_0}{|x_0 - y_0|}\right),$$

where we have used the fact that $\omega(t) = t^\alpha$. Hence, recalling that $x \otimes x = xx^T$, we can calculate

$$\begin{aligned} B^2 = \quad & \left(J_1\alpha|x_0-y_0|^{\alpha-2}\right)^2 \left(I + (\alpha-2)\frac{x_0-y_0}{|x_0-y_0|} \otimes \frac{x_0-y_0}{|x_0-y_0|}\right) \cdot \\ & \cdot \left(I + (\alpha-2)\frac{x_0-y_0}{|x_0-y_0|} \otimes \frac{x_0-y_0}{|x_0-y_0|}\right) \\ = \quad & \left(J_1\alpha|x_0-y_0|^{\alpha-2}\right)^2 \left(I + \alpha(\alpha-2)\frac{x_0-y_0}{|x_0-y_0|} \otimes \frac{x_0-y_0}{|x_0-y_0|}\right). \end{aligned}$$

Thus, choosing $\varepsilon \coloneqq 1/(J_1\alpha|x_0-y_0|^{\alpha-2})$, we get

$$B + \varepsilon B^2 = J_1\alpha|x_0-y_0|^{\alpha-2}\left(2I + (1+\alpha)(\alpha-2)\frac{x_0-y_0}{|x_0-y_0|} \otimes \frac{x_0-y_0}{|x_0-y_0|}\right).$$

Step 3: *The eigenvalues of $X - Y$.*

For a unitary vector ξ, consider vectors of the form $(\xi, \xi) \in \mathbb{R}^{2d}$ and observe that from (3.3) we get

$$\begin{pmatrix}\xi\\ \xi\end{pmatrix}^T \begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix} \le \begin{pmatrix}\xi\\ \xi\end{pmatrix}^T \begin{pmatrix}B+\varepsilon B^2 & -B-\varepsilon B^2\\ -B-\varepsilon B^2 & B+\varepsilon B^2\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}. \tag{3.4}$$

As a consequence of (3.4), for any pair of unit vectors $(\xi,\xi) \in \mathbb{R}^{2d}$, we obtain $X - Y \le 0$. Moreover, we can estimate

$$\|X\|, \|Y\| \le 2J_1\alpha|x_0-y_0|^{\alpha-2}\left(4 + \alpha(1-\alpha)\right). \tag{3.5}$$

Furthermore, applying inequality (3.4) with the vector $(\xi, -\xi)$, where ξ is given by $\xi \coloneqq (x_0-y_0)/|x_0-y_0|$, we have

$$\langle (X-Y)\xi, \xi\rangle \le 4\langle (B+\varepsilon B^2)\xi, \xi\rangle = 4J_1\alpha^2|x_0-y_0|^{\alpha-2}(\alpha-1) < 0,$$

which means that at least one eigenvalue of $X - Y$ is negative and smaller than the constant

$$4J_1\alpha^2|x_0-y_0|^{\alpha-2}(\alpha-1). \tag{3.6}$$

Throughout the remainder of the paper, we will denote this eigenvalue by λ_0.

Step 4: *The choice of Γ.*

For simplicity, let us denote

$$\xi_{x_0} \coloneqq D_x\psi(x_0,y_0) \qquad \text{and} \qquad \xi_{y_0} \coloneqq D_y\psi(x_0,y_0),$$

and fix an arbitrary vector $\xi \in \mathbb{R}^d$ satisfying $|\xi| > \Gamma$. Observe that since $0 < |x_0-y_0| \le 2$, it is sufficient (at this point) to consider

$$J_1 > \frac{2^{9-\alpha}}{r\alpha} \ge \frac{2^8}{r\alpha|x_0-y_0|^{\alpha-1}}.$$

Now, from the definition of J_2 and the information in (3.2), we have

$$J_1\alpha|x_0-y_0|^{\alpha-1} > 4J_2\max\{|x_0-z_0|, |y_0-z_0|\}.$$

Using this estimate, we get

$$2J_1\alpha|x_0-y_0|^{\alpha-1} \geq |\xi_{x_0}| \geq \frac{J_2}{2}|x_0-y_0|^{\alpha-1}$$

and

$$2J_1\alpha|x_0-y_0|^{\alpha-1} \geq |\xi_{y_0}| \geq \frac{J_2}{2}|x_0-y_0|^{\alpha-1}.$$

Define $\nu_{x_0} := \xi_{x_0} + \xi$ and $\nu_{y_0} := \xi_{y_0} + \xi$. Thus, by choosing

$$\Gamma := \frac{9}{4}J_1\alpha|x_0-y_0|^{\alpha-1} > 1, \tag{3.7}$$

we obtain

$$|\nu_{x_0}| \geq \frac{J_1}{4}\alpha|x_0-y_0|^{\alpha-1} > 1, \tag{3.8}$$

and

$$|\nu_{y_0}| \geq \frac{J_1}{4}\alpha|x_0-y_0|^{\alpha-1} > 1. \tag{3.9}$$

Then, the inequalities in the viscosity sense are

$$0 \leq \Phi(x_0, |\nu_{x_0}|)\left[\mathrm{Tr}(X+J_2I) + (p-2)\left\langle (X+J_2I)\frac{\nu_{x_0}}{|\nu_{x_0}|}, \frac{\nu_{x_0}}{|\nu_{x_0}|}\right\rangle\right] + f(x_0),$$

and

$$0 \geq \Phi(y_0, |\nu_{y_0}|)\left[\mathrm{Tr}(X-J_2I) + (p-2)\left\langle (X-J_2I)\frac{\nu_{y_0}}{|\nu_{y_0}|}, \frac{\nu_{y_0}}{|\nu_{y_0}|}\right\rangle\right] + f(y_0).$$

For $\nu \neq 0$, with $\overline{\nu} = \nu/|\nu|$ and

$$A(\nu) := I + (p-2)\overline{\nu}\otimes\overline{\nu},$$

we can rewrite the inequalities above as

$$\mathrm{Tr}\left(A(\nu_{x_0})(X+J_2I)\right) \geq -\frac{f(x_0)}{\Phi(x_0, |\nu_{x_0}|)}$$

and

$$\mathrm{Tr}(A(\nu_{y_0})(Y-J_2I)) \leq -\frac{f(y_0)}{\Phi(y_0, |\nu_{y_0}|)}.$$

Step 5: *Estimates for the viscosity inequality.*

Notice that we can use the linearity of the trace operator to conclude that

$$\begin{aligned}\mathrm{Tr}(A(\nu_{x_0})(X+J_2I)) - \mathrm{Tr}(A(\nu_{y_0})(Y-J_2I)) = \\ = \underbrace{\mathrm{Tr}(A(\nu_{x_0})(X-Y))}_{(I)} + \underbrace{J_2\left[\mathrm{Tr}(A(\nu_{x_0})) + \mathrm{Tr}(A(\nu_{y_0}))\right]}_{(II)} \\ + \underbrace{\mathrm{Tr}((A(\nu_{x_0}) - A(\nu_{y_0}))Y)}_{(III)}.\end{aligned}$$

At this point, we estimate each term individually. To estimate (I), recall that the eigenvalues of $A(\nu_{x_0})$ belong to the interval $[\min\{1,p-1\}, \max\{1,p-1\}]$ and (3.6) hold true. Consequently

$$\mathrm{Tr}(A(\nu_{x_0})(X-Y)) \leq \sum_{i=1}^{d}\lambda_i(A(\nu_{x_0}))\lambda_i(X-Y) \leq \min\{1,p-1\}\lambda_0(X-Y)$$

$$\leq 4J_1\alpha^2 \min\{1,p-1\}|x_0-y_0|^{\alpha-2}(\alpha-1).$$

For the estimate of (II), observe that

$$\begin{aligned} J_2\left[\mathrm{Tr}(A(\nu_{x_0}))+\mathrm{Tr}(A(\nu_{y_0}))\right] &\leq J_2\left(\sum_{i=1}^{d}\lambda_i(A(\nu_{x_0}))+\sum_{i=1}^{d}\lambda_i(A(\nu_{y_0}))\right)\\ &\leq 2dJ_2\max\{1,p-1\}. \end{aligned}$$

At last, in order to estimate (III), note that we can write

$$\begin{aligned} A(\nu_{x_0})-A(\nu_{y_0}) &= (p-2)(\overline{\nu}_{x_0}\otimes\overline{\nu}_{x_0}-\overline{\nu}_{y_0}\otimes\overline{\nu}_{y_0}+\overline{\nu}_{y_0}\otimes\overline{\nu}_{x_0}-\overline{\nu}_{y_0}\otimes\overline{\nu}_{x_0})\\ &= (p-2)[(\overline{\nu}_{x_0}-\overline{\nu}_{y_0})\otimes\overline{\nu}_{x_0}-\overline{\nu}_{y_0}\otimes(\overline{\nu}_{y_0}-\overline{\nu}_{x_0})]. \end{aligned} \tag{3.10}$$

Moreover, from the definition ξ_{x_0} and ξ_{y_0} we have

$$|\nu_{x_0}-\nu_{y_0}| = 2J_2|(x_0-z_0)+(y_0-z_0)| \leq \frac{J_2}{4}.$$

Hence, combining this estimate with the information in (3.8) and (3.9), we get

$$\begin{aligned} |\overline{\nu}_{x_0}-\overline{\nu}_{y_0}| &= \left(\frac{\nu_{x_0}}{|\nu_{x_0}|}-\frac{\nu_{y_0}}{|\nu_{y_0}|}\right) \leq 2\max\left\{\frac{|\nu_{x_0}-\nu_{y_0}|}{|\nu_{x_0}|},\frac{|\nu_{x_0}-\nu_{y_0}|}{|\nu_{y_0}|}\right\}\\ &\leq 2\left(\frac{|\nu_{x_0}-\nu_{y_0}|}{|\nu_{x_0}|}+\frac{|\nu_{x_0}-\nu_{y_0}|}{|\nu_{y_0}|}\right)\\ &\leq \frac{8J_2}{J_1\alpha|x_0-y_0|^{\alpha-1}}. \end{aligned}$$

Thereby, from (3.5), (3.10), the fact that $|\overline{\nu}_{x_0}|=|\overline{\nu}_{y_0}|=1$, and the above estimate, we obtain

$$\begin{aligned} \mathrm{Tr}\left((A(\nu_{x_0})-A(\nu_{y_0}))Y\right) &\leq d\|Y\|\|A(\nu_{x_0})-A(\nu_{y_0})\|\\ &\leq 2d|p-2|\|Y\||\overline{\nu}_{x_0}-\overline{\nu}_{y_0}|\\ &\leq 4dJ_1\alpha|p-2|\left(4+\alpha(1-\alpha)\right)\\ &\quad\cdot|x_0-y_0|^{\alpha-2}|\overline{\nu}_{x_0}-\overline{\nu}_{y_0}|\\ &\leq 32dJ_2|p-2|\left(4+\alpha(1-\alpha)\right)|x_0-y_0|^{-1}. \end{aligned}$$

Therefore,

$$\begin{gathered} \mathrm{Tr}(A(\nu_{x_0})(X+J_2I))-\mathrm{Tr}(A(\nu_{y_0})(Y-J_2I)) \leq (I)+(II)+(III)\\ \leq 4J_1\alpha^2\min\{1,p-1\}|x_0-y_0|^{\alpha-2}(\alpha-1)\\ +2dJ_2\max\{1,p-1\}+32dJ_2|p-2|\left(4+\alpha(1-\alpha)\right)|x_0-y_0|^{-1}. \end{gathered}$$

On the other hand, using $A2$, *i.e.*, that the map $t\to\Phi(x,t)/t^{\gamma}$ is almost increasing, together with assumptions $A1$ and $A3$, and equations (3.8) and (3.9), we obtain

$$\begin{aligned} \mathrm{Tr}\left(A(\nu_{x_0})(X+J_2I)\right)-\mathrm{Tr}(A(\nu_{y_0})(Y-J_2I)) \geq\; & -\frac{f(x_0)}{\Phi(x_0,|\nu_{x_0}|)}\\ & +\frac{f(y_0)}{\Phi(y_0,|\nu_{y_0}|)} \end{aligned}$$

$$\begin{aligned} &\geq -\frac{MB\|f\|_{L^\infty(B_1)}}{|\nu_{x_0}|^\gamma} \\ &\quad -\frac{MB\|f\|_{L^\infty(B_1)}}{|\nu_{y_0}|^\gamma} \\ &\geq -2MB\|f\|_{L^\infty(B_1)}, \end{aligned}$$

where, in the last inequality, we have used that $\gamma \geq 0$.

Step 6: *Choosing the value of J_1.*

At this point, we use the estimate in **Step 5** to obtain

$$\begin{aligned} &4J_1\alpha^2 \min\{1, p-1\}|x_0 - y_0|^{\alpha-2}(\alpha - 1) + 2dJ_2 \max\{1, p-1\} \\ &+ 2MB\|f\|_{L^\infty(B_1)} + 32dJ_2|p-2|\left(4 + \alpha(1-\alpha)\right)|x_0 - y_0|^{-1} \geq 0. \end{aligned} \tag{3.11}$$

Observe that in this inequality only the first term is negative since $\alpha \in (0,1)$, and thus we can choose the value of J_1 larger than

$$\frac{dJ_2\left[\max\{1, p-1\} + 16dJ_2|p-2|\left(4 + \alpha(1-\alpha)\right)|x_0 - y_0|^{-1}\right] + MB\|f\|_{L^\infty(B_1)}}{2\alpha^2 \min\{1, p-1\}|x_0 - y_0|^{\alpha-2}(\alpha - 1)}.$$

So this choice leads the sum in (3.11) to be strictly negative, which is a contradiction. Therefore, there exist constants $J_1 > 0$ and $J_2 > 0$ such that $\mathbf{S} \leq 0$, and the result follows. □

Lemma 3.2. *Let $u \in C(B_1)$ be a normalized viscosity solution to* (3.1)*. Assume that $\gamma \geq 0$ and that $A1 - A3$ are in force. Suppose further that $|\xi| \leq \Gamma$. Then, there exist constants $\alpha \in (0,1)$ and $C > 0$ such that*

$$\|u\|_{C^\alpha_{loc}(B_1)} \leq C.$$

Proof. If $|Du| \geq 2\Gamma$, we have

$$|Du + \xi| \geq ||\Gamma| - |\xi|| \geq \Gamma > 1,$$

which, together with assumptions $A1$ and $A2$, implies that

$$\Phi(x, |Du + \xi|) \geq (MB)^{-1}|Du + \xi|^\gamma > (MB)^{-1}.$$

Consequently, the operator in (3.1) is a non-degenerate uniformly elliptic operator, and, moreover, $u \in C(B_1)$ is a viscosity solution to

$$\begin{cases} \mathcal{M}^+(D^2u) + MB|f| \geq 0 & \text{in} \quad B_1 \\ \mathcal{M}^-(D^2u) - MB|f| \leq 0 & \text{in} \quad B_1. \end{cases}$$

Therefore, from [20], we ensure the existence of a universal constant $\alpha \in (0,1)$ such that $u \in C^\alpha_{\text{loc}}(B_1)$, with the estimate

$$\|u\|_{C^{0,\alpha}_{loc}}(B_1) \leq C,$$

for a universal constant $C > 0$. □

The following lemma treats the singular case, where we only need to consider $\xi = 0$.

Lemma 3.3. *Let $u \in C(B_1)$ be a normalized viscosity solution to (3.1), with $|\xi| = 0$. Assume that $-1 < \gamma < 0$ and $A1 - A3$ are in force. Then, u is a locally Lipschitz continuous function in B_1. Moreover, there exists a constant $C > 0$ such that*

$$\|u\|_{C^{0,1}_{loc}(B_1)} \leq C.$$

Proof. The proof of this lemma follows closely the proof of Lemma 3.1 and thus we will only emphasise the main changes. First, the modulus of continuity considered in this case is given by

$$\omega(t) := \begin{cases} t - \omega_0 t^\alpha & \text{if} \quad t \leq t_0, \\ \omega(t_0) & \text{if} \quad t \geq t_0, \end{cases} \tag{3.12}$$

where $\alpha \in (1,2)$ and ω_0 is a positive constant such that

$$t_0 := \left(\frac{1}{\alpha\omega_0}\right)^{1/(\alpha-1)} > 2 \qquad \text{and} \qquad \omega_0 2^{\alpha-1} \leq 1/4. \tag{3.13}$$

We can compute

$$\omega'(t) := \begin{cases} 1 - \alpha\omega_0 t^{\alpha-1} & \text{if} \quad t \leq t_0, \\ 0 & \text{if} \quad t \geq t_0, \end{cases}$$

and

$$\omega''(t) := \begin{cases} -\alpha(\alpha-1)\omega_0 t^{\alpha-2} & \text{if} \quad t \leq t_0, \\ 0 & \text{if} \quad t \geq t_0. \end{cases}$$

As a consequence, we get

$$\omega''(t) \leq 0 \qquad \text{and} \qquad \frac{3}{4} \leq \omega'(t) \leq 1, \quad \forall t \in (0,2). \tag{3.14}$$

Next, in the **Step 2**, we apply Lemma 2.1 in order to ensure the existence of matrices X and Y such that

$$-\left(\frac{1}{\varepsilon} + \|B\|\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} B + \varepsilon B^2 & -B - \varepsilon B^2 \\ -B - \varepsilon B^2 & B + \varepsilon B^2 \end{pmatrix}, \tag{3.15}$$

where B is given by

$$B := J_1\left[\frac{\omega'(|x_0 - y_0|)}{|x_0 - y_0|} I + \left(\omega''(|x_0 - y_0|) - \frac{\omega'(|x_0 - y_0|)}{|x_0 - y_0|}\right)\frac{x_0 - y_0}{|x_0 - y_0|} \otimes \frac{x_0 - y_0}{|x_0 - y_0|}\right],$$

and

$$B^2 = J_1^2\left[\left(\frac{\omega'(|x_0 - y_0|)}{|x_0 - y_0|}\right)^2 I + \left(\left(\omega''(|x_0 - y_0|)\right)^2 - \left(\frac{\omega'(|x_0 - y_0|)}{|x_0 - y_0|}\right)^2\right)\frac{x_0 - y_0}{|x_0 - y_0|} \otimes \frac{x_0 - y_0}{|x_0 - y_0|}\right].$$

Similarly, as was done in **Step 3** of the proof of Lemma 3.1, we have $X-Y \leq 0$. Moreover, we can use (3.14) to obtain

$$\omega''(t) - \frac{\omega'(t)}{t} \leq 0 \qquad \text{and} \qquad \omega''(t) \leq 0,$$

for any $t \in (0,2)$. Therefore, we have

$$\begin{aligned} \|B\| &= J_1 \left\| \frac{\omega'(|x_0-y_0|)}{|x_0-y_0|} I + \left(\omega''(|x_0-y_0|) - \frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right) \right. \\ &\quad \left. \cdot \left(\frac{x_0-y_0}{|x_0-y_0|} \otimes \frac{x_0-y_0}{|x_0-y_0|} \right) \right\| \\ &\leq J_1 \frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}, \end{aligned}$$

and

$$\begin{aligned} \|B^2\| &= J_1^2 \left\| \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2 I + \left(\left(\omega''(|x_0-y_0|)\right)^2 - \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2 \right) \right. \\ &\quad \left. \cdot \left(\frac{x_0-y_0}{|x_0-y_0|} \otimes \frac{x_0-y_0}{|x_0-y_0|} \right) \right\| \\ &\leq J_1^2 \left[\left|\omega''(|x_0-y_0|)\right| + \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right) \right]^2. \end{aligned}$$

By choosing

$$\frac{1}{\varepsilon} := 2J_1 \left[|\omega''(|x_0-y_0|)| + \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right) \right],$$

using the fact that $\omega'(t) \leq 1$, we obtain

$$\begin{aligned} \|X\|, \|Y\| &\leq 4\|B+\varepsilon B^2\| \leq 4\|B\| + \frac{1}{\varepsilon} \\ &\leq 2J_1 \left(3|x_0-y_0|^{-1} + |\omega''(|x_0-y_0|)| \right). \end{aligned} \tag{3.16}$$

Moreover, for $\xi = (x_0-y_0)/|x_0-y_0|$, we obtain

$$\begin{aligned} \langle (X-Y)\xi, \xi \rangle &\leq 4\langle (B+\varepsilon B^2)\xi, \xi\rangle \\ &= 4J_1 \omega''(|x_0-y_0|) + 4J_1^2 \varepsilon \left(\omega''(|x_0-y_0|)\right)^2 \\ &\leq 4J_1 \left(\omega''(|x_0-y_0|) + \frac{1}{2} \left|\omega''(|x_0-y_0|)\right| \right) \\ &= 2J_1 \omega''(|x_0-y_0|). \end{aligned}$$

Since $|\xi| = 0$ we know that $|\xi_{x_0}| = |\nu_{x_0}|$ and $|\xi_{y_0}| = |\nu_{y_0}|$, as a consequence, from **Step 4**, we can take at this point $J_1 \gg 1$ satisfying

$$\frac{J_1}{2} \geq \frac{2J_2 \max\{|x_0-z_0|, |y_0-z_0|\}}{\omega'(|x_0-y_0|)}.$$

Then, using the previous estimate together with the information about $\omega'(t)$ in (3.14), we have

$$\frac{3J_1}{2} \geq |\nu_{x_0}| \geq \frac{3J_1}{8} \tag{3.17}$$

and

$$\frac{3J_1}{2} \geq |\nu_{y_0}| \geq \frac{3J_1}{8}. \tag{3.18}$$

Thus, the change in **Step 5** occurs first when we estimate (I). More precisely,

$$\operatorname{Tr}(A(\nu_{x_0})(X-Y)) \leq 2J_1\omega''(|x_0-y_0|)\min\{1,p-1\}. \tag{3.19}$$

We can use the value of J_1, and the equations (3.17) and (3.18), to obtain

$$\begin{aligned}
|\overline{\nu}_{x_0}-\overline{\nu}_{y_0}| = \left(\frac{\nu_{x_0}}{|\nu_{x_0}|}-\frac{\nu_{y_0}}{|\nu_{y_0}|}\right) &\leq 2\max\left\{\frac{|\nu_{x_0}-\nu_{y_0}|}{|\nu_{x_0}|},\frac{|\nu_{x_0}-\nu_{y_0}|}{|\nu_{y_0}|}\right\}\\
&\leq \frac{J_2}{2}\left(\frac{1}{|\nu_{x_0}|}+\frac{1}{|\nu_{y_0}|}\right)\\
&\leq \frac{8J_2}{3J_1}.
\end{aligned}$$

Therefore, from (3.16) and the previous estimate, we estimate (III) as

$$\begin{aligned}
\operatorname{Tr}((A(\nu_{x_0})-A(\nu_{y_0}))\,Y) &\leq 2d|p-2|\|Y\||\overline{\nu}_{x_0}-\overline{\nu}_{y_0}|\\
&\leq \frac{32J_2d}{3}|p-2|\left(3|x_0-y_0|^{-1}+|\omega''(|x_0-y_0|)|\right).
\end{aligned}$$

The last change needed in **Step 5** is done by using the information in $A2$, *i.e.*, that the map $t\to\Phi(x,t)/t^{\gamma}$ is almost increasing together with the assumption $A1$, $A3$, and equations (3.17) and (3.18). We have

$$\begin{aligned}
&\operatorname{Tr}\left(A(\nu_{x_0})(X+J_2I)\right)-\operatorname{Tr}(A(\nu_{y_0})(Y-J_2I))\\
&\geq -\frac{f(x_0)}{\Phi(x_0,|\nu_{x_0}|)}+\frac{f(y_0)}{\Phi(y_0,|\nu_{y_0}|)}\\
&\geq -\frac{M\|f\|_{L^\infty(B_1)}}{\Phi(x_0,1)|\nu_{x_0}|^{\gamma}}-\frac{M\|f\|_{L^\infty(B_1)}}{\Phi(y_0,1)|\nu_{y_0}|^{\gamma}}\\
&\geq -\frac{MB\|f\|_{L^\infty(B_1)}}{|\nu_{x_0}|^{\gamma}}-\frac{MB\|f\|_{L^\infty(B_1)}}{|\nu_{y_0}|^{\gamma}}\\
&\geq -\frac{2^{\gamma}MB\|f\|_{L^\infty(B_1)}}{(3J_1)^{\gamma}}-\frac{2^{\gamma}MB\|f\|_{L^\infty(B_1)}}{(3J_1)^{\gamma}}\\
&\geq -\frac{2^{\gamma+1}MB\|f\|_{L^\infty(B_1)}}{(3J_1)^{\gamma}}.
\end{aligned}$$

Thereby, in **Step 6**, we have

$$\begin{aligned}
&2J_1\omega''(|x_0-y_0|)\min\{1,p-1\}+2dJ_2\max\{1,p-1\}\\
&\quad+32J_2d|p-2||x_0-y_0|^{-1}+\frac{32J_2d}{3}|p-2||\omega''(|x_0-y_0|)|\\
&\quad+3^{-\gamma}2^{\gamma+1}MB\|f\|_{L^\infty(B_1)}J_1^{-\gamma}\geq 0,
\end{aligned}$$

recalling that $-1<\gamma<0$ and $\omega''(|x_0-y_0|)\le 0$, and knowing that J_1 grows faster than $J_1^{-\gamma}$. Hence, by taking $J_1 \gg 1$ large enough, depending only on $\gamma, \alpha, M, B, p, J_2$ and d, we get a contradiction and conclude the proof. $\square$

Proposition 3.1 (Compactness estimate)**.** *Let $u\in C(B_1)$ be a normalized viscosity solution to* (3.1)*, and assume that $A1-A3$ are in force.*

(1) If $\gamma\ge 0$ and ξ is an arbitrary vector in $\mathbb{R}^d$, then $u\in C^{\alpha}_{loc}(B_1)$, for some universal $\alpha\in(0,1)$. Moreover, there exists a constant $C>0$ such that

$$\|u\|_{C^{\alpha}_{loc}(B_1)}\le C.$$

(2) If $-1<\gamma<0$, then u is a Lipschitz continuous function and

$$\|u\|_{C^{0,1}_{loc}(B_1)}\le C.$$

Proof. The proof of this proposition is a direct consequence of the previous lemmas. More precisely, the first item is a consequence of Lemma 3.1 and Lemma 3.2, and the second item is a consequence of Lemma 3.3. $\square$

The next result provides a type of cancellation law that might be of independent interest.

Proposition 3.2 (Cancellation law)**.** *Let $u\in C(B_1)$ be a viscosity solution to*

$$-\Phi(x,|Du+\xi|)\left[\Delta u+(p-2)\left\langle D^2u\frac{Du+\xi}{|Du+\xi|},\frac{Du+\xi}{|Du+\xi|}\right\rangle\right]=0 \quad in \quad B_1,$$

where ξ is a arbitrary vector in $\mathbb{R}^d$, and consider $\gamma\ge 0$. Then, u solves

$$-\Delta u-(p-2)\left\langle D^2u\frac{Du+\xi}{|Du+\xi|},\frac{Du+\xi}{|Du+\xi|}\right\rangle=0 \quad in \quad B_1. \tag{3.20}$$

Proof. Let us define $v:=u+\xi\cdot x$, and note that v is a viscosity solution to

$$-\Phi(x,|Dv|)\left[\Delta v+(p-2)\left\langle D^2v\frac{Dv}{|Dv|},\frac{Dv}{|Dv|}\right\rangle\right]=0 \quad \text{in} \quad B_1.$$

Considering a test function $\varphi\in C^2(B_1)$ that touches v from above at x_0 and satisfies $|D\varphi(x_0)|\neq 0$, we have

$$-\Phi\big(x,|D\varphi(x_0)|\big)\left[\Delta\varphi(x_0)+(p-2)\left\langle D^2\varphi(x_0)\frac{D\varphi(x_0)}{|D\varphi(x_0)|},\frac{D\varphi(x_0)}{|D\varphi(x_0)|}\right\rangle\right]\ge 0.$$

Since $\Phi(x,|D\varphi(x_0)|)>0$, we obtain

$$-\Delta\varphi(x_0)-(p-2)\left\langle D^2\varphi(x_0)\frac{D\varphi(x_0)}{|D\varphi(x_0)|},\frac{D\varphi(x_0)}{|D\varphi(x_0)|}\right\rangle\ge 0 \quad \text{in} \quad B_1.$$

But, in the homogeneous case, it is sufficient to use test functions that satisfy $D\varphi(x_0)\neq 0$ (see [22, 23]), and we thus conclude that v is a subsolution to

$$-\Delta v-(p-2)\left\langle D^2v\frac{Dv}{|Dv|},\frac{Dv}{|Dv|}\right\rangle=0 \quad \text{in} \quad B_1.$$

Going back to the function u, we get

$$-\Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|}\right\rangle \geq 0 \quad \text{in} \quad B_1,$$

which shows that u is a subsolution to (3.20). A similar argument shows that u is also a supersolution. Therefore, u is a viscosity solution to (3.20). □

The following result is technically pivotal for our analysis.

Proposition 3.3 (Stability)**.** *Let (u_n) be a sequence of normalized viscosity solutions to*

$$\begin{aligned} &-\Phi_n\left(x, |Du_n+\xi_n|\right)\Bigg[\Delta u_n \\ &\quad + (p-2)\left\langle D^2 u_n \frac{Du_n+\xi_n}{|Du_n+\xi_n|}, \frac{Du_n+\xi_n}{|Du_n+\xi_n|}\right\rangle\Bigg] = f_n(x) \quad \textit{in } B_1, \end{aligned} \tag{3.21}$$

where each Φ_n satisfies the assumptions A1 and A2. Assume that there exists a function $u_\infty \in C(B_1)$ such that u_n converges to u_∞ uniformly locally in B_1. Suppose further that $\|f_n\|_{L^\infty(B_1)} \to 0$.

(1) If the sequence $(\xi_n)_{n\in\mathbb{N}}$ is bounded, then u_∞ is a viscosity solution to

$$-\Delta u_\infty - (p-2)\left\langle D^2 u_\infty \frac{Du_\infty+\xi_\infty}{|Du_\infty+\xi_\infty|}, \frac{Du_\infty+\xi_\infty}{|Du_\infty+\xi_\infty|}\right\rangle = 0 \quad \textit{in} \quad B_{8/9}. \tag{3.22}$$

(2) If the sequence $(\xi_n)_{n\in\mathbb{N}}$ is unbounded, then u_∞ is a viscosity solution to

$$-\Delta u_\infty - (p-2)\left\langle D^2 u_\infty e_\infty, e_\infty\right\rangle = 0 \quad \textit{in} \quad B_{8/9}, \tag{3.23}$$

with $|e_\infty| = 1$.

Proof. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a quadratic polynomial that touches u_∞ from above at $x_0 \in B_{8/9}$. Let us assume, without loss of generality, that $|x_0| = u_\infty(x_0) = 0$, and that the polynomial function φ is given by

$$\varphi(x) \coloneqq b\cdot x + \frac{1}{2}\langle Mx, x\rangle .$$

By hypotheses, we know that $u_n \to u_\infty$, and as a consequence, we can find a sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ given by

$$\varphi_n(x) \coloneqq u_n(x_n) + b\cdot(x-x_n) + \frac{1}{2}\langle M(x-x_n), x-x_n\rangle ,$$

and a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that φ_n converges to φ and x_n converges to 0. Moreover, for each $n\in\mathbb{N}$, the function φ_n touches u_n from above at the point x_n.

At this point, let us assume that the sequence $(\xi_n)_{n\in\mathbb{N}}$ is bounded. Consequently, up to a subsequence, we have that ξ_n converges to some ξ_∞ in

$B_{8/9}$. Since the function u_n is a viscosity solution to (3.21), we have that φ_n satisfies

$$-\Phi_n(x_n, |b+\xi_n|)\left[\mathrm{Tr}(M) + (p-2)\left\langle M\frac{b+\xi_n}{|b+\xi_n|}, \frac{b+\xi_n}{|b+\xi_n|}\right\rangle\right] \leq f_n(x_n). \tag{3.24}$$

First, recall that in the homogeneous case, it is sufficient to use test functions with non-zero gradient, which implies we can assume $|b+\xi_\infty| > 0$. Hence, for n large enough, we can assume $|b+\xi_n| \geq 2^{-1}|b+\xi_\infty| > 0$. Thus, using the hypotheses that each Φ_n satisfies $A1$ and $A2$, we obtain

$$\Phi_n(x_n, |b+\xi_n|) \geq (MB)^{-1}|b+\xi_n|^\gamma \geq (MB)^{-1}2^{-\gamma}|b+\xi_\infty|^\gamma,$$

when $|b+\xi_\infty| \geq 1$, and

$$\Phi_n(x_n, |b+\xi_n|) \geq (MB)^{-1}|b+\xi_n|^\mu \geq (MB)^{-1}2^{-\mu}|b+\xi_\infty|^\mu,$$

when $|b+\xi_\infty| < 1$. Therefore,

$$-\left[\mathrm{Tr}(M) + (p-2)\left\langle M\frac{b+\xi_n}{|b+\xi_n|}, \frac{b+\xi_n}{|b+\xi_n|}\right\rangle\right] \leq \frac{f_n(x_n)}{\Phi_n(x_n, |b+\xi_n|)},$$

which implies,

$$-\left[\mathrm{Tr}(M) + (p-2)\left\langle M\frac{b+\xi_n}{|b+\xi_n|}, \frac{b+\xi_n}{|b+\xi_n|}\right\rangle\right] \leq \frac{2^\mu MB\|f_n\|_{L^\infty(B_1)}}{\max\{|b+\xi_\infty|^\gamma, |b+\xi_\infty|^\mu\}},$$

By passing the limit $n \to \infty$, that u_∞ is a viscosity subsolution to

$$-\Delta u_\infty - (p-2)\left\langle D^2u_\infty\frac{Du_\infty+\xi_\infty}{|Du_\infty+\xi_\infty|}, \frac{Du_\infty+\xi_\infty}{|Du_\infty+\xi_\infty|}\right\rangle = 0 \quad \text{in} \quad B_{8/9}.$$

A similar argument shows that u_∞ is also a viscosity supersolution to (3.22), and hence a viscosity solution to (3.22).

On the other hand, assume that the sequence $(\xi_n)_{n\in\mathbb{N}}$ is unbounded. Let us define a normalized sequence $(e_n)_{n\in\mathbb{N}}$ given by $e_n := \xi_n/|\xi_n|$, and observe that e_n, up to a subsequence, converges to some e_∞ in $B_{8/9}$. Moreover, since ξ_n is unbounded we have

$$\frac{b}{|\xi_n|} \neq -e_n \qquad \text{and} \qquad |b+\xi_n| > 1,$$

for n large enough. Since u_n is a viscosity solution to (3.21), we have that φ_n satisfies (3.24), and as a consequence we get

$$-\mathrm{Tr}(M) - (p-2)\left\langle M\frac{b|\xi_n|^{-1}+e_n}{|b|\xi_n|^{-1}+e_n|}, \frac{b|\xi_n|^{-1}+e_n}{|b|\xi_n|^{-1}+e_n|}\right\rangle \leq \frac{f_n(x_n)}{\Phi_n(x_n, |b+\xi_n|)}$$

in B_1. As before, by using $A1$ and $A2$, we obtain

$$\begin{aligned} &-\mathrm{Tr}(M) - (p-2)\left\langle M\frac{b|\xi_n|^{-1}+e_n}{|b|\xi_n|^{-1}+e_n|}, \frac{b|\xi_n|^{-1}+e_n}{|b|\xi_n|^{-1}+e_n|}\right\rangle \\ &\leq \frac{MB\|f_n\|_{L^\infty(B_1)}}{|b+\xi_n|^\gamma} \leq MB\|f_n\|_{L^\infty(B_1)}. \end{aligned}$$

Once more, by passing to the limit $n \to \infty$, we obtain that u_∞ is a subsolution to (3.23). With an analogous argument, we can prove that u_∞ is a viscosity supersolution to (3.23). This concludes the proof. □

4. Gradient estimates

In this section, we prove Theorem 2.1, showing that viscosity solutions to (1.1) are locally of class $C^{1,\alpha}$, where

$$\alpha = \begin{cases} \min\left\{\alpha_0^-, \frac{1}{1+\mu}\right\} & \text{if } \gamma \geq 0 \\ \min\left\{\alpha_0^-, \frac{1}{1+\mu-\gamma}\right\} & \text{if } -1 < \gamma < 0. \end{cases} \tag{4.1}$$

We start with an approximation lemma, connecting solutions of our equation to solutions of the $p-$Laplace equation. Recall that α_0 accounts for the $C^{1,\alpha_0}-$regularity for $-\Delta_p u = 0$.

Lemma 4.1 (Approximation lemma I)**.** *Let $u \in C(B_1)$ be a normalized viscosity solution to* (3.1)*, and assume $A1-A3$ are in force. Suppose further that $\gamma \geq 0$. Given $\delta > 0$, there exists $\varepsilon > 0$ such that, if*

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon,$$

then we can find a function $h \in C^{1,\alpha_0}_{loc}(B_1)$ satisfying

$$\|u - h\|_{L^\infty(B_{9/10})} \leq \delta.$$

Proof. We argue by contradiction. Suppose that, for $\delta_0 > 0$ and sequences $(\Phi_n)_{n\in\mathbb{N}}$, $(\xi_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$, and $(f_n)_{n\in\mathbb{N}}$, we have

i) each f_n satisfies $A3$, and $\|f_n\|_{L^\infty} \to 0$;
ii) each $\Phi_n \in C(B_1) \times C([0,\infty])$ satisfies $A1$ and $A2$;
iii) each $u_n \in C(B_1)$ is a viscosity solution to

$$\begin{aligned} &-\Phi_n\left(x, |Du_n + \xi_n|\right)\Bigg[\Delta u_n \\ &\qquad + (p-2)\left\langle D^2 u_n \frac{Du_n + \xi_n}{|Du_n + \xi_n|}, \frac{Du_n + \xi_n}{|Du_n + \xi_n|}\right\rangle\Bigg] = f_n(x) \quad \text{in } B_1; \end{aligned}$$

but, nevertheless,

$$\|u_n - h\|_{L^\infty(B_{9/10})} > \delta_0, \tag{4.2}$$

for all $h \in C^{1,\alpha_0}(B_{9/10})$. Proposition 3.1 ensures the existence of a function $u_\infty \in C^\alpha(B_{9/10})$ such that, up to a subsequence, u_n converge to u_∞ locally uniformly. At this point, we split the proof into two cases.

Case 1: *The sequence $(\xi_n)_{n\in\mathbb{N}}$ is bounded.*

In this case, we can assume that $\xi_n \to \xi_\infty$, for some $\xi_\infty \in \mathbb{R}^d$. From the stability result, Proposition 3.3, we have that u_∞ is a viscosity solution to

$$-\Delta u_\infty - (p-2)\left\langle D^2 u_\infty \frac{Du_\infty + \xi_\infty}{|Du_\infty + \xi_\infty|}, \frac{Du_\infty + \xi_\infty}{|Du_\infty + \xi_\infty|}\right\rangle = 0 \quad \text{in} \quad B_{8/9}.$$

Hence, $u \in C^{1,\alpha_1}_{\text{loc}}(B_{8/10})$, for some $\alpha_1 \in (0,1)$ (see, for instance, [4, Lemma 3.2]). Therefore, choosing $h \equiv u_\infty$, we get a contradiction with (4.2) for n sufficiently large.

Case 2: *The sequence $(\xi_n)_{n\in\mathbb{N}}$ is unbounded.*

Once more, Proposition 3.3 shows that u_∞ solves

$$-\Delta u_\infty - (p-2)\left\langle D^2 u_\infty e_\infty, e_\infty \right\rangle = 0 \quad \text{in} \quad B_{8/9},$$

in the viscosity sense, where e_∞ is a unitary vector. Since the operator above is linear and uniformly elliptic, and has constant coefficients, it follows from [13, Corollary 5.7] that there exists a universal $\alpha_2 \in (0,1)$ such that $u_\infty \in C^{1,\alpha_2}(B_{8/10})$. Again, by choosing $h \equiv u_\infty$, we obtain a contradiction with (4.2) for n large enough. □

By using Lemma 4.1, we can find an affine function that is close to u, in a suitable sense, on a fixed small scale.

Proposition 4.1. *Let $u \in C(B_1)$ be a normalized viscosity solution to* (3.1)*. Suppose that $A1 - A3$ are in force. Suppose further that $\gamma \geq 0$. There exists $\varepsilon > 0$ such that, if*

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon,$$

then we can find an affine function $\ell(x) := a + b \cdot x$, with $|a|, |b| \leq C$, for a universal constant $C > 0$, such that

$$\|u - \ell\|_{L^\infty(B_\rho)} \leq \rho^{1+\alpha},$$

for every $\alpha \in (0, \alpha_0)$.

Proof. Let us fix $\delta > 0$ and $\rho > 0$ sufficiently small to be chosen later. From Lemma 4.1, there exists $\varepsilon > 0$ and a function $h \in C^{1,\alpha_0}_{loc}(B_1)$ satisfying

$$\|u - h\|_{L^\infty(B_{9/10})} \leq \delta.$$

Since $h \in C^{1,\alpha_0}$, there exists a positive constant C such that

$$\|h(x) - h(0) - Dh(0) \cdot x\|_{L^\infty(B_\rho)} \leq C\rho^{1+\alpha_0}.$$

Thereby, we can define the affine function $\ell := h(0) + Dh(0) \cdot x$ and use the triangle inequality to obtain

$$\begin{aligned}\|u - \ell\|_{L^\infty(B_\rho)} &\leq \|u - h\|_{L^\infty(B_\rho)} + \|h - \ell\|_{L^\infty(B_\rho)} \\ &\leq \delta + C\rho^{1+\alpha_0}.\end{aligned}$$

Now, for $0 < \alpha < \alpha_0$, we make the universal choices

$$\rho := \left(\frac{1}{2C}\right)^{\frac{1}{\alpha_0 - \alpha}} \qquad \text{and} \qquad \delta := \frac{\rho^{1+\alpha}}{2},$$

fixing also the value of $\varepsilon > 0$ through Lemma 4.1. Thus, we get

$$\|u - \ell\|_{L^\infty(B_\rho)} \leq \rho^{1+\alpha}.$$

□

We now iterate the previous result to find a sequence of affine functions approximating u on every small scale.

Proposition 4.2. *Let $u \in C(B_1)$ be a normalized viscosity solution to* (1.1) *and assume $A1 - A3$ are in force. There exists $\varepsilon > 0$ such that, if*

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon,$$

then there exists a sequence of affine function $(\ell_k)_{k\in\mathbb{N}}$, such that, for each $k \in \mathbb{N}$, ℓ_k is of the form

$$\ell_k(x) := a_k + b_k \cdot x,$$

with

$$|a_{k+1} - a_k| + \rho^k |b_{k+1} - b_k| \leq C\rho^{k(1+\alpha)},$$

for a universal constant $C > 0$, and

$$\|u - \ell_k\|_{L^\infty(B_{\rho^k})} \leq \rho^{k(1+\alpha)}.$$

Proof. We will split the proof into two cases, depending on the sign of the constant γ.

Case 1: Assume first that $\gamma \geq 0$. As usual, we will argue inductively. For $k = 1$, we set $\ell_2(x) = \ell_1(x) = \ell(x)$, where $\ell(\cdot)$ is the affine function from Proposition 4.1, and the result follows from it. Now, suppose that the conclusion holds for $k = 1, \ldots, n$, and let us prove the case $k = n + 1$. We introduce the auxiliary function $v_n : B_1 \to \mathbb{R}$, defined by

$$v_n(x) := \frac{u(\rho^n x) - \ell_n(\rho^n x)}{\rho^{n(1+\alpha)}}.$$

Notice that, from the induction hypothesis, we have $\|v_n\|_{L^\infty(B_1)} \leq 1$; moreover, v_n solves

$$\begin{aligned} &-\Phi_n\left(x, \left|Dv_n + \rho^{-n\alpha} b_n\right|\right) \Bigg[\Delta v_n \\ &\qquad + (p-2)\left\langle D^2 v_n \frac{Dv_n + \rho^{-n\alpha} b_n}{|Dv_n + \rho^{-n\alpha} b_n|}, \frac{Dv_n + \rho^{-n\alpha} b_n}{|Dv_n + \rho^{-n\alpha} b_n|} \right\rangle \Bigg] = f_n(x) \quad \text{in } B_1, \end{aligned}$$

in the viscosity sense, where

$$\Phi_n(x,t) := \frac{\Phi(\rho^n x, \rho^{n\alpha} t)}{\Phi(\rho^n x, \rho^{n\alpha})} \qquad \text{and} \qquad f_n(x) := \frac{\rho^{n(1-\alpha)} f(\rho^n x)}{\Phi(\rho^n x, \rho^{n\alpha})}. \tag{4.3}$$

Notice that $\Phi_n(\cdot,\cdot)$ and $f_n(\cdot)$ satisfy the hypotheses of Proposition 4.1. Indeed, we have

$$\begin{aligned} \|f_n\|_{L^\infty(B_1)} &\leq \frac{\rho^{n(1-\alpha)} M \|f\|_{L^\infty(B_1)}}{\Phi(x,1)\rho^{n\alpha\mu}} \\ &\leq M\|f\|_{L^\infty(B_1)} \rho^{n[1-\alpha(1+\mu)]} \\ &\leq M\varepsilon, \end{aligned}$$

where we have used the value of α chosen in (4.1). Moreover, a simple computation shows that the map $t \mapsto \Phi_n(x,t)/t^\gamma$ is almost increasing, the map $t \mapsto \Phi_n(x,t)/t^\mu$ is almost decreasing, and $\Phi_n(x,1) = 1$ for every x in B_1. Therefore, we can apply Lemma 4.1 to the function v_n, and obtain

$$\|v_n - \ell\|_{L^\infty(B_\rho)} \leq \rho^{1+\alpha}.$$

Rescaling back to the function u, we get

$$\|u - \ell_{n+1}\|_{L^\infty(B_{\rho^{n+1}})} \leq \rho^{(n+1)(1+\alpha)},$$

where

$$\ell_{n+1}(x) \coloneqq \ell_n(x) + \rho^{n(1+\alpha)}\ell(\rho^{-n}x).$$

Moreover, we can verify that

$$|a_{n+1} - a_n| + \rho^n |b_{n+1} - b_n| \leq C\rho^{n(1+\alpha)},$$

which concludes the inductive process.

Case 2: Now, assume $-1 < \gamma < 0$. Recall that from Proposition 2.1 we have that u is a viscosity solution to

$$\overline{\Phi}(x, |Du|)\Delta_p^{\mathrm{N}} u = \overline{f}(x) \quad \text{in} \quad B_1, \tag{4.4}$$

where

$$\overline{\Phi}(x,t) \coloneqq t^{-\gamma}\Phi(x,t) \quad \text{and} \quad \overline{f}(x) \coloneqq |Du|^{-\gamma} f(x),$$

for $x \in B_1$ and $t > 0$. Now, $A1$ and $A2$ imply that the map $t \mapsto \overline{\Phi}(x,t)$ is almost increasing and that the map $t \mapsto \overline{\Phi}(x,t)/t^{\mu-\gamma}$ is almost decreasing and since $\overline{\Phi}(x,1) = \Phi(x,1)$, we have

$$\overline{\Phi}(x,1) = 1 \quad \text{for all} \quad x \in B_1.$$

Moreover, we obtain from Lemma 3.3 that u is a Lipschitz continuous function, and as a consequence, the gradient Du is bounded almost everywhere. Thus, we can estimate

$$\begin{aligned}
\|\overline{f}(x)\|_{L^\infty(B_{9/10})} &= |Du|^{-\gamma} \cdot \|f\|_{L^\infty(B_{9/10})} \\
&\leq \|f\|_{L^\infty(B_1)} \cdot \|u\|^{-\gamma}_{C^{0,1}(B_{9/10})}.
\end{aligned}$$

Therefore, $\bar{\Phi}$ and $\bar{f}$ satisfy the assumptions $A1$-$A3$, and we can apply the same inductive argument as in the **Case 1** to the equation (4.4). Once again, we define the function

$$v_n(x) \coloneqq \frac{u(\rho^n x) - \ell_n(\rho^n x)}{\rho^{n(1+\alpha)}},$$

which solves, in the viscosity sense,

$$\begin{aligned}
&\overline{\Phi}_n\left(x, \left|Dv_n + \rho^{-n\alpha} b_n\right|\right) \Bigg[\Delta v_n \\
&\qquad + (p-2)\left\langle D^2 v_n \frac{Dv_n + \rho^{-n\alpha} b_n}{|Dv_n + \rho^{-n\alpha} b_n|}, \frac{Dv_n + \rho^{-n\alpha} b_n}{|Dv_n + \rho^{-n\alpha} b_n|}\right\rangle\Bigg] = \overline{f}_n(x) \quad \text{in } B_1,
\end{aligned}$$

where

$$\overline{\Phi}_n(x,t) \coloneqq \frac{\overline{\Phi}(\rho^n x, \rho^{n\alpha} t)}{\overline{\Phi}(\rho^n x, \rho^{n\alpha})} \quad \text{and} \quad \overline{f}_n(x) \coloneqq \frac{\rho^{n(1-\alpha)}\overline{f}(\rho^n x)}{\overline{\Phi}(\rho^n x, \rho^{n\alpha})}.$$

Observe that since the map $t \mapsto \overline{\Phi}(x,t)/t^{\mu-\gamma}$ is almost decreasing, we have

$$\begin{aligned}
\|\overline{f}_n\|_{L^\infty(B_9/10)} &= \frac{\|\overline{f_n}\|_{L^\infty(B_{9/10})}}{\overline{\Phi}(\rho^n x, \rho^{n\alpha})} \\
&\leq \frac{\rho^{n(1-\alpha)} M \|\overline{f}\|_{L^\infty(B_{9/10})}}{\overline{\Phi}(\rho^n x, 1)\rho^{n\alpha(\mu-\gamma)}} \\
&\leq M\|\overline{f}\|_{L^\infty(B_{9/10})}\rho^{n[1-\alpha(1+\mu-\gamma)]} \\
&\leq M\varepsilon,
\end{aligned}$$

where we have again used the value of α chosen in (4.1). Now, note that

$$\overline{\Phi}_n(x,1) = \frac{\overline{\Phi}(\rho^n x, \rho^{n\alpha})}{\overline{\Phi}(\rho^n x, \rho^{n\alpha})} = 1, \quad \text{for all} \quad x \in B_1,$$

the map $t \mapsto \overline{\Phi}_n(x,t)/t^{\gamma}$ is almost increasing, and the map $t \mapsto \overline{\Phi}_n(x,t)/t^{\mu-\gamma}$ is almost decreasing. Therefore, we can apply the Lemma 4.1 and repeat the remainder of **Case 1** to obtain the desired result. □

We are now ready to provide proof of our first main result.

Proof of the Theorem 2.1. To begin with, Proposition 4.2 ensures the existence of a sequence of affine functions $(\ell_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$,

$$|a_{n+1} - a_n| + \rho^n|b_{n+1} - b_n| \leq C\rho^{n(1+\alpha)}$$

and

$$\|u - \ell_n\|_{L^\infty(B_{\rho^n})} \leq \rho^{n(1+\alpha)}.$$

Therefore, the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences, and as a consequence, there exist a_∞ and b_∞ such that $a_n \to a_\infty$ and $b_n \to b_\infty$, as $n \to \infty$. Let us define the affine function $\ell_\infty(x) := a_\infty + b_\infty \cdot x$, and observe that

$$|a_n - a_\infty| \leq C\rho^{n(1+\alpha)} \qquad \text{and} \qquad |b_n - b_\infty| \leq C\rho^{n\alpha}.$$

At this point, for any $0 < r \ll 1$ sufficiently small, we can take $n \in \mathbb{N}$ such that $\rho^{n+1} < r \leq \rho^n$. Consequently,

$$\begin{aligned}
\|u - \ell_\infty\|_{L^\infty(B_r)} &\leq \|u - \ell_n\|_{L^\infty(B_{\rho^n})} + \|\ell_n - \ell_\infty\|_{L^\infty(B_{\rho^n})} \\
&\leq \rho^{n(1+\alpha)} + |a_n - a_\infty| + |b_n - b_\infty| \cdot |x| \\
&\leq \rho^{n(1+\alpha)} + C\rho^{n(1+\alpha)} + C\rho^{n\alpha} \cdot \rho^n \\
&\leq C\rho^{n(1+\alpha)} \\
&\leq Cr^{1+\alpha},
\end{aligned}$$

which implies the $C^{1,\alpha}-$regularity. □

5. Sobolev estimates

In this section, we derive Sobolev estimates for viscosity solutions to the $p-$Laplace equation

$$-\Delta_p u = 0 \quad \text{in} \quad B_1. \tag{5.1}$$

Recall that, from [22], viscosity solutions to the equation above are also viscosity solutions to

$$-\Delta_p^{\mathrm{N}} u = 0 \quad \text{in} \quad B_1. \tag{5.2}$$

From now on, we will work with this equation.

Let $u \in C(B_1)$ be a viscosity solution to (5.2). We argue similarly to [3, 5] and consider the regularized operator

$$\begin{cases} -\Delta v_\varepsilon - (p-2)\dfrac{\langle D^2 v_\varepsilon D v_\varepsilon, D v_\varepsilon \rangle}{|Dv_\varepsilon|^2+\varepsilon^2} + \lambda v_\varepsilon = \lambda u & \text{in} \quad B_{3/4} \\ v_\varepsilon = u & \text{on} \quad \partial B_{3/4}. \end{cases} \tag{5.3}$$

for some $\lambda > 0$. We added the zero-order term in the equation above to avoid issues with the uniqueness of solutions. Now, since we have a uniformly elliptic operator with no singularities, classical results ensure that solutions v_ε are C^2 classical solutions (see, for instance [19]). It is immediate to check that v_ε is bounded, independently of ε, by using the maximum principle.

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let us consider the operator

$$F(D^2 u, x) := \Delta u + (p-2)\frac{\langle D^2 u D v_\varepsilon, D v_\varepsilon \rangle}{|Dv_\varepsilon|^2+\varepsilon^2},$$

and notice that it is uniformly elliptic, with ellipticity constants

$$\lambda = \min\{1, p-1\} \qquad \text{and} \qquad \Lambda = \max\{1, p-1\}.$$

Moreover, the oscillation of F,

$$\theta_F(x) := \sup_{M \in \mathcal{S}(d)} \frac{F(M,x) - F(M,0)}{\|M\|},$$

is such that

$$\theta_F(x) \leq 2|p-2|.$$

Since, from (5.3), v_ε solves

$$F(D^2 v_\varepsilon, x) = \lambda(v_\varepsilon - u) \in L^\infty(B_{3/4})$$

(recall that both u and v_ε are bounded), we can apply classic $W^{2,q}-$estimates (see [12], [13]) to guarantee that for a given $q \in [1, \infty)$, there exist $\tilde{\varepsilon} := \tilde{\varepsilon}(q) > 0$, such that if

$$|p-2| \leq \tilde{\varepsilon},$$

then $v_\varepsilon \in W^{2,q}$. Moreover, there exists a universal constant $C > 0$ such that

$$\|v_\varepsilon\|_{W^{2,q}(B_{1/2})} \leq C\left(\|v_\varepsilon\|_{L^\infty(B_{3/4})} + \lambda\|v_\varepsilon - u\|_{L^\infty(B_{3/4})}\right). \tag{5.4}$$

Finally, from the fact that $v_\varepsilon \to u$ locally uniformly, we also have that $u \in W^{2,q}$ and satisfies the estimate

$$\|u\|_{W^{2,q}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

□

Finally, we prove the corollary.

Proof of Corollary 2.1. The proof follows directly from Theorem 2.1 and Theorem 2.2, since for given $-1 < \gamma \leq \mu$ and $|p-2| \leq \tilde{\varepsilon}$, we can take $\alpha_0 > \frac{1}{1+\mu}$ if $\gamma \geq 0$ and $\alpha_0 > \frac{1}{1+\mu-\gamma}$ if $\gamma < 0$. □

Acknowledgments. This publication is based upon work supported by King Abdullah University of Science and Technology (KAUST) under Award No. ORFS-CRG12-2024-6430, and was partially financed by CAPES—Brazil, under Finance Code 001. C. Alcantara was partially supported by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) – Brasil and by Stone Instituição de Pagamento S.A., under the Project StoneLab. M. Santos is partially supported by the Portuguese government through FCT-Fundação para a Ciência e a Tecnologia, I.P., under the projects https://doi.org/10.54499/UID/04561/2025 and the Scientific Employment Stimulus - Individual Call (CEEC Individual), DOI identifier https://doi.org/10.54499/2023.08772.CEECIND/CP2831/CT0003.

Data Availability: Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

Conflict-of-Interest Statement: The authors have no conflict of interest to declare for this article.

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