TAXICAB AND EUCLIDEAN METRICS ON PERMUTATIONS WITH PRESCRIBED PEAK SET

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Abstract. We study the behavior of the taxicab and Euclidean metrics on permutations with a prescribed peak set. Given a set $S \subseteq \{1, \ldots, n\}$, we compute the diameter of the set $P(S; n)$ of permutations having peak set S with respect to these metrics. For the taxicab metric, we determine the pairs of permutations attaining the diameter and provide a characterization of those peak sets S for which the diameter of $P(S; n)$ is maximal or minimal, in terms of suitable partitions of $[n]$. For the Euclidean metric, the diameter of $\mathfrak{S}_n$ is attained by complementary permutations. We analyze how the peak set restriction affects this phenomenon and show that the diameter of $P(S; n)$ depends on both the number of peaks and their relative positions. As a consequence, we obtain an explicit formula for the Euclidean diameter of $P(S; n)$. Finally, we show that these results admit natural extensions to signed permutations of type B, yielding explicit formulas for the diameters of the corresponding peak classes.

1. Introduction

Permutations are fundamental combinatorial objects with a rich structure that has been studied from many perspectives. In this work, we focus on permutations as elements of the symmetric group, viewing it as a metric subspace of $\mathbb{R}^n$ under the taxicab and the Euclidean metric.

Let $\mathfrak{S}_n$ denote the symmetric group on $1, 2, \ldots, n$, consisting of all permutations of n elements. Given a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$, an index $i \in \{2, 3, \ldots, n-1\}$ is called a *peak* of σ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$. The *peak set* of σ is defined as

$$\mathrm{Peak}(\sigma) = \{i \in \{2, \ldots, n-1\} : \sigma_{i-1} < \sigma_i > \sigma_{i+1}\},$$

and for a fixed set $S \subseteq \{2, \ldots, n-1\}$, we denote by $P(S; n)$ the collection of all permutations in $\mathfrak{S}_n$ whose peak set is exactly S. A set S is called *admissible* if $P(S; n) \neq \emptyset$. For example, if $\sigma = 5\mathbf{8}32\mathbf{4}1\mathbf{7}6 \in S_8$, then $\mathrm{Peak}(\sigma) = \{2, 5, 7\}$.

The sets $P(S; n)$ were first studied by Nyman [12], who showed that sums of permutations sharing the same peak set form a subalgebra of the group algebra of $\mathfrak{S}_n$ over $\mathbb{Q}$. This algebraic perspective was subsequently enriched by Billey, Burdzy, and Sagan [3], who established the remarkable enumerative formula

$$|P(S; n)| = 2^{n-|S|-1} p_S(n),$$

where $p_S(n)$ is a polynomial in n, now known as the *peak polynomial* of S. The study of peak polynomials has since generated a substantial body of work, addressing questions of positivity, interlacing of roots, and combinatorial interpretations [2, 5, 8, 13]. Related problems have also been investigated in the setting of signed permutations and other Coxeter groups [1, 6].

Key words and phrases. Taxicab metric, euclidean metric, peak set, diameter.

The authors acknowledge partial financial support by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) under the scope of the projects UID/00324/2025 (https://doi.org/10.54499/UID/00324/2025) (Centre for Mathematics of the University of Coimbra).

Independently, permutations have long served as a natural model for ranking ordered collections of objects, and the notion of distance between permutations has been extensively investigated [4, 7]. Different metrics provide distinct ways of quantifying how far two permutations are from one another, often revealing important structural properties of the symmetric group. Classical metrics on $\mathfrak{S}_n$ include the Hamming distance, which counts the number of different coordinates; the Kendall–Tau distance, which measures the number of different pairwise inversions; and the ℓ_∞ (Chebyshev) metric, which records the largest absolute coordinate discrepancy between two permutations. More recently, these metrics have been studied in the context of permutations and signed permutations with a prescribed peak set, partly motivated by applications to data representation and information encoding problems [9, 10, 11]. In particular, a natural problem is to determine the minimum and maximum possible values that these metrics can attain on subsets of $\mathfrak{S}_n$ consisting of permutations with a prescribed peak set [1, 7].

In this article, we study two further metrics on sets of permutations with prescribed peak sets: the *Euclidean metric* and the *taxicab* (or *Manhattan*) *metric*. Recall that for $\sigma, \tau \in \mathfrak{S}_n$, the taxicab distance is defined by

$$d_1(\sigma,\tau) = \sum_{i=1}^{n} |\sigma_i - \tau_i|,$$

while the Euclidean distance is given by

$$d_2(\sigma,\tau) = \left(\sum_{i=1}^{n} (\sigma_i - \tau_i)^2\right)^{1/2}.$$

These metrics on $\mathfrak{S}_n$ have proved useful in the study of extremal distance problems, permutation statistics, and the geometry of permutation classes.

Both metrics belong to the family of ℓ_p distances on $\mathbb{R}^n$ restricted to the finite set $\mathfrak{S}_n$. They capture complementary aspects of the discrepancy between two permutations: the taxicab metric measures the total absolute entrywise deviation, whereas the Euclidean metric penalizes large individual deviations more strongly through quadratic aggregation.

In this work, we investigate the metric structure of the sets $P(S;n)$ under the taxicab and Euclidean metrics. More precisely, we study the extremal values of these distances on the set $P(S;n)$ of permutations with prescribed peak set S. For the taxicab metric, we characterize the admissible peak sets for which the diameter is maximal and relate this problem to suitable balanced decompositions of the peak set. For the Euclidean metric, we obtain an explicit formula for the diameter in terms of the number of peaks and the number of blocks in the decomposition of S. Surprisingly, the two metrics exhibit markedly different behaviour: while the taxicab diameter depends on the existence of certain balanced compositions of n, the Euclidean diameter is determined solely by the parameters S and b, where b denotes the number of blocks in the decomposition of S.

Finally, in the last section, we extend our analysis to signed permutations of the Coxeter group of type B. Exploiting the close relationship between peak sets in $\mathfrak{S}_n$ and $\mathfrak{S}_n^B$, we derive explicit formulas for the diameters of the corresponding peak classes $P_B(S;n)$ under the taxicab and Euclidean metrics.

2. Preliminaries

In this section, we introduce the notation and concepts used throughout the paper. Given a positive integer $n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$. A partition of $[n]$ is a collection of non-empty disjoint subsets of $[n]$, whose union is $[n]$. A composition of n is an ordered sequence of positive integers $(\lambda_1, \dots, \lambda_m)$ such that

$$n = \lambda_1 + \cdots + \lambda_m .$$

To each composition of n, we can naturally associate a partition of $[n]$ into blocks of consecutive integers.

Let $\mathfrak{S}_n$ denote the symmetric group on $[n]$. A permutation $\sigma \in \mathfrak{S}_n$ is written in one-line notation as

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_n,$$

where $\sigma(i) = \sigma_i$ for each $i \in [n]$. An index $i \in \{2, \dots, n-1\}$ is called a *peak* of σ if

$$\sigma_{i-1} < \sigma_i > \sigma_{i+1}.$$

The set of all peaks of σ is denoted by $\mathrm{Peak}(\sigma)$.

Definition 2.1. Given an integer $n \geq 3$ and a set $S \subseteq \{2, 3, \dots, n-1\}$, we denote the collection of all permutations in $\mathfrak{S}_n$ that possess the same peak set S by

$$P(S; n) = \{\sigma \in \mathfrak{S}_n \mid Peak(\sigma) = S\} \subseteq \mathfrak{S}_n.$$

The set S is said to be an *admissible peak set* whenever $P(S; n) \neq \emptyset$.

Note that if $S \subseteq \{2, \dots, n-1\}$ has no consecutive elements, we can construct a permutation with peak set S by assigning the largest available values to the positions indexed by S, and filling the remaining positions with the smallest values in increasing order. It follows that a set of peaks S is admissible if and only if $S \subseteq \{2, \dots, n-1\}$ and there are no consecutive integers in S.

For permutations $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ and $\tau = \tau_1 \tau_2 \cdots \tau_n$ in $\mathfrak{S}_n$, the d_k metric (or ℓ_k-distance) is defined by

$$d_k(\sigma, \tau) = \left(\sum_{i=1}^{n} |\sigma_i - \tau_i|^k \right)^{1/k}.$$

In this paper, we will only be interested in metrics d_1, the taxicab distance, and d_2, the Euclidean distance.

Given a subset $A \subseteq \mathfrak{S}_n$, its diameter with respect to the metric d_k is

$$\mathrm{diam}_k(A) = \max\{d_k(\sigma, \tau) : \sigma, \tau \in A\}.$$

The following lemma shows that the metrics d_1 and d_2 are invariant under right multiplication in $\mathfrak{S}_n$.

Lemma 2.2. *Let $\sigma, \tau \in \mathfrak{S}_n$. Then, for any $\pi \in \mathfrak{S}_n$,*

$$d_k(\sigma, \tau) = d_k(\sigma\pi, \tau\pi)$$

and consequently $d_k(\sigma, \tau) = d_k(e, \tau\sigma^{-1})$, where e is the identity.

Proof. Since π is a permutation of $[n]$, right multiplication by π simply relabels the values of σ and τ. Consequently, the multiset

$$\big\{|\sigma(j) - \tau(j)|, j \in [n]\big\}$$

coincides with

$$\big\{(\sigma\pi)(j) - (\tau\pi)(j)|, j \in [n]\big\}.$$

Since d_k depends only on the sum of the k-powers of these quantities, it follows that

$$d_k(\sigma, \tau) = d_k(\sigma\pi, \tau\pi),$$

from which the second identity follows. $\square$

If $\tau \in P(S;n)$ then the permutation τ' obtained by swapping the integers 1 and 2 is still a permutation in $P(S;n)$ with $d_k(\tau,\tau') = 2^{1/k}$. Since the distance between two permutations cannot be 1, because they differ in at least two positions, then for any admissible peak set S, the minimum value for the distance between two different permutations is

$$\min\{d_k(\sigma,\tau) : \sigma,\tau \in P(S;n) \text{ and } \sigma \neq \tau\} = 2^{1/k}\,.$$

To obtain a formula for the maximum values, *i.e.* for the diameter of $P(S;n)$ under the metrics d_1 and d_2, we need to study both cases separately and look into the structure of an admissible peak set, which will be done in the next two sections.

Since an admissible peak set S has no consecutive integers, it can be uniquely written as a partition whose blocks are maximal subsets of S consisting of *consecutive admissible peaks*. That is

$$S = S_1 \cup S_2 \cup \cdots \cup S_b, \tag{2.1}$$

where each block S_i is written as

$$S_i = \{t_i, t_i+2, t_i+4, \ldots, t_i+2u_i\},$$

with $t_i + 2u_i < t_{i+1} - 2$ for all i. We call (2.1) the *decomposition* of S with b blocks. Example 2.1 below illustrates this concept.

Proposition 2.3. *The number of admissible peak sets of $\mathfrak{S}_n$ is equal to the number of compositions of n into odd parts. In particular, if $a(n)$ denotes the number of admissible peak sets in $\mathfrak{S}_n$, then $(a(n))_{n\geq 1}$ is the Fibonacci sequence.*

Proof. We construct a bijection between admissible peak sets $S \subseteq [n]$ and compositions of n into odd parts.

Let $S = S_1 \cup S_2 \cup \cdots \cup S_b$ be the decomposition of an admissible peak set S, where

$$S_i = \{t_i, t_i+2, t_i+4, \ldots, t_i+2u_i\},$$

for all i. To each such block S_i, we associate the subset of $[n]$,

$$I_i = \{t_i - 1, t_i, \ldots, t_i + 2u_i, t_i + 2u_i + 1\},$$

which has odd cardinality $2u_i + 3$. Since $t_i + 2u_i < t_{i+1} - 2$ for all i, these subsets are pairwise disjoint and are labelled in increasing order of their smallest element. The remaining elements, namely

$$[n] \setminus \bigcup_{i=1}^{b} I_i,$$

will be considered as one-element sets. Thus we obtain a partition of $[n]$ into disjoint subsets, each of odd cardinality. Recording the sizes of these subsets (I_i's and one-element sets) in increasing order of their smallest element yields a composition of n into odd parts.

Conversely, given a composition of n into odd parts,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_m, \quad \text{with each } \lambda_j \text{ odd},$$

we construct an admissible peak set as follows. Partition $[n]$ into consecutive blocks of sizes $\lambda_1, \ldots, \lambda_m$. For each block of size at least three place peaks at positions

$$\{a+1,\, a+3,\, \ldots,\, a+2r-1\},$$

where a is the smallest element of the block and $2r+1 > 1$ is the number of elements of the block. This produces a set with no consecutive integers and avoiding 1 and n, hence an admissible peak set.

It is straightforward to verify that these two constructions are inverse to each other, yielding a bijection between admissible peak sets and compositions of n into odd parts.

Finally, it is well known that the number of compositions of n into odd parts is the n-th Fibonacci number F_n, which completes the proof. $\square$

The proof of the previous proposition shows that a decomposition $S = S_1 \cup S_2 \cup \cdots \cup S_b$ of an admissible set of peaks of $[n]$ corresponds to a composition of n into odd parts. We denote this composition by $\mathrm{comp}(S)$.

Example 2.1. Let $n = 20$, and let $S = \{2, 4, 6, 11, 14, 16, 18\}$. Then, $S = \{2, 4, 6\} \cup \{11\} \cup \{14, 16, 18\}$ is the decomposition of S with 3 blocks. This decomposition corresponds to the partition

$$[20] = \{1, 2, 3, 4, 5, 6, 7\} \cup \{8\} \cup \{9\} \cup \{10, 11, 12\} \cup \{13, 14, 15, 16, 17, 18, 19\} \cup \{20\}$$

and consequently to the following composition $20 = 7 + 1 + 1 + 3 + 7 + 1 \equiv \mathrm{comp}(S)$ of 20 into odd parts.

Given an admissible peak set S, we next define two permutations $e[S]$ and $w_0[S]$ in $P(S; n)$ which will be useful in computing the diameter of this set. Note that when S is the empty set, $e[\emptyset] = e = 1\,2\,\cdots\,n-1\,n$ is the identity and $w_0[\emptyset] = w_0 = n\,n-1\,\cdots\,2\,1$ is the longest element of the symmetric group.

Definition 2.4. [7] For an admissible peak set $S \subseteq \{2, 3, \ldots, n-1\}$, we define permutations $e[S]$ and $w_0[S]$ as follows:

- $e[S]$ is obtained by swapping the entries at positions k and $k+1$ in e for each $k \in S$.
- $w_0[S]$ is obtained by swapping the entries at positions $k-1$ and k in w_0 for each $k \in S$.

Since any admissible set S contains no consecutive entries, these permutations are well-defined as the order of the swaps does not matter. More explicitly, we have that for $i \in \{1, 2, \ldots, n\}$,

$$e[S]_i = \begin{cases} i+1 & \text{if } i \in S, \\ i-1 & \text{if } i-1 \in S, \\ i & \text{otherwise}, \end{cases} \quad \text{and} \quad w_0[S]_i = \begin{cases} (n+1-i)+1 & \text{if } i \in S, \\ (n+1-i)-1 & \text{if } i+1 \in S, \\ n+1-i & \text{otherwise}. \end{cases}$$

Example 2.2. Let $n = 7$, and let $S = \{2, 5\}$. Then, $e[S] = 1\,\mathbf{3}\,2\,4\,\mathbf{6}\,5\,7$ and $w_0[S] = 6\,\mathbf{7}\,5\,3\,\mathbf{4}\,2\,1$.

3. The Taxicab metric on permutations

The main goal of this section is to compute the diameter of $P(S; n)$ with respect to the taxicab metric d_1. We also characterize the peak sets S for which $\mathrm{diam}_1(P(S; n)) = \mathrm{diam}_1(\mathfrak{S}_n)$.

Given a permutation $\sigma \in \mathfrak{S}_n$, define

$$N(\sigma) = \{\tau \in \mathfrak{S}_n : d_1(\tau', \sigma) \leq d_1(\tau, \sigma), \ \text{ for all } \tau' \in \mathfrak{S}_n\}$$

to be the set of farthest neighbors of σ. The distance between σ and any of this permutations is the eccentricity of σ. By Lemma 2.2 all the permutations in $\mathfrak{S}_n$ have the same eccentricity, and that value is $\mathrm{diam}_1(\mathfrak{S}_n)$.

Proposition 3.1. *Let $n \geq 1$ be a positive integer and $\sigma \in \mathfrak{S}_n$.*

1. *For n even, $\tau \in N(\sigma)$ if and only if for all i, $\sigma_i \leq \frac{n}{2} \Leftrightarrow \tau_i > \frac{n}{2}$.*
2. *For n odd, $\tau \in N(\sigma)$ if and only if for all i, $\sigma_i \leq \frac{n+1}{2} \Leftrightarrow \tau_i \geq \frac{n+1}{2}$ or for all i, $\sigma_i < \frac{n+1}{2} \Leftrightarrow \tau_i > \frac{n+1}{2}$.*

Proof. We consider n even. The other case is analogous.

($\Rightarrow$) Suppose that there exists an index i such that $\sigma_i, \tau_i \leq \frac{n}{2}$. Then there also exists an index j such that $\sigma_j, \tau_j > \frac{n}{2}$. Define the permutation τ' by $\tau'_i = \tau_j$, $\tau'_j = \tau_i$ and $\tau'_k = \tau_k$ otherwise. We conclude that $d_1(\sigma, \tau) < d_1(\sigma, \tau')$, contradicting the fact that $\tau \in N(\sigma)$.

($\Leftarrow$) If $\tau \in \mathfrak{S}_n$ satisfies $\sigma_i \leq \frac{n}{2} \Leftrightarrow \tau_i > \frac{n}{2}$ for all i, then

$$d_1(\sigma, \tau) = \sum_{i=1}^{n} |\sigma_i - \tau_i| = 2\left(\sum_{j \in B} j - \sum_{i \in A} i\right) = \frac{n(3n+2)}{4} - \frac{n(n+2)}{4} = \frac{n^2}{2},$$

where $A = \{1, \ldots, \frac{n}{2}\}$ and $B = \{\frac{n}{2}+1, \ldots, n\}$. From the first implication we know that every permutation in $N(\sigma)$ fulfills the condition, and then this value is exactly the eccentricity of σ, which implies that $\tau \in N(\sigma)$. □

In the proof of the previous proposition we have obtained the diameter for $\mathfrak{S}_n$ for the taxicab metric. In particular, we have $diam_1(\mathfrak{S}_n) = \max\{d_1(e, \tau) \mid \tau \in \mathfrak{S}_n\} = d_1(e, w_0)$.

Proposition 3.2. *The diameter of $\mathfrak{S}_n$ for the taxicab metric is*

$$diam_1(\mathfrak{S}_n) = \begin{cases} \frac{n^2}{2} & \text{if } n \text{ is even} \\ \frac{n^2-1}{2} & \text{if } n \text{ is odd} \end{cases}.$$

The next proposition identifies the possible values of the diameter of $P(S; n)$.

Proposition 3.3. *Let S be an admissible peak set of $[n]$. Then either*

$$\mathrm{diam}_1(P(S;n)) = \mathrm{diam}_1(\mathfrak{S}_n) \qquad \text{or} \qquad \mathrm{diam}_1(P(S;n)) = \mathrm{diam}_1(\mathfrak{S}_n) - 2.$$

Proof. Since $P(S; n) \subseteq \mathfrak{S}_n$, we have $\mathrm{diam}_1(P(S;n)) \leq \mathrm{diam}_1(\mathfrak{S}_n)$. It remains to prove that $\mathrm{diam}_1(P(S;n))$ is even and

$$\mathrm{diam}_1(P(S;n)) \geq \mathrm{diam}_1(\mathfrak{S}_n) - 2.$$

Consider the permutations $e[S], w_0[S] \in P(S; n)$. By definition of diameter, we have

$$\mathrm{diam}_1(P(S;n)) \geq d(e[S], w_0[S]) = \sum_{i=1}^{n} |e[S]_i - w_0[S]_i|.$$

We compare this value with

$$\mathrm{diam}_1(\mathfrak{S}_n) = d(e, w_0) = \sum_{i=1}^{n} |i - (n-i+1)| = \sum_{i=1}^{n} |2i - n - 1|.$$

Write $f(i) := 2i - n - 1$.

Let $S = S_1 \cup S_2 \cup \cdots \cup S_b$ be the decomposition of S with b blocks, where for each i,

$$S_i = \{t_i, t_i+2, t_i+4, \ldots, t_i+2u_i\}.$$

For each such block S_i, one checks directly from the definitions of $e[S]$ and $w_0[S]$ that

$$e[S]_j - w_0[S]_j = f(j) \quad \text{for all } j \notin \{t_i - 1, t_i + 2u_i + 1\}.$$

Write $a = t_i$ and $c = t_i + 2u_i$, the first and the last peak of S_i. Thus only the indices $a-1$ and $c+1$ may contribute differently. A direct computation yields

$$e[S]_{a-1} - w_0[S]_{a-1} = f(a-1) + 1 \quad \text{and} \quad e[S]_{c+1} - w_0[S]_{c+1} = f(c+1) - 1.$$

Hence the contribution of the block S_i differs from the maximal one by

$$\Delta(B) = \big(|f(a-1)+1| - |f(a-1)|\big) + \big(|f(c+1)-1| - |f(c+1)|\big). \tag{3.1}$$

Since f is strictly increasing, we distinguish two situations:

(1) If $a-1$ and $c+1$ lie on the same side of the midpoint of $[n]$, then $\Delta(B) = 0$.
(2) If $a-1$ and $c+1$ lie on opposite sides of the midpoint, then $\Delta(B) = -2$.

Because the intervals $[a-1, c+1]$ corresponding to distinct blocks are disjoint, at most one block interval can cross the midpoint. Therefore

$$d_1(e[S], w_0[S]) = \mathrm{diam}_1(\mathfrak{S}_n) + \sum_B \Delta(B) \geq \mathrm{diam}_1(\mathfrak{S}_n) - 2.$$

Finally, for any $\sigma, \tau \in \mathfrak{S}_n$, the value $d_1(\sigma, \tau)$ is even, since

$$\sum_{i=1}^{n} |\sigma_i - \tau_i| \equiv \sum_{i=1}^{n} (\sigma_i - \tau_i) = 0 \pmod 2.$$

Hence $\mathrm{diam}_1(P(S;n))$ is even, and the only possible values in the interval

$$[\mathrm{diam}_1(\mathfrak{S}_n) - 2, \mathrm{diam}_1(\mathfrak{S}_n)]$$

are precisely $\mathrm{diam}_1(\mathfrak{S}_n)$ and $\mathrm{diam}_1(\mathfrak{S}_n) - 2$. □

Corollary 3.4. *Let S be an admissible peak set of $[n]$.*

(1) If $n = 2k+1$ is odd and ($k+1 \notin S$ or $\{k, k+2\} \not\subseteq S$) then $\mathrm{diam}_1(P(S;n)) = \mathrm{diam}_1(\mathfrak{S}_n)$.
(2) If $n = 2k$ is even and $k, k+1 \notin S$ then $\mathrm{diam}_1(P(S;n)) = \mathrm{diam}_1(\mathfrak{S}_n)$.

Proof. In both cases, we have $d_1(e[S], w_0[S]) = \mathrm{diam}_1(\mathfrak{S}_n)$ since $\Delta(B) = 0$, for all blocks of the decomposition of S, with $\Delta(B)$ as defined in equation (3.1) of Proposition 3.3. □

Note that $d_1(e[S], w_0[S])$ is not necessarily the diameter of $P(S;n)$. The diameter of $P(S;n)$ is attained in one of two possible cases, and we shall derive conditions on the peak set S that determine which case occurs.

Theorem 3.5. *Let S be an admissible peak set of $[n]$. The diameter $\mathrm{diam}_1(P(S;n)) = \mathrm{diam}_1(\mathfrak{S}_n)$ if and only if there exist two disjoint subsets $A, B \subseteq [n]$ such that $A \cup B = [n]$ and:*

(i) $0 \leq |B| - |A| \leq 1$;
(ii) *for every $s \in S$, one has*

$$s \in A \iff s-1 \in A \iff s+1 \in A;$$

(which implies that the same condition is true for the set B.)

(iii) *if $s \in S \cap A$, $i \in A$, and there is no element of S between s and i, then every integer between s and i belongs to A.*
(The same condition must hold for the set B.)

Before writing the proof of this theorem, we provide some intuition for condition (iii). The condition requires that every maximal sequence of consecutive integers between 2 and $n-1$ in A contains at least one peak.

Proof. ($\Rightarrow$) Assume first that $\operatorname{diam}_1(P(S;n)) = \operatorname{diam}_1(\mathfrak{S}_n)$. Then there exist permutations $\sigma, \tau \in P(S;n)$ such that

$$d_1(\sigma,\tau) = \operatorname{diam}_1(\mathfrak{S}_n).$$

By Proposition 3.1, this is equivalent to the condition that, for all $i \in [n]$,

$$\sigma_i \leq \frac{n}{2} \iff \tau_i > \frac{n}{2}.$$

Define

$A = \{i \in [n] : \sigma_i \leq \frac{n}{2}\} = \{i \in [n] : \tau_i > \frac{n}{2}\}$ and $B = \{i \in [n] : \sigma_i > \frac{n}{2}\} = \{i \in [n] : \tau_i \leq \frac{n}{2}\}$. Clearly, $A \cup B = [n]$ and $A \cap B = \emptyset$, and condition (i) follows immediately.

Next we show that condition (ii) holds. Let $s \in S$. Since σ has a peak at s, we have

$$\sigma_{s-1} < \sigma_s > \sigma_{s+1}.$$

If $s \in A$, then $\sigma_s \leq \frac{n}{2}$, hence

$$\sigma_{s-1}, \sigma_{s+1} < \sigma_s \leq \tfrac{n}{2},$$

which implies $s-1, s+1 \in A$. The converse implication follows symmetrically by exchanging A and B and using permutation τ instead of σ.

Finally, we prove condition (iii). Let $s \in S \cap A$ and $i \in A$, and suppose there exists j between s and i such that $j \in B$. Then

$$\sigma_j > \tfrac{n}{2} \geq \sigma_s, \sigma_i.$$

Thus, the maximum value of σ on the interval between s and i is attained at some ℓ in that interval, implying that ℓ is a peak of σ. This contradicts the assumption that no element of S lies between s and i. Hence all intermediate indices belong to A. Replacing σ by τ yields the corresponding statement for B.

($\Leftarrow$) Conversely, assume there exist disjoint subsets $A, B \subseteq [n]$ such that $A \cup B = [n]$ satisfying conditions (i)–(iii). We construct permutations $a, b \in \mathfrak{S}_n$ such that

$$a, b \in P(S;n) \quad \text{and} \quad d_1(a,b) = \operatorname{diam}_1(\mathfrak{S}_n).$$

Condition (i) implies

$$|A| = \left\lfloor \tfrac{n}{2} \right\rfloor, \qquad |B| = \left\lceil \tfrac{n}{2} \right\rceil.$$

Write

$$\begin{aligned} S \cap A &= \{s_1 < \cdots < s_k\}, \quad A \setminus S = \{u_1 < \cdots < u_m\}, \\ S \cap B &= \{t_1 < \cdots < t_\ell\}, \quad B \setminus S = \{v_1 < \cdots < v_p\}. \end{aligned}$$

Define the permutation $a = a_1 a_2 \cdots a_n$ as follows:

(1) assign to $s_i \in S \cap A$ the values

$$a_{s_i} = \left\lfloor \tfrac{n}{2} \right\rfloor - i + 1;$$

(2) assign to $u_j \in A \setminus S$ the values

$$a_{u_j} = j;$$

(3) assign to $t_i \in S \cap B$ the values

$$a_{t_i} = n - i + 1;$$

(4) assign to $v_j \in B \setminus S$ the values

$$a_{v_j} = \left\lfloor \tfrac{n}{2} \right\rfloor + p - j + 1;$$

The permutation a just defined may have a peak outside S, namely in transitions from indices in A to indices in B. To avoid this problem, we perform local adjustments:

(5) If $d-1 \in A$; $d, d+1, \ldots, d+q \in B$; $a_d = c, a_{d+1} = c-1, \ldots, a_{d+q} = c-q$ and ($d+q+1 \in S$ **or** $d+q=n$), then set $a_d := c-q, a_{d+1} := c-q+1, \ldots, a_{d+q} := c$. [This ensures that by reversing decreasing segments in B, when necessary, we do not have new peaks outside S.]

See Algorithm 1 and Example 3.1 for details on this construction.

Define b analogously by interchanging the roles of the sets A and B.

By construction,

$$a_i \leq \tfrac{n}{2} \iff b_i > \tfrac{n}{2},$$

and hence, by Proposition 3.1,

$$d_1(a,b) = \mathrm{diam}_1(\mathfrak{S}_n).$$

It remains to show that $a, b \in P(S;n)$ and, since the proof for both cases is the same, we consider only permutation a.

We start by showing that $S \subseteq \mathrm{Peak}(a)$. If $s \in S \cap A$, then by (ii), $s-1, s+1 \in A$, and the assignments ensure

$$a_{s-1} < a_s > a_{s+1}.$$

The same argument applies for $s \in S \cap B$.

Next, we show that $\mathrm{Peak}(a) \subseteq S$. If $s \in A \setminus S$, then values assigned on $A \setminus S$ are strictly increasing, and all transitions involving A preserve monotonicity, so s cannot be a peak.

If $s \in B \setminus S$, then values on $B \setminus S$ are strictly decreasing, and comparisons with neighboring elements prevent the peak condition.

The local adjustment step ensures that when transitioning from A to B the permutation is increasing until the next peak, and thus no unintended peaks are introduced.

Therefore, $\mathrm{Peak}(a) = S$, and similarly $\mathrm{Peak}(b) = S$. This completes the proof. □

Example 3.1. Let $n = 20$ and consider the set of peaks $S = \{2, 4, 10, 12, 14, 19\}$. Consider the partition of $[20]$ given by

$$A = \{1, \mathbf{2}, 3, \mathbf{4}, 5, 16, 17, 18, \mathbf{19}, 20\} \quad \text{and} \quad B = \{6, 7, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, 15\}.$$

We construct permutations $a, b \in P(S;20)$ following the procedure described in the proof of Theorem 3.5 (see also Algorithm 1).

Construction of a:

Step 1: Assign values to $S \cap A = \{2, 4, 19\}$.

$$a_2 = 10, \quad a_4 = 9, \quad a_{19} = 8.$$

Step 2: Assign values to $A \setminus S = \{1, 3, 5, 16, 17, 18, 20\}$.

$$a_1 = 1,\ a_3 = 2,\ a_5 = 3,\ a_{16} = 4,\ a_{17} = 5,\ a_{18} = 6,\ a_{20} = 7.$$

Step 3: Assign values to $S \cap B = \{10, 12, 14\}$.

$$a_{10} = 20, \quad a_{12} = 19, \quad a_{14} = 18.$$

Algorithm 1: Construction of the permutation $a \in \mathfrak{S}_n$ with peak set S

`// Initial Assignment`

for $i = 1$ **to** k **do**

  $a_{s_i} \leftarrow \lfloor n/2 \rfloor - i + 1$;

for $j = 1$ **to** m **do**

  $a_{u_j} \leftarrow j$;

for $i = 1$ **to** ℓ **do**

  $a_{t_i} \leftarrow n - i + 1$;

for $j = 1$ **to** p **do**

  $a_{v_j} \leftarrow \lfloor n/2 \rfloor + p - j + 1$;

`// Step 2: Local adjustment in set B`

if $d - 1 \in A$ ***and*** $\{d, d+1, \ldots, d+q\} \subseteq B$ **then**

  `// Check if values are decreasing`

  if $a_d = c, a_{d+1} = c - 1, \ldots, a_{d+q} = c - q$ **then**

    `// Adjustment condition: next is a peak or end of string`

    if $d + q + 1 \in S$ ***or*** $d + q = n$ **then**

      $a_d \leftarrow c - q, a_{d+1} \leftarrow c - q + 1, \ldots, a_{d+q} \leftarrow c$;

Step 4: Assign values to $B \setminus S = \{6, 7, 8, 9, 11, 13, 15\}$.

$$a_6 = 17,\ a_7 = 16,\ a_8 = 15,\ a_9 = 14,\ a_{11} = 13,\ a_{13} = 12,\ a_{15} = 11.$$

So far, the permutation a is

$$a = (1, \mathbf{10}, 2, \mathbf{9}, 3, 17, 16, 15, 14, \mathbf{20}, 13, \mathbf{19}, 12, \mathbf{18}, 11, 4, 5, 6, \mathbf{8}, 7).$$

Note that an unintended peak was created at position 6.

Step 5: Local adjustment. The block $\{6, 7, 8, 9\} \subseteq B$ initially receives a decreasing assignment

$$(17, 16, 15, 14),$$

which is reversed to avoid creating an unintended peak at position 6, yielding

$$(14, 15, 16, 17).$$

Thus, the permutation a is

$$a = (1, \mathbf{10}, 2, \mathbf{9}, 3, 14, 15, 16, 17, \mathbf{20}, 13, \mathbf{19}, 12, \mathbf{18}, 11, 4, 5, 6, \mathbf{8}, 7).$$

The permutation b is obtained by interchanging the roles of A and B (see Figure 1), yielding

$$b = (17, \mathbf{20}, 16, \mathbf{19}, 15, 1, 2, 3, 4, \mathbf{10}, 5, \mathbf{9}, 6, \mathbf{8}, 7, 12, 13, 14, \mathbf{18}, 11).$$

		△		△						△		△		△					△	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	1	10	2	9	3	14	15	16	17	20	13	19	12	18	11	4	5	6	8	7
b	17	20	16	19	15	1	2	3	4	10	5	9	6	8	7	12	13	14	18	11

FIGURE 1. Partition $[20] = A \cup B$ and permutations a, b. Construction: assign descending values in $S \cap A$ and $S \cap B$, increasing values in $A \setminus S$, decreasing values in $B \setminus S$, and apply local reversal on consecutive blocks to avoid unintended peaks.

By construction,

$$a_i \leq \tfrac{n}{2} \iff b_i > \tfrac{n}{2},$$

hence by Proposition 3.1, $d_1(a,b) = \mathrm{diam}_1(\mathfrak{S}_{20})$, with $\mathrm{Peak}(a) = \mathrm{Peak}(b) = S$.

The previous theorem characterizes the peak sets S for which $P(S;n)$ attains the maximal diameter in terms of the existence of a certain partition of $[n]$. We next relate the existence of such a partition to the existence of a particular composition of n.

Definition 3.6. A composition of n,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$$

is called *balanced* if it is possible to split its parts into two disjoint subsets A' and B' of $[m]$ such that:

$$0 \leq \sum_{i \in B'} \lambda_i - \sum_{j \in A'} \lambda_j \leq 1\,.$$

For instance, the composition $14 = 5+1+3+3+2$ is balanced, since $5+2 = 1+3+3$ with the sets $A' = \{1,5\}$ and $B' = \{2,3,4\}$ giving the balance. On the other hand, the composition $14 = 6+2+4+2$ is not balanced.

Corollary 3.7. *Let S be an admissible peak set of $[n]$. If $\mathrm{diam}_1(P(S;n)) = \mathrm{diam}_1(\mathfrak{S}_n)$, then the corresponding composition $\mathrm{comp}(S)$ is balanced.*

Proof. By Theorem 3.5, there are sets A and B which split $[n]$ into two sets. Let $[n] = I_1 \cup I_2 \cup \cdots \cup I_m$ be the partition of $[n]$ into consecutive blocks of odd size $\lambda_1, \ldots, \lambda_m$ with $n = \lambda_1 + \cdots + \lambda_m$, the composition $\mathrm{comp}(S)$, as seen in the proof of Proposition 2.3. Condition (ii) of Theorem 3.5 ensures that for all j, either $I_j \subseteq A$ or $I_j \subseteq B$. Define

$$A' = \{j \in [m] \,:\, I_j \subseteq A\} \quad \text{and} \quad B' = \{i \in [m] \,:\, I_i \subseteq B\}\,.$$

It is clear that $|A| = \sum_{j \in A'} \lambda_j$ and that $|B| = \sum_{i \in B'} \lambda_i$, and condition (i) of Theorem 3.5 ensures that $\mathrm{comp}(S)$ is balanced. $\square$

Corollary 3.8. *Let S be an admissible peak set of $[n]$ such that for all $i \in \{2, \ldots, n-1\}$, $\{i-1, i, i+1\} \cap S \neq \emptyset$, i.e. the distance between two consecutive peaks is at most three, the first peak is 2 or 3 and the last peak is $n-2$ or $n-1$.*

The corresponding composition $\mathrm{comp}(S)$ is balanced if and only if $\mathrm{diam}_1(P(S;n)) = diam_1(\mathfrak{S}_n)$.

Proof. The fact that $\mathrm{diam}_1(P(S;n)) = diam_1(\mathfrak{S}_n)$ implies that $\mathrm{comp}(S)$ is balanced is the previous corollary.

To prove the reverse implication, suppose that $\mathrm{comp}(S)$, $n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$, is balanced with sets A' and B' giving the balance. Moreover, since there are no three consecutive elements in $[n] \setminus S$, for all $j \neq 1, m, \quad \lambda_j > 1$. Let $[n] = I_1 \cup I_2 \cup \cdots \cup I_m$ be the partition of $[n]$ into consecutive blocks of odd sizes $\lambda_1, \ldots, \lambda_m$, associated with S. None of the sets I_i is a singleton, except, possibly, the first or the last one.

The sets:

$$A = \bigcup_{i \in A'} I_i \quad \text{and} \quad B = \bigcup_{j \in B'} I_j$$

fulfill conditions (i)–(iii) from Theorem 3.5.

The first condition is a direct consequence of the fact that the composition is balanced.

For every i, $I_i = \{t_i - 1, t_i, \ldots, t_i + 2u_i, t_i + 2u_i + 1\}$ and $I_i \cap S = \{t_i, t_i + 2 \ldots, t_i + 2u_i\}$, which implies that for every $s \in S$, $s \in I_i$ if and only if $s - 1 \in I_i$ if and only if $s + 1 \in I_i$. Since the sets A and B are unions of I_i's the second condition of Theorem3.5 follows.

Finally, the fact that the only singleton blocks are the ones at the endpoints trivializes condition (iii). □

Notice that the cardinal of a peak S is at most $\lceil n/2 - 1 \rceil$. We will show that, in most cases, peak sets S of maximum cardinality do not yield a maximal diameter for $P(S; n)$.

Proposition 3.9. *Let S be an admissible peak set of $[n]$ such that $|S| = \lceil n/2 - 1 \rceil$.*

(1) If $n \not\equiv 2 \pmod 4$, then $\mathrm{diam}_1(P(S; n)) = \mathrm{diam}_1(\mathfrak{S}_n) - 2$.

(2) If $n = 4k+2$, then $\mathrm{diam}_1(P(S; n)) = \mathrm{diam}_1(\mathfrak{S}_n)$ *if and only if* $S = \{2, 4, \ldots, 2k, 2k+3, 2k+5, \ldots 4k+1\}$.

Proof. *(1)* For odd n and $|S| = \frac{n-1}{2}$, the set of peaks is the set of even numbers smaller than n. Consequently, $[n]$ cannot be partitioned into two sets satisfying the conditions of Proposition 3.5, since those conditions force all elements of $[n]$ to lie in the same set (either A or B), yielding $\big||A| - |B|\big| > 1$.

(2) For even n, the maximality of $|S|$ implies that the composition of n, $\mathrm{comp}(S)$, must have exactly two parts, $n = \lambda_1 + \lambda_2$.

If $n = 4k$, then $|S| = 2k - 1$, and since λ_1, λ_2 are odd numbers, we have either $\lambda_1 \geq 2k + 1$ or $\lambda_2 \geq 2k + 1$. This means that $\mathrm{comp}(S)$ is not balanced and, by Corollary 3.7, $\mathrm{diam}_1(P(S; n))$ is not maximal.

If $n = 4k+2$, then $\mathrm{comp}(S)$ is balanced if and only if $\lambda_1 = \lambda_2 = 2k+1$. Again by Corollary 3.7, this implies that the diameter is maximal if and only if $S = \{2, 4, \ldots, 2k, 2k + 3, 2k + 5, \ldots 4k + 1\}$. □

Remark. Define the sequence $(a(n))_{n\geq 3}$ of the number of admissible peak sets whose diameter is maximal,

$$a(n) := |\{S\, :\, \mathrm{diam}_1\,(P(S; n)) = \mathrm{diam}_1(\mathfrak{S}_n)\}|.$$

An implementation of Algorithm 1 allows us to obtain the first terms of this sequence:

$$1, 1, 4, 6, 9, 13, 29, 43, 74, 114, 213, 325, 556, 871, 1517, 2376, 4003, 6387, 10673, 16993,$$
$$28067, 45050, 74140, 119087, 194617, 313907, 511562, \ldots$$

To the best of our knowledge, this sequence is not currently listed in the On-Line Encyclopedia of Integer Sequences [14].

4. Euclidean Distance

We now turn our attention to the Euclidean metric d_2. As we did for the metric d_1, our objective is to determine the diameter of $P(S; n)$ which will be computed using the decomposition of S into blocks.

The next lemma shows that maximizing the distance between σ and τ is equivalent to making the sums $\sigma_i + \tau_i$ as close as possible to $n + 1$, for every i. This relationship will be useful in deriving of a formula for the diameter of $P(S; n)$.

Lemma 4.1. *Let $\sigma, \tau \in \mathfrak{S}_n$. Then*

$$d_2(\sigma, \tau)^2 + \sum_{i=1}^{n} \big((n + 1) - (\sigma_i + \tau_i)\big)^2 = \frac{n(n^2 - 1)}{3}.$$

Proof. We will prove that

$$\sum_{i=1}^{n}(\sigma_i-\tau_i)^2+\sum_{i=1}^{n}\big((n+1)-(\sigma_i+\tau_i)\big)^2=\frac{n(n^2-1)}{3}.$$

Using the identity $(a-b)^2+\big((n+1)-(a+b)\big)^2=2(a^2+b^2)+(n+1)^2-2(n+1)(a+b)$, we may write

$$\sum_{i=1}^{n}(\sigma_i-\tau_i)^2+\sum_{i=1}^{n}\big((n+1)-(\sigma_i+\tau_i)\big)^2=\sum_{i=1}^{n}\Big(2(\sigma_i^2+\tau_i^2)+(n+1)^2-2(n+1)(\sigma_i+\tau_i)\Big).$$

Since σ and τ are permutations of $\mathfrak{S}_n$,

$$\sum_{i=1}^{n}\sigma_i=\sum_{i=1}^{n}\tau_i=\sum_{i=1}^{n}i=\frac{n(n+1)}{2},$$

and

$$\sum_{i=1}^{n}\sigma_i^2=\sum_{i=1}^{n}\tau_i^2=\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}.$$

Hence

$$\begin{aligned} d(\sigma,\tau)^2+\sum_{i=1}^{n}\big((n+1)-(\sigma_i+\tau_i)\big)^2 &= 4\sum_{i=1}^{n}i^2+n(n+1)^2-4(n+1)\sum_{i=1}^{n}i \\ &= \frac{2}{3}n(n+1)(2n+1)-(n+1)^2n=\frac{n(n^2-1)}{3}, \end{aligned}$$

which is equal to $\mathrm{diam}_2(\mathfrak{S}_n)^2$. ☐

This lemma shoes that the maximum distance in $\mathfrak{S}_n$ with respect to the metric d_2 is attained by complementary permutations. That is, $\sigma,\tau\in\mathfrak{S}_n$ satisfy

$$d_2(\sigma,\tau)=\mathrm{diam}_2(\mathfrak{S}_n),$$

if and only if $\tau_i+\sigma_i=n+1$ for all i. The diameter of $\mathfrak{S}_n$ follows directly.

Theorem 4.2. *The diameter of* $\mathfrak{S}_n$ *is*

$$\mathrm{diam}_2(\mathfrak{S}_n)=\sqrt{\frac{n(n^2-1)}{3}}.$$

Corollary 4.3. *Let S be an admissible peak set of* $[n]$. *Then,* $\mathrm{diam}_2(P(S;n))=\mathrm{diam}_2(\mathfrak{S}_n)$ *if and only if* $S=\emptyset$.

Proof. Since $e,w_0\in P(\emptyset;n)$ are complementary permutations, it follows that $\mathrm{diam}_2(P(\emptyset;n))=\mathrm{diam}_2(\mathfrak{S}_n)$.

On the other hand, if σ,τ are complementary, then $\sigma_i<\sigma_{i+1}$ if and only if $\tau_i>\tau_{i+1}$, which makes it impossible for both permutations to be in $P(S;n)$ unless S is the empty set. ☐

In contrast with the taxicab metric d_1, the last result shows that the diameter of $P(S;n)$ with respect to d_2 is maximal only when $S=\emptyset$. We will show that in general, the d_2 diameter of $P(S;n)$ depends on the number of peaks and on their relative positions.

If σ has a peak in position i, then it is clear that $2\sigma_i-(\sigma_{i-1}+\sigma_{i+1})\geq 3$. The next lemma shows that for consecutive peaks, this difference must on average be closer to 4.

Lemma 4.4. *Let S be an admissible peak set of $[n]$ and $B \subseteq S$ a block of consecutive peaks. Let $\sigma \in P(S;n)$. Then,*

$$2\left(\sum_{p\in B}\sigma_p\right) - \left(\sum_{p\in B}\sigma_{p-1}+\sigma_{p+1}\right) \geq 4|B|-1$$

Proof. Let $p \in B$. Since p is a peak of σ, we have

$$\sigma_p > \sigma_{p-1}, \qquad \sigma_p > \sigma_{p+1}.$$

Because σ is a permutation, all entries are distinct. Therefore the two neighbors of a peak cannot both equal $\sigma_p - 1$. Consequently, for every peak $p \in B$,

$$\sigma_{p-1} + \sigma_{p+1} \leq (\sigma_p - 1) + (\sigma_p - 2) = 2\sigma_p - 3.$$

That is,

$$2\sigma_p - (\sigma_{p-1} + \sigma_{p+1}) \geq 3.$$

Note that if for all peaks $p \in B$ the difference $2\sigma_p - (\sigma_{p-1} + \sigma_{p+1})$ is at least 4, the result follows.

Suppose that there exists a peak $p \in B$ such that $2\sigma_p - (\sigma_{p-1} + \sigma_{p+1}) = 3$. Then this difference must be at least 4 for the adjacent peaks. Indeed, if equality held for consecutive peaks

$$p \quad \text{and} \quad p+2,$$

then the shared intermediate position $p+1$ would satisfy both

$$\sigma_{p+1} = \sigma_p - 1 \quad \text{and} \quad \sigma_{p+1} = \sigma_{p+2} - 2 \quad \text{(or vice-versa)}.$$

But then, $\sigma_{p+3} = \sigma_{p+2} - 1 = \sigma_p$, which is impossible in a permutation.

Thus, whenever a peak p satisfies $2\sigma_p - (\sigma_{p-1} + \sigma_{p+1}) = 3$ then the adjacent peak $p+2$ must satisfy $2\sigma_{p+2} - (\sigma_{p+1} + \sigma_{p+3}) \geq 4$. Moreover, using a similar argument we can show that, if $q \in B$ is another peak whose difference is 3, then there is a peak $q' \in B$ between p and q whose difference is $2\sigma_{q'} - (\sigma_{q'-1} + \sigma_{q'+1}) \geq 5$. Thus, except for the peak p, the difference for the other peaks is on average at least 4. Therefore,

$$2\left(\sum_{p\in B}\sigma_p\right) - \left(\sum_{p\in B}\sigma_{p-1}+\sigma_{p+1}\right) \geq 4|B|-1. \qquad \square$$

Lemma 4.5. *Let $m \geq 1$ and let $a_1, \ldots, a_m, b_1, \ldots, b_{m-1}, c, d \in \mathbb{R}$ satisfy*

$$-\sum_{i=1}^{m} a_i + \sum_{i=1}^{m-1} b_i + \frac{c+d}{2} \geq 4m - 1.$$

Then

$$\sum_{i=1}^{m} a_i^2 + \sum_{i=1}^{m-1} b_i^2 + \left(c^2 + d^2\right) \geq 8m - 2.$$

Proof. Define the integer $N = 2m + 1$ and consider the vectors

$$x = (a_1, \ldots, a_m, b_1, \ldots, b_{m-1}, c, d) \quad \text{and} \quad v = (-1, \ldots, -1, 1, \ldots, 1, 1/2, 1/2)$$

in $\mathbb{R}^N$ so that the hypothesis is equivalent to the inner product inequality

$$\langle x, v\rangle \geq 4m - 1.$$

Note also that $\|v\|^2 = 2m - \frac{1}{2}$. By the Cauchy-Schwarz inequality applied to x and v, we get

$$\langle x, v\rangle^2 \leq \|x\|^2 \cdot \|v\|^2.$$

Therefore,

$$\|x\|^2 \geq \frac{\langle x, v\rangle^2}{\|v\|^2} \geq \frac{(4m-1)^2}{2m-1/2} = 8m - 2.$$

Expanding $\|x\|^2$ gives precisely

$$\sum_{i=1}^{m} a_i^2 + \sum_{i=1}^{m-1} b_i^2 + \left(c^2 + d^2\right) \geq 8m - 2. \qquad \square$$

Theorem 4.6. *Let S be an admissible peak set of $[n]$. Then*

$$\mathrm{diam}_2(P(S;n))^2 = \mathrm{diam}_2(\mathfrak{S}_n)^2 - 8|S| + 2b,$$

where b is the number of blocks.

Proof. We first prove that for every $\sigma, \tau \in P(S;n)$, $d_2(\sigma,\tau)^2 \leq \mathrm{diam}_2(\mathfrak{S}_n)^2 - 8|S| + 2b$. We start by assuming that S has only one block $B = \{p, p+2, \ldots, q\}$, where $q = p + 2k$ for some $k \geq 0$. In this case, we need to prove that

$$d_2(\sigma,\tau)^2 \leq \mathrm{diam}_2(\mathfrak{S}_n)^2 - 8|B| + 2. \tag{4.1}$$

Since by Lemma 4.1 we have

$$d_2(\sigma,\tau)^2 = \mathrm{diam}_2(\mathfrak{S}_n)^2 - \sum_{i=1}^{n} \left(n + 1 - (\sigma_i + \tau_i)\right)^2,$$

the inequality (4.1) is equivalent to $\displaystyle\sum_{i=1}^{n} \left(n + 1 - (\sigma_i + \tau_i)\right)^2 \geq 8|B| - 2$.

Note that by Lemma 4.4, with $y_i := \sigma_i + \tau_i$, we have

$$2\sum_{j\in B} y_j - \sum_{j\in B}(y_{j-1} + y_{j+1}) \geq 2(4|B| - 1).$$

Equivalently, since the terms in positions $p-1$ and $q+1$ are the only ones not counted twice, we may write this last equation as

$$2\sum_{j\in B} y_j - 2\sum_{j\in B\setminus\{q\}} y_{j+1} - (y_{p-1} + y_{q+1}) \geq 8|B| - 2.$$

Dividing both sides by 2, we get

$$\sum_{j\in B} y_j - \sum_{j\in B\setminus\{q\}} y_{j+1} - \frac{y_{p-1} + y_{q+1}}{2} \geq 4|B| - 1. \tag{4.2}$$

Now, putting $x_i := (n+1) - y_i$ and replacing in (4.2), we get

$$\sum_{j\in B}(n + 1 - x_j) - \sum_{j\in B\setminus\{q\}} (n + 1 - x_{j+1}) - \frac{2(n+1) - (x_{p-1} + x_{q+1})}{2} \geq 4|B| - 1.$$

From here we get

$$-\sum_{j\in B} x_j + \sum_{j\in B\setminus\{q\}} x_{j+1} + \frac{x_{p-1} + x_{q+1}}{2} \geq 4|B| - 1. \tag{4.3}$$

Applying Lemma 4.5 to the inequality (4.3), we obtain

$$\sum_{j\in B} x_j^2 + \sum_{j\in B\setminus\{q\}} x_{j+1}^2 + \left(x_{p-1}^2 + x_{q+1}^2\right) \geq 8|B| - 2.$$

Since the indices B, $\{j+1 : j \in B \setminus \{q\}\}$ and $\{p-1, q+1\}$ are pairwise disjoint, we have

$$\sum_{i=1}^{n} x_i^2 \geq \sum_{j\in B} x_j^2 + \sum_{j\in B\setminus\{q\}} x_{j+1}^2 + \left(x_{p-1}^2 + x_{q+1}^2\right).$$

Therefore,

$$\sum_{i=1}^{n} \left(n+1-(\sigma_i+\tau_i)\right)^2 = \sum_{i=1}^{n} x_i^2 \geq 8|B| - 2.$$

For the general case, where $S = B_1 \cup \cdots \cup B_b$ has b blocks, we have that

$$\begin{aligned}\sum_{i=1}^{n} \left(n+1-(\sigma_i+\tau_i)\right)^2 \geq& \sum_{k=1}^{b} (8|B_k| - 2)\\ =& 8|S| - 2b.\end{aligned}$$

We then conclude that $d_2(\sigma,\tau)^2 \leq \mathrm{diam}_2(\mathfrak{S}_n)^2 - 8|S| + 2b$.

To show that the diameter $\mathrm{diam}_2\left(P(S;n)\right)^2 = \mathrm{diam}_2(\mathfrak{S}_n)^2 - 8|S| + 2b$, we must find two permutations $\sigma, \tau \in P(S;n)$ such that $d_2(\sigma,\tau)^2 = \mathrm{diam}_2(\mathfrak{S}_n)^2 - 8|S| + 2b$. Indeed, for every admissible peak set S, the d_2 distance of the permutations $e[S]$ and $w_0[S]$ is precisely $\mathrm{diam}_2(\mathfrak{S}_n)^2 - 8|S| + 2b$ as we will see:

$$\begin{cases} e[S]_i + w_0[S]_i = (i+1) + (n+1-i) + 1 = n+3, & \text{if } i \in S \\ e[S]_i + w_0[S]_i = (i-1) + (n+1-i) - 1 = n-1, & \text{if } i-1, i+1 \in S \\ e[S]_i + w_0[S]_i = (i-1) + (n+1-i) = n, & \text{if } i-1 \in S \text{ and } i+1 \notin S \\ e[S]_i + w_0[S]_i = i + (n+1-i) - 1 = n, & \text{if } i+1 \in S \text{ and } i-1 \notin S \\ e[S]_i + w_0[S]_i = i + (n+1-i) = n+1, & \text{otherwise} \end{cases}$$

Note that in the third equation, the index $i-1$ is the last index of a block, and in the fourth equation $i+1$ is the first index of a block. It follows that

$$\sum_{i=1}^{n} \left(n+1-(e[S]_i + w_0[S]_i)\right)^2 = 4[S] + 4(|S| - b) + 2b = 8|S| - 2b,$$

proving that $\mathrm{diam}_2\left(P(S;n)\right)^2 = \mathrm{diam}_2(\mathfrak{S}_n)^2 - 8|S| + 2b$. □

Example 4.1. Let $n = 20$ and consider the peak set $S = \{2,4\} \cup \{10,12,14\} \cup \{19\}$ with $b = 3$ blocks, used in Example 3.1. Then,

$$\mathrm{diam}_2\left(P(S;n)\right)^2 = \frac{20(20^2-1)}{3} - 8 \cdot 6 + 2 \cdot 3 = 2618.$$

Notice that for some admissible peak sets S, it is possible to find a pair of permutations $\{\sigma,\tau\} \neq \{e[S], w_0[S]\}$ such that $d_2(\sigma,\tau) = \mathrm{diam}_2(P(S;n))$. For instance, for $S = \{4\} \subseteq [7]$, the permutations $\sigma = (3,4,6,\mathbf{7},5,2,1)$ and $\tau = (5,4,1,\mathbf{3},2,6,7)$ have distance $d_2(\sigma,\tau)^2 = 106 = \mathrm{diam}_2(P(S;7))^2$.

Corollary 4.7. *Let S be an admissible peak set of $[n]$ of maximum cardinality.*

(1) If n is odd, then $\operatorname{diam}_2(P(S;n))^2 = \dfrac{n^3-13n+18}{3}$.

(2) If n is even, then

$$\operatorname{diam}_2(P(S;n))^2 = \frac{n^3-13n+30}{3} \quad \textit{or} \quad \operatorname{diam}_2(P(S;n))^2 = \frac{n^3-13n+36}{3}.$$

Proof. The result follows from the previous theorem, noticing that when n is odd an admissible peak set of maximum cardinality S has cardinal $|S| = (n-1)/2$ and only one block, and for n even, the peak set S has cardinal $|S| = n/2 - 1$ with one or two blocks. □

5. Diameters for sets of sign permutations

In this section, we extend our results to signed permutations, that is, to the Coxeter group of type B. Let $\mathfrak{S}_n^B$ denote the Coxeter group of type B_n, consisting of all bijections

$$\sigma : \{\pm 1, \pm 2, \ldots, \pm n\} \longrightarrow \{\pm 1, \pm 2, \ldots, \pm n\}$$

satisfying $\sigma(-i) = -\sigma(i)$ for every $i \in [n]$. As usual, we write signed permutations in one-line notation as $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, where $\sigma_i = \sigma(i)$. A signed permutation $\sigma \in \mathfrak{S}_n^B$ is said to have a peak at position $i \in \{2, \ldots, n-1\}$ whenever $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$. The set of all peaks of σ is denoted by $\operatorname{Peak}(\sigma)$.

For a set $S \subseteq \{2, \ldots, n-1\}$, we define $P^B(S;n) = \{\sigma \in \mathfrak{S}_n^B : \operatorname{Peak}(\sigma) = S\}$. As in the case of ordinary permutations, the set S is called an *admissible peak set* whenever $P^B(S;n) \neq \emptyset$. The notions of admissible peak set and peak classes $P^B(S;n)$ are therefore completely analogous to those introduced for the symmetric group $\mathfrak{S}_n$. In particular, every admissible peak set for $\mathfrak{S}_n$ is also admissible for $\mathfrak{S}_n^B$.

The one-line representation gives a natural injection from $\mathfrak{S}_n^B$ into $\mathbb{R}^n$. Viewing $\mathfrak{S}_n$ and $\mathfrak{S}_n^B$ as subspaces of $\mathbb{R}^n$, we can see $\mathfrak{S}_n$ as a subset of $\mathfrak{S}_n^B$, and consequently $P(S;n)$ as a subset of $P^B(S;n)$.

Given two sign permutations σ and τ in $\mathfrak{S}_n^B$ they are at maximal distance in respect to the d_1 metric if for all $i \in [n]$, $\sigma_i > 0$ if and only if $\tau_i < 0$. Using the same reasoning as in Lemma 3.1 and the fact that σ_i and τ_i have opposite signs, we conclude that the diameter of $\mathfrak{S}_n^B$ with respect to d_1 is

$$\sum_{i=1}^{n} |\sigma_i - \tau_i| = \sum_{i=1}^{n} |\sigma_i| + \sum_{i=1}^{n} |\tau_i| = 2\frac{n(n+1)}{2} = n^2 + n.$$

For an admissible peak set S, there is always a permutation $\sigma \in P(S;n) \subseteq P^B(S;n)$. Similarly, there is a permutation $\tau \in P^B(S;n)$ such that $\tau_i < 0$ for all $i \in [n]$. From these facts we can conclude the following result.

Theorem 5.1. *Let S be an admissible peak set of $[n]$. Then,* $\operatorname{diam}_1(P^B(S;n)) = \operatorname{diam}_1(\mathfrak{S}_n^B)$.

We turn now our attention to the d_2 metric. To compute the diameter of $\mathfrak{S}_n^B$, we first state a lemma similar to Lemma 4.1. Using the identity $(a-b)^2 + (a+b)^2 = 2a^2 + 2b^2$, it is straightforward to conclude the next result.

Lemma 5.2. *Let $\sigma, \tau \in \mathfrak{S}_n^B$. Then*

$$d_2(\sigma,\tau)^2 + \sum_{i=1}^{n} (\sigma_i + \tau_i)^2 = \frac{2}{3} n(n+1)(2n+1).$$

It follows that the square of diameter of $\mathfrak{S}_n^B$ with respect to d_2 is attained when $\tau_i = -\sigma_i$ for every $i \in [n]$, that is:

$$\sum_{i=1}^{n} (\sigma_i - \tau_i)^2 = \sum_{i=1}^{n} (2\sigma_i)^2 = \frac{2}{3} n(n+1)(2n+1).$$

Using the again the Lemmas 5.2, 4.4 and 4.5, we obtain a explicitly formula for the diameter of $P^B(S;n)$.

Theorem 5.3. *Let S be an admissible peak set of $[n]$. Then*

$$\operatorname{diam}_2(P^B(S;n))^2 = \operatorname{diam}_2(\mathfrak{S}_n^B)^2 - 8|S| + 2b,$$

where b is the number of blocks.

The proof that $\operatorname{diam}_2(P^B(S;n))^2 \leq \operatorname{diam}_2(\mathfrak{S}_n^B)^2 - 8|S| + 2b$, follows closely the proof of Theorem 4.6. Given an admissible peak set S, there are always at least two pairs of sign permutations at maximal distance relatively to the d_2 metric, showing that the remaining inequality is true. One such pair is $e[S]$ and $\overline{e}[S]$, where $e[S]$ is define as in $\mathfrak{S}_n$ and $\overline{e}[S]$ is obtained from $\overline{e} = \overline{1}\,\overline{2} \cdots \overline{n}$ as follows:

- $\overline{e}[S]$ is obtained by swapping the entries at positions $k-1$ and k in $\overline{e}$ for each $k \in S$.

The other pair is constructed similarly using the longest element w_0^B of $\mathfrak{S}_n^B$ instead of the identity permutation (see [1]).

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